Tracking of inputs, states and parameters of linear structural dynamic systems^{☆,☆☆}

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Abstract

This paper presents a novel algorithm for joint input-state-parameter estimation in structural dynamics. The algorithm is derived from an existing smoothing algorithm. In each time step, the system model adopted in the joint input-state-parameter estimation is linearized around the current state, yielding an algorithm similar to the extended Kalman filter. It is shown that adopting a time delay in the estimation can significantly reduce the estimation error, especially in case data originates from sensors that are not collocated with the estimated inputs. Analytical expressions for the sensitivities of the system matrices with respect to unknown parameters are derived for the case of a linear underlying state-space model. These sensitivities are derived for models expressed in physical coordinates, models expressed in modal coordinates, and modally reduced-order models with a quasi-static correction to account for the contribution of the out-of-band modes. The proposed methodology is verified using numerical simulations and validated using data obtained from a laboratory experiment on a steel beam with I-shaped cross section.

Keywords: structural health monitoring, joint input-state-parameter estimation, modally reduced order model, data fusion

1. Introduction

Condition monitoring of civil engineering structures often involves the tracking of unknown states, inputs, and parameters. Many methods that allow for the identification of one or more of these quantities of interest have been developed. Most of these methods, however, require some prior knowledge on the inputs or parameters, in order to be able to estimate the remaining quantities. For example, many system identification methods assume a white noise model for the forces applied to the structure [1]. State and input estimation methods on the other hand generally assume the system parameters to be known [2, 3, 4]. In cases where these assumptions are violated or when no prior knowledge is available, the system states, inputs, and parameters should be jointly

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estimated from the dynamic response of the system. The problem of joint input-state-parameter estimation has been recently addressed in the literature. Naets et al. [5] propose to augment the state vector with the unknown inputs and parameters. The states of the augmented system are then determined by application of the extended Kalman filter (EKF). In order to make the filtering algorithm applicable for online applications, a parametric model order reduction technique (pMOR) is applied, where the sensitivities of the system matrices with respect to the unknown parameters are computed and stored prior to the joint input-state-parameter estimation. Azam et al. [6] augment the state vector with the unknown parameters and use an unscented version of the dual Kalman filter (DKF) proposed in [3] to estimate the states, parameters, and inputs. The algorithms proposed by Naets et al. [5] and Azam et al. [6] require the selection of a regularization matrix/parameter, which in most cases is not trivial.

In order to overcome the selection of a regularization parameter, Wan et al. [7] have recently proposed an extension of the so-called Gillijns and De Moor filter (GDF) [8]. After augmenting the state vector with the unknown system parameters and linearizing the system around the current state and input values, the state, input, and parameter estimation is performed in a minimum variance and unbiased way. Two numerical examples show the ability of the algorithm to track the inputs, states, and parameters of a linear system over time. The authors conclude, however, that collocated acceleration measurements, i.e. measurements that coincide with the unknown inputs, are required for accurate input estimation. An alternative technique for joint input-state estimation for non-linear systems and linear systems with unknown parameters was recently proposed by Song in [9], where an unscented minimum variance unbiased estimation scheme was proposed.

This paper presents a novel algorithm for joint input-state-parameter estimation. The algorithm is obtained by extending the smoothing algorithm that was recently proposed by Maes and Lombaert in [10]. Similar to the case of joint input-state estimation, adopting a time-delay in the estimation (so-called smoothing) allows to significantly reduce the estimation uncertainty introduced by measurement noise when all data originate from sensors that are not collocated with the estimated inputs. The estimation does not require the choice of a regularization parameter/matrix. The presented algorithm also extends the joint input-state-parameter estimation algorithm proposed in [7]. When the number of time steps L adopted in the estimation equals 0, both algorithms are equivalent. The presented algorithm can be applied to any problem of forced vibration, including a system excited by imposed base motion as considered in earthquake engineering applications.

After the derivation of the joint input-state-parameter estimation algorithm, analytical expressions for the required sensitivities of the system matrices with respect to the unknown system parameters are derived for the case of a linear underlying state-space model. The expressions are provided for models expressed in physical coordinates, models expressed in modal coordinates, and modally reduced-order models with a quasi-static correction to account for the contribution of out-of-band modes. In the three cases, it is assumed that a finite element model, describing the dynamic behavior of the structure, is available. The proposed methodology is verified using numerical simulations. Finally, a validation is performed using data obtained from a laboratory experiment on a steel beam with I-shaped cross section. .

The outline of the paper is as follows. Section 2 introduces the discrete-time state-space system model description that is adopted in the joint input-state-parameter estimation. Next, Section 3 presents the extension of the smoothing algorithm. Section 4 derives the analytical sensitivities adopted in the estimation. Section 5 presents a verification based on numerical simulation. Section 6 provides a validation of the proposed methodology using data obtained from a laboratory

experiment. Finally, Section 7 concludes the work.

2. System model

Consider the following linear continuous-time combined deterministic-stochastic state-space description of a system:

$$\dot{\mathbf{x}}_t(t) = \mathbf{A}_c(\boldsymbol{\theta}(t))\mathbf{x}_t(t) + \mathbf{B}_c(\boldsymbol{\theta}(t))\mathbf{p}(t) + \mathbf{w}_t(t) \quad (1)$$

$$\mathbf{d}(t) = \mathbf{G}_c(\boldsymbol{\theta}(t))\mathbf{x}_t(t) + \mathbf{J}_c(\boldsymbol{\theta}(t))\mathbf{p}(t) + \mathbf{v}(t) \quad (2)$$

where $\mathbf{x}_t \in \mathbb{R}^{n_s}$ is the dynamic state vector, $\mathbf{p} \in \mathbb{R}^{n_p}$ is the vector of unknown inputs, $\boldsymbol{\theta} \in \mathbb{R}^{n_\theta}$ is the vector of unknown parameters, and $\mathbf{d} \in \mathbb{R}^{n_d}$ is the output vector. The vectors $\mathbf{w}_t \in \mathbb{R}^{n_s}$ and $\mathbf{v} \in \mathbb{R}^{n_d}$ represent the process noise and measurement noise, respectively.

After time discretization, the following discrete-time combined deterministic-stochastic state-space description is obtained:

$$\mathbf{x}_{t[k+1]} = \mathbf{A}_t(\boldsymbol{\theta}_{[k]})\mathbf{x}_{t[k]} + \mathbf{B}_t(\boldsymbol{\theta}_{[k]})\mathbf{p}_{[k]} + \mathbf{w}_{t[k]} \quad (3)$$

$$\mathbf{d}_{[k]} = \mathbf{G}_t(\boldsymbol{\theta}_{[k]})\mathbf{x}_{t[k]} + \mathbf{J}_t(\boldsymbol{\theta}_{[k]})\mathbf{p}_{[k]} + \mathbf{v}_{[k]} \quad (4)$$

where $\mathbf{x}_{t[k]} = \mathbf{x}_t(k\Delta t)$, $\mathbf{p}_{[k]} = \mathbf{p}(k\Delta t)$, $\mathbf{w}_{t[k]} = \mathbf{w}_t(k\Delta t)$, $\mathbf{d}_{[k]} = \mathbf{d}(k\Delta t)$, and $\mathbf{v}_{[k]} = \mathbf{v}(k\Delta t)$, with $k = 1, \dots, N-1$, with Δt the sampling time step, and N the total number of samples. The matrices $\mathbf{A}_t$, $\mathbf{B}_t$, $\mathbf{G}_t$, and $\mathbf{J}_t$ are the discrete-time system matrices. The noise processes $\mathbf{w}_{t[k]}$ and $\mathbf{v}_{[k]}$ are assumed to be zero mean and white, with covariance matrices $\mathbf{Q}_t$, $\mathbf{R}$, and $\mathbf{S}_t$ defined by:

$$\mathbb{E} \left\{ \begin{pmatrix} \mathbf{w}_{t[k]} \\ \mathbf{v}_{[k]} \end{pmatrix} \begin{pmatrix} \mathbf{w}_{t[l]}^T & \mathbf{v}_{[l]}^T \end{pmatrix} \right\} = \begin{bmatrix} \mathbf{Q}_t & \mathbf{S}_t \\ \mathbf{S}_t^T & \mathbf{R} \end{bmatrix} \delta_{[k-l]} \quad (5)$$

with $\mathbf{R} > 0$, $\begin{bmatrix} \mathbf{Q}_t & \mathbf{S}_t \\ \mathbf{S}_t^T & \mathbf{R} \end{bmatrix} \geq 0$, and $\delta_{[k]} = 1$ for $k = 0$ and 0 otherwise.

In the remainder of this paper, a first-order hold (FOH) discretization scheme is assumed. The authors prefer the use of the FOH time discretization scheme over the commonly used zero-order hold (ZOH) discretization scheme, since the FOH scheme generally leads to a more accurate time discretization of the system. Note, however, that the FOH discretization scheme requires a larger computational effort than the ZOH scheme. The expressions for the case of the ZOH time discretization scheme are not provided here, but can be readily obtained from the expressions given for the case of FOH.

In this case of FOH time discretization, the discrete-time system matrices are derived from the continuous-time system matrices as follows [11]:

$$\mathbf{A}_t = \exp(\mathbf{A}_c \Delta t) \quad (6)$$

$$\mathbf{B}_t = \frac{1}{\Delta t} \mathbf{A}_c^{-2} (\mathbf{A}_t - \mathbf{I}_{n_s})^2 \mathbf{B}_c \quad (7)$$

$$\mathbf{G}_t = \mathbf{G}_c \quad (8)$$

$$\mathbf{J}_t = \mathbf{J}_c + \mathbf{G}_t \left(\frac{1}{\Delta t} \mathbf{A}_c^{-2} (\mathbf{A}_t - \mathbf{I}_{n_s}) - \mathbf{A}_c^{-1} \right) \mathbf{B}_c \quad (9)$$

In the following, a random walk model for the system parameters $\boldsymbol{\theta}$ is assumed, i.e. $\boldsymbol{\theta}_{[k+1]} = \boldsymbol{\theta}_{[k]} + \mathbf{w}_{\theta[k]}$, where $\mathbf{w}_{\theta[k]} \in \mathbb{R}^{n_t}$ is a zero mean and white process with covariance matrix $\mathbf{Q}_{\theta}$, defined by:

$$\mathbb{E} \left\{ \mathbf{w}_{\theta[k]} \mathbf{w}_{\theta[l]}^T \right\} = \mathbf{Q}_{\theta} \delta_{[k-l]} \quad (10)$$

The assumption of a random walk model for the system parameters allows rewriting the system (3–4) in its augmented form:

$$\mathbf{x}_{[k+1]} = \mathbf{f}(\mathbf{x}_{[k]}, \mathbf{p}_{[k]}) + \mathbf{w}_{[k]} \quad (11)$$

$$\mathbf{d}_{[k]} = \mathbf{h}(\mathbf{x}_{[k]}) + \mathbf{J}(\boldsymbol{\theta}_{[k]})\mathbf{p}_{[k]} + \mathbf{v}_{[k]} \quad (12)$$

with

$$\mathbf{f}(\mathbf{x}_{[k]}, \mathbf{p}_{[k]}) = \mathbf{A}(\boldsymbol{\theta}_{[k]})\mathbf{x}_{[k]} + \mathbf{B}(\boldsymbol{\theta}_{[k]})\mathbf{p}_{[k]} \quad (13)$$

$$\mathbf{h}(\mathbf{x}_{[k]}) = \mathbf{G}(\boldsymbol{\theta}_{[k]})\mathbf{x}_{[k]} \quad (14)$$

where $\mathbf{x}_{[k]} = \begin{bmatrix} \mathbf{x}_{t[k]}^T & \boldsymbol{\theta}_{[k]}^T \end{bmatrix}^T$, $\mathbf{w}_{[k]} = \begin{bmatrix} \mathbf{w}_{t[k]}^T & \mathbf{w}_{\theta[k]}^T \end{bmatrix}^T$, and the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{G}$, and $\mathbf{J}$ are given by:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n_t} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_t \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = [\mathbf{G}_t \quad \mathbf{0}], \quad \mathbf{J} = \mathbf{J}_t \quad (15)$$

In order to distinguish between the representations of the system in equations (3–4) and (11–12), these are referred to in the following as the underlying dynamic system and its augmented form, respectively. It should be pointed out that even when the underlying dynamic system is linear, its augmented form will generally be non-linear due to the potential products between the augmented states.

3. Joint input-state-parameter estimation

3.1. Smoothing algorithm

The system under consideration is described by Eqs. (3) and (4). Consider a vector $\mathbf{d}_{L[k]} \in \mathbb{R}^{(L+1)n_d}$ that contains the response $\mathbf{d}_{[k]}$ over $L + 1$ consecutive time steps ($L \geq 0$):

$$\mathbf{d}_{L[k]} \triangleq \begin{bmatrix} \mathbf{d}_{[k]} \\ \mathbf{d}_{[k+1]} \\ \vdots \\ \mathbf{d}_{[k+L]} \end{bmatrix} \quad (16)$$

Similar definitions are used for the input vector $\mathbf{p}_{L[k]}$ and the noise vectors $\mathbf{w}_{L[k]}$ and $\mathbf{v}_{L[k]}$. Under the assumption that the unknown system parameters are constant over the time interval $[k\Delta t, (k+L)\Delta t]$, it is readily obtained from Eqs. (11) – (14) and (16) that:

$$\mathbf{d}_{L[k]} = \mathbf{h}_L(\mathbf{x}_{[k]}) + \mathcal{H}_L(\boldsymbol{\theta}_{[k]})\mathbf{p}_{L[k]} + \mathcal{N}_L(\boldsymbol{\theta}_{[k]})\mathbf{w}_{L-1[k]} + \mathbf{v}_{L[k]} \quad (17)$$

with $\mathbf{h}_L(\mathbf{x}_{[k]}) = \mathcal{O}_L(\boldsymbol{\theta}_{[k]})\mathbf{x}_{[k]}$, where $\mathcal{O}_L \in \mathbb{R}^{(L+1)n_d \times n_s}$ and $\mathcal{H}_L \in \mathbb{R}^{(L+1)n_d \times (L+1)n_p}$ denote the extended observability matrix and the Toeplitz matrix, respectively, defined by:

$$\mathcal{O}_L \triangleq \begin{bmatrix} \mathbf{G} \\ \mathbf{GA} \\ \mathbf{GA}^2 \\ \vdots \\ \mathbf{GA}^L \end{bmatrix}, \quad \mathcal{H}_L \triangleq \begin{bmatrix} \mathbf{J} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{GB} & \mathbf{J} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{GAB} & \mathbf{GB} & \mathbf{J} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{GA}^{L-1}\mathbf{B} & \mathbf{GA}^{L-2}\mathbf{B} & \mathbf{GA}^{L-3}\mathbf{B} & \dots & \mathbf{J} \end{bmatrix} \quad (18)$$

The matrix $\mathcal{N}_L$ is defined recursively as follows:

$$\mathcal{N}_0 \triangleq \mathbf{0}, \quad \mathcal{N}_1 \triangleq \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}, \quad \mathcal{N}_k \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathcal{O}_{k-1} & \mathcal{N}_{k-1} \end{bmatrix} \quad (k \geq 2) \quad (19)$$

where $\mathcal{N}_0 \in \mathbb{R}^{n_d \times n_s}$ and $\mathcal{N}_k \in \mathbb{R}^{(k+1)n_d \times kn_s}$ for $k > 0$. The vector $\mathbf{w}_{L-1[k]}$ is defined in accordance to Eq. (16), with $\mathbf{w}_{-1[k]} \triangleq \mathbf{w}_{[k]}$.

The process noise vector $\mathbf{w}_{[k]}$ and measurement noise vector $\mathbf{v}_{[k]}$ are assumed to be zero mean and white, with known covariance matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$, where:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_\theta \end{bmatrix} \text{ and } \mathbf{S} = \begin{bmatrix} \mathbf{S}_t \\ \mathbf{0} \end{bmatrix} \quad (20)$$

Joint input-state-parameter estimation consists of estimating the forces $\mathbf{p}_{[k]}$ and the augmented states $\mathbf{x}_{[k]}$, from a set of response measurements $\mathbf{d}_{L[k]}$. A state estimate $\hat{\mathbf{x}}_{[k|l]}$ is defined as an estimate of $\mathbf{x}_{[k]}$, given the output sequence $\mathbf{d}_{[n]}$, with $n = 0, 1, \dots, l$. An input estimate $\hat{\mathbf{p}}_{[k|l]}$ is defined similarly.

Consider the following two-step recursive smoother:

$$\hat{\mathbf{p}}_{[k|k+L]} = \mathbf{M}_{L[k]} (\mathbf{d}_{L[k]} - \mathbf{h}_L(\hat{\mathbf{x}}_{[k|k+L-1]})) \quad (21)$$

$$\hat{\mathbf{x}}_{[k+1|k+L]} = \mathbf{f}(\hat{\mathbf{x}}_{[k|k+L-1]}, \mathbf{0}) + \mathbf{K}_{L[k]} (\mathbf{d}_{L[k]} - \mathbf{h}_L(\hat{\mathbf{x}}_{[k|k+L-1]})) \quad (22)$$

The gain matrices $\mathbf{M}_{L[k]}$ and $\mathbf{K}_{L[k]}$ are determined such that both the input estimates $\hat{\mathbf{p}}_{[k|k+L]}$ and the state estimates $\hat{\mathbf{x}}_{[k+1|k+L]}$ are minimum variance and unbiased (MVU), hereby approximating the state equation (11) and output equation (17) using a first order Taylor series expansion. The derivation of the algorithm is similar to the derivation of the extended Kalman filter (EKF) [12], where a similar linearization is applied.

The first step in Eq. (21), hereafter referred to as the “input estimation step”, yields an estimate of the unknown input vector $\mathbf{p}_{[k]}$, given the measured output up to time step $k + L$. The second step in Eq. (22), referred to as the “state estimation step”, yields an estimate of the state vector $\mathbf{x}_{[k+1]}$, given the measured output up to time step $k + L$.

The algorithm presented here is an extension of the smoothing algorithm presented by two of the authors in [10] to non-linear systems. In case of a linear system description, the algorithm presented herein boils down to the algorithm presented in [10], as indicated in the following. In addition, if the time delay L considered in the estimation equals 0, the algorithm is equivalent to the filtering algorithm presented in [7], with the exception that the algorithm presented in [7] assumes the process noise $\mathbf{w}_{[k]}$ and measurement noise $\mathbf{v}_{[k]}$ to be uncorrelated, i.e. $\mathbf{S} = \mathbf{0}$.

The first non-linearity in the state estimation step is dealt with by using a first order Taylor series expansion of the state equation (11), which is evaluated at the mean estimate of the state and input vectors $\hat{\mathbf{x}}_{[k|k+L-1]}$ and $\hat{\mathbf{p}}_{[k|k+L-1]}$, respectively:

$$\mathbf{x}_{[k+1]} \approx \mathbf{f}(\hat{\mathbf{x}}_{[k|k+L-1]}, \hat{\mathbf{p}}_{[k|k+L-1]}) + \bar{\mathbf{A}}_{[k]}(\mathbf{x}_{[k]} - \hat{\mathbf{x}}_{[k|k+L-1]}) + \bar{\mathbf{B}}_{[k]}(\mathbf{p}_{[k]} - \hat{\mathbf{p}}_{[k|k+L-1]}) + \mathbf{w}_{[k]} \quad (23)$$

where $\bar{\mathbf{A}}_{[k]}$ and $\bar{\mathbf{B}}_{[k]}$ are the time-dependent Jacobian matrices of the state equation:

$$\bar{\mathbf{A}}_{[k]} \triangleq \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_{[k|k+L-1]}, \hat{\mathbf{p}}_{[k|k+L-1]}}, \quad \bar{\mathbf{B}}_{[k]} \triangleq \left. \frac{\partial \mathbf{f}}{\partial \mathbf{p}} \right|_{\hat{\mathbf{x}}_{[k|k+L-1]}, \hat{\mathbf{p}}_{[k|k+L-1]}} \quad (24)$$

In addition, it is easily verified that:

$$\mathbf{f}(\hat{\mathbf{x}}_{[k|k+L-1]}, \hat{\mathbf{p}}_{[k|k+L-1]}) = \mathbf{f}(\hat{\mathbf{x}}_{[k|k+L-1]}, \mathbf{0}) + \bar{\mathbf{B}}_{[k]} \hat{\mathbf{p}}_{[k|k+L-1]} \quad (25)$$

leading to:

$$\mathbf{x}_{[k+1]} \approx \mathbf{f}(\hat{\mathbf{x}}_{[k|k+L-1]}, \mathbf{0}) + \bar{\mathbf{A}}_{[k]}(\mathbf{x}_{[k]} - \hat{\mathbf{x}}_{[k|k+L-1]}) + \bar{\mathbf{B}}_{[k]} \mathbf{p}_{[k]} + \mathbf{w}_{[k]} \quad (26)$$

The non-linearity in the output equation (17) is similarly dealt with, by adopting a first order Taylor series expansion that is evaluated at the mean estimate of the state vector $\hat{\mathbf{x}}_{[k|k+L-1]}$:

$$\begin{aligned} \mathbf{d}_{L[k]} &\approx \mathbf{h}_L(\hat{\mathbf{x}}_{[k|k+L-1]}) + \bar{\mathbf{O}}_{L[k]}(\mathbf{x}_{[k]} - \hat{\mathbf{x}}_{[k|k+L-1]}) + \mathcal{H}_L(\hat{\boldsymbol{\theta}}_{[k|k+L-1]}) \mathbf{p}_{L[k]} \dots \\ &\dots + \mathcal{N}_L(\hat{\boldsymbol{\theta}}_{[k|k+L-1]}) \mathbf{w}_{L-1[k]} + \mathbf{v}_{L[k]} \end{aligned} \quad (27)$$

where $\bar{\mathbf{O}}_{L[k]}$ is the time-dependent Jacobian matrix of the output equation:

$$\bar{\mathbf{O}}_{L[k]} \triangleq \left. \frac{\partial \mathbf{h}_L}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_{[k|k+L-1]}} \quad (28)$$

Combination of Eqs. (21) and (27) yields the error on the input estimate $\hat{\mathbf{p}}_{[k|k+L]}$, denoted by $\tilde{\mathbf{p}}_{[k|k+L]}$:

$$\begin{aligned} \tilde{\mathbf{p}}_{[k|k+L]} &\triangleq \mathbf{p}_{[k]} - \hat{\mathbf{p}}_{[k|k+L]} \\ &= (\check{\mathbf{I}}_{n_p} - \mathbf{M}_{L[k]} \mathcal{H}_{L[k]}) \mathbf{p}_{L[k]} - \mathbf{M}_{L[k]} (\bar{\mathbf{O}}_{L[k]} \tilde{\mathbf{x}}_{[k|k+L-1]} + \mathcal{N}_{L[k]} \mathbf{w}_{L-1[k]} + \mathbf{v}_{L[k]}) \end{aligned} \quad (29)$$

with $\check{\mathbf{I}}_{n_p} \triangleq [\mathbf{I}_{n_p} \quad \mathbf{0} \quad \dots \quad \mathbf{0}] \in \mathbb{R}^{n_p \times (L+1)n_p}$, where $\mathbf{I}_{n_p} \in \mathbb{R}^{n_p \times n_p}$ denotes an identity matrix, $\mathcal{H}_{L[k]} = \mathcal{H}_L(\hat{\boldsymbol{\theta}}_{[k|k+L-1]})$, and $\mathcal{N}_{L[k]} = \mathcal{N}_L(\hat{\boldsymbol{\theta}}_{[k|k+L-1]})$.

Similarly, combination of Eqs. (22), (26) and (17) yields the error on the state estimate $\hat{\mathbf{x}}_{[k+1|k+L]}$, denoted by $\tilde{\mathbf{x}}_{[k+1|k+L]}$:

$$\begin{aligned} \tilde{\mathbf{x}}_{[k+1|k+L]} &\triangleq \mathbf{x}_{[k+1]} - \hat{\mathbf{x}}_{[k+1|k+L]} \\ &= (\bar{\mathbf{A}}_{[k]} - \mathbf{K}_{L[k]} \bar{\mathbf{O}}_{L[k]}) \tilde{\mathbf{x}}_{[k|k+L-1]} + (\check{\mathbf{B}}_{[k]} - \mathbf{K}_{L[k]} \mathcal{H}_{L[k]}) \mathbf{p}_{L[k]} \dots \\ &\dots + (\check{\mathbf{I}}_{n_s} - \mathbf{K}_{L[k]} \mathcal{N}_{L[k]}) \mathbf{w}_{L-1[k]} - \mathbf{K}_{L[k]} \mathbf{v}_{L[k]} \end{aligned} \quad (30)$$

with $\check{\mathbf{B}}_{[k]} \triangleq [\bar{\mathbf{B}}_{[k]} \quad \mathbf{0} \quad \dots \quad \mathbf{0}] \in \mathbb{R}^{n_s \times (L+1)n_p}$ and $\check{\mathbf{I}}_{n_s} \triangleq [\mathbf{I}_{n_s} \quad \mathbf{0} \quad \dots \quad \mathbf{0}] \in \mathbb{R}^{n_s \times Ln_s}$ for $L > 0$ and $\check{\mathbf{I}}_{n_s} \triangleq \mathbf{I}_{n_s}$ for $L = 0$, with $\mathbf{I}_{n_s} \in \mathbb{R}^{n_s \times n_s}$ an identity matrix.

The expressions for the gain matrices $\mathbf{M}_{L[k]}$ and $\mathbf{K}_{L[k]}$ as well as the error covariance matrices $\mathbf{P}_{p[k|k+L]}$ and $\mathbf{P}_{x[k+1|k+L]}$ are immediately obtained from the expressions for the error on the input and state estimates in Eqs. (29) and (30), following the derivation provided in [10]. Moreover, the expressions for the error on the input and state estimates provided in [13] after replacing the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{O}_L$, $\mathbf{H}_L$, and $\mathbf{N}_L$ with their time-dependent equivalents $\bar{\mathbf{A}}_{[k]}$, $\bar{\mathbf{B}}_{[k]}$, $\bar{\mathbf{O}}_{L[k]}$, $\mathbf{H}_{L[k]}$, and $\mathbf{N}_{L[k]}$, respectively, are identical to those obtained in Eqs. (29) and (30). As a result, the expressions for the gain matrices and error covariance matrices presented in [13] can be straightforwardly adjusted to the presented case of joint input-state-parameter estimation. A summary of the resulting smoothing algorithm is provided in the following.

The smoothing algorithm is initialized using an initial state estimate vector $\hat{\mathbf{x}}_{[0|L-1]}$ and its error covariance matrix $\mathbf{P}_{x[0|L-1]}$. The estimate $\hat{\mathbf{x}}_{[0|L-1]}$ is assumed unbiased and independent of the noise processes $\mathbf{w}_{[k]}$ and $\mathbf{v}_{[k]}$ for all k . The algorithm proceeds by computing the force and state estimates recursively in two steps, i.e. the input estimation step and state estimation step:

Input estimation

$$\mathbf{P}_{xw[k]} = \sum_{i=1}^{\min(k,L)} \left(\left(\prod_{j=1}^{i-1} \bar{\mathbf{A}}_{[k-j]} - \mathbf{K}_{L[k-j]} \bar{\mathbf{O}}_{L[k-j]} \right) \left((\check{\mathbf{I}}_{n_s} - \mathbf{K}_{L[k-i]} \mathbf{N}_{L[k-i]}) \mathbf{Q}_L^i - \mathbf{K}_{L[k-i]} \mathbf{S}_L^{-i\top} \right) \right) \quad (31)$$

$$\mathbf{P}_{xv[k]} = \sum_{i=1}^{\min(k,L)} \left(\left(\prod_{j=1}^{i-1} \bar{\mathbf{A}}_{[k-j]} - \mathbf{K}_{L[k-j]} \bar{\mathbf{O}}_{L[k-j]} \right) \left((\check{\mathbf{I}}_{n_s} - \mathbf{K}_{L[k-i]} \mathbf{N}_{L[k-i]}) \mathbf{S}_L^i - \mathbf{K}_{L[k-i]} \mathbf{R}_{L+1}^i \right) \right) \quad (32)$$

$$\begin{aligned} \bar{\mathbf{R}}_{[k]} = & \bar{\mathbf{O}}_{L[k]} \mathbf{P}_{x[k|k+L-1]} \bar{\mathbf{O}}_{L[k]}^\top + \mathbf{R}_{L+1}^0 + \mathbf{N}_{L[k]} \mathbf{Q}_L^0 \mathbf{N}_{L[k]}^\top + \mathbf{N}_{L[k]} \mathbf{S}_L^0 + \mathbf{S}_L^{0\top} \mathbf{N}_{L[k]}^\top \dots \\ & \dots + \bar{\mathbf{O}}_{L[k]} \mathbf{P}_{xw[k]} \mathbf{N}_{L[k]}^\top + \mathbf{N}_{L[k]} \mathbf{P}_{xw[k]}^\top \bar{\mathbf{O}}_{L[k]}^\top + \bar{\mathbf{O}}_{L[k]} \mathbf{P}_{xv[k]} + \mathbf{P}_{xv[k]}^\top \bar{\mathbf{O}}_{L[k]}^\top \end{aligned} \quad (33)$$

$$\mathbf{M}_{L[k]} = \check{\mathbf{I}}_{n_p} \left(\mathbf{H}_{L[k]}^\top \bar{\mathbf{R}}_{[k]}^{-1} \mathbf{H}_{L[k]} \right)^\dagger \mathbf{H}_{L[k]}^\top \bar{\mathbf{R}}_{[k]}^{-1} \quad (34)$$

$$\hat{\mathbf{p}}_{[k|k+L]} = \mathbf{M}_{L[k]} (\mathbf{d}_{L[k]} - \mathbf{h}_L(\hat{\mathbf{x}}_{[k|k+L-1]})) \quad (35)$$

$$\mathbf{P}_{p[k|k+L]} = \mathbf{M}_{L[k]} \bar{\mathbf{R}}_{[k]} \mathbf{M}_{L[k]}^\top \quad (36)$$

State estimation

$$\begin{aligned}\bar{\mathbf{S}}_{[k]} &= \bar{\mathbf{A}}_{[k]} \mathbf{P}_{\mathbf{x}[k|k+L-1]} \bar{\mathbf{O}}_{L[k]}^T + \check{\mathbf{I}}_{n_s} \mathbf{Q}_L^0 \mathcal{N}_{L[k]}^T + \check{\mathbf{I}}_{n_s} \mathbf{S}_L^0 + \bar{\mathbf{A}}_{[k]} \mathbf{P}_{\mathbf{xw}[k]} \mathcal{N}_{L[k]}^T \dots \\ &\dots + \check{\mathbf{I}}_{n_s} \mathbf{P}_{\mathbf{xw}[k]}^T \bar{\mathbf{O}}_{L[k]}^T + \bar{\mathbf{A}}_{[k]} \mathbf{P}_{\mathbf{xv}[k]}\end{aligned}\quad (37)$$

$$\bar{\mathbf{T}}_{[k]} = \bar{\mathbf{A}}_{[k]} \mathbf{P}_{\mathbf{x}[k|k+L-1]} \bar{\mathbf{A}}_{[k]}^T + \mathbf{Q} + \bar{\mathbf{A}}_{[k]} \mathbf{P}_{\mathbf{xw}[k]} \check{\mathbf{I}}_{n_s}^T + \check{\mathbf{I}}_{n_s} \mathbf{P}_{\mathbf{xw}[k]}^T \bar{\mathbf{A}}_{[k]}^T \quad (38)$$

$$\mathbf{K}_{L[k]} = \bar{\mathbf{S}}_{[k]} \bar{\mathbf{R}}_{[k]}^{-1} - \left(\bar{\mathbf{S}}_{[k]} \bar{\mathbf{R}}_{[k]}^{-1} \mathcal{H}_{L[k]} - \check{\mathbf{B}}_{[k]} \right) \left(\mathcal{H}_{L[k]}^T \bar{\mathbf{R}}_{[k]}^{-1} \mathcal{H}_{L[k]} \right)^\dagger \mathcal{H}_{L[k]}^T \bar{\mathbf{R}}_{[k]}^{-1} \quad (39)$$

$$\hat{\mathbf{x}}_{[k+1|k+L]} = \mathbf{f}(\hat{\mathbf{x}}_{[k|k+L-1]}, \mathbf{0}) + \mathbf{K}_{L[k]} (\mathbf{d}_{L[k]} - \mathbf{h}_L(\hat{\mathbf{x}}_{[k|k+L-1]})) \quad (40)$$

$$\mathbf{P}_{\mathbf{x}[k+1|k+L]} = \mathbf{K}_{L[k]} \bar{\mathbf{R}}_{[k]} \mathbf{K}_{L[k]}^T - \mathbf{K}_{L[k]} \bar{\mathbf{S}}_{[k]}^T - \bar{\mathbf{S}}_{[k]} \mathbf{K}_{L[k]}^T + \bar{\mathbf{T}}_{[k]} \quad (41)$$

where † denotes the Moore Penrose pseudo inverse of a matrix, $\sum_{i=1}^j (\cdot) \triangleq \mathbf{0}$ for $i > j$, and $\prod_{i=1}^j (\cdot) \triangleq \mathbf{I}$ for $i > j$. The extended noise covariance matrices $\mathbf{Q}_L^i \in \mathbb{R}^{Ln_s \times Ln_s}$, $\mathbf{R}_{L+1}^i \in \mathbb{R}^{(L+1)n_d \times (L+1)n_d}$ and $\mathbf{S}_L^i \in \mathbb{R}^{Ln_s \times (L+1)n_d}$ are defined as:

$$\begin{aligned}\mathbf{Q}_L^i &\triangleq \mathbb{E} \left[\mathbf{w}_{L-1[k-i]} \mathbf{w}_{L-1[k]}^T \right] = \text{diag}_L^i(\mathbf{Q}, \mathbf{Q}, \dots, \mathbf{Q}) \\ \mathbf{R}_{L+1}^i &\triangleq \mathbb{E} \left[\mathbf{v}_{L[k-i]} \mathbf{v}_{L[k]}^T \right] = \text{diag}_{L+1}^i(\mathbf{R}, \mathbf{R}, \dots, \mathbf{R}) \\ \mathbf{S}_L^i &\triangleq \mathbb{E} \left[\mathbf{w}_{L-1[k-i]} \mathbf{v}_{L[k]}^T \right] = [\text{diag}_L^i(\mathbf{S}, \mathbf{S}, \dots, \mathbf{S}), \mathbf{0}] \quad \text{for } i \geq 0 \text{ and } L > 0 \\ &= [\mathbf{0}, \text{diag}_L^{i+1}(\mathbf{S}, \mathbf{S}, \dots, \mathbf{S})] \quad \text{for } i < 0 \text{ and } L > 0 \\ &= \mathbf{S} \quad \text{for } i = 0 \text{ and } L = 0\end{aligned}\quad (42)$$

where $\text{diag}_j^i(\cdot)$ denotes a block matrix with the j matrices between the brackets on the i -th block diagonal below the main diagonal and zeros elsewhere.

Although the focus of this paper is on linear structures with unknown parameters, one could also apply the presented smoothing algorithm in the more general case of non-linear structures. The application, however, is limited to analytical (or smooth) systems, where the state-space equations (1) and (2) are continuous, i.e., no singular points exist. This is a necessary condition for the linearization of the system as adopted in this work.

3.2. Observability considerations

In the selection of the sensor layout, one should consider the observability of the system [2]. This is a theoretical property of the system that determines whether or not the unknown quantities (states, inputs, and parameters) can be estimated given a specific set of measurements. If a state, an input, or a parameter is unobservable, it cannot be expected that any algorithm would be successful in properly estimating their value.

Whereas the observability of time-invariant linear systems is well understood [2, 14], the concept of observability of nonlinear systems and, therefore, the design of a sensor network for joint input-state-parameter estimation, remains a challenge. A linearized sensitivity analysis of the problem was proposed by Naets et al. in [5]. Firstly, it was concluded that the number of measurements n_d cannot be smaller than the sum of the number of unknown inputs n_p and the number of unknown parameters n_θ , i.e., $n_d \geq n_p + n_\theta$. Secondly, it was found that the joint input-state-parameter estimation cannot be performed based on acceleration data only. Displacement and/or strain data

is required, in order to provide information on the static force components and, as such, stabilize the identification. This finding is in line with what was found by Maes et al. [14] for the case of joint input-state estimation. Furthermore, the authors have recently developed an analytical method to investigate the observability of nonlinear systems in presence of unmeasured inputs, which can also be applied to the case of joint input-state-parameter estimation [15].

3.3. Asymptotic behavior of the smoothing algorithm

An in depth study on the asymptotic behavior of the EKF is provided by Ljung in [16]. It is shown that the EKF may diverge, even if the parameter estimates are close to their true values. In addition, it is found that a bias of the parameter estimates is inevitable in the case where the assumed noise characteristics, i.e. the noise covariance matrices, do not represent the actual noise statistics. Although the authors did not encounter instability issues for any of the examples discussed next, the reader should be aware that instabilities and bias errors might also cause problems for the application for the presented algorithm. An in depth study on the stability of the algorithm and a possible bias of the input, state, and parameter estimates, however, lies outside the scope of this paper.

3.4. Steady-state initialization

In [10], a steady-state initialization of the recursive smoothing algorithm is proposed, which consists of a pre-computation of the gain matrices, allowing to significantly reduce the calculation time in the case of a linear state-space model. In the considered case of a nonlinear state-space model, however, the gain matrices cannot be assumed constant in time, since the system description is time-variant. Nevertheless, the authors suggest to perform a steady-state initialization of the algorithm, hereby assuming the initial values of the parameters $\hat{\boldsymbol{\theta}}_{[0|L-1]}$. This avoids divergence of the algorithm in the first time steps, where the gain matrices are subject to large changes. After the steady-state initialization, small variations in the system parameters cause the gain matrices to change. Under the assumption that the variation in system parameters is reasonably slow, however, it can be assumed that the gain matrices at any time are close to their optimal value.

The steady-state initialization starts from the user-defined value of the error covariance matrix $\mathbf{P}_{x[0|L-1]}$. In this sense, it is clear that the selected value affects the initialization phase. When the algorithm is stable, however, the gain matrices at the end of the steady-state initialization are independent from the value assigned to $\mathbf{P}_{x[0|L-1]}$. The selected value only determines the number of iterations in the initialization phase before the steady-state is reached. As such, the choice of $\mathbf{P}_{x[0|L-1]}$ does not affect the final outcome of the estimation. In the following numerical and experimental studies, the covariance matrix $\mathbf{P}_{x[0|L-1]}$ is always assigned zero values.

4. Computation of the Jacobian matrices

The analytical expressions for the Jacobian matrices $\bar{\mathbf{A}}_{[k]}$, $\bar{\mathbf{B}}_{[k]}$, and $\bar{\mathbf{O}}_{L[k]}$ are obtained by application of the chain rule, starting from their definition in Eqs. (24) and (28). This section provides an overview of the different steps in the derivation. The explicit dependence of the nonlinear system functions and system matrices on the states and parameters is omitted to shorten the notation.

4.1. General framework

This subsection lists the first steps in the calculation of the Jacobian matrices that are generally applicable when dealing with a linear underlying dynamic system.

The first order partial derivatives of the nonlinear state and output functions with respect to the states and inputs are given by:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \mathbf{A} + \begin{bmatrix} \mathbf{0}_{n_s \times n_{st}} & \frac{\partial \mathbf{f}'}{\partial \boldsymbol{\theta}} \end{bmatrix} \quad (43)$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{p}} = \mathbf{B} \quad (44)$$

$$\frac{\partial \mathbf{h}_L}{\partial \mathbf{x}} = \boldsymbol{\mathcal{O}}_L + \begin{bmatrix} \mathbf{0}_{(L+1)n_d \times n_{st}} & \frac{\partial \mathbf{h}'_L}{\partial \boldsymbol{\theta}} \end{bmatrix} \quad (45)$$

with

$$\frac{\partial \mathbf{f}'}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial \mathbf{f}'}{\partial \theta_1} & \cdots & \frac{\partial \mathbf{f}'}{\partial \theta_{n_t}} \end{bmatrix} \quad (46)$$

$$\frac{\partial \mathbf{h}'_L}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial \mathbf{h}'_L}{\partial \theta_1} & \cdots & \frac{\partial \mathbf{h}'_L}{\partial \theta_{n_t}} \end{bmatrix} \quad (47)$$

and

$$\frac{\partial \mathbf{f}'}{\partial \theta_l} = \frac{\partial \mathbf{A}}{\partial \theta_l} \mathbf{x}_{[k]} + \frac{\partial \mathbf{B}}{\partial \theta_l} \mathbf{p}_{[k]} \quad (48)$$

$$\frac{\partial \mathbf{h}'_L}{\partial \theta_l} = \frac{\partial \boldsymbol{\mathcal{O}}_L}{\partial \theta_l} \mathbf{x}_{[k]} \quad (49)$$

The partial derivatives of the system matrices $\mathbf{A}$ and $\mathbf{B}$ and the extended observability matrix $\boldsymbol{\mathcal{O}}_L$ with respect to a parameter θ_l are given by:

$$\frac{\partial \mathbf{A}}{\partial \theta_l} = \begin{bmatrix} \frac{\partial \mathbf{A}_t}{\partial \theta_l} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (50)$$

$$\frac{\partial \mathbf{B}}{\partial \theta_l} = \begin{bmatrix} \frac{\partial \mathbf{B}_t}{\partial \theta_l} \\ \mathbf{0} \end{bmatrix} \quad (51)$$

$$\frac{\partial \boldsymbol{\mathcal{O}}_L}{\partial \theta_l} = \begin{bmatrix} \frac{\partial \mathbf{G}}{\partial \theta_l} \\ \frac{\partial \mathbf{G}}{\partial \theta_l} \mathbf{A} + \mathbf{G} \frac{\partial \mathbf{A}}{\partial \theta_l} \\ \vdots \\ \frac{\partial \mathbf{G}}{\partial \theta_l} \mathbf{A}^L + \mathbf{G} \sum_{q=1}^L \mathbf{A}^{q-1} \frac{\partial \mathbf{A}}{\partial \theta_l} \mathbf{A}^{L-q} \end{bmatrix} \quad (52)$$

with

$$\frac{\partial \mathbf{G}}{\partial \theta_l} = \begin{bmatrix} \frac{\partial \mathbf{G}_t}{\partial \theta_l} & \mathbf{0} \end{bmatrix} \quad (53)$$

The partial derivatives of the system matrices $\mathbf{A}_t$, $\mathbf{B}_t$ and $\mathbf{G}_t$ with respect to a parameter θ_l

are given by:

$$\frac{\partial \mathbf{A}_t}{\partial \theta_l} = \frac{\partial}{\partial \theta_l} (\exp(\mathbf{A}_c \Delta t)) \quad (54)$$

$$\begin{aligned} \frac{\partial \mathbf{B}_t}{\partial \theta_l} = & -\frac{1}{\Delta t} \left(\mathbf{A}_c^{-1} \frac{\partial \mathbf{A}_c}{\partial \theta_l} \mathbf{A}_c^{-2} + \mathbf{A}_c^{-2} \frac{\partial \mathbf{A}_c}{\partial \theta_l} \mathbf{A}_c^{-1} \right) (\mathbf{A}_t - \mathbf{I}_{n_s})^2 \mathbf{B}_c \dots \\ & \dots + \frac{1}{\Delta t} \mathbf{A}_c^{-2} \frac{\partial \mathbf{A}_t}{\partial \theta_l} (\mathbf{A}_t - \mathbf{I}_{n_s}) \mathbf{B}_c + \frac{1}{\Delta t} \mathbf{A}_c^{-2} (\mathbf{A}_t - \mathbf{I}_{n_s}) \frac{\partial \mathbf{A}_t}{\partial \theta_l} \mathbf{B}_c \dots \\ & \dots + \frac{1}{\Delta t} \mathbf{A}_c^{-2} (\mathbf{A}_t - \mathbf{I}_{n_s})^2 \frac{\partial \mathbf{B}_c}{\partial \theta_l} \end{aligned} \quad (55)$$

$$\frac{\partial \mathbf{G}_t}{\partial \theta_l} = \frac{\partial \mathbf{G}_c}{\partial \theta_l} \quad (56)$$

The partial derivatives of the matrix exponential occurring in Eq. (54) cannot be straightforwardly computed using the chain rule, unless the matrices $\mathbf{A}_c$ and $\frac{\partial \mathbf{A}_c}{\partial \theta_l}$ commute. This is generally not the case. Following [17], the first order partial derivative of the matrix exponential is obtained from the following expression:

$$\begin{bmatrix} \exp(\mathbf{A}_c \Delta t) & \frac{\partial}{\partial \theta_l} (\exp(\mathbf{A}_c \Delta t)) \\ \mathbf{0} & \exp(\mathbf{A}_c \Delta t) \end{bmatrix} = \exp \left(\begin{bmatrix} \mathbf{A}_c & \frac{\partial \mathbf{A}_c}{\partial \theta_l} \\ \mathbf{0} & \mathbf{A}_c \end{bmatrix} \Delta t \right) \quad (57)$$

The partial derivatives of the system matrices $\mathbf{A}_c$, $\mathbf{B}_c$ and $\mathbf{G}_c$ with respect to a parameter θ_l depend on the type of model that is assumed in the estimation. In the following subsections, a distinction is made between a (full-order) model expressed in physical coordinates (Section 4.2), a model expressed in modal coordinates (Section 4.3), and a modally reduced order model with quasi-static correction (Section 4.4).

4.2. Models expressed in physical coordinates

In the case where a full order model expressed in physical coordinates is used, the state vector $\mathbf{x}_t$ contains the nodal displacements and velocities corresponding to all nodes in the model. The matrices $\mathbf{A}_c$, $\mathbf{B}_c$, $\mathbf{G}_c$, and $\mathbf{J}_c$ in the case of viscous damping are given by:

$$\mathbf{A}_c = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1} \mathbf{K} & -\mathbf{M}^{-1} \mathbf{C}_v \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \mathbf{S}_p \end{bmatrix} \quad (58)$$

$$\mathbf{G}_c = [\mathbf{S}_{d,d} - \mathbf{S}_{d,a} \mathbf{M}^{-1} \mathbf{K} \quad \mathbf{S}_{d,v} - \mathbf{S}_{d,a} \mathbf{M}^{-1} \mathbf{C}_v], \quad \mathbf{J}_c = \mathbf{S}_{d,a} \mathbf{M}^{-1} \mathbf{S}_p \quad (59)$$

where $\mathbf{M} \in \mathbb{R}^{n_{\text{dof}} \times n_{\text{dof}}}$, $\mathbf{C}_v \in \mathbb{R}^{n_{\text{dof}} \times n_{\text{dof}}}$, and $\mathbf{K} \in \mathbb{R}^{n_{\text{dof}} \times n_{\text{dof}}}$ are respectively the mass matrix, the viscous damping matrix, and the stiffness matrix of the structure, obtained from the finite element (FE) model, with n_{dof} the number of degrees of freedom in the model. The matrix $\mathbf{S}_p \in \mathbb{R}^{n_{\text{dof}} \times n_p}$ is a selection matrix specifying the force locations. The matrices $\mathbf{S}_{d,a} \in \mathbb{R}^{n_d \times n_{\text{dof}}}$, $\mathbf{S}_{d,v} \in \mathbb{R}^{n_d \times n_{\text{dof}}}$, and $\mathbf{S}_{d,d} \in \mathbb{R}^{n_d \times n_{\text{dof}}}$ are selection matrices indicating the degrees of freedom corresponding to the acceleration, velocity and displacement or strain measurements, respectively.

The following expressions are obtained for the partial derivatives of the system matrices $\mathbf{A}_t$,

$\mathbf{B}_t$ and $\mathbf{G}_t$ with respect to a parameter θ_l :

$$\frac{\partial \mathbf{A}_c}{\partial \theta_l} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{-1} \frac{\partial \mathbf{M}}{\partial \theta_l} \mathbf{M}^{-1} \mathbf{K} - \mathbf{M}^{-1} \frac{\partial \mathbf{K}}{\partial \theta_l} & \mathbf{M}^{-1} \frac{\partial \mathbf{M}}{\partial \theta_l} \mathbf{M}^{-1} \mathbf{C}_v - \mathbf{M}^{-1} \frac{\partial \mathbf{C}_v}{\partial \theta_l} \end{bmatrix} \quad (60)$$

$$\frac{\partial \mathbf{B}_c}{\partial \theta_l} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}^{-1} \frac{\partial \mathbf{M}}{\partial \theta_l} \mathbf{M}^{-1} \mathbf{S}_p \end{bmatrix} \quad (61)$$

$$\frac{\partial \mathbf{G}_c}{\partial \theta_l} = \begin{bmatrix} \mathbf{S}_{d,a} \mathbf{M}^{-1} \frac{\partial \mathbf{M}}{\partial \theta_l} \mathbf{M}^{-1} \mathbf{K} - \mathbf{S}_{d,a} \mathbf{M}^{-1} \frac{\partial \mathbf{K}}{\partial \theta_l} & \mathbf{S}_{d,a} \mathbf{M}^{-1} \frac{\partial \mathbf{M}}{\partial \theta_l} \mathbf{M}^{-1} \mathbf{C}_v - \mathbf{S}_{d,a} \mathbf{M}^{-1} \frac{\partial \mathbf{C}_v}{\partial \theta_l} \end{bmatrix} \quad (62)$$

Under the assumption that all parameters θ are physical quantities related to mass, damping, and stiffness, the derivatives $\frac{\partial \mathbf{M}}{\partial \theta_l}$, $\frac{\partial \mathbf{C}_v}{\partial \theta_l}$, and $\frac{\partial \mathbf{K}}{\partial \theta_l}$ can be computed straightforwardly.

4.3. Models expressed in modal coordinates

In the case where a modal model is used instead of a model expressed in physical coordinates, the state vector $\mathbf{x}_t$ contains the modal displacements and velocities corresponding to all modes included in the model and the matrices $\mathbf{A}_c$, $\mathbf{B}_c$, $\mathbf{G}_c$, and $\mathbf{J}_c$ are given by:

$$\mathbf{A}_c = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Omega}^2 & -\mathbf{C}_m \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} \mathbf{0} \\ \mathbf{\Phi}^T \mathbf{S}_p \end{bmatrix} \quad (63)$$

$$\mathbf{G}_c = [\mathbf{S}_{d,d} \mathbf{\Phi} - \mathbf{S}_{d,a} \mathbf{\Phi} \mathbf{\Omega}^2 \quad \mathbf{S}_{d,v} \mathbf{\Phi} - \mathbf{S}_{d,a} \mathbf{\Phi} \mathbf{C}_m], \quad \mathbf{J}_c = \mathbf{S}_{d,a} \mathbf{\Phi} \mathbf{\Phi}^T \mathbf{S}_p \quad (64)$$

where $\mathbf{\Omega} \in \mathbb{R}^{n_m \times n_m}$ is a diagonal matrix containing the terms ω_j on its diagonal, with ω_j the natural frequency corresponding to mode j . The scalar n_m represents the number of modes accounted for in the system model, which equals n_{dof} in case of a full order model. $\mathbf{\Phi} \in \mathbb{R}^{n_{\text{dof}} \times n_m}$ is a matrix containing the mass normalized mode shapes ϕ_j as columns. Finally, $\mathbf{C}_m \in \mathbb{R}^{n_m \times n_m}$ is the modal damping matrix, given by $\mathbf{C}_m = \mathbf{\Phi}^T \mathbf{C}_v \mathbf{\Phi}$. In the case of proportional damping, $\mathbf{C}_m = \mathbf{\Gamma}$, where $\mathbf{\Gamma}$ is a diagonal matrix containing the terms $2\xi_j \omega_j$ on its diagonal, with ξ_j the modal damping ratio corresponding to mode j .

The following expressions are obtained for the partial derivatives of the system matrices $\mathbf{A}_c$, $\mathbf{B}_c$ and $\mathbf{G}_c$ with respect to a parameter θ_l :

$$\frac{\partial \mathbf{A}_c}{\partial \theta_l} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -2\mathbf{\Omega} \frac{\partial \mathbf{\Omega}}{\partial \theta_l} & -\frac{\partial \mathbf{C}_m}{\partial \theta_l} \end{bmatrix} \quad (65)$$

$$\frac{\partial \mathbf{B}_c}{\partial \theta_l} = \begin{bmatrix} \mathbf{0} \\ \frac{\partial \mathbf{\Phi}^T}{\partial \theta_l} \mathbf{S}_p \end{bmatrix} \quad (66)$$

$$\frac{\partial \mathbf{G}_c}{\partial \theta_l} = \left[\mathbf{S}_{d,d} \frac{\partial \mathbf{\Phi}}{\partial \theta_l} - \mathbf{S}_{d,a} \frac{\partial \mathbf{\Phi}}{\partial \theta_l} \mathbf{\Omega}^2 - 2\mathbf{S}_{d,a} \mathbf{\Phi} \mathbf{\Omega} \frac{\partial \mathbf{\Omega}}{\partial \theta_l} \quad \mathbf{S}_{d,v} \frac{\partial \mathbf{\Phi}}{\partial \theta_l} - \mathbf{S}_{d,a} \frac{\partial \mathbf{\Phi}}{\partial \theta_l} \mathbf{C}_m - \mathbf{S}_{d,a} \mathbf{\Phi} \frac{\partial \mathbf{C}_m}{\partial \theta_l} \right] \quad (67)$$

Analytical expressions for the modal sensitivities, i.e. the first partial derivative of the natural frequencies ω_j and the mode shape vectors ϕ_j with respect to the parameters, are obtained by application of Nelson's adjoint method [18, 19]. The modal sensitivities are given by:

$$\frac{\partial \omega_j}{\partial \theta_l} = \frac{1}{2\omega_j} \phi_j^T \left(\frac{\partial \mathbf{K}}{\partial \theta_l} - \omega_j^2 \frac{\partial \mathbf{M}}{\partial \theta_l} \right) \phi_j \quad (68)$$

$$\frac{\partial \phi_j}{\partial \theta_l} = (\mathbf{I} - \phi_j \phi_j^T \mathbf{M}) \mathbf{D}_j^{*-1} \mathbf{F}_{j,l}^* - \frac{1}{2} \phi_j \phi_j^T \frac{\partial \mathbf{M}}{\partial \theta_l} \phi_j \quad (69)$$

where $\mathbf{D}_j^*$ is used to denote the modified matrix derived from the matrix $\mathbf{D}_j$ in Eq. (70) by replacing the elements of the r -th column and the r -th row by zeros and the (r, r) -th element by 1, where r denotes the element of the mode shape vector ϕ_j with the highest absolute value. Similarly, $\mathbf{F}_{j,l}^*$ is used to denote the modified vector derived from the vector $\mathbf{F}_{j,l}$ in Eq. (71) by replacing the r -th element by 0.

$$\mathbf{D}_j = \mathbf{K} - \omega_j^2 \mathbf{M} \quad (70)$$

$$\mathbf{F}_{j,l} = -(\mathbf{I} - \mathbf{M}\phi_j\phi_j^T) \left(\frac{\partial \mathbf{K}}{\partial \theta_l} - \omega_j^2 \frac{\partial \mathbf{M}}{\partial \theta_l} \right) \phi_j \quad (71)$$

The derivative of the modal damping matrix $\mathbf{C}_m$ with respect to a parameter θ_l is obtained as:

$$\frac{\partial \mathbf{C}_m}{\partial \theta_l} = \frac{\partial \Phi}{\partial \theta_l}^T \mathbf{C}_v \Phi + \Phi^T \frac{\partial \mathbf{C}_v}{\partial \theta_l} \Phi + \Phi^T \mathbf{C}_v \frac{\partial \Phi}{\partial \theta_l} \quad (72)$$

In the case of proportional damping, one obtains the following expression:

$$\frac{\partial \mathbf{C}_m}{\partial \theta_l} = \frac{\partial \Gamma}{\partial \theta_l} = 2\Xi \frac{\partial \Omega}{\partial \theta_l} + 2\Omega \frac{\partial \Xi}{\partial \theta_l} \quad (73)$$

where $\Xi \in \mathbb{R}^{n_m \times n_m}$ is a diagonal matrix containing the modal damping ratios ξ_j on its diagonal.

4.4. Modally reduced order models with quasi-static correction

For most civil and mechanical engineering structures, it is very hard or even impossible to develop a model that accurately describes the dynamic behavior in a broad frequency range. For example, a finite element model is typically developed to obtain an accurate representation of the dynamic system behavior in a frequency range with finite bandwidth. For linear structures, it is computationally much more efficient to use a modally reduced order model instead of a full order model for representing the dynamic behavior of a system in a predefined frequency range. Modal truncation can lead to large errors in the system inversion, however [20]. To avoid these, Maes and Lombaert [20] have developed a technique that allows for quasi-static correction when applying model order reduction in combination with state-space modeling. The presented technique introduces the quasi-static contribution of the out-of-band modes by means of so-called dummy modes. The number of dummy modes generally needs to be equal to the number of forces applied to the structure n_p . The expressions for the system matrices in the case where dummy modes are introduced in the system description have been derived in [20] and are here repeated to compute the derivatives of the system matrices with respect to the parameters.

If the system description includes n_m structural modes and n_p dummy modes, the dynamic state vector $\mathbf{x}_t \in \mathbb{R}^{2(n_m+n_p)}$ contains the modal displacements $\mathbf{z} \in \mathbb{R}^{n_m}$ and modal velocities $\dot{\mathbf{z}} \in \mathbb{R}^{n_m}$ corresponding to the n_m structural modes, as well as the modal displacements $\mathbf{z}' \in \mathbb{R}^{n_p}$ and modal velocities $\dot{\mathbf{z}}' \in \mathbb{R}^{n_p}$ corresponding to the dummy modes, respectively:

$$\mathbf{x}_t = [\mathbf{z}^T \quad \mathbf{z}'^T \quad \dot{\mathbf{z}}^T \quad \dot{\mathbf{z}}'^T]^T \quad (74)$$

The modified expressions for the system matrices $\mathbf{A}_c$ and $\mathbf{B}_c$ are given by:

$$\mathbf{A}_c = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_m} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_p} \\ -\Omega^2 & \mathbf{0} & -\mathbf{C}_m & \mathbf{0} \\ \mathbf{0} & -\Omega'^2 & \mathbf{0} & -\Gamma' \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \Phi^T \mathbf{S}_p \\ \Phi'^T \mathbf{S}_p \end{bmatrix} \quad (75)$$

where $\mathbf{\Omega}' \in \mathbb{R}^{n_p \times n_p}$ is a diagonal matrix containing the natural frequencies ω'_l on its diagonal, with ω'_l the natural frequency assigned to dummy mode l . $\mathbf{\Gamma}' \in \mathbb{R}^{n_p \times n_p}$ is a diagonal matrix as well, containing the terms $2\xi'_l \omega'_l$ on its diagonal, with ξ'_l the modal damping ratio assigned to dummy mode l . Finally, $\mathbf{\Phi}'_p \in \mathbb{R}^{n_p \times n_p}$ is a diagonal matrix containing on its diagonal the dummy mode shape components at the input locations, ϕ'_{pl} , which are computed as:

$$\phi'_{pl} = \omega'_l \sqrt{\mathbf{S}_{pl}^T (\mathbf{K}^{-1} - \mathbf{\Phi} \mathbf{\Omega}^{-2} \mathbf{\Phi}^T) \mathbf{S}_{pl}} \quad (76)$$

where $\mathbf{S}_{pl}$ represents the l -th column of the selection matrix $\mathbf{S}_p$. The reader is referred to [20] for an in-depth discussion on the choice of the natural frequency and modal damping ratio assigned to the dummy modes. Finally, note that the dummy modes are assumed to be decoupled from the n_m structural modes and each other.

The expressions for the system matrices $\mathbf{G}_c$ and $\mathbf{J}_c$ are given by:

$$\mathbf{G}_c = [\mathbf{S}_{d,d} [\mathbf{\Phi} \quad \mathbf{\Phi}'] - \mathbf{S}_{d,a} [\mathbf{\Phi} \mathbf{\Omega}^2 \quad \mathbf{\Phi}' \mathbf{\Omega}'^2] \quad \mathbf{S}_{d,v} [\mathbf{\Phi} \quad \mathbf{\Phi}'] - \mathbf{S}_{d,a} [\mathbf{\Phi} \mathbf{C}_m \quad \mathbf{\Phi}' \mathbf{\Gamma}']]^T \quad (77)$$

$$\mathbf{J}_c = \mathbf{S}_{d,a} \mathbf{\Phi} \mathbf{\Phi}^T \mathbf{S}_p + \mathbf{S}_{d,a} \mathbf{\Phi}' \mathbf{\Phi}'_p \quad (78)$$

where $\mathbf{\Phi}' \in \mathbb{R}^{n_{\text{dof}} \times n_p}$ is a matrix containing as columns the dummy mode shape vectors $\phi'_{dl} \in \mathbb{R}^{n_{\text{dof}}}$, which are computed as:

$$\phi'_{dl} = \frac{\omega'_l{}^2}{\phi'_{pl}} ((\mathbf{K}^{-1} - \mathbf{\Phi} \mathbf{\Omega}^{-2} \mathbf{\Phi}^T) \mathbf{S}_{pl}) \quad (79)$$

Note that $\phi'_{pl} = \mathbf{S}_{pl}^T \phi'_{dl}$.

Based on Eqs. (75) – (78), the following expressions are derived for the derivatives of the system matrices $\mathbf{A}_c$, $\mathbf{B}_c$ and $\mathbf{G}_c$ with respect to a parameter θ_l :

$$\frac{\partial \mathbf{A}_c}{\partial \theta_l} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -2\mathbf{\Omega} \frac{\partial \mathbf{\Omega}}{\partial \theta_l} & \mathbf{0} & -\frac{\partial \mathbf{C}_m}{\partial \theta_l} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \frac{\partial \mathbf{B}_c}{\partial \theta_l} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \frac{\partial \mathbf{\Phi}^T}{\partial \theta_l} \mathbf{S}_p \\ \frac{\partial \mathbf{\Phi}'_p}{\partial \theta_l} \end{bmatrix} \quad (80)$$

$$\frac{\partial \mathbf{G}_c}{\partial \theta_l} = \begin{bmatrix} \left(\mathbf{S}_{d,d} \left[\frac{\partial \mathbf{\Phi}}{\partial \theta_l} \quad \frac{\partial \mathbf{\Phi}'}{\partial \theta_l} \right] - \mathbf{S}_{d,a} \left[\frac{\partial \mathbf{\Phi}}{\partial \theta_l} \mathbf{\Omega}^2 + 2\mathbf{\Phi} \mathbf{\Omega} \frac{\partial \mathbf{\Omega}}{\partial \theta_l} \quad \frac{\partial \mathbf{\Phi}'}{\partial \theta_l} \mathbf{\Omega}'^2 \right] \right)^T \\ \left(\mathbf{S}_{d,v} \left[\frac{\partial \mathbf{\Phi}}{\partial \theta_l} \quad \frac{\partial \mathbf{\Phi}'}{\partial \theta_l} \right] - \mathbf{S}_{d,a} \left[\frac{\partial \mathbf{\Phi}}{\partial \theta_l} \mathbf{C}_m + \mathbf{\Phi} \frac{\partial \mathbf{C}_m}{\partial \theta_l} \quad \frac{\partial \mathbf{\Phi}'}{\partial \theta_l} \mathbf{\Gamma}' \right] \right)^T \end{bmatrix}^T \quad (81)$$

In the derivation of Eqs. (80) and (81), it is assumed that the natural frequencies and modal damping ratios assigned to the dummy modes only depend on the upper frequency limit ω_u considered in the analysis [20] and are therefore independent of the parameters $\boldsymbol{\theta}$.

The analytical expressions for the modal sensitivities, i.e. the first derivative of the natural frequencies ω_j and the mode shape vectors ϕ_j with respect to the parameters, are given in Section 4.3. The derivatives of the dummy modes with respect to a parameter θ_l are given by:

$$\frac{\partial \phi'_{pn}}{\partial \theta_l} = \frac{\omega'_n{}^2}{2\phi'_{pn}} \mathbf{S}_{pn}^T \left(-\mathbf{K}^{-1} \frac{\partial \mathbf{K}}{\partial \theta_l} \mathbf{K}^{-1} + 2\mathbf{\Phi} \mathbf{\Omega}^{-3} \frac{\partial \mathbf{\Omega}}{\partial \theta_l} \mathbf{\Phi}^T - \frac{\partial \mathbf{\Phi}}{\partial \theta_l} \mathbf{\Omega}^{-2} \mathbf{\Phi}^T - \mathbf{\Phi} \mathbf{\Omega}^{-2} \frac{\partial \mathbf{\Phi}^T}{\partial \theta_l} \right) \mathbf{S}_{pn} \quad (82)$$

and

$$\frac{\partial \phi'_{dn}}{\partial \theta_l} = \frac{\omega_n'^2}{\phi'_{pn}} \left(-\mathbf{K}^{-1} \frac{\partial \mathbf{K}}{\partial \theta_l} \mathbf{K}^{-1} + 2\mathbf{\Phi} \mathbf{\Omega}^{-3} \frac{\partial \mathbf{\Omega}}{\partial \theta_l} \mathbf{\Phi}^T - \frac{\partial \mathbf{\Phi}}{\partial \theta_l} \mathbf{\Omega}^{-2} \mathbf{\Phi}^T - \mathbf{\Phi} \mathbf{\Omega}^{-2} \frac{\partial \mathbf{\Phi}^T}{\partial \theta_l} \right) \mathbf{S}_{pn} - \frac{\phi'_{dn}}{\phi'_{pn}} \frac{\partial \phi'_{pn}}{\partial \theta_l} \quad (83)$$

5. Verification

The proposed methodology is demonstrated and verified using numerical simulations. The structure under consideration is the four storey shear frame represented in Fig. 1. A simple mass-spring-damper model is used to represent the dynamic behavior. The floor mass m_j and column stiffness k_j assigned to each storey j is listed in Table 1. The four natural frequencies obtained from the model are 2.34 Hz, 5.60 Hz, 8.36 Hz, and 10.16 Hz.

Fig. 1: Four-storey shear building model (left) and its mass, stiffness, and damping matrices (right).

Storey	m_j [kg]	k_j [kN/m]
1	5000	6000
2	4000	5000
3	3000	4000
4	2000	3000

Table 1: Floor mass m_j and column stiffness k_j assigned to the four storeys of the shear building model.

The stiffness of the first and second storey, k_1 and k_2 , are modified by a factor $\theta_1 = 0.8$ and $\theta_2 = 0.9$, respectively. After the stiffness modification, the four natural frequencies are 2.18 Hz, 5.39 Hz, 8.02 Hz, and 10.00 Hz. The (modified) structure is excited at the top floor using a series of impulse forces. The force time history consists of three consecutive Ricker wavelets and is given by:

$$p(t) = \sum_{j=1}^3 A_j (1 - 2\pi^2 f_j^2 (t - t_j)^2) \exp(-\pi^2 p_j^2 (t - t_j)^2) \quad (84)$$

For each of the three Ricker wavelets with index j , f_j is the dominating frequency, A_j is the maximum force amplitude, and t_j is the time where the maximum force amplitude is reached. In the presented example, $f_j = 2.5$ Hz and $A_j = 100$ N, for $j = 1, 2, 3$, and $t_1 = 0.5$ s, $t_2 = 1.3$ s, and $t_3 = 2.1$ s. The corresponding force amplitude and frequency spectrum are shown in Fig. 2. Note that the dominating frequency of the three Ricker wavelets f_j is chosen such that mainly the first two modes of the structure are excited.

Fig. 2: (a) Time history and (b) amplitude of the Ricker wavelet force.

The floor displacements $d_1 - d_4$ and floor accelerations $a_1 - a_4$ are obtained from a frequency domain calculation, hereby adopting a sampling frequency of 200 Hz and a time window of 100 s. The large time window is applied to avoid leakage of the response signals. Only the first five seconds of the calculated response time histories are considered in the following analysis. All signals obtained from the forward calculation are polluted with additive sensor noise, which is drawn independently from a normal distribution with a zero mean value and a standard deviation of 10^{-5} m for the displacements and 10^{-3} m/s² for the accelerations. In the following, a subset of these data is used in the joint input-state-parameter estimation. A distinction is made between two different sensor configurations, which are referred to as configurations S_1 and S_2 (Table 2). Sensor configuration S_1 consists of one acceleration a_4 and one displacement d_4 , both collocated with the estimated force p , one non-collocated acceleration a_2 , and one non-collocated displacement d_2 . In sensor configuration S_2 , the collocated acceleration a_4 is replaced by a non-collocated acceleration a_3 .

Configuration	Sensors
S_1	a_2, a_4, d_2, d_4
S_2	a_2, a_3, d_2, d_4

Table 2: Overview of the sensor configurations used in the verification.

First, the joint input-state-parameter estimation is performed using a full-order model expressed in physical coordinates (Section 4.2). In the application of the smoothing algorithm, the initial state $\hat{\mathbf{x}}_{[0|L-1]}$ is assigned zero values for the displacements and velocities and a unit value is assigned for both parameters, corresponding to the original structure before modification. Due to the perturbation of the model in terms of the parameters θ_l , the corresponding model uncertainty should be reflected in the process noise covariance matrix $\mathbf{Q}$, through the choice of $\mathbf{Q}_\theta$. Since no unmodeled excitation or modeling errors are present, the entries of the noise covariance matrices $\mathbf{Q}_t$ and $\mathbf{S}_t$, which correspond to process noise on the dynamic system states, are assigned zero values (see also [13]). The parameter covariance matrix $\mathbf{Q}_\theta$ is assumed to be diagonal, with the same variance $\sigma_{\mathbf{Q},\theta}^2 = 10^{-10}$ for all parameters θ_l . The choice of the parameter covariance $\sigma_{\mathbf{Q},\theta}^2$ is discussed in the remainder of this section. The noise covariance matrix $\mathbf{R}$ is assumed diagonal. The standard deviations chosen to model the measurement noise in the algorithm are $\sigma_{\mathbf{R},d} = 10^{-5}$ m and $\sigma_{\mathbf{R},a} = 10^{-3}$ m/s² for the displacements and accelerations, respectively, corresponding to the noise added to the simulated data.

Fig. 3: Time history of (a) the force $p(t)$, (b) parameter θ_1 , and (c) parameter θ_2 for **sensor configuration S_1** and assuming a **model expressed in physical coordinates** (dash-dotted red: estimate for $L = 0$, solid blue: estimate for $L = 10$, dashed black: true values).

Fig. 3 compares the estimated forces and parameters obtained with delay $L = 0$ and $L = 10$ for sensor configuration S_1 . For both delays, a very fast convergence of both parameters towards their actual values is observed. For the case of smoothing ($L = 10$), it is typically observed that the convergence is initiated faster than for the case of filtering ($L = 0$). This is due to the fact that the smoother uses future information (L steps) in the estimation for a given time step k . In both cases, a small error in the peak force is obtained for the first impulse (3.9% for $L = 0$ and 0.8% for $L = 10$). This is mainly because the parameters have not converged to their correct value yet. After convergence, very accurate force estimates are obtained for both $L = 0$ and $L = 10$.

Fig. 4 compares the estimated forces and parameters obtained with delay $L = 1$ and $L = 10$ for sensor configuration S_2 . The results are intentionally not shown for $L = 0$, because the system is theoretically not invertible for $L < 1$ [10, 14]. This is due to the fact that none of the outputs in the considered case of non-collocated acceleration measurements is instantaneously affected by the input $p(t)$. Similar to sensor configuration S_1 , a very fast convergence of both parameters towards their actual values is observed for both $L = 1$ and $L = 10$. It is also observed that the input estimates in the considered case of non-collocated acceleration measurements heavily depend on the delay L assumed in the estimation. This was previously observed for the case of joint input-state estimation in [10]. For $L = 1$, the input estimate is found to be highly sensitive to modeling errors, which in this example are due to errors in the system parameters. After the parameters have converged to their true values, the estimation errors introduced by measurement noise remain very large. For $L = 10$, the sensitivity to both modeling errors and measurement noise is significantly reduced and very accurate force estimates are obtained for the entire measurement period. Further increasing the delay L assumed in the estimation is not beneficial in this case, since it does not significantly reduce the errors on the force estimates, meanwhile increasing the calculation time as well as the time required for the parameters to converge towards their true values.

Fig. 4: Time history of (a) the force $p(t)$, (b) parameter θ_1 , and (c) parameter θ_2 for **sensor configuration S_2** and assuming a **model expressed in physical coordinates** (dash-dotted red: estimate for $L = 1$, solid blue: estimate for $L = 10$, dashed black: true values).

The value assigned to $\sigma_{Q,\theta}^2$, characterizing the variance of the parameters θ_l , determines the convergence rate of the parameter estimates. This is illustrated in Fig. 5, which shows the convergence of the parameters for different values of the variance $\sigma_{Q,\theta}^2$. A small variance results in slow convergence and low sensitivity of the parameter estimates to noise. A large variance results in faster convergence but also a larger sensitivity of the parameter estimates to noise. In the limit, a zero variance results in the case where the parameter estimates do not change during the estimation, whereas a too large variance may lead to large noise sensitivity and, ultimately, divergence of the algorithm. For applications where a slow variation of the system parameters is expected, one can use a relatively small variance $\sigma_{Q,\theta}^2$ to prevent those issues.

Fig. 5: Time history of (a) parameter θ_1 and (b) parameter θ_2 for **sensor configuration S_2** and assuming a **model expressed in physical coordinates** for $L = 10$ (dotted green: $\sigma_{Q,\theta}^2 = 10^{-8}$, solid blue: $\sigma_{Q,\theta}^2 = 10^{-10}$, dash-dotted red: $\sigma_{Q,\theta}^2 = 10^{-12}$, dashed black: true values).

Introducing a time delay in the estimation can also lead to reduced estimation errors when performing so-called response estimation. In this process, the estimated states $\hat{\mathbf{x}}_{[k|k+L-1]}$ and inputs $\hat{\mathbf{p}}_{[k|k+L]}$ are used to predict the response $\mathbf{d}_{e[k]}$ at unmeasured locations, hereby adopting the following prediction step [21]:

$$\hat{\mathbf{d}}_{e[k]} = \mathbf{G}_{t_e}(\hat{\boldsymbol{\theta}}_{[k|k+L-1]})\hat{\mathbf{x}}_{t[k|k+L-1]} + \mathbf{J}_{t_e}(\hat{\boldsymbol{\theta}}_{[k|k+L-1]})\hat{\mathbf{p}}_{[k|k+L]} \quad (85)$$

where $\mathbf{G}_{t_e}$ and $\mathbf{J}_{t_e}$ are the system matrices that relate the response $\mathbf{d}_{e[k]}$ to the system states and the system inputs, respectively. As an example, Fig. 6 shows the estimated top floor acceleration a_4 obtained for sensor configuration S_2 and delays $L = 1$ and $L = 10$. Similarly to the input estimation, it can be concluded here that the increased delay in the case of non-collocated acceleration measurements leads to a reduced sensitivity to modeling errors and measurement noise.

Fig. 6: (a) Time history and (b) time history detail of the estimated acceleration $\hat{a}_4$ for **sensor configuration** S_2 and assuming a **model expressed in physical coordinates** (dash-dotted red: estimate for $L = 1$, solid blue: estimate for $L = 10$, dashed black: true values).

As a second step, the joint input-state-parameter estimation is performed using a modally reduced order model. At this point, a distinction is made between (1) a modal model without quasi-static correction to account for out-of-band modes (Section 4.3) and (2) a modal model with quasi-static correction by means of a dummy mode (Section 4.4). In both cases, the model includes the dynamic contribution of the first two modes of the structure that dominate the dynamic response in the frequency range between 0 and 6 Hz ($n_m = 2$). As in the previously considered case of a full order model expressed in physical coordinates, the initial state vector $\hat{\mathbf{x}}_{[0|L-1]}$ is assigned zero values for the dynamic system states (modal displacements and velocities) and unit values for both parameters. The noise covariance matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ are also composed similarly. The entries of the noise covariance matrices $\mathbf{Q}_t$ and $\mathbf{S}_t$ are assigned zero values. For the augmented states $\boldsymbol{\theta}$, a variance $\sigma_{Q,\theta}^2 = 10^{-10}$ is assigned to the diagonal of the process noise covariance matrix $\mathbf{Q}_\theta$. The noise covariance matrix $\mathbf{R}$ represents the same measurement noise and is therefore assumed identical to the case of the full order model expressed in physical coordinates.

Fig. 7 compares the estimated forces and parameters obtained for both modal models in the case of sensor configuration S_1 with delay $L = 10$. For the model without dummy mode correction, no clear convergence of the parameter estimates towards the true parameter values is observed in the time period where the impact forces are applied (0 to 2.9 s). In the subsequent free decay, the dynamic response of the structure is mainly dominated by its first two modes that are correctly accounted for in the model, resulting in a monotonic convergence of the parameter estimates towards their true value. In addition to the large errors in the parameter estimates, significant errors in the estimated force time history are observed (up to 15%). These errors are mainly due to incorrect parameter values and the contribution of the higher modes which has been disregarded

in the model (see also [20]). For the model with dummy mode correction, a fast convergence of the parameter estimates towards the true parameter values is initiated right after the first impact. It is also observed that force estimates become more accurate as the parameter estimates converge. This indicates that the quasi-static correction is important for both the parameter and input estimation.

The importance of the quasi-static correction becomes even more pronounced if the initial parameter estimates equal the true parameter values, i.e. $\hat{\theta}_{1[0|L-1]} = 0.8$ and $\hat{\theta}_{2[0|L-1]} = 0.9$. In absence of modeling errors, one would expect that the parameter estimates stay close to the true parameter values, with small oscillations that are attributed to measurement noise. As shown in Fig. 8, this is definitely not the case when a modal model without quasi-static correction is used. Indeed, very large oscillations in the parameter values are observed. For the model with quasi-static correction, the variation of the parameters is much smaller.

Fig. 7: Time history of (a) the force $p(t)$, (b) parameter θ_1 , and (c) parameter θ_2 for **sensor configuration S_1** and assuming a **modal model** that includes the first two modes ($n_m = 2$) without quasi-static correction (solid blue line) and with quasi-static correction (dash-dotted red line), for $L = 10$ and assuming the **initial parameter values corresponding to the structure before modification**. The true force and parameter values are given by a dashed black line.

In a final verification example, the proposed joint input-state-parameter estimation algorithm is applied to identify the modal damping ratio corresponding to the first mode of the four-storey building in Fig. 1. In this case, the masses m_j and stiffnesses k_j are assumed known and equal to the values given in Table 1. The modal damping ratios corresponding to modes 2, 3, and 4 are also assumed known and are obtained from the viscous damping matrix $\mathbf{C}_v (= \mathbf{K}/1000)$ as follows:

$$\xi_j = \frac{\phi_j^T \mathbf{C}_v \phi_j}{2\omega_j} \quad (86)$$

This results in the following values: $\xi_2 = 1.76\%$, $\xi_3 = 2.63\%$, and $\xi_4 = 3.19\%$.

The modal damping ratio corresponding to mode 1 is identified. Its target value is also obtained from Eq. (86) and equals $\xi_1 = 0.73\%$. In the identification, the modal damping ratio ξ_1 is

Fig. 8: Time history of (a) the force $p(t)$, (b) parameter θ_1 , and (c) parameter θ_2 for **sensor configuration** S_1 and assuming a **modal model** that includes the first two modes ($n_m = 2$) without quasi-static correction (solid blue line) and with quasi-static correction (dash-dotted red line), for $L = 10$ and assuming the **initial parameter values corresponding to the structure after modification**. The true force and parameter values are given by a dashed black line.

parametrized using an exponential mapping of the parameter θ_1 :

$$\xi_1 = \xi_{1,0} \exp(\theta_1) \quad (87)$$

where $\xi_{1,0}$ is an initial guess of the modal damping ratio ξ_1 , which is here assumed $\xi_{1,0} = 0.5\%$. The use of an exponential mapping allows to avoid non-physical negative damping values ($\xi_1 > 0$). A similar exponential mapping could also be used for the previously considered examples, as to avoid negative stiffness values.

The partial derivative of the modal damping ratio ξ_1 with respect to the parameter θ_1 , which is required for the evaluation of Eq. (73), is given by:

$$\frac{\partial \xi_1}{\partial \theta_1} = \xi_{1,0} \exp(\theta_1) \quad (88)$$

The response of the structure to the previously considered impact loading at the top floor is calculated in the frequency domain. The resulting acceleration and displacement signals are polluted with additive sensor noise, hereby using the same noise characteristics. Only sensor configuration S_1 is considered (Table 2), but the results obtained for configuration S_2 are similar. Two different models are considered in the joint input-state-parameter estimation. The first model is a full-order modal model that includes the dynamic contribution of the four modes of the structure ($n_m = 4$). The second model is a modally reduced order model that includes the dynamic contribution of the first two modes ($n_m = 2$) and a quasi-static correction for the out-of-band modes by means of a single dummy mode. For both models, the initial state vector $\hat{\mathbf{x}}_{[0|L-1]}$ is assigned zero values, i.e. the structure is assumed at rest and the initial modal damping ratio estimate equals $\xi_{1,0}$ (see Eq. (87) for $\theta_1 = 0$). The entries of the noise covariance matrices $\mathbf{Q}_t$ and $\mathbf{S}_t$ are assigned zero

values. For the augmented state θ_1 , a variance $\sigma_{Q,\theta}^2 = 10^{-8}$ is assigned, with $\mathbf{Q}_\theta = \sigma_{Q,\theta}^2$. The noise covariance matrix $\mathbf{R}$ is identical to the one assumed in previous cases. Finally, a delay $L = 10$ is assumed in the estimation.

Fig. 9 compares the estimated forces and modal damping ratios obtained for both models. Very accurate force estimates are obtained for both models throughout the entire time interval considered. The modal damping ratio estimates show a monotonic convergence towards the true damping ratio $\xi_1 = 0.73\%$ for the full-order modal model. For the modally reduced order model with dummy mode correction, the convergence is characterized by larger variations, especially during the impacts, but the estimate finally converges towards the true damping ratio. This illustrates the ability of the presented algorithm to identify the modal damping ratios of a structure and, eventually, track them over time.

Fig. 9: Time history of (a) the estimated force $p(t)$ and (b) the estimated modal damping ratio ξ_1 for sensor configuration S_1 and assuming a modal model that includes the all modes ($n_m = 4$) (solid blue line) and a modal model with 2 modes and a quasi-static correction (dash-dotted red line), for $L = 10$. The true force and damping ratio values are given by a dashed black line.

6. Validation

This section presents a validation of the proposed methodology based on laboratory experiments that have been performed on an IPE100 steel beam (Fig. 10). The beam has a length of 3 m. Steel plates of dimensions 150 mm $\times$ 150 mm $\times$ 15 mm are welded to its ends. The beam has a total mass of 30.2 kg. The beam is suspended to steel frames by very flexible springs as to approximate free-free boundary conditions. As a result, the modes of the beam in bending and torsion are decoupled from its rigid body modes.

Two different configurations of the beam are considered. In the first configuration, the empty beam is loaded by a series of impact loads. In the second configuration, a local point mass, which consists of four magnets with a total mass of 272 g, is added to the beam. Thereafter, a similar series of impact loads is applied. The data obtained from the experiments on the empty beam are used to extract the experimental modal characteristics of the structure before adding mass. These modal characteristics are used to build a dynamic system model of the structure. The data obtained from both experiments are used to validate the developed joint input-state-parameter estimation approach. The validation consists of estimating the added point mass and comparing the estimated forces to the directly measured impact forces.

Fig. 10: (a) Overview picture of the IPE100 steel beam, (b) the instrumented impact hammer, and (c) the 4 magnets that compose the added mass.

6.1. Measurement setup

The beam is instrumented with 17 uniaxial accelerometers that measure the vertical acceleration at the upper flange of the beam (Fig. 11). Six accelerometers of type PCB 353B34 (sensitivity 100 mV/g) and eleven accelerometers of type PCB 338B35 (sensitivity 100 mV/g) have been used. In addition, the beam is instrumented with two uniaxial strain gauges of type HBM 1-LY11-10/350 (resistance 350 Ω), that are glued at the center of the bottom flange. The beam is excited in the vertical direction using an instrumented impact hammer of type PCB 086C03 (mass 0.136 kg).

The data acquisition is performed by means of a National Instruments (NI) cDAQ-9178 chassis in combination with four NI 9234 ICP input modules, to measure the accelerations and the applied force, and one NI 9237 strain input module. A sampling frequency of 6400 Hz is used in the data acquisition.

Fig. 11: Overview of the instrumentation of the IPE100 steel beam. The accelerometers are denoted by a_j , the strain gauges by s_j , the hammer force by $p(t)$, and the added magnet mass by m_m .

6.2. System model

A detailed finite element (FE) model of the beam has been developed using the FE program ANSYS [22]. The model used here has previously been described by Anastasopoulos et al. in [23]. The model is composed of eight-node SOLID45 elements with three translational degrees of freedom at each node. A Young's modulus of 197 GPa, material density of 7850 kg/m³, and Poisson's ratio of 0.3 are assumed in the model. The springs are modeled by means of mass-less LINK180 elements, hereby assuming a spring stiffness of 1350 N/m [24].

The modal characteristics (natural frequencies and mode shapes) obtained from the FE model are compared to those obtained from an experimental modal analysis. In order to obtain the experimental modal characteristics, the response of the (empty) beam under a sequence of impact loads is processed together with the measured force signal using the data-driven combined

deterministic-stochastic subspace identification (CSI-data) algorithm [25]. The driving point acceleration is employed as reference output in the identification. Table 3 shows the identified modal characteristics and a comparison to the FE model. Except for the rigid body modes of the beam, where a relatively large difference in natural frequency is observed, a very good overall agreement between the identified and calculated modal characteristics is obtained, both in terms of the natural frequencies and the mode shapes, as is deduced from the small relative errors in natural frequency $\epsilon_f = (f_j - \tilde{f}_j)/\tilde{f}_j$ and MAC-values [26] that approach 1 for all bending modes and the lower torsional modes (orders 1 to 6). For the higher torsional modes (orders 7 and 8), the MAC values are very sensitive to small variations in the sensor locations, due to the very short wavelengths characterizing the mode shapes.

No. ID	f_{id} [Hz]	ξ_{id} [%]	No. FEM	f_{fem} [Hz]	ϵ_f [%]	MAC [-]	Description
1	1.9	0.14	1	2.0	7.51	1.00	RB mode - vertical translation
2	2.8	0.15	3	3.0	6.89	1.00	RB mode - rotation around y
3	5.0	0.34	2	2.4	-52.13	0.99	RB mode - rotation around x
4	30.8	0.07	5	30.4	-1.20	0.99	1st torsional mode
5	61.0	0.10	7	60.5	-0.85	1.00	1st vertical bending mode
6	71.9	0.75	8	71.0	-1.28	0.94	2nd torsional mode
7	128.2	0.22	10	126.8	-1.07	0.99	3rd torsional mode
8	174.1	0.02	12	171.6	-1.42	1.00	2nd vertical bending mode
9	202.1	0.12	13	200.1	-0.98	0.82	4th torsional mode
10	294.5	0.19	15	292.3	-0.75	0.81	5th torsional mode
11	342.4	0.02	16	337.1	-1.54	0.95	3rd vertical bending mode
12	405.2	0.13	18	404.0	-0.30	0.93	6th torsional mode
13	533.7	0.18	21	535.6	0.35	0.56	7th torsional mode
14	558.2	0.04	22	548.4	-1.77	0.92	4th vertical bending mode
15	685.8	0.27	25	686.6	0.12	0.62	8th torsional mode
16	808.8	0.06	27	795.5	-1.65	0.89	5th vertical bending mode

Table 3: Comparison of the experimentally identified and calculated modal characteristics.

A modally reduced-order discrete-time state-space model is constructed by combination of the modal characteristics obtained from the FE model and the system identification. The model includes the 16 modes listed in Table 3 ($n_m = 16$). For each mode, the mass-normalized mode shape is obtained from the FE model. The natural frequencies and modal damping ratios are obtained from the system identification. The contribution of out-of-band modes to the response in the frequency range of interest (0 to 700 Hz) is included by means of a single dummy mode (Section 4.4). A natural frequency of 1400 Hz and a modal damping ratio of 0.01% are assigned to the dummy mode. Fig. 12 shows the frequency response function (FRF) that relates the driving point acceleration a_{17} to the force p , as obtained from the state-space model. A comparison to the FRF obtained from the model without quasi-static correction shows the importance of the dummy mode correction, especially near the antiresonance frequencies.

A point mass m_m representing the magnets is added directly to the modally reduced-order model of the empty beam. In the considered case where the added mass only constitutes a fraction of the total structural mass, this extension of the modal model yields a good approximation of the system dynamics in the frequency band of interest, without requiring additional modes. The result is a combined modal-physical model. Both the translational and rotational inertia of the magnets are accounted for. The expressions for the system matrices $\mathbf{A}_c$, $\mathbf{B}_c$, $\mathbf{G}_c$, and $\mathbf{J}_c$ are

Fig. 12: Frequency response function that relates acceleration a_{17} to force p , for the model with dummy mode correction (solid blue line) and the model without dummy mode correction (dash-dotted red line). The contribution of the dummy mode is indicated with a dashed black line.

obtained from Eqs. (58) and (59), after replacing $\mathbf{M}$ by $\mathbf{I}_{n_m+1} + [\Phi \ \Phi']^T \mathbf{S}_m \mathbf{M}_m \mathbf{S}_m^T [\Phi \ \Phi']$, $\mathbf{K}$ by $\text{diag}(\Omega^2, \Omega'^2)$, $\mathbf{C}_v$ by $\text{diag}(\Gamma, \Gamma')$, $\mathbf{S}_p$ by $[\Phi \ \Phi']^T \mathbf{S}_p$, $\mathbf{S}_{d,d}$ by $\mathbf{S}_{d,d} [\Phi \ \Phi']$, $\mathbf{S}_{d,v}$ by $\mathbf{S}_{d,v} [\Phi \ \Phi']$, and $\mathbf{S}_{d,a}$ by $\mathbf{S}_{d,a} [\Phi \ \Phi']$, where ‘diag’ denotes a block diagonal matrix with the matrices between brackets on its diagonal. The selection matrix $\mathbf{S}_m \in \mathbb{R}^{n_{\text{dof}} \times 6}$ extracts the three translations and three rotations of the point where the magnets are connected to the beam, and $\mathbf{M}_m \in \mathbb{R}^{6 \times 6}$ is a matrix containing on its diagonal the translational and rotational inertia of the magnets with respect to the connection point on the beam and zeros elsewhere. The partial derivatives of the system matrices with respect to an unknown parameter θ_j , which in this case consists of the added mass m_m , are obtained from Eqs. (60) – (62).

The major advantage of adding the mass directly to the modal model of the empty beam is the significant reduction in calculation effort. Indeed, the modal characteristics of the empty beam and therefore Ω , Γ , and Φ are independent from the mass m_m . As a result, the sensitivity calculation does not require an eigenvalue decomposition.

6.3. Joint input-state-parameter estimation

The joint input-state-parameter estimation algorithm presented in Section 3.1 is applied to jointly identify the force applied at the impact location and the mass m_m added at the location where the magnets are attached to the structure. The estimation procedure is applied for the two configurations considered, i.e. for the empty beam before adding the mass ($m_m = 0$ kg) and after adding the mass ($m_m = 0.272$ kg). Two different sensor configurations are considered, which are referred to as configurations S_1 and S_2 . In the first sensor configuration S_1 , the data obtained from all 17 accelerometers $a_1 - a_{17}$ and the 2 strain gauges s_1 and s_2 are adopted in the estimation. In the second sensor configuration S_2 , the acceleration data is limited to four sensors a_1 , a_5 , a_9 , and a_{13} , meanwhile considering the same two strain signals s_1 and s_2 . The two sensor configurations are summarized in Table 4. Note that sensor configuration S_2 does not include a sensor that is collocated with the applied force. Finally, the estimation is performed considering two different values for the time delay, $L = 0$ and $L = 5$.

For the application of the smoothing algorithm, the initial state $\hat{\mathbf{x}}_{[0|L-1]}$ is assigned zero values for all entries. This assumes that the structure is at rest at the start of the measurement interval (the modal displacements and velocities are all zero). In addition, the initialization assumes the original structure before modification with the zero value assigned to the uncertain parameter $\theta = m_m$.

Configuration	Sensors
S_1	$a_1 - a_{17}, s_1, s_2$
S_2	$a_1, a_5, a_9, a_{13}, s_1, s_2$

Table 4: Overview of the sensor configurations adopted in the validation of the joint input-state-parameter estimation procedure using data obtained from the steel beam with I-shaped cross section.

The measured response and force signals are all digitally lowpass filtered by means of a 10th-order Butterworth filter with a cutoff frequency of 700 Hz and subsequently digitally highpass filtered by means of a 4th-order Butterworth filter with a cutoff frequency of 10 Hz. Both filters are applied in the forward and reverse direction to remove all phase distortion.

Since modeling errors are assumed to be small, the entries of the noise covariance matrices $\mathbf{Q}_t$ and $\mathbf{S}_t$ that correspond to process noise on the dynamic system states are assigned zero values. For the augmented state θ , a variance $\sigma_{\mathbf{Q},\theta}^2 = 10^{-12} \text{ kg}^2$ is assigned to the diagonal of the covariance matrix $\mathbf{Q}_\theta$. The noise covariance matrix $\mathbf{R}$ is assumed diagonal. The standard deviations chosen to model the measurement noise assumed by the algorithm are $\sigma_{\mathbf{R},s} = 10^{-6}$ and $\sigma_{\mathbf{R},a} = 10^{-2} \text{ m/s}^2$ for the strains and accelerations, respectively.

Fig. 13 shows the results of the joint input-state-parameter estimation for the case of the empty beam. For sensor configuration S_1 (Figs. 13a–13c), the estimated forces are in very close agreement with the measured force, both for $L = 0$ and $L = 5$. The estimated mass θ is nearly identical for both delays and remains close to the actual added mass $m_m = 0 \text{ kg}$. For sensor configuration S_2 (Figs. 13d–13f), a clear difference in the estimated forces is obtained between the two delays $L = 0$ and $L = 5$. For $L = 0$, the estimated force is characterized by estimation errors characterized by large oscillations that tend to decay exponentially with time after every impact. For $L = 5$, the estimated force is in close agreement with the measured force. Although the impact force is less accurately estimated than for sensor configuration S_1 , it is again found that the introduction of a time delay can significantly improve the force estimation accuracy in the case where no collocated acceleration measurements are made. As observed from Fig. 13f, however, the introduction of a delay does not significantly affect the parameter estimation. Both for $L = 0$ and $L = 5$, the parameter value tends to converge towards a value of about -50 g. This small error in the estimated mass can be attributed to a bias in the initial model, for example due to errors in the assumed modal characteristics.

Fig. 14 shows the results of the joint input-state-parameter estimation for the case of the beam with added point mass. For sensor configuration S_1 (Figs. 14a–14c), the estimated forces are in very close agreement with the measured force, both for $L = 0$ and $L = 5$. As for the case of the empty beam, nearly identical parameter estimates are obtained for both delays. The parameter estimates are characterized by a clear shift following the first impact, hereafter converging towards a value of about 210 g during the subsequent free decay. Although this value is slightly smaller than the actual added mass (272 g), a clear change in the parameter value is obtained as compared to the case of the empty beam (Fig. 13). This illustrates the ability of the algorithm to track small structural changes. After convergence of the parameter, large shifts in the parameter value following the next impacts are observed as well, which can be attributed to modeling errors. After every shift, however, a clear convergence towards the same value of about 210 g is achieved. For sensor configuration S_2 (Figs. 14d–14f), the estimated force time history is characterized by large oscillations when no delay is adopted in the estimation ($L = 0$), yielding a result which is similar

Fig. 13: Results of the joint input-state-parameter estimation for the case of the **empty beam** ($m_m = 0$ kg) and considering sensor configuration S_1 (a–c) and sensor configuration S_2 (d–f). (a,d) Time history and (b,e) detail of the time history of the identified force for $L = 0$ (blue solid line) and $L = 5$ (red dash-dotted line) and comparison to the measured force time history (black dashed line). (c,f) Time history of the identified added mass for $L = 0$ (blue solid line) and $L = 5$ (red dash-dotted line) and comparison to the actual added mass (black dashed line).

to the case of the empty beam. For $L = 5$, the estimated force is in close agreement with the measured force. The parameter estimates are again similar for $L = 0$ and $L = 5$. Although a larger variation is observed, the parameter value tends to converge towards a value which is similar to the value obtained for sensor configuration S_1 .

7. Conclusions

This paper introduces a smoothing algorithm for joint input-state-parameter estimation. The algorithm is obtained by extending an existing smoothing algorithm that was previously developed for joint input-state estimation. In addition to the algorithm derivation, analytical expressions for the sensitivities of the system matrices with respect to the estimated system parameters are derived. These expressions are essential in the application of the joint input-state-parameter estimation algorithm. The expressions are derived for (1) linear models expressed in physical coordinates, (2) models expressed in modal coordinates, and (3) modally reduced-order models with a quasi-static correction to account for the contribution of the out-of-band modes.

It is shown by numerical simulations and a validation study based on laboratory experiments that the introduction of a time delay in the estimation allows to significantly reduce the estimation errors when compared to commonly adopted filtering. This is especially beneficial when data originates from sensors that are not collocated with the estimated inputs/forces. Furthermore, it is shown that a quasi-static correction in case of modal truncation is essential for an accurate joint input-state-parameter estimation. Finally, the validation study also shows the ability of the algorithm to track small structural changes for a case where modeling errors are inherently present. Future work concerns the investigation of the system observability and the sensitivity of the results to the choice of the noise covariance matrices, which act as tuning parameters in the estimation.

Fig. 14: Results of the joint input-state-parameter estimation for the case of the **beam with added mass** ($m_m = 0.272$ kg) and considering sensor configuration S_1 (a–c) and sensor configuration S_2 (d–f). (a,d) Time history and (b,e) detail of the time history of the identified force for $L = 0$ (blue solid line) and $L = 5$ (red dash-dotted line) and comparison to the measured force time history (black dashed line). (c,f) Time history of the identified added mass for $L = 0$ (blue solid line) and $L = 5$ (red dash-dotted line) and comparison to the actual added mass (black dashed line).

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