On the m-dimensional Cayley-Hamilton theorem and its application to an algebraic decision problem inferred from the $\mathcal{H}_2$ norm analysis of delay systems [⋆]

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Abstract

We consider a recursion formula for multi-dimensional powers of a finite set of matrices, which can be interpreted as a natural generalization of the celebrated Cayley-Hamilton theorem, and we show how it allows to solve an algebraic decision problem on a semigroup of matrices, which bears similarities to the observability problem of a switched linear system. This problem appears in the computation of the $\mathcal{H}_2$ norm of a stable system described by a class of linear time-invariant delay differential equations (DDAEs) with multiple delays. The $\mathcal{H}_2$ norm of a DDAE may not be finite even if there are seemingly no direct feedthrough terms. We show that necessary and sufficient conditions for a finite $\mathcal{H}_2$ norm consist of an infinite number of linear equations to be satisfied, inducing the algebraic decision problem, and that using the generalized Cayley-Hamilton theorem checking these conditions can be turned into a check of a finite number of equations. We conclude with some comments on the computation of the $\mathcal{H}_2$ norm whenever it is finite and by stating an open problem.

Key words: Decision problem on semigroup of matrices; Cayley-Hamilton theorem; time-delay systems; $\mathcal{H}_2$ norm.

1 Introduction

Finitely generated semigroups of matrices play an important role in systems and control. These are (often infinite) sets of matrices, which consist of all the arbitrary products of matrices that are taken from a finite set of matrices of the same dimension. Given such a finite set of matrices (the *generators*), many different questions can be asked about the set that they generate. As a natural example, let us consider a discrete-time switched system: given a set $\mathcal{M} = \{A_1, \ldots, A_m\} \subset \mathbb{R}^{n\times n}$, the corresponding switched system is described by the following equation:

$$x(k+1) = A_{\sigma(k)}x(k), \quad \sigma(k) \in \{1, \ldots, m\}.$$

That is, such a system has linear dynamics, but the matrix applied to the system is not uniquely defined, and may take at every time k an arbitrary value in the set $\mathcal{M}$.

A natural question for a control theorist is whether such a system has bounded trajectories, whatever switching sequence occurs? The question can be translated into a property of the semigroup of matrices generated by $\mathcal{M}$: are all the products in the semigroup bounded by a constant K? This question is algorithmically undecidable as proved by Blondel and Tsitsiklis [2]. In fact, many simply-looking problems become very hard to solve when asked on a semigroup of matrices. The *mortality problem* is another striking example: given a finitely generated semigroup (described by its generators), is the zero matrix a member of the semigroup? Several other problems in control theory reduce to a problem on semigroups of matrices. Interestingly, continuous-time switched systems are also ruled by properties of the semigroup generated by their matrices, while the connection is less evident than for discrete time (see [1] for a celebrated result connecting stability properties of a continuous-time switched system with the Lie Algebra generated by the corresponding set of matrices). Other applications involving semigroups of matrices include consensus problems [11] or variable delays in wireless control networks [13]. Even though many natural problems become very hard when more than one matrix is involved in the dy-

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namics, in some situations an efficient algorithm can be found. This is famously the case of deciding consensus for a system of multi-agents communicating on time-varying topologies (see [3] and references therein for a survey of results on this problem).

In this paper, we consider a problem originating from delay differential algebraic equations (DDAEs), and show how it leads to an algorithmic decision problem on a semigroup of matrices, which can be stated as follows. Consider a set of generators $\mathcal{M} = \{A_1, \ldots, A_m\} \subset \mathbb{R}^{n\times n}$, and matrices $B \in \mathbb{R}^{n\times n_b}$, $C \in \mathbb{R}^{n_c\times n}$. Define the matrix polynomial

$$
\begin{aligned}
&P_{k_1,\ldots,k_m}(A_1,\ldots,A_m) := A_1 P_{k_1-1,k_2,\ldots,k_m}(A_1,\ldots,A_m)\\
&+A_2 P_{k_1,k_2-1,\ldots,k_m}(A_1,\ldots,A_m)+\ldots+\\
&+A_m P_{k_1,k_2,\ldots,k_m-1}(A_1,\ldots,A_m)
\end{aligned}
$$

for any $k_j \in \mathbb{Z}_+$, $j = 1,\ldots,m$, $P_{0,\ldots,0}(A_1,\ldots,A_m) := I$, and $P_{k_1,\ldots,k_m}(A_1,\ldots,A_m) := 0$ if any $k_j \in \mathbb{Z}_{--}$, $j = 1,\ldots,m$. Here, $\mathbb{Z}_+$ and $\mathbb{Z}_{--}$ denote the set of nonnegative integers including zero, and negative integers, respectively. We notice that $P_{k_1,\ldots,k_m}(A_1,\ldots,A_m)$, $k_j \in \mathbb{Z}_+$, is a matrix polynomial of degree $\kappa = k_1 + \ldots + k_m$ in m variables, which consist of the sum of all monomials of order k_j in A_j, $j = 1,\ldots,m$, and the number of monomials for a given m-tuple $(k_1,\ldots,k_m)$ is $\frac{(k_1+\ldots+k_m)!}{k_1!\cdots k_m!}$. We address the following problem:

Problem 1 *Find a finite test for determining that the following condition holds:*

$$
CP_{k_1,\ldots,k_m}(A_1,\ldots,A_m)B = 0,\ \forall k_j \in \mathbb{Z}_+,\ j = 1,\ldots,m, \tag{1}
$$

where $B \in \mathbb{R}^{n\times n_b}$ and $C \in \mathbb{R}^{n_c\times n}$ are different from zero.

For instance, in the case $m = 2$, condition (1) takes the form

$$
\begin{aligned}
CP_{0,0}B &= CB = 0,\\
CP_{1,0}B &= CA_1B = 0,\ CP_{0,1}B = CA_2B = 0,\\
CP_{2,0}B &= CA_1^2B = 0,\ CP_{1,1}B = C\left(A_1A_2 + A_2A_1\right)B = 0,\\
CP_{0,2}B &= CA_2^2B = 0,\ CP_{3,0}B = CA_1^3B = 0,\\
CP_{2,1}B &= C(A_1^2A_2 + A_1A_2A_1 + A_2A_1^2)B = 0,\ \ldots
\end{aligned} \tag{2}
$$

We note that problem can also be stated as follows: *is there a set of indices $k_1,\ldots,k_m \in \mathbb{Z}_+$, such that $CP_{k_1,\ldots,k_m}B \neq 0$?* While, to the best of our understanding, nothing could indicate a priori that the problem is efficiently tractable, we show that one can indeed provide an efficient, polynomial time, algorithm.

The classic Cayley-Hamilton theorem (CH theorem from now on) establishes that every square matrix $A \in \mathbb{R}^{n\times n}$ satisfies its own characteristic equation, i.e. $p(A) = 0$, where $p(x) := \det(xI - A)$. An implication of this is that every power $k \geqslant n$ of matrix A can be expressed as a linear combination of A^k, $k = 1,\ldots,n-1$. The classic CH theorem is useful if only one matrix A_1 is considered in Problem 1, i.e. $m = 1$. For this case, condition (1) reduces to

$$
CA_1^{k_1}B = 0,\ \forall k_1 \in \mathbb{Z}_+, \tag{3}
$$

and by the CH theorem a finite test for determining that (3) holds is that $CA_1^{k_1}B = 0$ for $k_1 = 0,1,\ldots,n-1$. Although condition (1) involves products of the form

$$
CA_j^kB = 0,\ j = 1,\ldots,m, \tag{4}
$$

for any $k \in \mathbb{Z}_+$, the reasoning for the case $m = 1$ cannot be applied, as shown in the next example.

Example 1 *Consider $m = 2$ and matrices*

$$
A_1 = \begin{pmatrix} 1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 & 0\\ 1 & 1 & 1\\ 1 & 1 & 1 \end{pmatrix},
$$

$$
B^T = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.
$$

One can check by direct calculation that

$$
CA_1^kB = 0,\ \text{ and } CA_2^kB = 0,
$$

for $k = 0,1,2$, therefore, by the CH theorem equation (4) holds for $j = 1,2$, for all $k \in \mathbb{Z}_+$. However,

$$
C\left(A_1A_2 + A_2A_1\right)B \neq 0.
$$

The cornerstone of our solution to Problem 1 is the m-dimensional CH theorem. Several "generalizations" of the classic CH theorem have been introduced in the last few decades, most of them aiming at studying mD systems ([7], [21], [19]). See also [14], [23], [15], and the references therein. The m-dimensional CH theorem that we present is a natural extension of the two-dimensional case introduced in [7] in the study of $2D$ systems, and of Theorem 2 in paper [15], which is devoted to the analysis of fractional systems. It is a generalization of the classic one in the sense that it presents a recursion formula for m-tuple powers of a block matrix. A similar generalization is presented in [21], requiring the definition of input and output of mD systems. Here, we avoid such definitions by using the ideas introduced by [22]. An appropriate construction of a block matrix consisting of matrices A_j, $j = 1\ldots,m$, allows us to use the m-dimensional CH theorem to express any matrix polynomial of the form $P_{k_1\ldots,k_m}(A_1,\ldots,A_m)$ of degree $\kappa \geqslant mn$ as a linear combination of matrix polynomials of the same class of degree $\kappa < mn$. This fact enables us to solve Problem 1, since we can test condition (1) by uniquely testing the products corresponding to matrix polynomials of order $\kappa < mn$.

Problem 1 is motivated by determining the conditions under which the transfer matrix of a difference equation with multiple delays is zero, which is related with the $\mathcal{H}_2$ norm analysis of DDAEs. The $\mathcal{H}_2$ norm is widely used as a performance index in the field of automatic control [24], and it has been object of study for time-delay systems in recent years (see [12], [20], [8] and the references therein). However, up to the best of the author's knowledge, there are no results addressing the analysis of the $\mathcal{H}_2$ norm of differential algebraic equations with multiple delays, despite its practical and theoretical relevance.

In contrast with the $\mathcal{H}_2$ norm of other classes of time-delay systems, the $\mathcal{H}_2$ norm of DDAE might be infinite even if the system has no seemingly feedthrough term or is stable (see [4], [9], for the one delay case). Solving Problem 1 allows us to provide a finite test for determining the finiteness of the $\mathcal{H}_2$ norm of differential algebraic equations with multiple delays.

The paper is organized as follows. In Section 2, we explain the motivation and background of Problem 1, and in Section 3, we introduce the m-dimensional CH theorem. The solution of Problem 1 is presented in Section 4. Moreover, we provide a more efficient algorithm in the case in which all the parameters are nonnegative. In Section 5, we show the link between Problem 1 and the finiteness analysis of the $\mathcal{H}_2$ norm of DDAE, leading to a tractable test for finiteness. We end the contribution with some final comments.

We adopt the following notation. The symbol $\mathbb{Z}$ denotes the set of integer numbers. The sets of real and positive real numbers including the zero are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. The sum over all indexes $(k_1, \dots, k_m)$ such that $k := (k_1, \dots, k_m) \in \Omega$, for a given set Ω, is denoted by $\sum\limits_{k\in\Omega}$. Multiple sums with the same lower limit i and upper limit j of the form $\sum\limits_{k_1=i}^{j} \cdots \sum\limits_{k_m=i}^{j}$ are denoted by $\sum\limits_{k_1,\dots,k_m=i}^{j}$ for shortness.

2 Background and motivation of the problem

Consider system

$$\begin{aligned} x(t) =& \sum_{j=1}^{m} A_j x(t-h_j) + Bu(t), \;\; t \geqslant 0, \\ y(t) =& Cx(t), \end{aligned} \tag{5}$$

where $0 < h_1 < \dots < h_m$ are the delays and matrices $A_1, \dots, A_m$, B and C as previously given. The delays might be rationally dependent or rationally independent. We say that h_j, $j = 1, \dots, m$, are rationally dependent delays if there exists a m-tuple $(c_1, \dots, c_m) \in \mathbb{Z}^m \setminus \{\vec{0}\}$ such that

$$\sum_{j=1}^{m} c_j h_j = 0. \tag{6}$$

If such m-tuple does no exist, then h_j, $j = 1, \dots, m$, are rationally independent delays, i.e. the only solution of (6) is $(c_1, \dots, c_m) = \vec{0}$.
The impulse response in the frequency domain of system (5) is given by the transfer matrix

$$G_a(s) := C\left(I - \sum_{j=1}^{m} A_j e^{-sh_j}\right)^{-1} B, \; s \in \mathbb{C}. \tag{7}$$

Determining whether G_a is zero is closely related with the analysis of the $\mathcal{H}_2$ norm of DDAE. More precisely, for given matrices A_j, B and C, $G_a(s) \equiv 0$ is a necessary and sufficient condition for the finiteness of the $\mathcal{H}_2$ norm of DDAE with multiple delays (this is formally proved in Section 5). In what follows, we establish a link between condition (1) and the condition under which $G_a(s) \equiv 0$. By considering the unit impulse in the input $u(t)$ in system (5), one obtains in the output

$$\begin{aligned} y(t) = C \sum_{k_1,\dots,k_m=0}^{\infty} & P_{k_1,\dots,k_m}(A_1, \dots, A_m) \cdot \\ & \cdot \delta\left(t - k_1 h_1 - \dots - k_m h_m\right) B, \end{aligned} \tag{8}$$

where $k_j \in \mathbb{Z}_+$, $j = 1, \dots, m$, and $\delta(t)$ is the Kronecker delta function (see Figure 1 for the sake of illustration). If the delays h_j, $j = 1, \dots, m$, are rationally in-

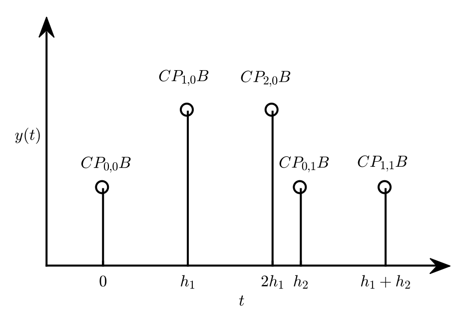

Fig. 1. Behavior of (8) on $t \in [0, h_1 + h_2]$ considering two delays. The values of $P_{k_1,k_2}(A_1, A_2)$ in the figure are as follows: $P_{00} = I$, $P_{1,0} = A_1$, $P_{2,0} = A_1^2$, $P_{01} = A_2$ and $P_{1,1} = A_1 A_2 + A_2 A_1$.

dependent, then it is not possible that for any m-tuples

$(k_1, \ldots, k_m) \in \mathbb{Z}_+^m$ and $(k_1^*, \ldots, k_m^*) \in \mathbb{Z}_+^m$ such that $(k_1, \ldots, k_m) \neq (k_1^\star, \ldots, k_m^\star)$ the equality

$$k_1 h_1 + \ldots + k_m h_m = k_1^* h_1 + \ldots + k_m^* h_m \tag{9}$$

holds. Hence, it follows from (8) that the impulse response of system (5) is determined by a sequence of products of the form

$$CP_{k_1,\ldots,k_m}(A_1, \ldots, A_m)B,\ k_j \in \mathbb{Z}_+,\ j = 1, \ldots, m,$$

and condition (1) is necessary and sufficient for $G_a(s) \equiv 0$.
In the case of rationally dependent delays only the sufficiency direction holds, i.e. condition (1) implies that $G_a(s) = 0$, but not the necessity. Indeed, in this case there exist two m-tuples $(k_1, \ldots, k_m) \in \mathbb{Z}_+^m$ and $(k_1^*, \ldots, k_m^*) \in \mathbb{Z}_+^m$ whose elements are not all zero such that $(k_1, \ldots, k_m) \neq (k_1^*, \ldots, k_m^*)$ and (9) holds. From (8), this implies that the output of system (5) at $t = k_1 h_1 + \ldots + k_m h_m$ contains at least terms of the form

$$CP_{k_1,\ldots,k_m}(A_1, \ldots, A_m)B + CP_{k_1^*,\ldots,k_m^*}(A_1, \ldots, A_m)B,$$

whose zero sum does not imply in general that each term is zero.
Thus, from the previous arguments, we arrive at the next proposition.

Proposition 1 *The following statements hold:*

(1) If the delays h_j, $j = 1, \ldots, m$, are rationally dependent, then condition (1) *is sufficient for $G_a(s) \equiv 0$.*
(2) If the delays h_j, $j = 1, \ldots, m$, are rationally independent then condition (1) *is necessary and sufficient for $G_a(s) \equiv 0$.*

The set of rationally independent delays is dense in $\mathbb{R}_+^m$, and in applications, model parameters are always subject to perturbations. If the delays correspond to independent parameters, it is natural to take into account small perturbations and test the most stringent criterion, corresponding to rationally independent delays. Hence, this criterion is necessary and sufficient if small delay perturbations are taken into account.
Sometimes delays might be rationally dependent as a result of mathematical modeling, e.g. when they depend on a smaller number of independent physical parameters, as in the case $(h_1, h_2, h_3) = (r_1, r_2, r_1 + r_2)$ with r_1 and r_2 the physical parameters. In such a case, it might not be possible to obtain rationally independent delays by perturbing the physical parameters. In general, if the delays $h_1, \ldots, h_m$ are rationally dependent, there always exist a smaller number p of rationally independent numbers $(r_1, \ldots, r_p)$ and a matrix $R \in \mathbb{Z}_+^{m \times p}$ of full column rank such that [17]

$$\begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix} = R \begin{pmatrix} r_1 \\ \vdots \\ r_p \end{pmatrix}.$$

With the following example we show how a delay-difference equation with rationally dependent delays can always be transformed into a delay-difference equation with rationally independent delay.

Example 2 *Consider system* (5) *with delays* $(h_1, h_2, h_3) = (r_1, r_2, r_1 + r_2)$*. We have that*

$$\begin{aligned} Bu(s) &= \left(I - A_1 e^{-sr_1} - A_2 e^{-sr_2} - A_3 e^{-s(r_1+r_2)}\right) x(s), \\ y(s) &= Cx(s). \end{aligned} \tag{10}$$

By using the change of variables

$$x_1(s) = x(s),\ \ x_2(s) = e^{-sr_2} x(s),$$

we rewrite (10) *as*

$$\begin{aligned} &\begin{pmatrix} B \\ 0 \end{pmatrix} u(s) = \\ &= \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} - \begin{pmatrix} A_1 e^{-sr_1} + A_2 e^{-sr_2} & A_3 e^{-sr_1} \\ e^{-sr_2} I & 0 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix}, \\ &y(s) = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix}. \end{aligned} \tag{11}$$

From (11)*, we obtain a transfer matrix of the form* (7) *which corresponds to the delay-difference equation*

$$\begin{aligned} x_l(t) &= \begin{pmatrix} A_1 & A_3 \\ 0 & 0 \end{pmatrix} x_l(t - h_1) \\ &\quad + \begin{pmatrix} A_2 & 0 \\ I & 0 \end{pmatrix} x_l(t - h_2) + \begin{pmatrix} B \\ 0 \end{pmatrix} u(t), \\ y(t) &= \begin{pmatrix} C & 0 \end{pmatrix} x_l(t). \end{aligned}$$

3 The m-dimensional CH theorem

In this section, we introduce the m-dimensional CH theorem. In order to do so, we consider the block matrix

$$D = \begin{pmatrix} D_{11} & \dots & D_{1m} \\ \vdots & & \vdots \\ D_{m1} & \dots & D_{mm} \end{pmatrix} \in \mathbb{R}^{mn \times mn}, \tag{12}$$

where $D_{ij} \in \mathbb{R}^{n \times n}$, and inspired by [7, 15], we introduce the next definition.

Definition 2 *Consider matrices*

$$D^{[1,0,\dots,0]} := \begin{pmatrix} D_{11} & \dots & D_{1m} \\ & 0 & \\ & 0 & \end{pmatrix}, \dots,$$

$$D^{[0,0,\dots,1]} := \begin{pmatrix} & 0 & \\ & 0 & \\ D_{m1} & \dots & D_{mm} \end{pmatrix}.$$

The m-tuple power of D, denoted by $D^{[k_1,\dots,k_m]}$, is defined as

$$\begin{aligned} D^{[k_1,\dots,k_m]} :=& D^{[1,\dots,0]} D^{[k_1-1,\dots,k_m]} + \\ &+ \dots + D^{[0,\dots,1]} D^{[k_1,\dots,k_m-1]}, \end{aligned} \tag{13}$$

where $k_j \in \mathbb{Z}_+$, $j = 1, \dots, m$, $D^{[0,\dots,0]} := I$, and $D^{[k_1,\dots,k_m]} := 0$ if any $k_j \in \mathbb{Z}_{--}$.

The following lemma, which relates a power of the block matrix D with a power in the sense of the previous definition, is key in the deduction of the main theorem of this section.

Lemma 3 *The equality*

$$D^j = \sum_{k \in \Omega_j} D^{[k_1,\dots,k_m]} \tag{14}$$

holds for any $j \in \mathbb{Z}_+$, where $k := (k_1, \dots, k_m)$ and $\Omega_j := \{k \in \mathbb{Z}^m : \sum_{i=1}^m k_i = j\}$.

PROOF. From (13), we have that for any $q \in \mathbb{Z}_+$

$$\begin{aligned} \sum_{k\in\Omega_q} D^{[k_1,\dots,k_m]} =& D^{[1,0,\dots,0]} \sum_{k\in\Omega_q} D^{[k_1-1,\dots,k_m]} \\ &+ D^{[0,1,\dots,0]} \sum_{k\in\Omega_q} D^{[k_1,k_2-1\dots,k_m]} + \dots + \\ &+ D^{[0,0,\dots,1]} \sum_{k\in\Omega_q} D^{[k_1,\dots,k_m-1]}. \end{aligned}$$

Notice that the function mapping $(k_1, \dots, k_m)$ into $(k_1 - 1, \dots, k_m)$ is a bijection from $\mathbb{Z}^m$ to $\mathbb{Z}^m$, and $(k_1, \dots, k_m) \in \Omega_q$ if and only if $(k_1 - 1, k_2, \dots, k_m) \in \Omega_{q-1}$. The same holds for $(k_1, k_2 - 1, \dots, k_m) \in \Omega_{q-1}$, and so on. Then, by considering the change of variable $(l_1, \dots, l_m) = (k_1 - 1, k_2, \dots, k_m)$ in the first sum, $(l_1, \dots, l_m) = (k_1, k_2 - 2, \dots, k_m)$ in the second one, and so on, and since $D^{[l_1,\dots,l_m]} = 0$ if any l_i is negative, the previous expression can be written as

$$\begin{aligned} \sum_{k\in\Omega_q} D^{[k_1,\dots,k_m]} &= D^{[1,0,\dots,0]} \sum_{l\in\Omega_{q-1}} D^{[l_1,\dots,l_m]} + \\ &+ D^{[0,1,\dots,0]} \sum_{l\in\Omega_{q-1}} D^{[l_1,l_2\dots,l_m]} + \dots + \\ &+ D^{[0,0,\dots,1]} \sum_{l\in\Omega_{q-1}} D^{[l_1,\dots,l_m]} = \\ &= \left(D^{[1,0,\dots,0]} + \dots + D^{[0,0,\dots,1]}\right) \sum_{l\in\Omega_{q-1}} D^{[l_1,\dots,l_m]} = \\ &= D \sum_{l\in\Omega_{q-1}} D^{[l_1,\dots,l_m]}, \end{aligned} \tag{15}$$

and the result directly follows by induction.

We are now in position to introduce the m-dimensional CH theorem. It is a generalization of the standard CH theorem in the sense that any m-tuple power of the matrix D as in Definition 2 such that $k_1 + \dots + k_m \geqslant mn$ can be expressed as a linear combination of m-tuple powers satisfying $k_1 + \dots + k_m < mn$ via a recursion formula (see [7] for the case $m = 2$). For the case where $k_1 = k_2 = \dots = k_m = n$ the recursion formula reduces to the formula in Theorem 2 of [15]. The presented proof is inspired by [22], and unlike the approach used in [21], the proof presented here does not require the concept of input nor output of any m-dimensional system.

Theorem 4 *Given a block matrix $D \in \mathbb{R}^{mn \times mn}$, partitioned as in (12), there exist $a_{k_1,\dots,k_m} \in \mathbb{R}$ such that*

$$D^{[n+p_1,\dots,n+p_m]} = - \sum_{\substack{k_1,\dots,k_m=0 \\ k\neq(n,\dots,n)}}^{n} a_{k_1,\dots,k_m} D^{[k_1+p_1,\dots,k_m+p_m]} \tag{16}$$

holds for any $(p_1, \dots, p_m) \in \mathbb{Z}^m$ satisfying $\sum_{i=1}^m p_i \geqslant 0$.

PROOF. We consider the function

$$f(x_1, \dots, x_m) := \det\left(\begin{pmatrix} x_1 I & \dots & 0 \\ & \ddots & \\ 0 & \dots & x_m I \end{pmatrix} - D\right).$$

We have

$$f(x_1,\dots,x_m)=\sum_{l_1,\dots,l_m=0}^{n} a_{l_1,\dots,l_m}x_1^{l_1}\dots x_m^{l_m},$$

where $a_{n,\dots,n}=1$, and

$$g(x)=\det(xI-D)=\sum_{j=0}^{mn} a_j x^j.$$

One observes that

$$f(x,\dots,x)=g(x),$$

which implies that

$$a_j=\sum_{l\in\Omega_j} a_{l_1,\dots,l_m},\ j=1\dots,mn, \tag{17}$$

where $a_{l_1,\dots,l_m}=0$ whenever any $l_i>n$, or any $l_i<0$, $i=1,\dots,m$. By the classic CH theorem, we have that

$$\sum_{j=0}^{mn} a_j D^j=0.$$

Then, by substituting (17) in the previous expression and using (14), we obtain

$$\sum_{j=0}^{mn}\sum_{l\in\Omega_j}\sum_{k\in\Omega_j} a_{l_1,\dots,l_m}D^{[k_1,\dots,k_m]}=0, \tag{18}$$

where $k=(k_1,\dots,k_m)$ and $l=(l_1,\dots,l_m)$. Notice that the coefficients $a_{l_1,\dots,l_m}$ are of degree $n-l_i$, $i=1,\dots,m$, in elements of the i^{th} block row of D, i.e. in elements of $D^{[1,\dots,0]}$, $D^{[0,1\dots,0]}$, …, $D^{[0,\dots,1]}$. Similarly, observe that $D^{[k_1,\dots,k_m]}$ are homogeneous polynomials of degree k_i in elements of the i^{th} block row of D. Hence, the left hand side of (18) is a sum of monomials of degree $n-l_i+k_i$, $i=1,\dots,m$, in elements of the i^{th} block row of D. Restricting (18) to the terms of order n, i.e. $k_i=l_i$, $i=1,\dots,m$, with $k\in\Omega_j$, we obtain

$$\sum_{j=0}^{mn}\sum_{k\in\Omega_j} a_{k_1,\dots,k_m}D^{[k_1,\dots,k_m]}=0. \tag{19}$$

A slightly different proof of (19) can be found in [15]. The above argument are however instrumental for the case where $(p_1,\dots,p_0)\neq(0,\dots,0)$, addressed in what follows.

Take now any $(p_1,\dots,p_m)\in\mathbb{Z}^m$ satisfying the restriction $\sum_{i=1}^{m}p_i=0$. Restricting (18) to the terms of degree $n+p_i$ in the i^{th} block row of D, $i=1,\dots,m$, we have that $k_i=l_i+p_i$, $i=1,\dots,m$, with $k\in\Omega_j$, and get

$$\sum_{j=0}^{mn}\sum_{k\in\Omega_j} a_{k_1,\dots,k_m}D^{[k_1+p_1,\dots,k_m+p_m]}=0. \tag{20}$$

Thus, since $a_{n,\dots,n}=1$, by (19) and (20), we have that

$$D^{[n+p_1,\dots,n+p_m]}=-\sum_{\substack{k_1,\dots,k_m=0\\ k\neq(n,\dots,n)}}^{n} a_{k_1,\dots,k_m}D^{[k_1+p_1,\dots,k_m+p_m]} \tag{21}$$

holds for all $(p_1,\dots,p_m)\in\Omega_0$. Then, by using the fact that

$$\begin{aligned}&\sum_{j=0}^{mn}\sum_{k\in\Omega_j} a_{k_1,\dots,k_m}D^{[k_1+p_1,\dots,k_m+p_m]}=\\ &=D^{[1,\dots,0]}\sum_{j=0}^{mn}\sum_{k\in\Omega_j} a_{k_1,\dots,k_m}D^{[k_1+p_1-1,\dots,k_m+p_m]}+\\ &+\dots+D^{[0,\dots,1]}\sum_{j=0}^{mn}\sum_{k\in\Omega_j} a_{k_1,\dots,k_m}D^{[k_1+p_1,\dots,k_m+p_m-1]},\end{aligned}$$

we obtain by induction that (21) holds for any $(p_1,\dots,p_m)\in\mathbb{Z}^m$ such that $\sum_{i=1}^{m}p_i\geqslant 0$.

4 Solution of Problem 1

Let us consider the block matrix

$$A=\begin{pmatrix}A_1&\dots&A_1\\ A_2&\dots&A_2\\ \vdots&&\vdots\\ A_m&\dots&A_m\end{pmatrix}\in\mathbb{R}^{mn\times mn},$$

which contains m matrices A_j in the j^{th} block row. The solution to Problem 1, provided in the next theorem, follows from Theorem 4 and a relation between the powers of matrix A in the sense of Definition 2, and polynomials of the form $P_{k_1,\dots,k_m}(A_1,\dots,A_m)$.

Theorem 5 *Condition* (1)*, i.e.* $CP_{k_1,\dots,k_m}(A_1,\dots,A_m)B=0$ *for all* $(k_1,\dots,k_m)\in\mathbb{Z}_+^m$*, is satisfied if and only if*

$$CP_{k_1,\dots,k_m}(A_1,\dots,A_m)B=0, \tag{22}$$

for all $(k_1,\dots,k_m)\in\mathbb{Z}^m$ *such that* $\sum_{j=1}^{m}k_j<mn$.

PROOF. The necessity is obvious. Let us prove the sufficiency. It follows by induction that the equality

$$P_{k_1,\dots,k_m}(A_1,\dots,A_m) = \frac{1}{m}\mathbf{I}_m A^{[k_1,\dots,k_m]}\mathbf{I}_m^T, \qquad (23)$$

is satisfied for any $(k_1,\dots,k_m) \in \mathbb{Z}^m$, where matrix $\mathbf{I}_m \in \mathbb{R}^{n\times mn}$ denotes a block matrix whose columns are m identity matrices of dimension n, i.e. $\mathbf{I}_m := \begin{pmatrix} I & \dots & I\end{pmatrix}$. By Theorem 4, we have that the block matrix A satisfies

$$\frac{1}{m}C\mathbf{I}_m A^{[l_1,\dots,l_m]}\mathbf{I}_m^T B = \\ = -\frac{1}{m}C\mathbf{I}_m \sum_{\substack{k_1,\dots,k_m=0\\ k\neq(n,\dots,n)}}^{n} a_{k_1,\dots,k_m} A^{[k_1+l_1-n,\dots,k_m+l_m-n]}\mathbf{I}_m^T B$$

for any $l := (l_1,\dots,l_m) \in \mathbb{Z}^m$ such that $l_1+\dots+l_m \geqslant mn$. Consider $l \in \Omega_{mn}$, and notice that $k_1+\dots+k_m < mn$ implies that the sum of the indexes of the m-tuple power of matrix A on the right hand side of the previous expression is less than mn, i.e

$$\sum_{j=1}^{m}(k_j+l_j-n) < mn.$$

Hence, by condition (22) and equation (23), we get

$$CP_{l_1,\dots,l_m}(A_1,\dots,A_m)B = \\ = \frac{1}{m}C\mathbf{I}_m A^{[l_1,\dots,l_m]}\mathbf{I}_m^T B = 0, \ \forall l \in \Omega_{mn}.$$

Consider now $l \in \Omega_{mn+1}$, then by the same arguments and using previous equation, we obtain that

$$CP_{l_1,\dots,l_m}(A_1,\dots,A_m)B = \\ = \frac{1}{m}C\mathbf{I}_m A^{[l_1,\dots,l_m]}\mathbf{I}_m^T B = 0, \ \forall l \in \Omega_{mn+1}.$$

Thus, it follows by induction that

$$CP_{l_1,\dots,l_m}(A_1,\dots,A_m)B = 0 \ \forall l_j \in \mathbb{Z}_+, \ j=1,\dots,m.$$

The number of products of the form (22) that one has to test in Theorem 5 is

$$1+\sum_{j=1}^{mn-1}\frac{(j+m-1)!}{(m-1)!j!}. \qquad (24)$$

One observes that it increases as the number m does. The numerical complexity in the test can be considerably reduced for the particular case in which all the elements of the matrices are nonnegative. This is shown in the next theorem.

Theorem 6 *Consider matrices $A_1,\dots,A_m \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times n_b}$ and $C \in \mathbb{R}_+^{n_c\times n}$. The following statements are equivalent:*

(1) There exists $k_j \in \mathbb{Z}_+$, $j=1,\dots,m$, such that

$$CP_{k_1,\dots,k_m}(A_1,\dots,A_m)B \neq 0. \qquad (25)$$

(2) There exists $\hat{k} \in \mathbb{Z}_+$ such that

$$C(A_1+\dots+A_m)^{\hat{k}}B \neq 0. \qquad (26)$$

Moreover, the latter condition can be checked in $O(n^2)$ operations, and if it holds, we have $\hat{k} = k_1 + \dots + k_m \leqslant n$.

PROOF. *Item 1 ⇒ Item 2.* Suppose that (25) holds. This implies that one of the terms of

$$CP_{k_1,\dots,k_m}(A_1,\dots,A_m)B,$$

denoted by $CM_{k_1,\dots,k_m}(A_1,\dots,A_m)B$, must be different from zero. Now, by using the fact that

$$(A_1+\dots+A_m)^q = \sum_{l\in\Omega_q} P_{l_1,\dots,l_m}(A_1,\dots,A_m)$$

for any $q \in \mathbb{Z}_+$, and that all the elements in the sum are nonnegative,

$$C\left(A_1+\dots+A_m\right)^{k_1+\dots+k_m}B = \\ = C\sum_{l\in\Omega_{k_1+\dots+k_m}} P_{l_1,\dots,l_m}(A_1,\dots,A_m)B \\ \geqslant CM_{k_1,\dots,k_m}(A_1,\dots,A_m)B, \quad (27)$$

where $A \geqslant B$ denotes the entry-wise inequality. This concludes the first part of the proof.

Item 2 ⇒ Item 1. For the reverse implication, observe that the sum in equation (27) is made of nonnegative terms, and thus it is nonzero if and only if one of the terms is nonzero. Let this term be $CM_{k_1^*,\dots,k_m^*}(A_1,\dots,A_m)B$, of order $k_j^* \in \mathbb{Z}_+$ in A_j. We have that (25) holds with these values of k_j^*, $j=1,\dots,m$, which concludes the reverse direction.

Condition (26) can be checked efficiently as follows: First, construct the directed graph on n vertices corresponding to the matrix $A_1+A_2+\dots+A_m$. It is easy to see that condition (26) holds if and only if this graph admits a path from a vertex v_i to a vertex v_j such that Ce_i and B^Te_j are both different from zero. This can be easily verified by performing a breadth-first search in the graph. The length of such a path would be smaller than n (since there are n vertices in the graph) and would provide constructively a corresponding product such that $CA_{\sigma_1}\ldots A_{\sigma_{(k_1^*+\cdots+k_m^*)}}B \neq 0$. The indices $\sigma_1,\ldots,\sigma_{(k_1^*+\cdots+k_m^*)}$ of this product can be obtained by labeling the edges of the graph depending on which matrix has the corresponding entry different from zero (i.e., give label k to the edge from node i to node j if the matrix A_k has a nonzero (i,j)-entry), and defining σ as the sequence of labels in the obtained path.

5 Analysis of the finiteness of $\mathcal{H}_2$ norm of DDAE

We consider system

$$\begin{aligned} E\frac{d}{dt}\tilde{x}(t) &= \sum_{j=0}^{m}\tilde{A}_j\tilde{x}(t-h_j)+\tilde{B}u(t),\ t\geqslant 0,\\ y(t) &= \tilde{C}\tilde{x}(t), \end{aligned} \tag{28}$$

where $h_0=0$, $\tilde{x}(t)\in\mathbb{R}^{\tilde{n}}$, matrix $E\in\mathbb{R}^{\tilde{n}\times\tilde{n}}$ is possibly singular with rank $E=r$, $\tilde{A}_j\in\mathbb{R}^{\tilde{n}\times\tilde{n}}$, $j=0,\ldots,m$, $\tilde{B}\in\mathbb{R}^{\tilde{n}\times n_b}$, and $\tilde{C}\in\mathbb{R}^{n_c\times\tilde{n}}$. Systems of the form (28) are amenable for modeling interconnected systems, which allows one to describe linear time invariant retarded and neutral systems with delays in states, inputs and outputs, and systems with a nontrivial feedthrough (see [10, 18]).

A problem that arises in the study of the $\mathcal{H}_2$ norm of system (28) is that it might be infinite even if the system is stable (see [4] and [9] for the one delay case). This can be seen from the fact that this class of systems might hide nontrivial feedthrough terms, as shown in the next example.

Example 3 *Consider matrices*

$$E=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix},\ \tilde{A}_0=\begin{pmatrix} H_0 & 0\\ 0 & -I\end{pmatrix},\ \tilde{A}_1=\begin{pmatrix} H_1 & 0\\ 0 & 0\end{pmatrix},$$

$$\tilde{B}=\begin{pmatrix} B\\ I\end{pmatrix},\ \tilde{C}=\begin{pmatrix} C_1 & D\end{pmatrix},$$

where matrices H_0, H_1, B, C_1 and D are of appropriate dimensions. For these matrices, considering $\tilde{x}=\begin{pmatrix} x_1^T & x_2^T\end{pmatrix}^T$, system (28) can be written as

$$\begin{aligned} \dot{x}_1(t) =& H_0x_1(t)+H_1x_1(t-h)+Bu(t),\ \ t\geqslant 0\\ y(t) =& C_1x_1(t)+Du(t), \end{aligned}$$

where one can observe that there is a feedthrough term from u to y.

In this section, we provide conditions for the finiteness of the $\mathcal{H}_2$ norm based on the results presented in Section 4. We first provide some basic facts concerning system (28), and then address the finiteness analysis of the $\mathcal{H}_2$ norm.

5.1 *Basic facts*

Consider matrices $\begin{pmatrix} U_1 & U_2\end{pmatrix}$ and $\begin{pmatrix} V_1 & V_2\end{pmatrix}$, which are the left and right factor of the Singular Value Decomposition of matrix E, respectively, where $U_1\in\mathbb{R}^{n\times r}$, $U_2\in\mathbb{R}^{n\times n-r}$, $V_1\in\mathbb{R}^{n\times r}$, and $V_2\in\mathbb{R}^{n\times n-r}$. We consider the following assumption.

Assumption 7 *Matrix $U_2^T\tilde{A}_0V_2$ is nonsingular.*

This assumption implies that the differentiation index is one (semi-explicit DDAE), and guarantees well posedness of the equation ([5], [16]). The change of coordinates

$$\tilde{x}(t)=\begin{pmatrix} V_1 & V_2\end{pmatrix}\begin{pmatrix} x_1(t)\\ x(t)\end{pmatrix},\ x_1(t)\in\mathbb{R}^r,\ x(t)\in\mathbb{R}^{\tilde{n}-r},$$

and premultipication of (28) by $\begin{pmatrix} U_1 & U_2\end{pmatrix}^T$ allows us to rewrite system (28) as coupled delay differential equations and delay difference equation:

$$\begin{aligned} \widetilde{E}\frac{d}{dt}x_1(t) =& \sum_{j=0}^{m}A_j^{(11)}x_1(t-h_i)\\ &+\sum_{j=0}^{m}A_j^{(12)}x(t-h_j)+B_1u(t)\\ x(t) =& \sum_{j=0}^{m}A_j^{(21)}x_1(t-h_j)\\ &+\sum_{j=1}^{m}A_jx(t-h_j)+Bu(t)\\ y(t) =& C_1x_1(t)+Cx(t), \end{aligned} \tag{29}$$

where we have assumed without loss of generality that $U_2^T\tilde{A}_0V_2=-I$ (otherwise it can be achieved by another transformation), $\widetilde{E}=U_1^TEV_1$,

$$\begin{aligned} A_j^{(11)} =& U_1^T\tilde{A}_jV_1,\ \ A_j^{(12)}=U_1^T\tilde{A}_jV_2,\\ A_j^{(21)} =& U_2^T\tilde{A}_jV_1,\qquad A_j=U_2^T\tilde{A}_jV_2, \end{aligned}$$

$j=0,\ldots,m$, and

$$B_1=U_1^T\tilde{B},\ B=U_2^T\tilde{B},\ C_1=\tilde{C}V_1,\ C=\tilde{C}V_2.$$

Existence and uniqueness of solutions, and stability properties of system (29) are discussed in [5] and [16].

5.2 *Finiteness of the $\mathcal{H}_2$ norm*

The $\mathcal{H}_2$ norm of an exponentially stable system (29) is defined by

$$\|G\|_{\mathcal{H}_2} := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}(G^*(i\omega)G(i\omega))d\omega}, \qquad (30)$$

where G is the transfer matrix of system (29), given by

$$G(s) := \begin{pmatrix} C_1 & C \end{pmatrix} \cdot$$
$$\cdot \begin{pmatrix} s\widetilde{E} - \sum_{j=0}^m A_j^{(11)} e^{-sh_j} & -\sum_{j=0}^m A_j^{(12)} e^{-sh_j} \\ -\sum_{j=0}^m A_j^{(21)} e^{-sh_j} & I - \sum_{j=1}^m A_j e^{-sh_j} \end{pmatrix}^{-1} \begin{pmatrix} B_1 \\ B \end{pmatrix}.$$

We notice that the transfer matrix G is the same as the transfer matrix $\tilde{C}\left(sE - \sum_{j=0}^m \tilde{A}_j e^{-sh_j}\right)^{-1}\tilde{B}$ of system (28). Hence, when we refer to the $\mathcal{H}_2$ norm of system (29), we equivalently refer to the $\mathcal{H}_2$ norm of system (28). For $x_1(t) \equiv 0$, one observes that (29) reduces to (5) with transfer matrix G_a in (7). We have the following result.

Theorem 8 *Let system* (29) *be exponentially stable. The $\mathcal{H}_2$ norm of system* (29) *is finite if and only if* $G_a(s) \equiv 0$.

PROOF. Let us introduce the matrix

$$G_b(s) := \begin{pmatrix} C_1 & C \end{pmatrix} \begin{pmatrix} F^{-1}(s) & G_b^{12}(s) \\ G_b^{21}(s) & G_b^{22}(s) \end{pmatrix} \begin{pmatrix} B_1 \\ B \end{pmatrix},$$

where

$$\begin{aligned} F(s) :=& s\widetilde{E} - A_{11}(s) + A_{12}(s)A_{22}^{-1}(s)A_{21}(s), \\ G_b^{12}(s) :=& -F^{-1}(s)A_{12}(s)A_{22}(s), \\ G_b^{21}(s) :=& -A_{22}^{-1}(s)A_{21}(s)F^{-1}(s), \\ G_b^{22}(s) :=& A_{22}^{-1}(s)A_{21}(s)F^{-1}(s)A_{12}(s)A_{22}^{-1}(s), \end{aligned}$$

with

$$A_{11}(s) := \sum_{j=0}^m A_j^{(11)} e^{-sh_j}, \;\; A_{12}(s) := \sum_{j=0}^m A_k^{(12)} e^{-sh_j},$$
$$A_{21}(s) := \sum_{j=0}^m A_j^{(11)} e^{-sh_j}, \;\; A_{22}(s) := -I + \sum_{j=1}^m A_j e^{-sh_j}.$$

By applying the formula of block matrix inversion, we have that

$$G(i\omega) = G_b(i\omega) - G_a(i\omega),\; \omega \in \mathbb{R}.$$

As the transfer matrix G_b is strictly proper and $\|G_a\|_{\mathcal{H}_2}$ is either zero or infinite, it follows from the previous expression that $\|G\|_{\mathcal{H}_2}$ is finite if and only if $G_a(s) \equiv 0$.

Now, a test for determining the finiteness of (30) can be directly deduced from the results previously presented in Section 2 and Section 4. More precisely, combining Proposition 1 with Theorem 5 leads to the following result.

Corollary 9 *Let system* (29) *be exponentially stable. The following statements hold:*

(1) If the delays h_j, $j = 1, \ldots, m$, are rationally dependent, then $\|G\|_{\mathcal{H}_2}$ is finite if (22) *is satisfied.*
(2) If the delays h_j, $j = 1, \ldots, m$, are rationally independent, then $\|G\|_{\mathcal{H}_2}$ is finite if and only if (22) *is satisfied.*

Remark 1 *In the context of the finiteness check of the $\mathcal{H}_2$ norm of DDAE (28), Equation (5) corresponds to the (delay) difference part, which boils down to the algebraic constraints in the delay-free case. (D)DAEs are mainly applied in modeling interconnected systems, where the algebraic equations describe the interconnections (e.g. $u_1 = y_1$, $u_2 = -y_2$ for a feedback interconnection of two subsystems with inputs u_1, u_2 and outputs y_1, y_2). For such systems the dimensions of matrices A_i, $i = 1, \ldots, m$, are determined by the number of inputs and outputs. For high-dimensional control systems, the number of state variables is typically large, while the number of inputs and outputs is still limited. In addition, not all delays in (28) might be effectively present in (5), i.e. some of the matrices A_i might be equal to zero, because only delayed terms related to direct paths from inputs to outputs, that form cycles (e.g. a control loop) are present. Hence, for the application to $\mathcal{H}_2$ norm analysis, both the dimensions of matrices A_i and the number of delays in (5) are expected to be very small, even for high-dimensional systems.*

6 Closing remarks

We solved Problem 1 by using the m-dimensional CH theorem, which is a generalization of the classic one in the sense of a recursion formula for m-tuple powers of a block matrix. It enabled us to present a finite test to determine whether the $\mathcal{H}_2$ norm of differential algebraic equations with multiple delays is finite.
A direction of future research consists of improving the performance of our algorithm. Indeed, checking conditions in Theorem 5 by brute force requires checking a number of equalities equal to (24). See also Remark 1 about the computational feasibility of the test.
Another direction includes the computation of the $\mathcal{H}_2$ norm of system (29) whenever it is finite. We note that

if condition (22) is strengthened to $CMB = 0$ for any monomial M in $(A_1, \ldots, A_m)$, then there exists a similarity transformation such that

$$\left(\{A_j\}_{j=1}^m,\, B,\, C\right) \to \left(\left\{ \begin{pmatrix} A_{j1} & 0 \\ A_{j2} & A_{j3} \end{pmatrix} \right\}_{j=1}^m, \begin{pmatrix} 0 \\ B_c \end{pmatrix}, \begin{pmatrix} C_u & 0 \end{pmatrix} \right),$$

which allows us to transform system (29) to a neutral type system and use standard tools for $\mathcal{H}_2$ norm computation. Hence, the focus point of research is the gap between (22) and the strengthened condition.

Finally, although Problem 1 is motivated by the $\mathcal{H}_2$ norm analysis of DDAEs, we believe that the presented results might be of interest in addressing a wider class of problems. For instance, the class of products in matrix polynomials of the form $P_{k_1,\ldots,k_m}(A_1, \ldots, A_m)$ also appears in the $\mathbb{R}^n$-controllability and $\mathbb{R}^n$-observability analysis of retarded type systems (see, for instance, Chapter 2 in [6]).

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