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On auto-and cross-interdependence in interval field finite element analysis

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Summary

This paper discusses the concepts of auto- and cross-interdependence in interval field finite element analysis. In classic interval analysis, independent intervals are used to construct hyper-rectangular input spaces that correspond to the bounded uncertainty that is present on some model parameters. This is a direct result from the inability of modeling inter-dependence. Such assumption of complete independence might prove in some cases to be highly over-conservative. A first example is the modeling of spatial uncertainty, where the interdependence is governed by allowable spatial gradients of field realizations. Secondly, interdependence can also occur in case uncertainty in several structural quantities has the same root cause (e.g., the manufacturing process). Recent work by the authors introduced concepts for modeling dependence between intervals in a spatial and multivariate context. However, it is unclear how an analyst has to deal with multiple quantities that have a spatial uncertainty component and are furthermore interdependent. This paper presents an approach to link multiple interval fields using recently introduced convex hull pair constructions and inverse distance weighting interpolation. Two case studies to illustrate the new methodology are included, proving the flexibility of the methodology in the modeling of auto- and cross-interdependence between multiple interval scalars and/or interval fields.

KEYWORDS:

interval analysis, cross-dependence, interval fields, spatial uncertainty, non-probabilistic analysis

1 | INTRODUCTION

In engineering practice, design calculations are often performed with incomplete knowledge on the real values of the parameters of the structure under consideration, as the corresponding quantities are usually inherently variable (e.g., wind loads on a building), insufficiently quantified (e.g., not directly measurable quantities), or a combination of both¹. When a pure deterministic representation of those model quantities is used to design and optimize the structure, typically a large degree of conservatism is included to ensure reliability and safety. This however impairs the benefits that are obtained by performing the design optimization in the first place, and is hence economically not optimal. Therefore, recent trends in computer aided engineering and scientific computing focus on the application of advanced methodologies for the modeling, propagation and quantification of non-deterministic model quantities in a parametric or non-parametric way². Available tools include methods from the highly matured probabilistic framework (e.g., advanced Monte Carlo methods³ or techniques based on Polynomial Chaos Expansions⁴), possibilistic techniques such as interval⁵, fuzzy^{6,7} or convex arithmetical approaches⁸, or techniques based on the concepts of imprecise probability^{9,10}.

Following an interval finite element approach, the non-determinism in selected model quantities is represented as a bounded interval, and no assumption on the relative likelihood of parameter values within this interval is presupposed¹¹. This is often also referred to as uncertain-but-bounded quantities. One of the key challenges in the application of interval theoretical approaches for the modeling and propagation of uncertainty in numerical models is the by-definition orthogonality of multiple intervals. Indeed, since intervals are not capable of modeling interdependence between multiple parameters, typically results of an interval finite element computation are over-conservative as the interval bounds expand during multiple interval arithmetical operations¹². A large body of literature is therefore dedicated to limiting this over-conservatism by e.g., employing element-by-element approaches¹¹ or affine arithmetic¹³. A second challenge lies in the modeling of interdependence between multiple interval valued model quantities. To model such interdependence in a spatial or time-domain, interval fields were introduced by the authors in¹⁴, further expanded in¹⁵ and applied to modeling spatial uncertainty in additive manufactured components in¹⁶. The main idea of the interval field approach is to expand globally defined intervals over the model domain using an priori defined set of basis functions. Also, other methods for modeling interval fields have been introduced more recently by other authors based on affine arithmetic^{17,18}, a spatial averaging method¹⁹ or set-theoretical approaches for time-domain problems²⁰. Recently, also fuzzy fields were introduced²¹. In case dependence between multiple non-deterministic context is considered, Elishakoff et. al introduced several set-theoretical approaches to cope with dependence in a non-probabilistic way^{22,23,24,25}. Following the most basic approach, the interdependence is represented as a d -dimensional hyper-ellipsoid which should abide by some minimum volume property. Also extensions towards Lamé curves and other, nodal, convex sets were introduced in recent years^{26,27}. However, while providing the analyst with an intuitive tool, the explicit assumption is still that all parameters are governed by a single underlying dependence structure (e.g., ellipsoidal). A solution hereto could be to only consider pair-wise dependence, but this neglects possible higher-order dependence structures between multiple model quantities. The authors recently introduced a flexible yet powerful method to model inter-dependence between multiple intervals in a finite element context²⁸ based on convex hull pair constructions, a non-probabilistic counterpart for the probabilistic framework of copula pair constructions^{29,30}. Instead of aiming at the modeling of the full joint-dependence between all interval-valued parameters at the same time, the idea is to model bivariate admissible sets representing bivariate interdependence. Hereto, bivariate sets that are unconditional on other sets of interval scalars, as well as bivariate sets that are conditional on other scalars are combined via set-theoretical operations into a full admissible set. This allows modeling interdependence with great flexibility following a very intuitive approach.

This paper presents an extension of this method towards modeling cross-interdependence between interval scalars and interval fields, as well as between interval fields. The method allows for the first time an analyst to model multiple cross-interdependent spatially uncertain parameters in an interval context. The core idea is to introduce inter-dependence into the interval field series expansion by constructing an admissible set for the globally defined interval scalars in the interval field, following a $\mathcal{C}$ -vine admissible set decomposition²⁸. Also, the concept of interdependence in an interval context is deepened and extended towards modeling and quantifying non-linear interdependence. The paper is structured as follows. Section 2 explains the basics of interval finite element analysis and extends the framework with the recently introduced convex hull pair constructions to decompose the admissible set of dependent intervals. Section 3 starts by deepening the theoretical foundation of interval field finite element analysis via Inverse Distance Weighting interpolation and expanding on the nature of the auto-interdependence in such an interval field. Then, an approach to model cross-interdependence between interval fields via $\mathcal{C}$ -vine admissible set decomposition is presented. Sections 4 and 5 present two case studies to illustrate the developed approaches. Finally, section 6 gives the most important conclusions.

2 | COMPUTING WITH DEPENDENT INTERVALS

2.1 | Independent interval finite element method

Let $\mathcal{M}$ be a deterministic numerical model that is used to approximate $\mathbf{y} \in \mathbb{R}^{d_s}$, the solution of a (set of) differential equations, through a set of (usually) real-valued function operators $\mathbf{g} = \{g_i \mid i = 1, \dots, d_s\}$:

$$\mathcal{M}(\mathbf{x}) : \mathbf{y} = \mathbf{g}_i(\mathbf{x}), \quad g_i : \mathbb{R}^{d_i} \mapsto \mathbb{R}, i = 1, \dots, d_s \quad (1)$$

with $\mathbf{x} \in \mathcal{F} \subset \mathbb{R}^{d_i}$ the vector of model parameters and $\mathcal{F}$ the sub-domain of feasible parameters (e.g., non-negative contact stiffness). Following an interval approach, the uncertainty that is attributed to $\mathbf{x}$ is modeled as an interval vector $\mathbf{x}^I \in \mathcal{F}^I \subset \mathbb{R}^{d_i} = \mathbb{R}^{d_i} \times \mathbb{R}^{d_i} \times \dots \times \mathbb{R}^{d_i}$, with $\mathbb{R}^{d_i}$ the space of d_i -dimensional interval vectors and $\times$ denoting the Cartesian product.

Therefore, by construction, $\mathbf{x}^I$ spans a hyper-rectangular space in $\mathbb{R}^{d_i}$ and hence, all x_i , $i = 1, \dots, d_i$ are orthogonal, and as such independent.

The interval FE method generally aims at finding a solution set $\tilde{\mathbf{y}}$ containing the extreme realizations of $\mathbf{x}^I$, which is defined as:

$$\tilde{\mathbf{y}} = \{\mathbf{y} \mid \mathbf{x} \in \mathbf{x}^I, \mathbf{y} = \mathcal{M}(\mathbf{x})\} \quad (2)$$

In general, $\tilde{\mathbf{y}}$ spans a non-convex manifold in $\mathbb{R}^{d_s}$, since $\mathcal{M}$ provides possibly nonlinear coupling between at least a subset of y_i , $i = 1, \dots, d_s$. Therefore, $\tilde{\mathbf{y}}$ is commonly approximated by an uncertain realization set $\tilde{\mathbf{y}}_s$, which is obtained by propagating q well selected deterministic realizations $\mathbf{x}_i$ of $\mathbf{x}^I$:

$$\tilde{\mathbf{y}}_s = \{\mathbf{y} \mid \mathbf{x}_i \in \mathbf{x}^I, \mathbf{y}_i = \mathcal{M}(\mathbf{x}_i), i = 1, \dots, q\} \quad (3)$$

These q samples can for instance be generated for linear $\mathcal{M}(\mathbf{x})$ using the Cauchy deviates method³¹, or the Transformation Method for more general non-monotonic models³². As an alternative approach, also techniques based on affine arithmetic and the interval rational series expansion method have recently been proposed^{33,13}. In case only the hyper-rectangular approximation $y_1^I \times y_2^I \times \dots \times y_{d_s}^I$ of $\tilde{\mathbf{y}}_s$ is of interest, following optimization problem is usually solved⁵:

$$\begin{aligned} \underline{y}_i &= \min_{\mathbf{x} \in \mathbf{x}^I} g_i(\mathbf{x}) & i = 1, \dots, d_s \\ \bar{y}_i &= \max_{\mathbf{x} \in \mathbf{x}^I} g_i(\mathbf{x}) & i = 1, \dots, d_s \end{aligned} \quad (4)$$

Also series-expansion based methods have been presented in literature (e.g.,^{34,35}). However, these methods are found to be mostly accurate when the intervals are relatively small or when $\mathcal{M}(\mathbf{x})$ is *sufficiently* linear³⁶. (note: *sufficiently* implies that the need for linearity of $\mathcal{M}(\mathbf{x})$ depends on the width of the interval bounds that have to be propagated)

A general mathematical handle to the boundaries of $\tilde{\mathbf{y}}_s$ is provided by means of a convex approximation: the convex hull $C(\tilde{\mathbf{y}})$ of the realization set $\tilde{\mathbf{y}}_s$ ³⁷. $C(\tilde{\mathbf{y}})$ can be considered as a set of vertices bounding the uncertain realization set:

$$C(\tilde{\mathbf{y}}) = \left\{ \sum_{i=1}^q \beta_i \mathbf{y}_i \mid (\forall i : \beta_i \geq 0) \wedge \sum_{i=1}^q \beta_i = 1 ; \mathbf{y}_i \in \tilde{\mathbf{y}} \right\} \quad (5)$$

with $\boldsymbol{\beta}$ a vector of weighting factors, such that all elements β_i are non-negative and sum to one. Alternatively, $C(\tilde{\mathbf{y}})$ can also be represented as a set of d_s -dimensional linear inequalities that describe the boundaries of $C(\tilde{\mathbf{y}})$:

$$C(\tilde{\mathbf{y}}) \equiv [\mathbf{f}_1, \dots, \mathbf{f}_h]^T = \mathbf{A}\mathbf{y}^T - \mathbf{b} \geq 0 \quad (6)$$

with $\mathbf{A} \in \mathbb{R}^{h \times d_s}$, $\mathbf{b} \in \mathbb{R}^h$ and h the number of half-spaces f_i , $i = 1, \dots, h$. According to the Minkowsky-Weyl theorem, both representations are equivalent. Note that any other data-enclosing convex approximation (using for instance set-theoretical concepts), will always include more conservatism in the analysis. This can for instance be warranted to account for data scarcity as presented in³⁸.

2.2 | Admissible set decomposition for dependent interval analysis

In recent work by the authors²⁸, a methodology is presented to model high-dimensional dependence between interval scalars following a vine copula-like decomposition of the admissible set. For ease of reading, the method to model scalar dependence is explained first in detail, since the approach that is presented in this paper builds upon these results to model cross-interdependence between multiple interval fields.

As a basis for modeling dependence between a set of intervals $\mathbf{x}^I$, a set $\mathcal{D}$ is defined to limit the range of admissible parameter combinations. This set is denoted the admissible set, and as such, the analysis becomes more generally set-theoretical as multi-variate intervals, or interval vectors, are a very specific type of convex sets: hyper-rectangles. The admissible set decomposition method²⁸ is an efficient and versatile method for the decomposition of a general high-dimensional $\mathcal{D}$ into a set of conditional and unconditional bi-variate admissible sets, and is comparable to the probabilistic concept of copula pair constructions, which are also referred to as ‘vine copula’ or copula-pair constructions²⁹.

Specifically, it is proposed to formulate the $\mathfrak{D}$ -vine decomposition of the admissible set $\mathcal{D}_{1, \dots, d_i}(x_1^I, x_2^I, \dots, x_{d_i}^I)$ as:

$$\mathcal{D}_{1, \dots, d_i}(x_1^I, x_2^I, \dots, x_{d_i}^I) = \bigotimes_{k=1}^{d_i} x_k^I \bigcap_{j=1}^{d_i-1} \bigcap_{i=1}^{d_i-j} \mathcal{D}_{i, i+j|i+1, \dots, i+j+1} \quad (7)$$

where the operator $\otimes$ is used to denote the Cartesian product of the intervals, and hence, the first part of the equation describes the d_i dimensional hyper-cube of independent intervals. The dependence is included by computing intersections of this hyper-cubic space with (conditional) bivariate admissible sets that are either defined a priori by the analyst, fitted to existing data or determined in an inverse approach (as e.g., presented in^{37,39}). Similarly, a $\mathfrak{C}$ -vine decomposition is expressed as:

$$D_{1,\dots,d_i}(x_1^I, x_2^I, \dots, x_{d_i}^I) = \bigotimes_{k=1}^{d_i} x_k^I \bigcap_{j=1}^{d_i-1} \bigcap_{i=1}^{d_i-j} D_{j,j+i|1,\dots,j-1} \quad (8)$$

which is advantageous when the dependence of one parameter with all other considered parameters is easier to quantify than a set of pairwise dependencies (e.g. due to availability of data)⁴⁰. In that case, the lower-order dependence structure of the admissible set can be build completely around this central parameter.

Hence, the admissible combinations of parameter values inside the hyper-cubic domain $\mathbf{x}^I$ can be fully described by a set of bivariate admissible sets D_{ij} . Since the definition of a two-dimensional admissible set is much more intuitive from an analysts point of view, this is easier as compared to defining the full d_i -dimensional D . Furthermore, since the higher-order, conditional $D_{ij|k}$ can be made a function of x_k , this allows for the highly flexible modeling of complex dependence structures in an interval context.

This concept is also illustrated in figure 1 . This figure shows the unit cube $[0, 1]^3$, together with **three arbitrarily selected admissible sets** D_{12} , D_{23} and $D_{13|2}$. As can be noted, the set $D_{13|2}$ is not necessarily constant over x_2 , allowing for the definition of highly complicated dependence structures with a very limited set of parameters.

FIGURE 1 Decomposition of the $d = 3$ -dimensional dependence using the proposed admissible set decomposition via a $\mathfrak{D}$ -vine structure²⁸ with bi-variate admissible sets D_{12} , D_{23} and $D_{13|2}$, where the latter can be modeled as a function of x_2 . The hyper-cube is defined with $\underline{x}_1 = \underline{x}_2 = \underline{x}_3 = 0$ and $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 1$.

Note that, while employing a similar decomposition structure as copula pair constructions in a probabilistic context, the underlying meaning is fundamentally different. Instead of decomposing a multivariate density function into its marginals and a set of bivariate copula, this method presents an approach to limit the hypercube $x_1^I \times \dots \times x_d^I$ of admissible values by computing intersections with two-dimensional admissible sets. While similar, still no inference of the likelihood of certain values within this set is made or aimed at. As such, the interval paradigm is not violated following this approach. The admissible set can then be propagated towards a convex approximation of the uncertain realization set (see eq. (5)) through the numerical model using either a constrained global optimization approach, **by using the halfspaces bounding D as inequality constraints**, or the Extended Transformation Method (the reader is referred to²⁸ for more information on these techniques, as well as a more rigorous explanation of the methodology and underlying ideas).

3 | COMPUTING WITH CROSS- INTERDEPENDENT INTERVAL FIELDS

3.1 | Interval field finite element computations

Following a classical interval finite element approach, only independent intervals can be propagated through $\mathcal{M}(\mathbf{x})$. In case a model quantity $\mathbf{x}(\mathbf{r})$ is considered that has a spatial nature over a discretized d_d -dimensional model domain $\Omega \subset \mathbb{R}^{d_d}$ with spatial parameter $\mathbf{r} \in \Omega$ following the classical interval framework, usually two approaches can be applied as visualized in figure 2 . A first naive approach attributes a single interval to each discrete element $\Omega_e \subset \Omega, e = 1, \dots, N_e$, with Ω_e for instance corresponding to the N_e finite elements in case $\mathcal{M}(\mathbf{r})$ is a finite element model. This however provides the analyst with spurious results, as unrealistic realizations are explicitly taken into account in the analysis since intervals are independent by nature, and hence, realizations of the uncertain field with large spatial discontinuities are explicitly included in the analysis. Furthermore, this inflates the dimension of the hyper-rectangular space drastically, which makes application to even medium-sized models intractable due to the so-called *curse of dimensionality*. On the other hand, an analyst can attribute a single interval $[\min_{\Omega}(\mathbf{x}(\mathbf{r})); \max_{\Omega}(\mathbf{x}(\mathbf{r}))]$ that captures the extremes over Ω to the spatially uncertain parameter. While very efficient, this neglect of the spatial nature of the uncertain model quantity might yield severely over-conservative results, especially when the uncertain model quantity is not homogeneously uncertain over the model domain¹⁵.

FIGURE 2 L: fully independent interval uncertainty, where $\underline{x}$ and $\bar{x}$ are attributed to each $\Omega_e \in \Omega$ with the corresponding spurious realizations; R: effect of considering a single interval in case the parameter is not homogeneously uncertain over Ω

In recent work, the authors therefore introduced the concept of *interval fields* as an interval counterpart to the established framework of Random Fields⁴¹. Following the explicit formulation introduced in¹⁴, an interval field $\mathbf{x}^I(\mathbf{r}) : \Omega \times \mathbb{R}^{n_b} \mapsto \mathbb{R}$ is defined as:

$$\mathbf{x}^I(\mathbf{r}) = \sum_{i=1}^{n_b} \boldsymbol{\psi}_i(\mathbf{r}) \alpha_i^I \quad (9)$$

and can be interpreted as the superposition of $i = 1, \dots, n_b \in \mathbb{N}$ base functions $\boldsymbol{\psi}_i(\mathbf{r}) : \Omega \mapsto [0, 1]$, which represent a set of spatial uncertainty *patterns*, scaled by independent interval scalars $\alpha_i^I \in \mathbb{R}$, which represent magnitude of the uncertainty. This section re-iterates on these recent developments and provides a more theoretical foundation to the nature of the dependence between local intervals in $\mathbf{x}^I(\mathbf{r}_i)$ that is modeled by the basis functions. This foundation is later on used to model and explain cross-interdependence between multiple interval fields.

In this paper, the basis functions are constructed using inverse distance weighting interpolation⁴², as first applied in previous work of the authors (see e.g.,¹⁵ or¹⁶). Following this approach, the base functions $\boldsymbol{\psi}_i(\mathbf{r}) : \mathbb{R}^{N_e} \mapsto \mathbb{R}$ are discretized over the N_e elements $\Omega_e \subset \Omega$ of the FE mesh and constructed according to:

$$\boldsymbol{\psi}_i(\mathbf{r}) = \frac{w_i(\mathbf{r})}{\sum_{j=1}^{n_b} w_j(\mathbf{r})} \quad (10)$$

with $w_i(\mathbf{r})$ a weight function that depends on the Euclidean distance $d()$ between so-called control points $\hat{\mathbf{r}}$ and the rest of the domain:

$$w_i(\mathbf{r}) = \frac{1}{[d(\hat{\mathbf{r}}, \mathbf{r})]^p} \quad (11)$$

with $p \in \mathbb{R}^+$ a scalar that controls the gradients of the basis functions. Also other measures of distance than the Euclidean distance can be included in this equation. Note that the basis functions are not orthogonal:

$$\langle \psi_i(\mathbf{r}), \psi_j(\mathbf{r}) \rangle = \int_{\Omega_e} \psi_i(\mathbf{r}) \psi_j(\mathbf{r}) \neq \delta_{ij} \quad (12)$$

with $\langle \bullet, \bullet \rangle$ the inner product operator and δ_{ij} the Kronecker Delta, which has implications for the modeling of the dependence between interval scalars. Finally, the interval field is then explicitly defined by attributing an interval scalar α_i^I to each control point:

$$x^I(\mathbf{r}) = \sum_{i=1}^{n_b} \frac{w_i(\mathbf{r})}{\sum_{j=1}^{n_b} w_j(\mathbf{r})} \alpha_i^I \quad (13)$$

Hence, these local interval scalars α_i^I , attributed to a location $\hat{\mathbf{r}}$ are interpolated to $\mathbf{r} \in \Omega_{e,\hat{\mathbf{r}}}$, which is the model domain excluding the control point locations. The local dependence of intervals in $\Omega_{e,\hat{\mathbf{r}}}$ is therefore controlled by the local value of all basis functions $\psi_i(\mathbf{r})$, $i = 1, \dots, n_b$ and the dependent intervals in $\Omega_{e,\hat{\mathbf{r}}}$ as such are a linear combination of α_i^I at $\hat{\mathbf{r}}$. Therefore, the admissible set of possible parameter values (i.e., the set of admissible realizations of the interval field $x^I(\mathbf{r})$) is limited by the basis functions of the interval field¹⁵.

The independence between the globally defined interval scalars at the control points stems from the fact that all $\psi_i(\mathbf{r})$ behave as unity vectors in the subset $\mathcal{R} = \{\hat{\mathbf{r}}\} \subset \Omega_e$. When considering the local intervals of the interval field, evaluated at the control points, it can be argued that the interval scalars are projected on an orthogonal basis $\mathcal{B}_{\hat{\mathbf{r}}} \in \mathbb{R}^{n_b \times n_b}$:

$$\mathcal{B}_{\hat{\mathbf{r}}} = \begin{Bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{Bmatrix} \quad (14)$$

stemming directly from the interval field series expansion shown in eq. (9), with the basis functions defined for instance as in eq. (10). Indeed, considering eq. (10), one can see that for $\mathbf{r}_i = \hat{\mathbf{r}}_j$, $\psi_j(\mathbf{r}_i) = 1$ whereas $\psi_j(\hat{\mathbf{r}}_j) = 0$ when evaluating the basis function at all other control point locations $\mathbf{r}_j$. As such, the interval scalars defined at $\hat{\mathbf{r}}$ remain fully orthogonal, and hence, independent.

However, when local intervals, obtained by evaluating the interval field shown in eq. (9) of the model domain $\mathbf{r}_i \subset \hat{\mathcal{R}}$, with $\hat{\mathcal{R}}$ all locations in Ω_e excluding the control point locations, this orthogonality no longer holds. Indeed, the basis $\mathcal{B}_{\hat{\mathbf{r}}}$ for the subset of the model locations without the control points of the field, on which the interval scalars are projected to obtain their local value in the domain is for these locations no longer orthogonal. Specifically, for $\Omega_{e,\hat{\mathbf{r}}}$ it is defined as d:

$$\mathcal{B}_{\hat{\mathbf{r}}} = \begin{Bmatrix} \psi_1(\mathbf{r}_1) & \psi_2(\mathbf{r}_1) & \dots & \psi_{n_b}(\mathbf{r}_{e-n_b}) \\ \psi_1(\mathbf{r}_2) & \psi_2(\mathbf{r}_2) & \dots & \psi_{n_b}(\mathbf{r}_{e-n_b}) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(\mathbf{r}_{e-n_b}) & \psi_2(\mathbf{r}_{e-n_b}) & \dots & \psi_{n_b}(\mathbf{r}_{e-n_b}) \end{Bmatrix} \quad (15)$$

This is a direct result from the interval field property shown in eq. (12). Since the basis functions are not orthogonal, any $\mathcal{B}_{\hat{\mathbf{r}}} \subset \mathcal{B}_{\hat{\mathbf{r}}}$ does not form an orthogonal basis for the projection of the global interval scalars α^I . Hence, local dependence between the interval scalars is enforced in the interval field as the projection of a hyper-rectangle onto a non-orthogonal basis generally renders a convex polytope. This polytope can be considered as an admissible set $\mathcal{D}$ limiting the admissible combination of parameter values throughout Ω_e . Since the degree of non-orthogonality is definable by both the controlling scalar p and the number n_b and location $\hat{\mathbf{r}}$ of the control points in eq. (10), the degree of local dependence in $x^I(\mathbf{r})$ can be tailored completely by the analyst.

Finally, since according to eq. (14) the interval scalars α_i^I remain independent, commonly applied techniques for the propagation of (independent) interval uncertainty remain applicable, and the uncertain realization set is defined as:

$$\tilde{\mathbf{y}}_s = \{ \mathbf{y} \mid \mathbf{x}_i(\mathbf{r}) \in x^I(\mathbf{r}), \mathbf{y}_i = \mathcal{M}(\mathbf{x}_i(\mathbf{r})), i = 1, \dots, q \} \quad (16)$$

3.2 | Admissible set decomposition for cross-dependence in interval fields

In case an analyst is faced with multiple spatially uncertain parameters that are to be modeled using the interval field framework, as explained in section 3, the assumption that both interval fields are fully independent might not be justified. A possible case when interdependence cannot be safely ignored when the uncertainty in the parameters that are modeled stem from a common root-cause (e.g., local uncertainty on the width and thickness of a welded beam) In that context, a measure of cross-dependence should be introduced in the analysis to limit over-conservatism in the obtained results.

In this paper, we propose to use a $\mathfrak{C}$ -vine coupling, as described in eq. (8), to couple multiple interval fields. For brevity of notation and to illustrate the main ideas, the methodology is explained to model the coupling between two interval fields. Note that such explanation is without loss of generality, as adding more interval fields to the analysis is a straightforward expansion of the explained ideas. These two fields are defined as:

$$^{(1)}x^I(\mathbf{r}) = \sum_{i=1}^{n_{b1}} \alpha_i^{I(1)} \psi_i(\mathbf{r}) \quad (17)$$

with $^{(1)}\psi_i(\mathbf{r}) : \mathbb{R}^{N_e} \mapsto \mathbb{R}$ the basis functions discretized over Ω_e and $\alpha_i^I \in \mathbb{IR}$ the corresponding interval scalars and

$$^{(2)}x^I(\mathbf{r}) = \sum_{i=1}^{n_{b2}} \beta_i^{I(2)} \psi_i(\mathbf{r}) \quad (18)$$

with $^{(2)}\psi_i(\mathbf{r}) : \mathbb{R}^{N_e} \mapsto \mathbb{R}$ the basis functions discretized over Ω_e and $\beta_i^I \in \mathbb{IR}$ the corresponding interval scalars. Note that following properties for two uncoupled interval fields hold:

$$D(\alpha^I) = \bigotimes_{i=1}^{n_{b1}} \alpha_i^I \quad (19a)$$

$$D(\beta^I) = \bigotimes_{i=1}^{n_{b2}} \beta_i^I \quad (19b)$$

$$D(\alpha^I, \beta^I) = \bigotimes_{i=1}^{n_{b1}} \alpha_i^I \bigotimes_{j=1}^{n_{b2}} \beta_j^I \quad (19c)$$

with $\bigotimes$ the Cartesian product operator. Hence, all globally defined interval scalars are fully orthogonal when no coupling between two interval fields is present. Note that due to the non-orthogonality of the functions inside Ω_e , the intervals evaluated at certain locations $\mathbf{r}_i \subset \Omega_e$ are dependent and limited by an admissible set, as explained in section 3.1.

To introduce dependence between these interval fields, it is proposed to employ the methodology introduced in section 2 to limit the admissible set of joint globally defined intervals. Specifically, $^{(1)}x^I(\mathbf{r})$ is used as a pivot point in a $\mathfrak{C}$ -vine with which the dependence of all scalars in $^{(2)}x^I(\mathbf{r})$ is modeled via a bi-variate admissible set:

$$D(\alpha_l^I, \beta_1^I, \beta_2^I, \dots, \beta_{n_{b2}}^I) = \bigotimes_{k=1}^{n_{b2}+1} x_{k,l}^I \bigcap_{j=1}^{n_{b2}} \bigcap_{i=1}^{n_{b2}+1-j} \mathcal{D}_{j,j+i|1,\dots,j-1,l} \quad (20)$$

with $l = 1, 2, \dots, n_{b1}$, $x_l^I = [x_l^I, \beta_1^I, \beta_2^I, \dots, \beta_{n_{b2}}^I]$, k a counter running over the entries of x_l^I and $\mathcal{D}_{j,j+i|1,\dots,j-1,l}$ a bivariate admissible set. In the context of modeling dependence between two interval fields, this expansion is simplified to retain only first-order (unconditional) admissible sets:

$$D(\alpha_l^I, \beta_1^I, \beta_2^I, \dots, \beta_{n_{b2}}^I) = \bigotimes_{k=1}^{n_{b2}+1} x_{k,l}^I \bigcap_{i=2}^{n_{b2}+1} \mathcal{D}_{1,i} \quad (21)$$

In a probabilistic context, cross-correlation between random fields is usually modeled using concepts based on correlation, which form a linearization of the dependence structure between the random fields. Following a non-probabilistic concept, this corresponds to a convex admissible set. In this paper, we therefore propose to use a diagonal-band copula-like function to construct the bivariate admissible sets. To model negative dependence between two interval scalars, this function is defined as:

$$D_{ij} = 1 - \mathcal{H}(|x_i + x_j - 1| - \theta) \quad \forall x_i, x_j \in \mathbf{x}^I \quad (22)$$

with $\mathcal{H}$ the well-known Heaviside function, and θ a parameter to control the width of the diagonal band, and hence, the dependence between x_i and x_j . Equation 21 for the modeling of cross-interdependence then becomes:

$$D(\alpha_1^I, \beta_1^I, \beta_2^I, \dots, \beta_{n_{b2}}^I) = \bigotimes_{k=1}^{n_{b2}+1} x_{k,l}^I \bigcap_{i=2}^{n_{b2}+1} 1 - \mathcal{H}(|x_1 + x_i - 1| - \theta_{1i}) \quad (23)$$

with θ_i the dependence between x_1 and x_i , $i = 1, \dots, n_{b2} + 1$. Conversely, to model positive linear dependence between two intervals, the dependence function is defined as:

$$D_{ij} = 1 - \mathcal{H}(|x_2 - x_1| - \theta) \quad \forall x_i, x_j \in \mathbf{x}^I \quad (24)$$

and hence, equation 21 becomes:

$$D(\alpha_1^I, \beta_1^I, \beta_2^I, \dots, \beta_{n_{b2}}^I) = \bigotimes_{k=1}^{n_{b2}+1} x_{k,l}^I \bigcap_{i=2}^{n_{b2}+1} 1 - \mathcal{H}(|x_i - x_1 - 1| - \theta_{1i}) \quad (25)$$

Both bivariate admissible sets are visualized in figure 3 when applied to two arbitrary parameters x_1 and x_2 . In the context of coupling two interval fields, x_1 and x_2 can be interpreted as two globally defined interval scalars of the interval field.

FIGURE 3 Illustration of a positive (striped lines) and negative (full lines) bivariate admissible sets based on the Heaviside function.

In case the same θ is used to model the dependence between each $\alpha_l^I, l = 1, \dots, n_{b1}$ and $\beta_k^I, k = 1, \dots, n_{b2}$ the coupling is defined to be homogeneous and the interval fields are considered to show homogeneous cross-interdependence. Indeed, since the coupling between all globally defined interval scalars is the same and the fact that it are the basis functions $\psi_i(\mathbf{r})$ that control the local dependence, the coupling is indeed homogeneous in the sense that the same degree of interdependence exists for all $\alpha^I(\mathbf{r}_i)$ and $\beta^I(\mathbf{r}_j)$. Conversely, two interval fields are defined to be inhomogeneously coupled in case not all θ are the same.

In case these diagonal-band copula-like functions are used to model cross-interdependence, following property holds:

$$\theta_{1i} + \theta_{1j} > 1 \Rightarrow D_{ij} \subset \beta_i \times \beta_j \quad (26)$$

which is a direct result of the geometric shape of the constructed admissible set $\mathcal{D}$. Indeed, in case the sum of any two dependence parameters is more than one, the introduction of such strong cross-interdependence automatically influences the auto-interdependence of the interval field as well. This is a result from the fact that this corresponds to a convex hull in $n_{b1} + n_{b2}$ dimensions with a potentially very high aspect ratio resulting from the intersection of the hyper-rectangular space of the interval scalars with bivariate admissible sets, e.g., according to eq. (23), where the ‘width’ of these intersections is dictated by θ_{ij} . In case eq. (26) is violated, these intersections limit the joint admissibility of the interval scalars in such a way that the extreme values in the respective interval scalars no longer belong to the joint-admissible set. Hence, the basis in the control points (see eq. (14) can no longer be orthogonal, which in its turn alters the auto-interdependence of the interval field. Note that also other, possibly non-linear or non-convex dependence functions can be applied to model cross-interdependence between two interval

fields. Note furthermore that in case of non-convex admissible sets, the Extended Transformation Method, as presented in²⁸ can no longer be applied. In that case, the analyst has to resort to global optimization schemes.

3.3 | A measure for auto- and cross-interdependence in interval fields

In the context of random fields, usually the concept of auto-correlation is applied to quantify the correlation of different random variables within the field. In an interval context however, this is not feasible since no relative likelihood within the interval bounds is quantified. Therefore it is proposed to translate the *interdependency index*, as introduced by Giannini and Hanss⁴³ in the context of fuzzy model responses, to the description of spatial auto- and cross-interdependence in interval fields.

Specifically, it is proposed to define the degree of interdependence $\mathbb{D}_{ij} \in [0, 1]$ between intervals $\mathbf{x}_i^I$ and $\mathbf{x}_j^I$ at two locations $\mathbf{r}_i$ and $\mathbf{r}_j$ in Ω_e as:

$$\mathbb{D}_{ij} = 1 - \frac{\int_C(\mathbf{x}_i^I, \mathbf{x}_j^I) d\mathbf{x}_i d\mathbf{x}_j}{\int_{(\bar{\mathbf{x}}_i - \underline{\mathbf{x}}_i) \times (\bar{\mathbf{x}}_j - \underline{\mathbf{x}}_j)} d\mathbf{x}_i d\mathbf{x}_j} \quad (27)$$

with $i \neq j$ and $i, j = 1, \dots, N_e$ and C the convex hull of the admissible set of parameter values for $\mathbf{x}_i^I$ and $\mathbf{x}_j^I$. As such, the auto-interdependence degree between any two responses y_i and y_j can be described as the ratio of (multidimensional) volume of the convex hull that bounds there admissible combinations to the (multidimensional) volume of the hyper-rectangular space spanned by their Cartesian product. It can be seen that in case $\mathbf{x}_i^I$ and $\mathbf{x}_j^I$ are fully independent (and as such, orthogonal), following equality holds by construction of the Cartesian product:

$$C(\mathbf{x}_i^I, \mathbf{x}_j^I) = (\bar{\mathbf{x}}_i - \underline{\mathbf{x}}_i) \times (\bar{\mathbf{x}}_j - \underline{\mathbf{x}}_j) \quad (28)$$

and hence, $\mathbb{D}_{ij} = 0$. This situation for instance occurs when considering the dependence of $\alpha_i^I(\hat{\mathbf{r}}_i)$ and $\alpha_j^I(\hat{\mathbf{r}}_j)$. Conversely, in the limit case where y_i and y_j are fully dependent, $C(\mathbf{x}_i^I, \mathbf{x}_j^I)$ will collapse to a line with infinitely small volume, and hence, $\mathbb{D}_{ij} = 1$. To compute the auto-interdependence of local interval scalars in an interval field, $\mathbf{x}_i^I$ and $\mathbf{x}_j^I$ in eq. (27) represent intervals **from the same set of interval field realizations, evaluated at each $\Omega_e \subset \Omega$** . Conversely, when computing cross-interdependence between two **different interval fields**, $\mathbf{x}_i^I$ and $\mathbf{x}_j^I$ respectively represent local intervals from $^{(1)}\mathbf{x}^I(\mathbf{r})$ and $^{(2)}\mathbf{x}^I(\mathbf{r})$, **evaluated at either the same location in the model domain, or between two different locations in the model domain**.

These metrics can be applied to assess the influence of the basis functions of an interval field on the local coupling between interval scalars, investigating the cross-coupling between interval fields. **Furthermore, the metric can also be applied to assess the dependence between output responses of a finite element model subjected to interval valued parametric uncertainties to investigate their interdependence**. However, it should be noted that in case the interdependence, be it auto-interdependence or cross-interdependence, is non-convex, these metrics may give a biased estimation of the true interdependence. **Such non-convex interdependence is especially an issue when considering dependence between multiple model responses $\mathbf{y}$, as the model $\mathcal{M}$ might provide non-linear coupling between the responses, depending on the physics being modeled.**

4 | CASE STUDY1: COUPLING OF AN INTERVAL SCALAR TO AN INTERVAL FIELD

Two case studies are performed to illustrate the presented methods for modeling dependence in an interval field context. The first case study concerns the modeling of dependence between an interval scalar and an interval field, whereas the second case study illustrates the application of the method to the modeling of cross-interdependence between two interval fields. Both case studies deal with one-dimensional interval fields for the purpose of visualization. Note that this does not limit the generality of the methods to 1-dimensional problems, as adding more spatial or time dimensions to the interval field expansion is only represented in the basis functions $\psi_i(\mathbf{r})$ (see eq. (9)) and not in the interval scalars, between which the dependence is modeled.

4.1 | Dependence between one scalar and a field

This case study illustrates the application of the admissible set decomposition for the modeling of the dependence between an interval scalar $\beta^I \in \mathbb{R}$ and a one-dimensional interval field $\mathbf{x}^I(\mathbf{r})$.

First, a homogeneous coupling is considered, and the effect hereof on the cross- and auto-interdependence of the interval field, as well as the corresponding realizations, is studied. Then, the effect of introducing inhomogeneous coupling into the problem is considered.

4.1.1 | Homogeneous dependence

A one-dimensional domain Ω , that discretized into 10 distinct elements Ω_e , is considered. Within this domain, $x^I(\mathbf{r})$ is modeled as an interval field consisting of 3 basis functions with corresponding control points:

$$x^I(\mathbf{r}) = \sum_{i=1}^3 \frac{w_i(\mathbf{r})}{\sum_{j=1}^3 w_j(\mathbf{r})} \alpha_i^I \quad (29)$$

with w_j the weight functions corresponding to each of the control points $\hat{\mathbf{r}}_j$, computed via inverse distance weighting interpolation. For illustrative purposes, and without loss of generality, the control points are considered to be $\hat{\mathbf{r}}_1 = \Omega_2$, $\hat{\mathbf{r}}_2 = \Omega_5$ and $\hat{\mathbf{r}}_3 = \Omega_8$. All interval scalars are considered to be unit intervals (i.e., $\underline{\cdot} = 0$, $\bar{\cdot} = 1$). These assumptions do not limit the generality of the case study, as all intervals can easily be scaled to a unit interval and the basis functions do not contribute to the cross-interdependence. In this study, β^I and $x^I(\mathbf{r})$ are considered to be cross-interdependent with a linear, positive, homogeneous dependence. Hence, a diagonal-band copula-like dependence function is used to parameterize the cross-interdependence:

$$D_{1j} = 1 - \mathcal{H}(|\beta + \alpha_j - 1| - \theta_{1j}) \quad \forall \beta \in \beta^I, \alpha_j \in \alpha^I, j = 1, \dots, 3 \quad (30)$$

The effect of the degree of dependence θ_{1j} on the modeling of the cross-interdependence is studied by gradually increasing the value of θ_{1j} , where $\theta_{11} = \theta_{12} = \theta_{13}$ (i.e., $D(\beta^I, \alpha_1^I) = D(\beta^I, \alpha_2^I) = D(\beta^I, \alpha_3^I)$).

A first important consideration that has to be made is that as long as $\theta_{1i} + \theta_{1j} \leq 1$, the auto-interdependence of $x^I(\mathbf{r})$ is not influenced by including cross-interdependence with β^I . This is illustrated in figure 4, which shows the local cross-interdependence between the local interval values $x^I(\mathbf{r}_i)$ and the interval scalar β^I . The degree of local cross-interdependence is computed using eq. (27). It can be noted that indeed, for this case, the auto-interdependence of local interval scalars within the interval field is not altered as long as all $\theta \leq 0.5$. This can be seen since the cross-interdependence remains constant for all $x^I(\mathbf{r}_i)$. As such, the analyst can separately model the auto- and cross-interdependence between an interval and an interval field, which enables a highly flexible tailoring of the modeled uncertainty. For the case that $\theta_{1j} \in]0.5, 1[$, the cross-interdependence becomes inhomogeneous over Ω , indicating that the auto-interdependence is altered by the modeling of dependence between β^I and α^I . This implies that no homogeneous cross-interdependence between an interval scalar and an interval field can be modeled in this range of θ values. Finally, when all $\theta_{1j} = 1$, the auto-interdependence becomes again homogeneous. It is easy to see that this is a direct result from the fact that all local interval scalars in α^I are fully dependent on β^I and hence, lie on a line.

To further illustrate this point, figure 5 shows bi-variate intersections of the 10-dimensional dependent input space that is modeled by the interval field $x^I(\mathbf{r})$. These intersections are created by projecting the 10-dimensional space on a selection of $\{\Omega_{e,i}, \Omega_{e,j}\}$ pairs. The dependence in this input space is governed both by the basis functions $\psi_i(\mathbf{r})$ in the interval field series expansion (eq. (9)), as by the $\mathfrak{C}$ -vine decomposition based coupling between the interval field and the interval scalar β^I . As can be noted, in case all $\theta_{1j} > 0.5$, the auto-interdependence of the interval field is altered since the volume of the convex hull representing the admissible set that is defined by the interval field is reduced with respect to the case where no cross-interdependence is included in the model. Furthermore, on a local level, even the extreme values of the local interval scalars are affected due to the large degree of dependence on other interval realizations.

These phenomena are a direct result from violating the inequality in eq. 26. Indeed, in case the sum of any two $\theta_{i,j}$ becomes larger than 1, the union of the bivariate admissible sets with the 3-dimensional hypercube α^I becomes very shallow (see also figure 5). More importantly, the extreme values in the respective interval scalars no longer belong to the joint-admissible set, when the inequality is violated. Hence, the basis in the control points (see eq. (14)) can no longer be orthogonal, which in its turn alters the auto-interdependence of the interval field.

4.1.2 | Heterogeneous dependence

The effect of including heterogeneous cross-interdependence between $x^I(\mathbf{r})$ and β^I is studied by altering only one θ_{1i} while the other two are kept constant at a value of $\theta = 0.3$. The resulting local degree of cross-interdependence for each $\Omega_e \in \Omega$ is illustrated in figure 6. This figure illustrates that, depending on the location of the control points $\hat{\mathbf{r}}$, the analyst can tailor the local cross-interdependence of the interval field towards the separate interval scalar β with only a minor influence on the

FIGURE 4 Local cross-interdependence $\mathbb{D}_{ij}$ (see eq. (27)) between local values of the interval field with the dependent interval scalar. The legend shows the value for θ that was used in the homogeneous coupling.

FIGURE 5 Realizations of the interval field, evaluated at elements 1 to 5. The convex hulls represent the admissible sets that are generated due to the induced coupling with interval scalar β^I .

cross-interdependence of the other local interval scalars in Ω . Obviously, for high values of θ , some joint influence is inevitable, especially when $\theta_{1i} + \theta_{1j} > 1$, but also due to the auto-interdependence imposed by the basis functions.

FIGURE 6 Local cross-interdependence $\mathbb{D}_{ij}$ (see eq. (27)) between local values of the interval field with the dependent interval scalar when heterogeneous cross-interdependence is modeled by altering θ_{1j} between β^I and one α_j^I . **Left:** Variation in $\theta_{1,1}$ (i.e., between the β^I and α_1^I), **Middle:** variation in $\theta_{1,2}$, **Right:** variation in $\theta_{1,3}$.

5 | CASE STUDY 2: MODELING OF CROSS-DEPENDENCE BETWEEN TWO INTERVAL FIELDS

This case study illustrates the application of the admissible set decomposition for the modeling of the dependence between two one-dimensional interval fields $^{(1)}x^I(\mathbf{r})$ and $^{(2)}x^I(\mathbf{r})$, as defined in eq. (17) and (18). The base functions $^{(1)}\psi_i(\mathbf{r})$ and $^{(2)}\psi_i(\mathbf{r})$ are constructed using inverse distance weighting interpolation as explained in eq. (13) over a uniformly distributed grid consisting of Ω_e , $e = 1, \dots, 100$. For this case study, the control points are selected as: $^{(1)}\hat{\mathbf{r}}_1 = 10$, $^{(1)}\hat{\mathbf{r}}_1 = 40$ and $^{(1)}\hat{\mathbf{r}}_1 = 70$ for $^{(1)}x^I(\mathbf{r})$ and $^{(2)}\hat{\mathbf{r}}_1 = 20$ and $^{(2)}\hat{\mathbf{r}}_1 = 70$ for $^{(2)}x^I(\mathbf{r})$. The cross-interdependence is modeled as the intersection of two $\mathcal{C}$ -vine admissible set decomposition using diagonal-band copula-like bivariate admissible sets, where both interval scalars from $^{(2)}x^I(\mathbf{r})$ are used as a pivot point to model dependence with all interval scalars in $^{(1)}x^I(\mathbf{r})$:

$$\mathcal{D}(\alpha_1^I, \alpha_2^I, \alpha_3^I, \beta_1^I, \beta_2^I) = \bigotimes_{k=1}^{n_{b2}+1} x_{k,l}^I \bigcap_{j=1}^3 \bigcap_{i=1}^2 1 - \mathcal{H}(|\beta_i - \alpha_i - 1| - \theta_{ji}) \quad (31)$$

Twelve combinations of θ_{ji} are considered, which are listed in table 1 .

The realizations of the interval fields, in case the cross-dependence is modeled homogeneously with θ_{12} are shown in figure 7 . These realizations are generated using the Extended Transformation Method, as presented in²⁸. The top figure shows $^{(1)}x^I(\mathbf{r})$ both with (denoted $^{(1)}x^I(\mathbf{r}) \mid ^{(2)}x^I(\mathbf{r})$) and without the modeled cross-interdependence with $^{(2)}x^I(\mathbf{r})$. Similarly, the bottom figure shows $^{(2)}x^I(\mathbf{r})$ both with and without the modeled cross-interdependence with $^{(1)}x^I(\mathbf{r})$. A first conclusion that can be drawn from this figure is that the extreme bounds of the interval field are in this case not impacted by imposing cross-dependence between both fields. In fact, this also holds when the globally defined interval scalars are not limited to the unit-cube as presented in this case. This is explained by the fact that the transformation from any hyper-rectangle to the unit cube is an affine transformation. Second, it is clear that, when cross-interdependence is introduced in the modeling of the interval field, additional realizations are included to cope with the additional limitations on the admissible set of joint globally defined intervals. This stems from the increased complexity of the admissible set that models the possible joint combinations of the set that jointly contains α^I

TABLE 1 Considered dependence parameter vectors θ

n^0	$\theta = [\theta_{11}, \theta_{21}, \theta_{31}, \theta_{12}, \theta_{22}, \theta_{32}]$
θ_1	[1.0, 0.0, 0.0, 0.0, 0.0, 0.0]
θ_2	[0.0, 1.0, 0.0, 0.0, 0.0, 0.0]
θ_3	[0.0, 0.0, 1.0, 0.0, 0.0, 0.0]
θ_4	[0.0, 0.0, 0.0, 1.0, 0.0, 0.0]
θ_5	[0.0, 0.0, 0.0, 0.0, 1.0, 0.0]
θ_6	[0.0, 0.0, 0.0, 0.0, 0.0, 1.0]
θ_7	[1.0, 0.0, 0.0, 1.0, 0.0, 0.0]
θ_8	[0.0, 1.0, 0.0, 0.0, 1.0, 0.0]
θ_9	[0.0, 0.0, 1.0, 0.0, 0.0, 1.0]
θ_{10}	[1.0, 1.0, 1.0, 0.0, 0.0, 0.0]
θ_{11}	[0.0, 0.0, 0.0, 1.0, 1.0, 1.0]
θ_{12}	[0.6, 0.6, 0.6, 0.6, 0.6, 0.6]

and β^I . Evidently, this has implications on the computational cost of propagating dependent intervals, as was illustrated in an interval scalar context in²⁸. Similarly, figure 8 shows $^{(1)}x^I(\mathbf{r})$ and $^{(2)}x^I(\mathbf{r})$ in case their cross-interdependence is modeled using $0.86 * \theta_{11}$ in the $\mathfrak{C}$ -vine decomposition. As can be seen, no noticeable influence of cross-interdependence is present in $^{(2)}x^I(\mathbf{r})$ when $\mathbf{r} \in [0, 30]$. The influence of the cross-interdependence from there on increases towards the point where the realizations of $^{(2)}x^I(\mathbf{r})$ become highly dependent on $^{(1)}x^I(\mathbf{r})$ due to the high value of θ in those areas.

FIGURE 7 Realizations of $^{(1)}x^I(\mathbf{r})$ and $^{(2)}x^I(\mathbf{r})$ with and without the inclusion of **homogeneous** cross-interdependence.

Figure 9 shows the cross-interdependence functions $\mathbb{D}(\mathbf{r}_1, \mathbf{r}_2, \theta_{ij})$, obtained as a result from inducing the 12 different θ vectors in the cross-interdependence model shown in eq. (31). It can be seen that a highly tailored modeling of the degree of

cross-interdependence is attainable using the proposed method. Considering $\theta_1 - \theta_6$, it can be seen that a very strong cross-interdependence in localized areas of the joint domain can be modeled. These areas correspond to the respective neighborhoods of the control points $\mathbf{f}$ between which a bivariate admissible set is defined on the corresponding global interval scalars. For these specific parameter combinations, it can be seen in table 1 that this always is just one pair, explaining why only one peak is visible in the shown cross-interdependence functions $\mathbb{D}(\mathbf{r}_1, \mathbf{r}_2, \theta_{ij})$.

Since the domains of cross-interdependence are mutually complementary, they can be superposed to combine more intricate cross-dependence functions. $\theta_7 - \theta_9$ illustrate the effect of tuning the $\mathfrak{C}$ -vine decomposition to a single interval scalar of $^{(1)}x^I(\mathbf{r})$ (respectively α_1^I , α_2^I and α_3^I). As can be seen, in that case, the cross inter-dependence is constant for the combinations of an interval β^I , obtained by evaluating $^{(2)}x^I(\mathbf{r})$ at a location $^{(2)}\mathbf{r}$ with all possible intervals α^I that are obtained by evaluating $^{(1)}x^I(\mathbf{r})$ over the entire model domain. When the $\mathfrak{C}$ -vine decomposition is tuned towards one scalar of $^{(2)}x^I(\mathbf{r})$, similar conclusions are obtained, as shown in cases θ_{10} and θ_{11} . A fully homogeneous cross-interdependence between the two interval fields can be obtained by using the same θ_{ji} for all bi-variate admissible sets, as illustrated in θ_{12} .

Finally, $^{(1)}x^I(\mathbf{r})$ and $^{(2)}x^I(\mathbf{r})$ are applied to a practical example of a clamped square steel plate to model respectively spatial uncertainty in Young's modulus as the density. The plate has dimensions of 1 m by 1 m and has a thickness of 5 mm. The analyst is interested in the first five eigenfrequencies of the plate. The dynamics are modeled using a Finite Element model, in which the model domain Ω is discretized over a 100×100 grid, using 2D bending plate elements and a consistent mass matrix. The model domain is parameterized by a vector $\mathbf{r} = [r_1 r_2]$, where r_1 denotes the direction perpendicular to the Dirichlet boundary condition that models the clamping of the plate and r_2 the parallel direction. For the analysis, the corresponding interval scalars are rescaled as $\alpha_1^I = [210, 230]$, $\alpha_2^I = [190, 200]$ and $\alpha_3^I = [200, 210]$ GPa for $^{(1)}x^I(\mathbf{r})$ and $\beta_1^I = [7500, 7600]$ kg/m³ and $\beta_2^I = [7200, 7400]$ kg/m³ for $^{(2)}x^I(\mathbf{r})$. Poisson's coefficient is selected as 0.3. It is furthermore assumed that the local uncertain parameter has a constant value over the r_2 -direction for consistency with the interval fields that are defined above. Obviously, this is not a physically realistic assumption. Nonetheless, it does not limit the generality of the results shown below, as making the uncertain parameters also a function of r_2 only adds dimensions to $^{(1)}\boldsymbol{\psi}(\mathbf{r})$ and $^{(2)}\boldsymbol{\psi}(\mathbf{r})$. For this case study, both $\boldsymbol{\theta} = [0, 0, 0, 0, 0, 0]$ and $\boldsymbol{\theta} = [0.8, 0.6, 0.35, 0.7, 0.4, 0.9]$ are applied to model cross-dependence between $^{(1)}x^I(\mathbf{r})$ and $^{(2)}x^I(\mathbf{r})$ illustrating the effect of both fully independent and cross-interdependent interval fields on the corresponding eigenvalues of the clamped plate.

Figure 10 shows the effect of propagating the $^{(1)}x^I(\mathbf{r})$ and $^{(2)}x^I(\mathbf{r})$ on two-dimensional cross-sections of the five dimensional convex hull consisting of the first five eigenfrequencies of the clamped plate. These results are obtained by applying the Extended Transformation Method²⁸. Following this procedure, 124 deterministic finite element analyses were required for the propagation

FIGURE 8 Realizations of $^{(1)}x^I(\mathbf{r})$ and $^{(2)}x^I(\mathbf{r})$ with and without the inclusion of **heterogeneous** cross-interdependence.

FIGURE 9 Cross-interdependence functions $\mathbb{D}(\mathbf{r}_1, \mathbf{r}_2, \theta_{ij})$ depicting the degree to which the two interval fields are cross dependent for nine different sets of θ . The color coding depicts the degree of cross-interdependence as introduced in eq. (27)

of the cross-interdependent interval fields, as compared to 32 deterministic analyses for the independent interval fields. As can be noted, the convex hull of eigenfrequencies, obtained by propagating cross-interdependent interval fields, is a subset of the convex hull that is obtained by propagating independent interval fields. This illustrates the fact that, by ignoring cross-interdependence, over-conservative results are obtained in an interval analysis.

6 | CONCLUSIONS

This paper discusses the concepts of auto- and cross-interdependence in interval field finite element analysis. An overview of recent work in the spatial modeling of dependence between interval scalars is given and a deeper theoretical explanation

FIGURE 10 Two-dimensional cross-sections of the five dimensional convex hull consisting of the first five eigenfrequencies of the clamped plate obtained by propagating $^{(1)}x^I(\mathbf{r})$ and $^{(2)}x^I(\mathbf{r})$ as independent interval fields (blue), as well as cross-interdependent interval fields (red)

towards the modeling of spatial dependence in interval fields using inverse distance weighting interpolation is provided. The paper presents a flexible yet powerful approach to model cross-interdependence between either interval scalars and interval fields on the one hand and between interval fields on the other, based on the recently introduced admissible set decomposition in combination with the Extended Transformation Method. Specifically, a $\mathfrak{C}$ -vine decomposition is applied to limit the joint admissible set of globally defined interval scalars in the interval field series expansions, which allows for a flexible tailoring of the local degree of dependence. The provided case studies illustrate the performance of the method and highlight the possibility to model the cross-interdependence as either homogeneous as in homogeneous. Furthermore, it is clear that a large set of possible dependence structures can be constructed for modeling cross dependence between interval fields, as it is shown that the influence zone of one modeled bivariate dependence is in fact localized. One limitation of the method is that the cross-interdependence between an interval scalar and an interval field cannot be modeled homogeneously for the full domain of possible dependencies. This is however a direct implication of the construction of the interval field. Since the method allows for a parameterization of the cross-interdependence between intervals and interval fields, it is possible to quantify these dependence structures based on experimental data. Future work will therefore focus on such quantification in either a direct or inverse approach (e.g., based on the work presented in³⁷ or³⁹).

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Conflict of interest

The authors declare no potential conflict of interests.

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