ARTICLE TYPE

Characterization and optimization of the smoothed spectral abscissa for time-delay systems

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Summary

We introduce a new robust stability measure for systems with multiple point-wise delays, which is called smoothed spectral abscissa, consistently with the existing measure for delay-free systems. Its main characteristics are that it is smooth with respect to the system parameters, and it provides a trade-off between the decay rate of the system solution and the $\mathcal{H}_2$ norm of a transfer matrix related with the system. The smoothed spectral abscissa is implicitly defined in terms of the $\mathcal{H}_2$ norm of an auxiliary system, and its computation is based on the so-called delay Lyapunov matrix. We show that these features make the smoothed spectral abscissa suitable for the design of robust controllers by using standard gradient-based optimization techniques, and exploiting a novel characterization of the derivatives of the delay Lyapunov matrix with respect to the system parameters.

KEYWORDS:

Time-delay systems, $\mathcal{H}_2$ norm, smoothed spectral abscissa, delay Lyapunov matrix

1 | INTRODUCTION

We consider time-delay systems of the form

$$\dot{x}(t) = \sum_{i=0}^m A_i(p)x(t - h_i), \quad t \geq 0, \quad (1)$$

where $0 = h_0 < h_1 < \dots < h_m$ are the delays, and $A_0(p), \dots, A_m(p) \in \mathbb{R}^{n \times n}$, smoothly depend on the system parameters, denoted by the vector $p = (p_1, \dots, p_k)^T \in \mathbb{R}^k$. We assume without loss of generality that $t_0 = 0$ and consider the initial conditions φ from the space of piecewise continuous functions such that existence and uniqueness of solutions is guaranteed.

The stabilization of systems of the form (1) is a topic of relevance as delay differential equations have proved to be adequate for modelling a variety of physical phenomena,^{1,2} and techniques within both frequency and time domain approach have been developed for the design of controllers in the last decades (see, for instance, the books^{3,4}).

Some methods within the frequency domain approach that have been useful in the design of controllers are based on the optimization of the spectral abscissa and pseudospectral abscissa.^{5,3} For the sake of simplicity, we recall these concepts by considering the delay-free differential equation

$$\dot{x}(t) = A(p)x(t), \quad t \geq 0. \quad (2)$$

In this case, the spectral abscissa and the pseudospectral abscissa are respectively given by

$$\alpha(p) = \sup \{ \Re(s) : \det(sI - A(p)) \},$$

and

$$\alpha_\epsilon(p) := \sup \{ \Re(s) : s \in \Lambda_\epsilon \},$$

where Λ_ϵ is known as the pseudospectrum, and defined as

$$\begin{aligned} \Lambda_\epsilon &:= \{ s \in \mathbb{C} : \det(sI - (A(p) + X)) = 0, \text{ for some } X \in \mathbb{C}^{n \times n} \text{ with } \|X\|_2 \leq \epsilon \} \\ &= \left\{ s \in \mathbb{C} : \|(sI - A(p))^{-1}\|_2 \geq \frac{1}{\epsilon} \right\}. \end{aligned}$$

The pseudospectral abscissa $\alpha_\epsilon(p)$ provides a worst-case stability measure of (2) when the system includes uncertainties that are ϵ -bounded. One observes that the pseudospectral abscissa can be written as

$$\begin{aligned} \alpha_\epsilon(p) &= \sup \left\{ \gamma : \|(\gamma + i\omega)I - A(p)\|_2 \geq \frac{1}{\epsilon}, \text{ for some } \omega \in \mathbb{R} \right\} \\ &= \sup \left\{ \gamma : \|(sI - (A(p) - \gamma I))^{-1}\|_{\mathcal{H}_\infty} \geq \frac{1}{\epsilon} \right\}. \end{aligned} \quad (3)$$

Note also that $\|(sI - (A(p) - \gamma I))^{-1}\|_{\mathcal{H}_\infty}$ is not finite if $\gamma \leq \alpha(p)$, i.e. if the matrix $A - \gamma I$ is not Hurwitz, and finite if $\gamma > \alpha(p)$. Optimizing the function $p \mapsto \alpha_\epsilon(p)$ provides a trade-off between the optimum of the spectral abscissa of (2) and the $\mathcal{H}_\infty$ norm of $(sI - A(p))^{-1}$.⁵ A characteristic of the pseudospectral abscissa is that it is a non-smooth function of the system parameters p , requiring the use of special software for the optimization of non-smooth functions.

Inspired by the relation of the pseudospectral abscissa with the $\mathcal{H}_\infty$ norm of a shifted transfer matrix, the following stability measure of (2), which is called *smoothed spectral abscissa* and is related with the $\mathcal{H}_2$ norm in analogy with (3), is presented in Vanbiervliet et al.:⁶

$$\begin{aligned} \tilde{\alpha}_\epsilon(p) &:= \sup \left\{ \gamma : \|(sI - (A(p) - \gamma I))^{-1}\|_{\mathcal{H}_2}^2 \geq \frac{1}{\epsilon} \right\} \\ &= \left\{ \gamma : \|(sI - (A(p) - \gamma I))^{-1}\|_{\mathcal{H}_2}^2 = \frac{1}{\epsilon} \right\}. \end{aligned}$$

The main characteristics of $\tilde{\alpha}_\epsilon(p)$ is that it is a smooth function of the system parameters, and analogously with the pseudospectral abscissa, optimizing this function provides a trade-off between optimizing the spectral abscissa of (2) and the $\mathcal{H}_2$ norm of $(sI - A(p))^{-1}$. In this case, the parameter $\epsilon > 0$ determines the smoothness degree of the function $p \mapsto \tilde{\alpha}_\epsilon(p)$.

Whereas the concept of the pseudospectral abscissa has been generalized and its optimization has proved to be a powerful tool for the design of robust controllers for time-delay systems,^{7 8 9 10} no study of the smoothed spectral abscissa for systems of the form (1) has been reported in the literature before. This paper is motivated by this gap and the relevance of both the decay rate of the system solution and the $\mathcal{H}_2$ norm as stability measures in the design of controllers for time-delay systems.^{11 12 13 14}

Based on the ideas introduced for delay-free systems in Vanbiervliet et al.,⁶ and Diehl et al.,¹⁵ we introduce the *smoothed spectral abscissa* of system (1), and show that it fulfils the following: i) it approximates the spectral abscissa of the system, providing an upper bound that converges to the spectral abscissa as the “smoothness parameter” ϵ goes to zero; ii) it is smooth with respect to the system parameters; and iii) it provides a trade-off between the decay rate of the system solution and the $\mathcal{H}_2$ norm of a transfer function related with the system, depending on the parameter ϵ . We exploit these characteristics and state the optimization of the smoothed spectral abscissa problem as an unconstrained minimization problem, which allows designing robust controllers by solving it.

Similar as the delay-free case, we show that the smoothed spectral abscissa of system (1) is given by an implicit relation with the $\mathcal{H}_2$ norm of a shifted transfer matrix. We use this relation in conjunction with the delay Lyapunov matrix based formula of the $\mathcal{H}_2$ norm of time-delay systems¹⁶ in order to compute it (the reader is referred to the book by Kharitonov¹⁷ for a comprehensive study and applications of the delay Lyapunov matrix). The smoothness property allows us to solve the optimization of the smoothed spectral abscissa problem within a gradient-based optimization framework, where the delay Lyapunov matrix plays a key role, since it enables us to obtain explicit expressions for the gradient computation. Unlike previous formulae of the sensitivity of the delay Lyapunov matrix with respect to the system parameters introduced in Gomez et al.,^{18 19} here we present a simpler characterization that avoids the numerical computation of some integral terms. It is worthy mentioning that the optimization of the smoothed spectral abscissa can be solved with standard optimization techniques, in contrast to the minimization of the spectral and pseudospectral abscissa, which are in general non-smooth functions of the system parameters.³

The remainder of the paper is organized as follows. The formal definition of the smoothed spectral abscissa and its properties are introduced in Section 2. Section 3 is split in two subsections: we first formulate the optimization problem and explain the trade-off between the minimization of the spectral abscissa of the system and the $\mathcal{H}_2$ norm; and then, we show that the

smoothed spectral abscissa and its gradient can be expressed in terms of the delay Lyapunov matrix. The required formulae for the computation of the delay Lyapunov matrix and its derivatives are addressed in Section 4. We illustrate the theoretical results applied to the stabilization of a triple integrator with a delay-based feedback controller in Section 5, and we end the contribution with some final comments in Section 6. A preliminary version of this paper is presented in²⁰, where the proofs of Lemma 1, Theorem 1 and Theorem 4 are only sketched, and the computation of the derivatives of the delay Lyapunov matrix with respect to the system parameters is carried out in line with the schemes proposed in Gomez et al,^{18 19} which are different from the proposed in Section 4, as previously explained.

Throughout the paper, we use the following notation. The Euclidian norm is denoted by $\|\cdot\|$, and the trace of a matrix A is represented by $\text{tr}(A)$. The symbol ∇g stands for the gradient of a function $g : \mathbb{R}^n \mapsto \mathbb{R}$ and $\frac{\partial P(t, p)}{\partial p}$ stands for the matrix obtained by taking the partial derivative of each element of a matrix-function P with respect to p . The notation $j = \overline{1, p}$ means that j takes integer values from 1 to p . The classical Kronecker product of two matrices A and B , and the (column-wise) vectorization of a matrix A are denoted by $A \otimes B$ and $\text{vec}(A)$, respectively. The notation $\{\text{vec}(A_j)^T\}_{j=\overline{1, 2m}}$ represents a row block vector with $2m$ vectors and whose j^{th} block is determined by the vectorized matrix A_j , i.e. $(\text{vec}(A_1)^T \dots \text{vec}(A_{2m})^T)$. The symbol $\text{mat}(a, j)$ denotes the reshaping of the j^{th} vector a_j , $j = \overline{1, p}$, of a block vector $a = (a_1^T \dots a_p^T)$, into a square matrix A_j such that $a_j = \text{vec}(A_j)$, i.e. $\text{mat}(a, j) = A_j$. The identity and null matrix of appropriate dimensions are represented by I and 0 , respectively.

In what follows, we make no distinction between the notation previously used for the spectral abscissa ($\alpha(p)$) and the smoothed spectral abscissa ($\tilde{\alpha}_\epsilon(p)$) of system (2), and the notation used for the analogous definitions of system (1), as both definitions for delay-free systems are indeed particular cases of the time-delay case.

2 | THE SMOOTHED SPECTRAL ABCISSA

In this section, we present the formal definition of the smoothed spectral abscissa of system (1). Let us introduce first some basic definitions and auxiliary results.

We say that system (1) is exponentially stable if there exists positive constants β and σ such that for every initial condition φ its solution fulfils

$$\|x(t, \varphi)\| \leq \beta e^{-\sigma t} \cdot \sup_{\theta \in [-h_m, 0]} \|\varphi(\theta)\|, \quad t \geq 0.$$

The spectrum of system (1) is defined as

$$\Lambda := \{s \in \mathbb{C} : \det H(s) = 0\},$$

where

$$H(s) := sI - \sum_{i=0}^m A_i(p) e^{-sh_i}.$$

The spectral abscissa of system (1) is defined as the supremum of the real part of its spectrum:

$$\alpha(p) := \sup \{\Re(s) : \det H(s) = 0, s \in \mathbb{C}\}.$$

It is well known that system (1) is exponentially stable if and only if the spectral abscissa of the system is strictly less than zero, i.e., $\alpha(p) < 0$, and that although the matrices of system (1) smoothly depend on $p \in \mathbb{R}^k$, the function $p \mapsto \alpha(p)$ may not be smooth as a result of multiple rightmost eigenvalues.³

We consider now for $\gamma \in \mathbb{R}$ system

$$\dot{z}(t) = (A_0(p) - \gamma I) z(t) + \sum_{i=1}^m A_i(p) e^{-\gamma h_i} z(t - h_i), \quad t \geq 0, \quad (4)$$

which is obtained by substituting $x(t) = e^{\gamma t} z(t)$ in (1), inducing a shift of the spectrum Λ by $-\gamma$. The fundamental matrix of (4), denoted by $K_\gamma(t) \in \mathbb{R}^{n \times n}$, satisfies²¹

$$\frac{d}{dt} K_\gamma(t) = K_\gamma(t) (A_0(p) - \gamma I) + \sum_{i=1}^m K_\gamma(t - h_i) A_i(p) e^{-\gamma h_i}, \quad t \geq 0, \quad (5)$$

with the initial condition

$$K_\gamma(0) = I, \quad \text{and} \quad K_\gamma(t) = 0, \quad \forall t < 0. \quad (6)$$

For given matrices Q_1 and Q_2 of compatible dimensions, we introduce the function

$$f(p, \gamma) := \int_0^\infty \text{tr} \left\{ Q_1^T K_\gamma^T(t) Q_2^T Q_2 K_\gamma(t) Q_1 \right\} dt, \quad (7)$$

where $Q_2 K_\gamma(t) Q_1$ is the impulse response of system

$$\begin{aligned} \dot{z}(t) &= (A_0(p) - \gamma I) z(t) + \sum_{i=1}^m A_i(p) e^{-\gamma h_i} z(t - h_i) + Q_1 u(t), \quad t \geq 0, \\ y(t) &= Q_2 z(t). \end{aligned}$$

Throughout the paper we assume that matrices Q_1 and Q_2 satisfy

$$\text{rank} \left(H(s_d) Q_1 \right) = n, \quad (8)$$

and

$$\text{rank} \left(\begin{matrix} H(s_d) \\ Q_2 \end{matrix} \right) = n, \quad (9)$$

where s_d is a rightmost root of the spectrum Λ of system (1). Conditions (8) and (9) can be understood as controllability and observability conditions for the dominant mode of system (1).

We note that function $f(p, \gamma)$ is the square $\mathcal{H}_2$ norm of the transfer matrix

$$T_\gamma(p, s) := Q_2 \left(sI - (A_0(p) - \gamma I) - \sum_{i=1}^m A_i(p) e^{-\gamma h_i} e^{-s h_i} \right)^{-1} Q_1,$$

and if system (4) is exponentially stable, it can be expressed as

$$f(p, \gamma) = \|T_\gamma(p, s)\|_{\mathcal{H}_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left(T_\gamma^*(p, i\omega) T_\gamma(p, i\omega) \right) d\omega. \quad (10)$$

For $\gamma = 0$, we have

$$f(p, 0) = \|T(p, s)\|_{\mathcal{H}_2}^2, \quad (11)$$

where

$$T(p, s) := Q_2 H^{-1}(s) Q_1. \quad (12)$$

In the following lemma, we present some properties of the function $f(p, \gamma)$ that enable us to introduce the definition of the smoothed spectral abscissa, which is given by an implicit relation of $f(p, \gamma)$ with the argument γ , and its properties. The proof is presented in Appendix A.

Lemma 1. The following assertions hold:

1. If $\gamma > \alpha(p)$, then system (4) is exponentially stable.
2. $f(p, \gamma) < \infty$ for all $(p, \gamma) \in \mathbb{R}^k \times \mathbb{R}$ such that $\gamma > \alpha(p)$.
3. The function $(p, \gamma) \mapsto f(p, \gamma)$ is smooth for all $(p, \gamma) \in \mathbb{R}^k \times \mathbb{R}$ such that $\gamma > \alpha(p)$.
4. The function $\gamma \mapsto f(p, \gamma)$ satisfies $\frac{\partial f(p, \gamma)}{\partial \gamma} < 0$ for all $(p, \gamma) \in \mathbb{R}^k \times \mathbb{R}$ such that $\gamma > \alpha(p)$.
5. For any $p \in \mathbb{R}^k$,

$$\lim_{\gamma \rightarrow \infty} f(p, \gamma) = 0,$$

and

$$\lim_{\gamma \rightarrow \alpha(p)+} f(p, \gamma) = \infty.$$

By Item 4 and Item 5 of Lemma 1, function $\gamma \mapsto f(p, \gamma)$ is monotonically decreasing on the interval $(\alpha(p), +\infty)$, and has a unique intersection with the constant function $1/\epsilon$, where $\epsilon > 0$. Hence, for a given value $\epsilon > 0$ there exists a function $\tilde{\alpha}_\epsilon : \mathbb{R}^k \rightarrow \mathbb{R}$, $p \mapsto \tilde{\alpha}_\epsilon(p)$, whose image is defined on $(\alpha(p), +\infty)$, such that

$$f(p, \tilde{\alpha}_\epsilon(p)) = \frac{1}{\epsilon}. \quad (13)$$

Furthermore, the function $p \mapsto \tilde{\alpha}_\epsilon(p)$ is smooth (this is proved in Theorem 1 below). We refer to this function as the smoothed spectral abscissa of system (1), and a formal definition is as follows.

Definition 1. The smoothed spectral abscissa function, associated with smoothing parameter $\epsilon > 0$, is the mapping $\mathbb{R}^k \ni p \mapsto \tilde{\alpha}_\epsilon(p)$ that uniquely solves equation (13) on the interval $(\alpha(p), +\infty)$.

Some properties of the function $\tilde{\alpha}_\epsilon(p)$ are presented next.

Theorem 1. The smoothed spectral abscissa $\tilde{\alpha}_\epsilon(p)$ satisfies the following:

1. The function $p \mapsto \tilde{\alpha}_\epsilon(p)$ is smooth.
2. $\tilde{\alpha}_\epsilon(p) > \alpha(p)$ for all $\epsilon > 0$.
3. $\lim_{\epsilon \rightarrow 0} \tilde{\alpha}_\epsilon(p) = \alpha(p)$.
4. The function $\epsilon \mapsto \tilde{\alpha}_\epsilon(p)$ is increasing for all $p \in \mathbb{R}^k$.

Proof. Item 1 is deduced from the implicit function theorem²² and the fact that, by Item 3 of Lemma 1, the function $f(p, \gamma) - 1/\epsilon$ is a smooth function of $p \in \mathbb{R}^k$. Item 2 follows from Definition 1. Item 3 is deduced from the fact that, from Item 2 and Item 4 of Lemma 1, the function $\gamma \mapsto f(p, \gamma)$ is finite and decreasing with respect to γ for $\gamma > \alpha(p)$, but $f(p, \tilde{\alpha}_\epsilon(p)) \rightarrow \infty$ as $\epsilon \rightarrow 0 +$. Finally, Item 4 is proved as follows. By differentiating equation (13) with respect to ϵ , we have

$$\frac{\partial f(p, \gamma)}{\partial \gamma} \frac{\partial \tilde{\alpha}_\epsilon}{\partial \epsilon} = -\frac{1}{\epsilon^2}.$$

By Item 4 of Lemma 1 and rearranging the previous equality, we get

$$\frac{\partial \tilde{\alpha}_\epsilon(p)}{\partial \epsilon} = -\left(\frac{\partial f(p, \gamma)}{\partial \gamma}\right)^{-1} \frac{1}{\epsilon^2} > 0,$$

which proves the assertion. $\square$

The concept of the smoothed spectral abscissa and its properties are illustrated in Figures 1 and 2. Figure 1 illustrates the behaviour of the function $f(p, \gamma)$ as a function of γ . The smoothed spectral abscissa is the value of γ for which the function $f(p, \gamma)$ equals ϵ^{-1} . Figure 2 depicts a comparison of the spectral abscissa and smoothed spectral abscissa for different values of ϵ of a scalar delay equation. There, one observes the properties presented in Theorem 1.

FIGURE 1 Behaviour of the function $\gamma \mapsto f(p, \gamma)$ and its relation with the smoothed spectral abscissa of system (1).

3 | OPTIMIZATION OF THE SMOOTHED SPECTRAL ABCISSA

In this section, we first formulate the optimization problem of the smoothed spectral abscissa, and we explain how its solution can be used for the stabilization of system (1), providing a trade-off between the minimum of the spectral abscissa and the $\mathcal{H}_2$

FIGURE 2 Spectral abscissa (continuous line) and smoothed spectral abscissa for different values of ϵ of the system $\dot{x}(t) = -x(t) + bx(t-1)$, $b \in [-2, 2]$. It is computed by solving equation (22) (Section 4).

norm of the transfer matrix (12). Then, we show that the smoothed spectral abscissa and its gradient can be expressed in terms of the delay Lyapunov matrix.

3.1 | Trade-off between the spectral abscissa and the $\mathcal{H}_2$ norm

The optimization of $\tilde{\alpha}_\epsilon(p)$ for a fixed value of ϵ can be formulated as the unconstrained minimization problem

$$\min_p \tilde{\alpha}_\epsilon(p). \quad (14)$$

Let

$$p^*(\epsilon) = \arg \min_p \tilde{\alpha}_\epsilon(p) \quad (15)$$

be a minimizer of (14). In order to obtain $p \in \mathbb{R}^k$ such that system (1) is exponentially stable, we can arbitrarily choose ϵ and solve (14). If we find $p^*(\epsilon)$ such that $\tilde{\alpha}_\epsilon(p^*) < 0$, then by Item 2 of Theorem 1 we have that $\alpha(p^*) < 0$ and system (1) is exponentially stable with its rightmost root bounded by the smoothed spectral abscissa. If the minimum of (14) is nonnegative, one can try to solve the minimization problem with a smaller ϵ , since the function $\tilde{\alpha}_\epsilon(p)$ is increasing with respect to ϵ by Item 4 of Theorem 1. The price to pay for ϵ too small is that the optimization problem may become more difficult, as $\tilde{\alpha}_\epsilon$ approaches the spectral abscissa, which is in general a non-smooth function of the system parameters (this is visualized in Figure 2). Notice that by Item 3 of Theorem 1, we have that $p^*(0)$ is an optimizer of the spectral abscissa $\alpha(p)$.

We explain now the relation of the minimization problem (14) with the minimization of the $\mathcal{H}_2$ norm of the transfer matrix (12). Let ϵ^* be such that

$$\tilde{\alpha}_{\epsilon^*}(p^*(\epsilon^*)) = 0.$$

Recall from equation (11) that $f(p, 0) = \|T(p, s)\|_{\mathcal{H}_2}^2$, and from Definition 1 that

$$f(p^*, \tilde{\alpha}_{\epsilon^*}(p^*)) = \frac{1}{\epsilon^*}.$$

Hence, $p^*(\epsilon^*)$ is an optimizer of the $\mathcal{H}_2$ norm of the transfer matrix (12), i.e.

$$p^*(\epsilon^*) = \arg \min_p \|T(p, s)\|_{\mathcal{H}_2}, \quad (16)$$

and

$$\|T(p^*, s)\|_{\mathcal{H}_2} = \sqrt{\frac{1}{\epsilon^*}}. \quad (17)$$

The previously explained results can be summarized as follows:

$$\tilde{\alpha}_\epsilon(p^*(\epsilon)) \xrightarrow{\epsilon \rightarrow 0} \min_p \alpha(p) \quad \text{and} \quad \tilde{\alpha}_\epsilon(p^*(\epsilon)) \xrightarrow{\epsilon \rightarrow \epsilon^*} \min_p \|T(p, s)\|_{\mathcal{H}_2}^2.$$

Thus, optimizing the function $p \mapsto \tilde{\alpha}_\epsilon(p)$ for values of ϵ on $(0, \epsilon^*]$ provides a trade-off between the minimization of the spectral abscissa of system (1) and the $\mathcal{H}_2$ norm of the transfer matrix (12). Figure 3 sketches the connection of the smoothed spectral abscissa with the spectral abscissa and the $\mathcal{H}_2$ norm of $T(p, s)$ evaluated at the minimizers $p^*(\epsilon)$ of the smoothed spectral abscissa for different values of ϵ . There, the previously described trade-off depending on $\epsilon \in (0, \epsilon^*]$ is illustrated. Notice that a similar behavior to the stability measures in Figure 3 can be observed in Figure 4 and Figure 5 in the example in Section 5.

FIGURE 3 Behavior of $\tilde{\alpha}_\epsilon(p^*(\epsilon))$, $\alpha(p^*(\epsilon))$ and $\|T(p^*(\epsilon), s)\|_{\mathcal{H}_2}$ as functions of ϵ . At $\epsilon = \epsilon^*$, we have $\tilde{\alpha}_\epsilon(p^*(\epsilon)) = 0$ and $\|T(p^*(\epsilon), s)\|_{\mathcal{H}_2} = \sqrt{1/\epsilon^*}$, and $\tilde{\alpha}_\epsilon(p^*(\epsilon)) \rightarrow \alpha(p^*(\epsilon))$ as $\epsilon \rightarrow 0$. It is important to mention that the behavior of $\alpha(p^*(\epsilon))$ and $\|T(p^*(\epsilon), s)\|_{\mathcal{H}_2}$ is not necessarily monotonic with respect to ϵ .

Problem (14) is in general non-convex, but since $\tilde{\alpha}_\epsilon(p)$ is a smooth function of p , it is smooth, unlike the minimization of the spectral and pseudospectral abscissa. It enables us to use classic gradient based techniques for its solution. Explicit expressions of the smoothed spectral abscissa and its gradient that are key for solving (14) are provided in the next subsection.

3.2 | The smoothed spectral abscissa and its gradient in terms of the delay Lyapunov matrix

We address the computation of the smoothed spectral abscissa and its gradient, both based on the so-called delay Lyapunov matrix of system (4), which is defined next:

Definition 2. ¹⁷ Matrix $U_\gamma(\tau, p)$, $\tau \in [-h_m, h_m]$, is called the delay Lyapunov matrix of system (4) associated with a symmetric matrix $W \in \mathbb{R}^{n \times n}$ if it satisfies the equations

$$\frac{\partial}{\partial \tau} U_\gamma(\tau, p) = U_\gamma(\tau, p)(A_0(p) - \gamma I) + \sum_{i=1}^m U_\gamma(\tau - h_i, p) A_i(p) e^{-\gamma h_i}, \quad \tau \in [0, h_m], \quad (18)$$

$$U_\gamma(\tau, p) = U_\gamma^T(-\tau, p), \quad \tau \in [-h_m, h_m], \quad (19)$$

$$-W = (A_0(p) - \gamma I)^T U_\gamma(0, p) + U_\gamma(0, p)(A_0(p) - \gamma I) + \sum_{i=1}^m (A_i^T(p) e^{-\gamma h_i} U_\gamma(h_i, p) + U_\gamma(-h_i, p) A_i(p) e^{-\gamma h_i}), \quad (20)$$

known as dynamic, symmetry and algebraic properties, respectively.

The existence and uniqueness of the delay Lyapunov matrix is stated in the next theorem.

Theorem 2. ¹⁷ Matrix $U_\gamma(\tau, p)$, $\tau \in [-h_m, h_m]$, exists and is unique if and only if system (4) satisfies the Lyapunov condition, that is, if its spectrum does not contain any root s_0 such that $s_0 + s_0 = 0$.

Notice that every exponentially stable system satisfies the Lyapunov condition, which implies that matrix $U_\gamma(\tau, p)$ of system (4) is well defined for every $\gamma \in \mathbb{R}$ and $p \in \mathbb{R}^k$ such that $\gamma > \alpha(p)$ (Item 1 of Lemma 1).

As previously mentioned, the function $f(p, \gamma)$ can be seen as the $\mathcal{H}_2$ norm of the transfer matrix $T_\gamma(p, s)$. This fact allows us to deduce the following theorem from equation (10) and Theorem 1 in Jarlebring et al. ¹⁶

Theorem 3. For all $(\gamma, p) \in \mathbb{R} \times \mathbb{R}^k$ such that $\gamma > \alpha(p)$, the equality

$$f(p, \gamma) = \text{tr} \left(Q_1^T U_\gamma(0, p) Q_1 \right) \quad (21)$$

is satisfied, where $U_\gamma(0, p)$ is the delay Lyapunov matrix of system (4) associated with matrix $Q_2^T Q_2$.

Proof. By Item 1 of Lemma 1, system (4) is exponentially stable, and equation (21) follows from (10) and Theorem 1 in Jarlebring et al.¹⁶ $\square$

With formula (21) at hand, for given $\epsilon > 0$ and vector of system parameters $p \in \mathbb{R}^k$, we can compute the smoothed spectral abscissa by finding γ such that

$$1/f(p, \gamma) - \epsilon = 0, \quad (22)$$

where $f(p, \gamma) = \text{tr} \left(Q_1^T U_\gamma(0, p) Q_1 \right)$, on the interval $(\alpha(p), +\infty)$. In order to do this, one can use for instance the Dekker-Brent method, or Newton-bisection method,²³ the latter requiring the derivative

$$\frac{\partial U_\gamma(0, p)}{\partial \gamma},$$

that can be computed by the scheme proposed in Section 4. The obtained value of γ corresponds to the smoothed spectral abscissa $\tilde{\alpha}_\epsilon(p)$ of system (1) (as illustrated in Figure 1).

A key element for solving (14) is the computation of the gradient of the smoothed spectral abscissa, for which an explicit expression is given in the next theorem.

Theorem 4. For a given $\epsilon > 0$, the gradient of the smoothed spectral abscissa $\tilde{\alpha}_\epsilon(p)$ with respect to p is

$$\nabla \tilde{\alpha}_\epsilon(p) = - \left(\text{tr} \left(Q_1^T \frac{\partial U_\gamma(0, p)}{\partial \gamma} Q_1 \right) \right)^{-1} \begin{pmatrix} \text{tr} \left(Q_1^T \frac{\partial U_\gamma(0, p)}{\partial p_1} Q_1 \right) \\ \vdots \\ \text{tr} \left(Q_1^T \frac{\partial U_\gamma(0, p)}{\partial p_k} Q_1 \right) \end{pmatrix}, \quad (23)$$

where $U_\gamma(0, p)$ is the delay Lyapunov matrix of system (4) associated with matrix $Q_2^T Q_2$, with $\gamma = \tilde{\alpha}_\epsilon(p)$.

Proof. From Definition 1, we have that $f(p, \tilde{\alpha}_\epsilon) = \frac{1}{\epsilon}$, then applying the chain rule, we obtain

$$\frac{\partial f(p, \gamma)}{\partial p_j} = - \frac{\partial f(p, \gamma)}{\partial \gamma} \frac{\partial \tilde{\alpha}_\epsilon(p)}{\partial p_j}, \quad j = \overline{1, k}.$$

Using the fact that matrix $U_\gamma(0, p)$ is smooth with respect to p ¹⁸ and equation (21), we arrive at

$$\frac{\partial \tilde{\alpha}_\epsilon(p)}{\partial p_j} = - \left(\text{tr} \left(Q_1^T \frac{\partial U_\gamma(0, p)}{\partial \gamma} Q_1 \right) \right)^{-1} \text{tr} \left(Q_1^T \frac{\partial U_\gamma(0, p)}{\partial p_j} Q_1 \right), \quad j = \overline{1, k},$$

and the assertion of the theorem follows from the definition of the gradient. $\square$

From Theorem 3 and Theorem 4, one can notice that computing the smoothed spectral abscissa and its gradient require the computation of the delay Lyapunov matrix and its derivatives with respect to the system parameters. This is addressed in the next section.

4 | COMPUTATION OF THE DELAY LYAPUNOV MATRIX AND ITS DERIVATIVES

In this section, we impose the following assumption:

Assumption 1. The delays h_i , $i = \overline{0, m}$, in system (1) are commensurate, expressed as $h_i = ih$, $i = \overline{0, m}$, where $h \geq 0$ is the basic delay.

We provide explicit formulae for the computation of $U_\gamma(0, p)$ and its derivatives. Unlike the formulae introduced in Gomez et al.^{18,19} for the computation of the sensitivity of $U_\gamma(0, p)$, it is worthy mentioning that here no integral term is explicitly required and the computation of both the delay Lyapunov matrix and its sensitivity can be carried out in only one stage. We use the

semianalytic approach, which consists of solving a delay-free boundary value problem explicitly in terms of matrix exponentials, deduced from (18), (19) and (20) (see section 3.4 in the book¹⁷).

Let us consider the vector of functions

$$v_\gamma(\tau, p) = \left\{ \text{vec}(U_\gamma(\tau + (m-j)h, p)) \right\}_{j=1,2m}^T, \quad \tau \in [0, h], \quad p \in \mathbb{R}^k,$$

whose j^{th} element is the vectorized segment of the delay Lyapunov matrix $U_\gamma(\tau + (m-j)h, p)$, $\tau \in [0, h]$. In order to lighten the notation, here we denote the partial derivatives of the vector $v_\gamma(\tau, p)$ with respect to τ by $v'_\gamma(\tau, p)$. By following the procedure presented in Section 3.4 in the book,¹⁷ which consists of imposing the dynamic property (18) for $\tau \in [0, h]$ and its analogous for $\tau \in [-h, 0]$ to every segment of the delay Lyapunov matrix, followed by vectorization, we obtain the ordinary differential equations system

$$v'_\gamma(\tau, p) = L(p, \gamma)v_\gamma(\tau, p), \quad \tau \in [0, h]. \quad (24)$$

Then, by using (19) and vectorizing (20), one obtains the boundary condition

$$M(p, \gamma)v_\gamma(0, p) + N(p, \gamma)v_\gamma(h, p) = - \begin{pmatrix} \text{vec}(Q_2^T Q_2) \\ 0 \end{pmatrix}. \quad (25)$$

Matrices $L(p, \gamma)$, $M(p, \gamma)$ and $N(p, \gamma)$ are given in Appendix B. From (24), one has that $v_\gamma(\tau, p) = e^{L(p, \gamma)\tau}v_\gamma(0, p)$, and by using it in (25), one arrives at

$$v_\gamma(0, p) = - \left(M(p, \gamma) + N(p, \gamma)e^{L(p, \gamma)h} \right)^{-1} \begin{pmatrix} \text{vec}(Q_2^T Q_2) \\ 0 \end{pmatrix},$$

where the inverse of the matrix on the right hand side exists whenever system (4) satisfies the Lyapunov condition.²⁴ Thus, matrix $U_\gamma(0, p)$ can be obtained from the last equation by devectorizing the m^{th} vector element of $v_\gamma(0, p)$, i.e.

$$U_\gamma(0, p) = \text{mat}(v_\gamma(0, p), m).$$

We now show that a similar procedure can be applied to the computation of the derivatives of the delay Lyapunov matrix. Taking the partial derivatives with respect to γ and p_j , $j = \overline{1, k}$, on both sides in (24) and (25), we get

$$\begin{aligned} \frac{\partial v'_\gamma(\tau, p)}{\partial \gamma} &= L(p, \gamma) \frac{\partial v_\gamma(\tau, p)}{\partial \gamma} + \frac{\partial L(p, \gamma)}{\partial \gamma} v_\gamma(\tau, p), \\ \frac{\partial v'_\gamma(\tau, p)}{\partial p_j} &= L(p, \gamma) \frac{\partial v_\gamma(\tau, p)}{\partial p_j} + \frac{\partial L(p, \gamma)}{\partial p_j} v_\gamma(\tau, p), \quad j = \overline{1, k}, \end{aligned} \quad (26)$$

for $\tau \in [0, h]$, and

$$\begin{aligned} M(p, \gamma) \frac{\partial v_\gamma(0, p)}{\partial \gamma} + \frac{\partial M(p, \gamma)}{\partial \gamma} v_\gamma(0, p) + N(p, \gamma) \frac{\partial v_\gamma(h, p)}{\partial \gamma} + \frac{\partial N(p, \gamma)}{\partial \gamma} v_\gamma(h, p) &= 0, \\ M(p, \gamma) \frac{\partial v_\gamma(0, p)}{\partial p_j} + \frac{\partial M(p, \gamma)}{\partial p_j} v_\gamma(0, p) + N(p, \gamma) \frac{\partial v_\gamma(h, p)}{\partial p_j} + \frac{\partial N(p, \gamma)}{\partial p_j} v_\gamma(h, p) &= 0. \end{aligned} \quad (27)$$

By rewriting equations (24) and (26) in matrix form, we arrive at the ordinary differential equations system

$$\chi'_\gamma(\tau, p) = \hat{L}(p, \gamma)\chi_\gamma(\tau, p), \quad \tau \in [0, h], \quad (28)$$

where

$$\chi_\gamma(\tau, p) = \left(v_\gamma^T(\tau, p), \frac{\partial v_\gamma^T(\tau, p)}{\partial \gamma}, \frac{\partial v_\gamma^T(\tau, p)}{\partial p_1}, \dots, \frac{\partial v_\gamma^T(\tau, p)}{\partial p_k} \right)^T,$$

and

$$\hat{L}(p, \gamma) = \begin{pmatrix} L & 0 & 0 & \dots & 0 \\ \frac{\partial L}{\partial \gamma} & L & 0 & \dots & 0 \\ \frac{\partial L}{\partial p_1} & 0 & L & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \frac{\partial L}{\partial p_k} & 0 & 0 & \dots & L \end{pmatrix}.$$

Combining (25) and (27), we arrive at the boundary condition

$$\hat{M}(p, \gamma)\chi_\gamma(0, p) + \hat{N}(p, \gamma)\chi_\gamma(h, p) = \hat{w}, \quad (29)$$

where, $\hat{w}^T = (-\text{vec}(Q_2^T Q_2)^T, 0)$,

$$\widehat{M}(p, \gamma) = \begin{pmatrix} M & 0 & 0 & \dots & 0 \\ \frac{\partial M}{\partial \gamma} & M & 0 & \dots & 0 \\ \frac{\partial M}{\partial p_1} & 0 & M & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \frac{\partial M}{\partial p_k} & 0 & 0 & \dots & M \end{pmatrix},$$

and

$$\widehat{N}(p, \gamma) = \begin{pmatrix} N & 0 & 0 & \dots & 0 \\ \frac{\partial N}{\partial \gamma} & N & 0 & \dots & 0 \\ \frac{\partial N}{\partial p_1} & 0 & N & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \frac{\partial N}{\partial p_k} & 0 & 0 & \dots & N \end{pmatrix}.$$

It follows from (28) that $\chi_\gamma(\tau, p) = e^{\widehat{L}(p, \gamma)\tau} \chi_\gamma(0, p)$. Then, substituting $\chi_\gamma(\tau, p)$ in the second term of (29), we obtain

$$\left(\widehat{M}(p, \gamma) + \widehat{N}(p, \gamma) e^{\widehat{L}(p, \gamma)h} \right) \chi_\gamma(0, p) = \hat{w}.$$

One can notice that the diagonal block terms of the lower triangular matrix $\widehat{M}(p, \gamma) + \widehat{N}(p, \gamma) e^{\widehat{L}(p, \gamma)h}$ are given by the matrix $M(p, \gamma) + N(p, \gamma) e^{L(p, \gamma)h}$, which is non-singular whenever system (4) satisfies the Lyapunov condition.²⁴ Hence, matrix $\widehat{M}(p, \gamma) + \widehat{N}(p, \gamma) e^{\widehat{L}(p, \gamma)h}$ is non-singular if system (4) satisfies the Lyapunov condition as well, and from the previous expression, we get

$$\chi_\gamma(0, p) = \left(\widehat{M}(p, \gamma) + \widehat{N}(p, \gamma) e^{\widehat{L}(p, \gamma)h} \right)^{-1} \hat{w}.$$

In this way, we arrive at the following proposition.

Proposition 1. If system (4) satisfies the Lyapunov condition, then the following equality holds:

$$\chi_\gamma(0, p) = \left(\widehat{M}(p, \gamma) + \widehat{N}(p, \gamma) e^{\widehat{L}(p, \gamma)h} \right)^{-1} \hat{w}.$$

The general procedure in order to compute the delay Lyapunov matrix and its derivatives with respect to the system parameters is summarized as follows:

1. For given matrices $A_i(p)$, $i = \overline{1, m}$, construct block matrices L , M and N as in Appendix B.
2. Obtain the partial derivatives of each element of matrices L , M and N with respect to γ and p_i , $i = \overline{1, k}$, and construct block matrices $\widehat{L}(p, \gamma)$, $\widehat{M}(p, \gamma)$ and $\widehat{N}(p, \gamma)$.
3. Compute $\chi_\gamma(0, p)$ from Proposition 1.
4. Finally, obtain the matrices of interest in Theorem 3 and Theorem 4 by devectorizing $\chi_\gamma(0, p)$ as follows:

$$\begin{aligned} U_\gamma(0, p) &= \text{mat} \left(v_\gamma(0, p), m \right) = \text{mat} \left(\chi_\gamma(0, p), m \right), \\ \frac{\partial U_\gamma(0, p)}{\partial \gamma} &= \text{mat} \left(\frac{v_\gamma(0, p)}{\partial \gamma}, m \right) = \text{mat} \left(\chi_\gamma(0, p), 3m \right), \\ \frac{\partial U_\gamma(0, p)}{\partial p_j} &= \text{mat} \left(\frac{v_\gamma(0, p)}{\partial p_j}, m \right) = \text{mat} \left(\chi_\gamma(0, p), (2j + 3)m \right), \quad j = \overline{1, k}. \end{aligned}$$

5 | EXAMPLE

We illustrate the previous results with the stabilization of a triple integrator with a delay-based controller.²⁵ We compute the smoothed spectral abscissa by solving equation (22) with the algorithm Dekker-Brent, implemented by the function `fzero` of MATLAB, and we solve the optimization problem (14) by using the BFGS (Broyden-Fletcher-Goldfar-Shanno) quasi-Newton method²⁶ implemented by the MATLAB function `fminunc` included in the optimization toolbox. The gradient of $\tilde{\alpha}_\epsilon(p)$ is provided by Theorem 4, and computed by using Proposition 1. We consider the weighting matrices $Q_1 = I$ and $Q_2 = I$, satisfying (8) and (9).

TABLE 1 Obtained results of the optimization of the smoothed spectral abscissa $\tilde{\alpha}_\epsilon(p)$ for different values of ϵ .

ϵ	$\tilde{\alpha}_\epsilon(p^*(\epsilon))$	$\ T(p^*(\epsilon), s)\ _{\mathcal{H}_2}$	$\alpha(p^*(\epsilon))$	$p^*(\epsilon)^T$		
				$(p_1,$	$p_2,$	$p_3)$
1	1.6792		0.3887	(−0.4571,	1.1158,	−1.2411)
e^{-1}	0.8002		0.2070	(−1.0057,	2.0354,	−1.3322)
e^{-2}	0.3946		0.0614	(−1.4325,	2.6488,	−1.4346)
e^{-3}	0.1499	7.8264	−0.0583	(−1.5262,	2.6292,	−1.269)
e^{-4}	−0.0076	7.1710	−0.1475	(−1.4139,	2.3335,	−1.0489)
e^{-5}	−0.1110	7.4131	−0.2053	(−1.2481,	2.0217,	−0.8742)
e^{-6}	−0.1812	7.9833	−0.2380	(−1.0956,	1.7685,	−0.7511)
e^{-7}	−0.2307	8.8105	−0.2671	(−0.9715,	1.5759,	−0.6655)

Example 1. Consider the state space representation of a triple integrator

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t), \\ y(t) &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t). \end{aligned} \quad (30)$$

A controller based on output derivatives approximation of the form

$$u(t) = p_1 y(t) + p_2 y(t-1) + p_3 y(t-2) \quad (31)$$

is proposed in Kharitonov et al²⁵. The closed-loop system (30)-(31) can be written as a system of the form (1) with two delays:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p_1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_2 & 0 & 0 \end{pmatrix} x(t-1) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_3 & 0 & 0 \end{pmatrix} x(t-2). \quad (32)$$

According with (12), we consider the transfer matrix related with system (32) as

$$T(p, s) = \left(sI - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p_1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_2 & 0 & 0 \end{pmatrix} e^{-s} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_3 & 0 & 0 \end{pmatrix} e^{-2s} \right)^{-1}. \quad (33)$$

We set the vector of system parameters as $p = (p_1, p_2, p_3)^T$, and look for p such that system (32) is exponentially stable. We optimize the function $\tilde{\alpha}_\epsilon(p)$ for different values of ϵ on the interval $[e^{-7}, 1]$ considering as starting point

$$p_0 = (-0.2693, 0.5184, -0.2496)^T.$$

Table 1 displays the obtained results for some values of ϵ . The fifth column shows the obtained optimizers, the fourth and second columns display the spectral and the smoothed spectral abscissa of system (32), respectively, and the third column shows the $\mathcal{H}_2$ norm of (33), all of them evaluated at the obtained minimizer $p^*(\epsilon)$. Notice that system (32) is not exponentially stable for the optimizers obtained in the first three rows, hence, the evaluation of the $\mathcal{H}_2$ norm of (33) at these optimizers is not finite. A comparison between the values of the third and fourth columns allows observing the trade-off between the minimization of the spectral abscissa of system (32) and the $\mathcal{H}_2$ norm of (33) depending on ϵ .

Figure 4 shows the spectral and smoothed spectral abscissa evaluated at $p^*(\epsilon)$ considering a finer grid of ϵ values. There, we observe that $\tilde{\alpha}_\epsilon(p^*)$ approaches $\alpha(p^*)$, and therefore also approaches a minimum of the spectral abscissa, as ϵ goes to zero. The searching of p can be stopped at $\epsilon = e^{-4}$ as $\tilde{\alpha}_\epsilon(p^*) < 0$, which implies that the obtained minimizer $p^*(\epsilon)$ makes system (32) exponentially stable (as corroborated by the computed spectral abscissa). It is important to mention that, although in this example $\alpha(p^*(\epsilon))$ monotonically grows with respect to the considered values of ϵ , in general there is no guarantee of this.

The shaded zone in Figure 4 depict the interval of ϵ where there is a minimum of the $\mathcal{H}_2$ norm of (33), that is, the interval where we can find $p^*(\epsilon^*)$ such that $\tilde{\alpha}_{\epsilon^*}(p^*) = 0$. By bisecting the minimum of $\tilde{\alpha}_\epsilon$ with respect to ϵ on this interval, we get

that $\tilde{\alpha}_{\epsilon^*}(p^*) \approx 0$ for $\epsilon^* \approx 0.0195$ and $p^*(\epsilon^*) = (-1.4236, 2.354, -1.0617)^T$. Then, by equation (16) and (17), $p^*(\epsilon^*)$ is a minimizer of the $\mathcal{H}_2$ norm of (33) and

$$\min \|T(p^*, s)\|_{\mathcal{H}_2} \approx \sqrt{\frac{1}{\epsilon^*}} = 7.1611.$$

Figure 5 displays the $\mathcal{H}_2$ norm of (33) evaluated at the minimizers $p^*(\epsilon)$ of $\tilde{\alpha}_\epsilon(p)$ for different values of ϵ on the interval $[e^{-7}, e^{-2.5}]$, for which system (32) is exponentially stable. The interval of ϵ on which the bisection is carried out is shaded and the value of the obtained minimum is marked with the blue line.

FIGURE 4 Smoothed spectral abscissa and spectral abscissa of system (32) evaluated at the minimizers of $\tilde{\alpha}_\epsilon(p)$ obtained for different values of ϵ .

FIGURE 5 $\mathcal{H}_2$ norm of the transfer matrix (33) evaluated at the minimizers of $\tilde{\alpha}_\epsilon(p)$ obtained for different values of ϵ .

6 | CONCLUSION

We introduced the concept of the smoothed spectral abscissa for retarded type systems, which is a smooth function of the system parameters, and showed that a control design approach based on minimizing this measure results in a compromise between optimizing the exponential decay rate of solutions and the $\mathcal{H}_2$ norm of the transfer matrix (12), as illustrated in Section 5. A useful tool in the control design approach are the formulae of the smoothed spectral abscissa and its gradient with respect to the system parameters in terms of the delay Lyapunov matrix. Finally, in Section 4, we assume that the delays are commensurate, which led

to a novel characterization of derivatives of the Lyapunov matrix allowing to compute matrix and derivatives simultaneously. For computing Lyapunov matrices of systems with incommensurate delays, we refer to the recent works.^{27 28 29 16}

ACKNOWLEDGMENTS

This work was supported by the projects C14/17/072 of the KU Leuven Research Council and G0A5317N of the Research Foundation- Flanders (FWO - Vlaanderen).

APPENDIX

A PROOF OF LEMMA 1

Proof of Lemma 1:. *Item 1.* Consider the quasipolynomials

$$q_1(p, s) = \det H(s) = \det \left(sI - \sum_{i=0}^m A_i(p)e^{-sh_i} \right), \quad s \in \mathbb{C},$$

and

$$q_2(p, s) = \det \left(sI - (A_0(p) - \gamma I) - \sum_{i=1}^m A_i(p)e^{-\gamma h_i}e^{-sh_i} \right), \quad s \in \mathbb{C},$$

of system (1) and system (4), respectively. Since

$$q_1(p, s) = q_2(p, s - \gamma),$$

we have that for a given $\gamma > \alpha(p)$, $\Re(s) - \gamma < 0$, for all $s \in \Lambda$, therefore system (4) is exponentially stable.

Item 2. From Item 1, if $\gamma > \alpha(p)$, system (4) is exponentially stable, and from equation (5), matrix K_γ admits an exponential bound of the form

$$\|K_\gamma(t)\| \leq \mu e^{-\beta t}, \quad t \geq 0,$$

where μ and β are positive constants depending on $p \in \mathbb{R}^k$ and $\gamma \in \mathbb{R}$. Therefore the improper integral in (7) is convergent and $f(p, \gamma)$ is finite for all $p \in \mathbb{R}^k$ and $\gamma \in \mathbb{R}$ such that $\gamma > \alpha(p)$.

Item 3. The smoothness of the function $(p, \gamma) \mapsto f(p, \gamma)$ follows from the fact that the matrices of system (4) smoothly depend on $p \in \mathbb{R}^k$ and γ , and that, by Item 2, the function $f(p, \gamma)$ is finite.

Item 4. Let us prove first that

$$K_\gamma(t) = e^{-\gamma t} K(t), \quad t \in \mathbb{R},$$

where $K(t)$ is the fundamental matrix of system (1) satisfying

$$\frac{d}{dt} K(t) = \sum_{i=0}^m K(t - h_i) A_i(p), \quad t \geq 0, \tag{A1}$$

with initial condition

$$K(0) = I, \quad \text{and} \quad K(t) = 0, \quad \forall t < 0. \tag{A2}$$

In order to do this, it is enough to prove that $e^{-\gamma t} K(t)$ satisfies (5) and (6). From equation (A1), we get

$$\begin{aligned} \frac{d}{dt} K_\gamma(t) &= \frac{d}{dt} (e^{-\gamma t} K(t)) = -\gamma e^{-\gamma t} K(t) + e^{-\gamma t} \frac{d}{dt} K(t) \\ &= -\gamma K_\gamma(t) + e^{-\gamma t} \sum_{i=0}^m K(t - h_i) A_i(p) \\ &= K_\gamma(t) (A_0(p) - \gamma I) + \sum_{i=1}^m K_\gamma(t - h_i) A_i(p) e^{-\gamma h_i}, \quad t \geq 0, \end{aligned}$$

and from (A2), we deduce that $e^{-\gamma t}K(t)$ satisfies (6). Hence, (7) can be written as

$$f(p, \gamma) = \int_0^\infty \text{tr} \{ Q_1^T K^T(t) Q_2^T Q_2 K(t) Q_1 \} e^{-2\gamma t} dt. \quad (\text{A3})$$

Differentiating with respect to γ , we obtain

$$\frac{\partial f(p, \gamma)}{\partial \gamma} = -2 \int_0^\infty \text{tr} \{ Q_1^T K^T(t) Q_2^T Q_2 K(t) Q_1 \} t e^{-2\gamma t} dt < 0.$$

Item 5. We prove first that $f(p, \gamma) \rightarrow 0$ as $\gamma \rightarrow \infty$. We rewrite the transfer matrix T_γ as

$$\begin{aligned} T_\gamma(p, i\omega) &= Q_1 \frac{1}{\gamma} \left(\left(1 + \frac{i\omega}{\gamma} \right) I - \frac{1}{\gamma} \left(\sum_{i=0}^m A_i e^{-i\omega h_i} e^{-\gamma h_i} \right) \right)^{-1} Q_2 \\ &= Q_1 \frac{1}{\gamma} \left(1 + \frac{i\omega}{\gamma} \right)^{-1} \left(I - \frac{1}{\gamma} \left(1 + \frac{i\omega}{\gamma} \right)^{-1} \left(\sum_{i=0}^m A_i e^{-i\omega h_i} e^{-\gamma h_i} \right) \right)^{-1} Q_2. \end{aligned}$$

From the previous expression, one can see that $\|T_\gamma(p, s)\|_{\mathcal{H}_2} \rightarrow 0$ as $\gamma \rightarrow \infty$, and the assertion follows from equation (10).

In order to prove that $f(p, \gamma) \rightarrow \infty$ as $\gamma \rightarrow \alpha(p)+$, let us prove first that if conditions (8) and (9) are satisfied, then

$$\int_{-\infty}^\infty \text{tr} (T^*(p, s_d + \eta + i\omega) T(p, s_d + \eta + i\omega)) d\omega \rightarrow \infty \text{ as } \eta \rightarrow 0+, \quad (\text{A4})$$

where $T(p, s)$ is given by (12). We say that the vectors $v \in \mathbb{C}^n$ and $w \in \mathbb{C}^n$ are right and left eigenvectors associated with a given eigenvalue $s_0 \in \Lambda$ if

$$H(s_0)v = 0, \text{ and } w^*H(s_0) = 0. \quad (\text{A5})$$

Let the geometric multiplicity of the rightmost eigenvalue s_d be denoted by q . Then, according to Keldysh's theorem (see Theorem 2.6 in³⁰ and Theorem 1.6.5 in³¹), we have that in a neighbourhood $\mathcal{U}$ of the rightmost eigenvalue s_d the following holds:

$$H^{-1}(s) = \frac{1}{(s - s_d)^\kappa} \sum_{i=1}^q v_0^i w_0^{i*} + R_1(s), \quad s \in \mathcal{U} \setminus \{s_d\},$$

where $R_1(s)$, $s \in \mathcal{U} \setminus \{s_d\}$, is such that $R_1(s)(s - s_d)^{\kappa-1}$ is an analytic function, v_0^i and w_0^i , $i = \overline{1, q}$, are right and left linearly independent eigenvectors associated with s_d , respectively, $\kappa := m_1 = \max \{m_l : l = \overline{1, L}\}$, where $m_1 \geq \dots \geq m_L$ are integer numbers, and $q = L$. The numbers m_l , $l = \overline{1, L}$, are called the partial multiplicities of H at s_d and the definition of each of them can be found in Definition 2.5 in³⁰ or in Definition 1.6.1 in³¹. Then, the following equality holds:

$$\begin{aligned} T(p, s) &= \frac{1}{(s - s_d)^\kappa} Q_2 \sum_{i=1}^q v_0^i w_0^{i*} Q_1 + Q_2 R_1(s) Q_1 \\ &= \frac{1}{(s - s_d)^\kappa} Q_2 (v_0^1 \dots v_0^q) \begin{pmatrix} w_0^{1*} \\ \vdots \\ w_0^{q*} \end{pmatrix} Q_1 + Q_2 R_1(s) Q_1. \end{aligned} \quad (\text{A6})$$

Now notice that (8) and (9) hold if and only if for every left and right eigenvectors $w \in \mathbb{C}^n$ and $v \in \mathbb{C}^n$ associated with $s_d \in \Lambda$,

$$w^* Q_1 \neq 0, \text{ and } Q_2 v \neq 0, \quad (\text{A7})$$

respectively. Hence, for every $(c_1, \dots, c_q) \neq 0$, we have that

$$(c_1 w_0^{1*} + \dots + c_q w_0^{q*}) Q_1 \neq 0, \text{ and } Q_2 (c_1 v_0^1 + \dots + c_q v_0^q) \neq 0,$$

which implies that

$$(Q_2 v_0^1 \dots Q_2 v_0^q) \text{ and } \begin{pmatrix} w_0^{1*} Q_1 \\ \vdots \\ w_0^{q*} Q_1 \end{pmatrix} \quad (\text{A8})$$

have full column rank and full row rank, respectively. Hence, if conditions (8) and (9) hold, then

$$Q_2 \sum_{i=1}^q v_0^i w_0^{i*} Q_1 \neq 0,$$

and it follows from (A6) that (A4) holds.

Then, by rewriting the transfer matrix T_γ in terms of T , we have

$$T_\gamma(p, s) = T(p, s + \gamma),$$

hence,

$$\int_{-\infty}^{\infty} \text{tr} \left(T_\gamma^*(p, i\omega) T_\gamma(p, i\omega) \right) d\omega = \int_{-\infty}^{\infty} \text{tr} \left(T^*(p, \gamma + i\omega) T(p, \gamma + i\omega) \right) d\omega,$$

and by (A4), we have that

$$\int_{-\infty}^{\infty} \text{tr} \left(T_\gamma^*(p, i\omega) T_\gamma(p, i\omega) \right) d\omega \rightarrow \infty \text{ as } \gamma \rightarrow \alpha(p) + .$$

Finally, by equation (10) and the definition of the $\mathcal{H}_2$ norm, we get that $f(p, \gamma) \rightarrow \infty$ as $\gamma \rightarrow \alpha(p) +$. $\square$

B MATRICES FOR THE COMPUTATION OF THE DELAY LYAPUNOV MATRIX

We present the matrices L , M and N used in the computation of the delay Lyapunov matrix $U_\gamma(0, p)$ and its partial derivatives with respect to γ and p_j , $j = \overline{1, k}$, in Proposition 1. In order to simplify the expressions, we omit the arguments p and γ and denote the matrices of system (4) as $\hat{A}_0 = A_0 - \gamma I$ and $\hat{A}_i = A_i e^{-\gamma h_i}$, $i = \overline{1, m}$.

Matrix L , M and N are as follows:

$$L = \begin{pmatrix} \hat{A}_0^T \otimes I_n & \dots & \hat{A}_{m-1}^T \otimes I_n & \hat{A}_m^T \otimes I_n & 0 & 0 \\ 0 & \hat{A}_0^T \otimes I_n & \dots & \dots & \hat{A}_m^T \otimes I_n & 0 \\ & & \ddots & & \ddots & \\ 0 & 0 & \hat{A}_0^T \otimes I_n & \dots & \dots & \hat{A}_m^T \otimes I_n \\ -I \otimes \hat{A}_m^T & \dots & \dots & -I \otimes \hat{A}_0^T & 0 & 0 \\ 0 & -I \otimes \hat{A}_m^T & \dots & \dots & -I \otimes \hat{A}_0^T & 0 \\ & & \ddots & & \ddots & \\ 0 & 0 & -I \otimes \hat{A}_m^T & \dots & \dots & -I \otimes \hat{A}_0^T \end{pmatrix}$$

$$M = \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline I_{(2m-1)n^2} & 0 \end{array} \right),$$

$$N = \left(\begin{array}{c|c} N_{11} & 0 \\ \hline 0 & -I_{(2m-1)n^2} \end{array} \right),$$

where

$$M_{11} = (I_n \otimes \hat{A}_{m-1}^T \dots I_n \otimes \hat{A}_1^T \ I_n \otimes \hat{A}_0^T + \hat{A}_0^T \otimes I_n \ \hat{A}_1^T \otimes I_n \dots \hat{A}_{m-1}^T \otimes I_n),$$

and

$$M_{12} = \hat{A}_m^T \otimes I_n, \ N_{11} = I_n \otimes \hat{A}_m^T.$$

For the sake of clarity here we denote the identity matrix of dimension n by I_n .

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