**The dynamic bending analysis of plates under thermal load using an
efficient wave-based method**

**Qiang Chen^{a,b,c,d}, Qingguo Fei^a, Hendrik Devriendt^{c,d}, Shaoqing Wu^{a,b}, Bert Pluymers^{c,d},
Wim Desmet^{c,d}**

^a *Institute of Aerospace Machinery and Dynamics, Southeast University, Nanjing, 211189, China*

^b *Department of Engineering Mechanics, Southeast University, Nanjing, 211189, China*

^c *KU Leuven, Department of Mechanical Engineering, Division PMA, Leuven, B-3001, Belgium*

^d *DMMS Lab, Flanders Make, Belgium*

Corresponding author: Qingguo Fei

Institute of Aerospace Machinery and Dynamics,

Southeast University, Nanjing, Jiangsu 211189, China

Email: qgfei@seu.edu.cn

Phone: +86-025-83792569

18 Abstract

19 The efficient Wave-Based Method (WBM) is extended to predict the bending vibration response of
20 thin-walled rectangular plates under thermal load. Governing equations of dynamic plate bending
21 problems considering thermal effects are first introduced. The general solution and particular solution
22 of governing equation are derived by the separate variable method and residue theorem, respectively.
23 Then novel wave functions and particular solution function are selected to establish the wave-based
24 model of dynamic plate bending problems under thermal load. The accuracy of WBM is verified by
25 finite element method (FEM) for a simply supported plate and a clamped plate. Then the influence of
26 thermal load on the dynamic bending responses of a clamped plate is investigated. Simulation results
27 prove that WBM achieves higher convergence rate than FEM. The thermal load significantly changes
28 the structural dynamic behaviors. When both the temperature-dependent material properties and
29 thermal stresses are considered, the increase of environment temperature leads to the resonant peak of
30 displacement response shifts toward lower frequency.

31

32 Keywords: Wave-based method; Thermal effect; Plate bending vibrations; Residue theorem;
33 Computation efficiency.

34

35 1. Introduction

36 Thin-walled structures used in aerospace engineering usually encounter aerodynamic heating and
37 broadband excitation during their service period [1]. The thermal load caused by aerodynamic heating
38 changes the structural stiffness and alters the structural dynamic behaviors [2]. Recent developments in
39 aerospace engineering [3, 4] have heightened the need for broadband vibration analysis of thin-walled
40 structures under thermal load. Therefore, it is essential to develop efficient simulation techniques that
41 are able to accurately predict the broadband dynamic response of thin-walled structures under thermal
42 load.

43 Simulation method in different frequency ranges for the thermal vibration analysis of thin-walled
44 structures is diverse. In the low-frequency range, deterministic approaches such as Finite Element
45 Method (FEM) and Boundary Element Method (BEM) are commonly used [5-9]. Jeyaraj et al. [10]
46 investigated the influence of thermal effect and boundary conditions on the vibro-acoustic response of a
47 plate-cavity coupled system by a combined FEM/BEM approach. Geng et al. [11, 12] investigated the
48 vibro-acoustic behavior of a clamped plate in thermal environment by a modal-based method, a
49 FEM/BEM technique, and the experimental method. Sharma et al.[13] investigated the vibro-acoustic
50 behavior of a layered panel structure under thermal load by a higher-order coupled FEM/BEM
51 approach. Wang and Chen [14] investigated the structural intensity of the cantilevered plate under
52 thermal load by using the modal expansions and FEM. Bui et al. [15, 16] developed a
53 consecutive-interpolation procedure (CIP) by using the 4-node quadrilateral element (CQ4), which has
54 better accuracy than the standard quadrilateral element, for structural dynamic problems. Deterministic
55 methods can accurately predict the thermal vibration response of thin-walled structures in the
56 low-frequency range. However, to describe the high-frequency vibration behavior of thin-walled
57 structures, enormous elements may be required. Therefore, the applications of deterministic methods
58 are limited to low-frequency ranges.

59 In the high-frequency range, the energy-based methods, which describe the vibration behavior in an
60 average or a statistical way, are widely adopted [17-20]. Among these energy-based methods, statistical
61 energy analysis (SEA, [21]) is a prominent technique. With a combination of the energy flow model
62 and the power injection method, Chen et al. [22] calculated the SEA parameter of thin-walled structure
63 in thermal environment. The proposed method is capable of extending the traditional SEA to the

64 thermal vibrational analysis. The influence of thermal effect and non-uniform temperature distribution
65 on the SEA parameters of a composite L-shaped plate was investigated. However, the energy response
66 calculated by SEA is space- and frequency-averaged for each subsystem of structure; local response,
67 which is commonly required in engineering design, cannot be obtained by SEA. Moreover,
68 assumptions in SEA, such as the mode count in a frequency band and the diffusivity of the excitation,
69 can only be satisfied above a certain frequency. Therefore, SEA is restricted to high-frequency ranges.

70 The frequency range between the low-frequency band and high-frequency band is regarded as
71 mid-frequency gap [23]. The mid-frequency dynamic analysis of thin-walled structures has drawn
72 much attention by researchers and several approaches were proposed [22-26]. These approaches can be
73 categorized into three classes [23]: the extension of the deterministic approaches towards higher
74 frequency, the extension of the statistical methods towards lower frequency, and the combination of
75 deterministic and statistical approaches. Wave-based method (WBM, [27]), developed based on the
76 indirect Trefftz method, is an effective deterministic approach to solve the mid-frequency dynamic
77 problems. The wave functions and particular solution function, rather than mode shape functions used
78 in FEM, are adopted in WBM to calculate the dynamic response of systems. It has successfully been
79 applied to acoustic simulations, plate bending problems, and vibro-acoustic analysis.

80 For the applications of WBM on dynamic plate bending analysis, Desmet [27] proposed a set of
81 wave functions to solve the plate bending vibration problems. The advantages of WBM on convergence
82 rate and computational cost are performed by comparing with FEM. Then Vanmaele et al. [28]
83 validated the performance of WBM for dynamic plate bending problems with different boundary
84 conditions. To achieve the convergence of WBM on the dynamic bending analysis of structure with
85 corner stress singularities, Vanmaele et al. [29] proposed a set of wave functions with corner functions
86 and improved the accuracy of WBM. Jonckheere et al. [30] applied WBM to the dynamic bending
87 analysis of plates under distributed deterministic and random excitation. The distinct particular
88 solutions for distributed loads were derived by a Hankel-based approach and a Fourier-based approach.
89 Klanner and Ellermann [31] extended WBM to the dynamic bending analysis of thick plates by using
90 the Mindlin plate theory. Chen et al. [32, 33] applied WBM to analyze the dynamic response of
91 cylindrical shells with discontinuity in thickness and cylindrical shells with nonuniform stiffener
92 distributions, respectively. Compared with FEM, WBM has better computational efficiency for
93 response prediction in the mid-frequency ranges. Compared with SEA, WBM can give the local

dynamic response, which is more useful for structural design and safety evaluation. Therefore, WBM could be a useful technique in the dynamic analysis of thin-walled structures under the broadband excitation. At present, the existed work mainly focuses on the application of WBM to the dynamic analysis of thin-walled structures in normal temperature environment. The applicability and feasibility of WBM for structural vibration problems under thermal load need further exploration.

The main purpose of this paper is to extend WBM to analyze bending vibrations of plates under thermal load. The research work is organized as follows. In Section 2, the governing equation of dynamic plate bending problems in thermal environment is introduced and the solutions of governing equation are derived. In Section 3, novel wave functions and particular solution function are selected and the wave-based model is established. In Section 4, numerical simulations are performed and the influence of thermal load on bending vibration response are discussed. Conclusions are drawn in the last Section.

2. Problem definition

2.1. Governing equation

The strain-stress relations of a thin plate under thermal load are expressed as [34, 35]

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} - \boldsymbol{\sigma}_T \Leftrightarrow \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = -\frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & (1-\mu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} - \frac{E\alpha\Delta T}{1-\mu} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad (1)$$

where $\boldsymbol{\sigma}$, $\mathbf{C}$, $\boldsymbol{\varepsilon}$, and $\boldsymbol{\sigma}_T$ are the stress vector, elastic coefficients matrix, strain vector, and thermal stress vector, respectively. ε_x and ε_y are the normal strains parallel to the x and y axes, respectively. γ_{xy} is the shear strain in the xy plane. σ_x and σ_y are the normal stresses. τ_{xy} is the shear stress. E is the Young's modulus. μ is the Poisson's ratio. α denote the linear thermal expansion coefficient. $\Delta T = T - T_0$ is the difference between the environment temperature T and the reference temperature T_0 .

The in-plane displacements of any point through the thickness of plate are expressed as [35]

$$u = z\theta_y = -z\frac{\partial w}{\partial x}, \quad v = -z\theta_x = -z\frac{\partial w}{\partial y} \quad (2)$$

where u , v , and w are the displacements of any point in the plate along the x , y , and z directions, respectively.

The strain-displacement relations are given as following [35]

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -2z\frac{\partial^2 w}{\partial x \partial y} \quad (3)$$

121 These relations can further be rewritten in matrix form as

$$122 \quad \boldsymbol{\varepsilon} = -z \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial y^2} & 2 \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}^T \quad (4)$$

123 Therefore, the strain-displacement relations in matrix form can be expressed as [14]

$$124 \quad \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = -\frac{Ez}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & 1-\mu \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} - \frac{E\alpha\Delta T}{1-\mu} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad (5)$$

125 By integrating Eq.(5) along the plate thickness, the membrane forces, the bending moments, and

126 the shear forces are given as[14]

$$127 \quad \begin{aligned} N_x &= -\frac{Eh}{1-\mu^2} \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) - N_T, N_y = -\frac{Eh}{1-\mu^2} \left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) - N_T, N_{xy} = -\frac{Eh}{1+\mu} \frac{\partial^2 w}{\partial x \partial y}, \\ M_x &= -D \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) - M_T, M_y = -D \left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) - M_T, M_{xy} = M_{yx} = -D(1-\mu) \frac{\partial^2 w}{\partial x \partial y}, \\ Q_x &= -D \left(\frac{\partial^3 w}{\partial x^3} + \mu \frac{\partial^3 w}{\partial x \partial y^2} \right) - \frac{\partial M_T}{\partial x} = -D \frac{\partial}{\partial x} \nabla^2 w - \frac{\partial M_T}{\partial x}, \\ Q_y &= -D \left(\frac{\partial^3 w}{\partial y^3} + \mu \frac{\partial^3 w}{\partial x^2 \partial y} \right) - \frac{\partial M_T}{\partial y} = -D \frac{\partial}{\partial y} \nabla^2 w - \frac{\partial M_T}{\partial y}, \end{aligned} \quad (6)$$

128 where N_x , N_y , and N_{xy} are the internal membrane forces, M_x and M_y are the bending moments, M_{xy} is the

129 twisting moment, Q_x and Q_y are the transverse shear forces, h is the thickness of plate, $D = \frac{Eh^3}{12(1-\mu^2)}$

130 is the bending stiffness of the plate, $N_T = \frac{E\alpha}{1-\mu} \int_{-h/2}^{h/2} \Delta T dz$ and $M_T = \frac{E\alpha}{1-\mu} \int_{-h/2}^{h/2} \Delta T z dz$ are the equivalent

131 membrane force and the equivalent bending moment, respectively. For a plate under uniform

132 temperature load, it gives $M_T = 0$. $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

133 For an infinitesimal plate element, the force balance equations in the three directions and the

134 moment balance equations around the x and y axes are respectively given as [11]

$$135 \quad \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (7)$$

$$136 \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (8)$$

$$137 \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + F - \rho h \frac{\partial^2 w}{\partial t^2} = 0 \quad (9)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \quad (10)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \quad (11)$$

where F is the applied excitation.

Substituting Eq.(6) into Eqs.(7) to (11), the governing equation of dynamic plate bending problems under thermal load can be simplified and expressed as

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} - N_c \nabla^2 w = F \quad (12)$$

where ρ is the density, $N_c = \frac{-E\alpha\Delta T h}{1-\mu}$ is the thermally induced force.

2.1.1. General solutions

For the case with harmonic excitation, the general solution of Eq.(12) can be derived by the separate variable method

$$w(x, y, t) = C e^{-jk_x x} e^{-jk_y y} e^{-j\omega t} = W(x, y) e^{-j\omega t} \quad (13)$$

where k_x and k_y are the wavenumbers in x - and y - directions, j is the imaginary unit, ω is the circular frequency, $f = \omega/2\pi$ is the frequency, C is the constant coefficient.

By substituting Eq.(13) into the homogeneous part of Eq.(12), we obtain

$$D(k_x^2 + k_y^2)^2 + N_c(k_x^2 + k_y^2) - \rho h \omega^2 = 0 \quad (14)$$

Assuming that k is the total wave numbers in the wave propagation angle θ , so k_x and k_y could be expressed as:

$$k_x = k \cos \theta, k_y = k \sin \theta \quad (15)$$

Substituting Eq.(15) into Eq.(14), it gives

$$Dk^4 + N_c k^2 - \rho h \omega^2 = 0 \quad (16)$$

The solutions of the Eq.(16) are

$$k = \pm k_{c1}, \pm j k_{c2} \quad (17)$$

$$\text{where } k_{c1} = \sqrt{-\frac{N_c}{2D} + \sqrt{\left(\frac{N_c}{2D}\right)^2 + \frac{\rho h \omega^2}{D}}}, k_{c2} = \sqrt{\frac{N_c}{2D} + \sqrt{\left(\frac{N_c}{2D}\right)^2 + \frac{\rho h \omega^2}{D}}} \quad \text{and} \quad k_x^2 + k_y^2 = k_{c1}^2, -k_{c2}^2.$$

2.1.2. Particular solution

The point force excitation is considered in this paper. Other types of excitations can be regarded as the summation of different point forces with amplitudes.

164 By substituting Eq.(13) into Eq.(12), it gives

$$165 \quad D\nabla^4 W(x, y) - N_c \nabla^2 W(x, y) - \rho h W(x, y) = F_0 \delta(x - x_F) \delta(y - y_F) \quad (18)$$

166 where F_0 is the amplitude of point force excitation F , point $p_F(x_F, y_F)$ is the location of F , $\delta(x - x_F)$ and
167 $\delta(y - y_F)$ are the Delta functions.

168 To calculate the particular solution of Eq.(18), the complex Fourier transform is applied on Eq.(18).

169 The Fourier transform of $W(x, y)$, which denoted as $\tilde{W}(\gamma)$, can be obtained as:

$$170 \quad \tilde{W}(\gamma) = \frac{F_0}{2\pi D} \frac{1}{(\gamma^2 - k_{c1}^2)(\gamma^2 + k_{c2}^2)} \quad (19)$$

171 Therefore, $W(x, y)$ can be obtained by using the inversion formula

$$172 \quad W(x, y) = \frac{F_0}{2\pi D} \int_0^{+\infty} \frac{f_0(\gamma r) \gamma d\gamma}{(\gamma^2 - k_{c1}^2)(\gamma^2 + k_{c2}^2)} \quad (20)$$

173 where $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ is the position vector.

174 Then, we can state Eq.(20) as

$$175 \quad W(x, y) = \frac{F_0}{4\pi D} \int_{-\infty}^{+\infty} \frac{H_0^{(1)}(\gamma r)}{(\gamma^2 - k_{c1}^2)(\gamma^2 + k_{c2}^2)} \gamma d\gamma = \frac{F_0}{4\pi D} \int_{-\infty}^{+\infty} f(\gamma) d\gamma \quad (21)$$

176 where $H_0^{(1)}$ is the zero-order Hankel function of the second kind [34].

177 The simplest way to solve Eq.(21) is to integral the equation in the complex plane. The integrand has
178 poles at $\gamma = \pm k_{c1}, \pm jk_{c2}$ and then $W(x, y)$ can be evaluated using the residue theorem [34]. The residue of
179 $f(\gamma)$ at $\gamma = \pm k_{c1}, \pm jk_{c2}$ can be respectively expressed as

$$180 \quad \text{Res}_{\gamma=k_{c1}} f(\gamma) = \frac{H_0^{(1)}(k_{c1}r)k_{c1}}{2k_{c1}(k_{c1}^2 + k_{c2}^2)} = \frac{H_0^{(1)}(k_{c1}r)}{2(k_{c1}^2 + k_{c2}^2)} \quad (22)$$

$$181 \quad \text{Res}_{\gamma=-k_{c1}} f(\gamma) = \frac{H_0^{(1)}(-k_{c1}r)(-k_{c1})}{-2k_{c1}(k_{c1}^2 + k_{c2}^2)} = \frac{H_0^{(1)}(-k_{c1}r)}{2(k_{c1}^2 + k_{c2}^2)} \quad (23)$$

$$182 \quad \text{Res}_{\gamma=jk_{c2}} f(\gamma) = \frac{H_0^{(1)}(jk_{c2}r)(jk_{c2})}{2jk_{c2}(-k_{c1}^2 - k_{c2}^2)} = -\frac{H_0^{(1)}(jk_{c2}r)}{2(k_{c1}^2 + k_{c2}^2)} \quad (24)$$

$$183 \quad \text{Res}_{\gamma=-jk_{c2}} f(\gamma) = \frac{H_0^{(1)}(-jk_{c2}r)(-jk_{c2})}{-2jk_{c2}(-k_{c1}^2 - k_{c2}^2)} = -\frac{H_0^{(1)}(-jk_{c2}r)}{2(k_{c1}^2 + k_{c2}^2)} \quad (25)$$

184 According to the residue theorem, the expression of particular solution $W_F(x, y)$ is given by

$$185 \quad W_F(x, y) = -\frac{jF_0}{4D} \frac{1}{(k_{sc1}^2 + k_{sc2}^2)} (H_0^{(2)}(k_{c1}r_F) - H_0^{(2)}(-jk_{c2}r_F)) \quad (26)$$

186 where $r_F = \sqrt{(x - x_F)^2 + (y - y_F)^2}$

187 2.2. Boundary conditions

188 The boundary of a convex plate domain Ω_s can be divided into three non-overlapping domains

189 $\Gamma_{W\theta} \cup \Gamma_{mQ} \cup \Gamma_{Wm}$, as shown in Fig. 1. The expressions in Tab. 1 [28] represent the most common
 190 boundary conditions for the plate.

Fig. 1 Boundaries at a convex plate domain

Tab. 1 Boundary conditions

Boundary Condition	Equations
clamped edge	$W(r) = \bar{W}(r), L_{\theta_n}[W(r)] = \bar{\theta}_n(r), r \in \Gamma_{W\theta}$
free or loaded edge	$L_{m_n}[W(r)] = \bar{m}_n(r), L_{Q_n}[W(r)] = \bar{Q}_n(r), r \in \Gamma_{mQ}$
simply supported edge	$W(r) = \bar{W}(r), L_{m_n}[W(r)] = \bar{m}_n(r), r \in \Gamma_{Wm}$

194 The $\bar{W}$, $\bar{\theta}_n$, $\bar{m}_n$ and $\bar{Q}_n$ are the prescribed displacement, the prescribed rotational displacement, the
 195 prescribed bending moment and the prescribed generalized shear force, respectively. The differential
 196 operators used in Tab. 1 are defined as:

$$197 \quad L_{\theta_n} = -\frac{\partial}{\partial \gamma_n}, L_{m_n} = -D \left(\frac{\partial^2}{\partial \gamma_n^2} + \mu \frac{\partial^2}{\partial \gamma_s^2} \right), L_{Q_n} = -D \frac{\partial}{\partial \gamma_n} \left(\frac{\partial^2}{\partial \gamma_n^2} + (2 - \mu) \frac{\partial^2}{\partial \gamma_s^2} \right) \quad (27)$$

198 where γ_n and γ_s are the in-plane normal and in-plane tangential directions of the boundary, respectively.

199 3. The wave-based method

200 In this section, the analysis procedure of WBM for the bending vibration analysis of a plate under
 201 thermal load is presented. First, the field variable is approximated by the expansion of wave functions
 202 and a particular solution function that exactly satisfy the governing equation. Secondly, the new sets of
 203 wave functions are selected to establish the wave-based (WB) model of the plate bending vibration
 204 problem under thermal load. Thirdly, by using the weighted residual formulation in boundary
 205 conditions, the participation factors of wave functions are obtained; then, the bending vibration
 206 response can be determined by WBM.

207 The convergence of WBM is theoretically guaranteed in case when the structure domain is convex.
 208 However, engineering structures are commonly non-convex domain. Therefore, for a non-convex
 209 structural domain, the structure should be divided into several convex subdomains to ensure
 210 convergence. The WB model can be constructed by using the weighted residual formulation at the
 211 boundary and interface conditions. In addition, it should be noticed that WBM is not applicable for
 212 structures under a non-uniform thermal load or a partially applied thermal load, since the wave
 213 functions and particular solution function used in WBM are all field functions that should exactly
 214 satisfy the governing equation.

215 3.1. Field variable expansions

216 The transverse displacement $W(x, y)$ is approximated by $\hat{W}(x, y)$ [27]:

$$217 \quad W(x, y) \approx \hat{W}(x, y) = \sum_{s=1}^{n_s} W_s \phi_s(x, y) + \hat{W}_F(x, y) \quad (28)$$

218 where $\phi_s(x, y)$ is the wave function, W_s is the participation factor of $\phi_s(x, y)$, n_s is the number of wave
 219 functions, $\hat{W}_F(x, y) = W_F(x, y)$ is the particular solution function introduced in Section 2.1.2.

220 3.2. Wave function

221 The wave function ϕ_s should satisfy the homogeneous part of Eq.(18). Therefore, the expression of
 222 wave functions is given as

$$223 \quad \phi_s(x, y) = e^{-j(k_x x + k_y y)} \quad \text{with} \quad k_x^2 + k_y^2 = k_{c1}^2, -k_{c2}^2 \quad (29)$$

224 As shown in Tab. 2, the selected wave functions are divided into two sets, in which the wave
 225 functions are respectively associated with cosine functions and sine functions. The first wavenumber of
 226 wave functions is determined based on the lengths L_x and L_y [27], which represent the dimensions of a
 227 smallest rectangular box circumscribing the considered plate domain Ω_s . The second wavenumber of
 228 wave functions is calculated by Eq.(29). It has been proven by Desmet [27] that the convergence of this
 229 kind of wave functions can be theoretically guaranteed for the convex domain.

230 According to Tab. 2, the total number of wave functions n_s is given as

$$231 \quad n_s = 4(n_{s1} + 1) + 4(n_{s2} + 1) + 4n_{s3} + 4n_{s4} \quad (30)$$

232 where

$$233 \quad \frac{n_{s1}}{L_x} \approx \frac{n_{s2}}{L_y} \approx \frac{n_{s3}}{L_x} \approx \frac{n_{s4}}{L_y} \geq T_1 \frac{\max(k_{c1}, k_{c2})}{\pi} \quad (31)$$

234 with n_{s1} , n_{s2} , n_{s3} , and n_{s4} are integer truncation values and T_1 is a user-defined truncation parameter to

235 ensure the convergence.

236 Tab. 2 Sets of wave functions

Set	Wave functions	Wavenumbers
1	$\phi_{s1}(x, y) = \cos(k_{x1}x) \exp(-jk_{y1}y)$	$k_{x1} = \frac{s_1\pi}{L_x}, k_{y1} = \begin{cases} \pm\sqrt{k_{c1}^2 - k_{x1}^2} \\ \pm j\sqrt{k_{c2}^2 + k_{x1}^2} \end{cases}, s_1 = 0, 1, \dots, n_{s1}$
	$\phi_{s2}(x, y) = \exp(-jk_{x2}x) \cos(k_{y2}y)$	$k_{y2} = \frac{s_2\pi}{L_y}, k_{x2} = \begin{cases} \pm\sqrt{k_{c1}^2 - k_{y2}^2} \\ \pm j\sqrt{k_{c2}^2 + k_{y2}^2} \end{cases}, s_2 = 0, 1, \dots, n_{s2}$
2	$\phi_{s3}(x, y) = \sin(k_{x3}x) \exp(-jk_{y3}y)$	$k_{x3} = \frac{s_3\pi}{L_x}, k_{y3} = \begin{cases} \pm\sqrt{k_{c1}^2 - k_{x3}^2} \\ \pm j\sqrt{k_{c2}^2 + k_{x3}^2} \end{cases}, s_3 = 1, 2, \dots, n_{s3}$
	$\phi_{s4}(x, y) = \exp(-jk_{x4}x) \sin(k_{y4}y)$	$k_{y4} = \frac{s_4\pi}{L_y}, k_{x4} = \begin{cases} \pm\sqrt{k_{c1}^2 - k_{y4}^2} \\ \pm j\sqrt{k_{c2}^2 + k_{y4}^2} \end{cases}, s_4 = 1, 2, \dots, n_{s4}$

237 3.3. Weighted residual formulation

238 As the wave functions and the particular solution function in Eq.(28) exactly satisfy the governing
 239 equation Eq.(18), approximate errors of Eq.(28) only occur on the boundary conditions. By minimizing
 240 these approximate errors on the boundary through the weighted residual formulation, the unknown
 241 participation factor W_s in Eq.(28) can be determined.

242 In the weighted residual formulation of structural boundary conditions, the residual error functions
 243 are given as:

$$244 R_W(r) = W(r) - \bar{W}(r) = 0, R_{\theta_n}(r) = L_{\theta_n}[W(r)] - \bar{\theta}_n(r) = 0, r \in \Gamma_{W\theta} \quad (32)$$

$$245 R_{m_n}(r) = L_{m_n}[W(r)] - \bar{m}_n(r) = 0, R_{\theta_n}(r) = L_{\theta_n}[W(r)] - \bar{Q}_n(r) = 0, r \in \Gamma_{mQ} \quad (33)$$

$$246 R_W(r) = W(r) - \bar{W}(r) = 0, R_{m_n}(r) = L_{m_n}[W(r)] - \bar{m}_n(r) = 0, r \in \Gamma_{Wm} \quad (34)$$

247 The boundary residuals can be set orthogonal with respect to the weighting functions $\tilde{W}$ as

$$248 \int_{\Gamma_{W\theta} \cup \Gamma_{Wm}} L_{Q_n}(\tilde{W}) R_W d\Gamma + \int_{\Gamma_{W\theta}} L_{m_n}(\tilde{W}) R_{\theta_n} d\Gamma - \int_{\Gamma_{Wm} \cup \Gamma_{mQ}} L_{\theta_n}(\tilde{W}) R_{m_n} d\Gamma - \int_{\Gamma_{mQ}} \tilde{W} R_{Q_n} d\Gamma = 0 \quad (35)$$

249 The Eq.(35) leads to a set of linear equations which can be used to calculate the unknown
 250 contribution factors W_s . Then, the participation factor vectors W_s will be substituted back into the field
 251 variable expansions in Eq.(28) to obtain the transverse displacement of structures under thermal load.

4. Numerical simulations

4.1. Numerical modeling

In order to investigate the applicability and feasibility of WBM for solving bending vibration problems of thin-walled structures under thermal load, numerical simulations on a rectangular plate demonstrated in Fig. 2 are conducted. The dimension of the plate is $l_1 \times l_2 \times h = 0.4m \times 0.3m \times 0.003m$.

Fig. 2 A plate exposed to the thermal load

The plate is made of titanium alloy with density $\rho=4420\text{kg/m}^3$ and Poisson's ratio $\nu=0.33$. As shown in Fig. 3, the elastic modulus E and the thermal expansion coefficient α are taken as temperature-dependent. A unit transverse harmonic force is applied at the center of the plate. The plate is exposed to a uniform thermal load T and the reference temperature T_0 is set to 20°C .

Fig. 3 Temperature-dependent material properties

For constrained structures under high temperature environment, the thermal buckling may occur and lead to the nonlinear dynamic behaviors of structures, which could not be accurately predicted by WBM. Therefore, to ensure the following linear analysis is promising, the critical buckling temperature for the plate T_{cr} is first calculated by FEM and the thermal vibrational analysis is conducted below this temperature.

4.2. Validation case

In this section, the advantage of WBM is demonstrated on computational efficiency over FEM. Simulations of a clamped plate and a simply supported plate are conducted in Section 4.2.1 and Section

273 4.2.2, respectively.

274 The FEM predictions are obtained by MSC. NASTRAN 2016. The finite element (FE) model is built
275 with the four-node quadrilateral shell element. The element type is set to CQUAD4. WBM is
276 implemented in MATLAB 2016a. All simulations are performed on a laptop with Intel Core i5 7200U
277 processor and 8GB RAM, running on Windows 10 64bit.

278 In order to guarantee the validity of the FE model and check the accuracy of the WB model, the
279 prediction error $\langle Er \rangle$, defined as the averaged error over $n_p = 4$ response points p_1 (0.1 m, 0.1 m), p_2
280 (0.2 m, 0.2 m), p_3 (0.2 m, 0.1 m), and p_F (0.2 m, 0.15 m), is proposed

$$281 \quad \langle Er \rangle = \frac{1}{n_p} \sum_{j=1}^{n_p} Er_j, Er_j = \left| \frac{W(x_j, y_j) - W^{ref}(x_j, y_j)}{W^{ref}(x_j, y_j)} \right| \quad (36)$$

282 where superscript 'ref' denotes the result obtained from the reference model.

283 4.2.1. Clamped plate

284 The critical buckling temperature for the clamped plate $T_{cr}=51.3^\circ\text{C}$ is first calculated by FEM.
285 Therefore, the valid temperature range for the clamped plate is set to $[20^\circ\text{C}, 51.2^\circ\text{C}]$. The clamped
286 plate is exposed to a uniform thermal load with $T=40^\circ\text{C}$.

287 Normally, at least 12 elements should be included in per wavelength to ensure accuracy of the FE
288 results. In this Section, FE model with 20 elements in per wavelength is used as references. The
289 corresponding element number at 2000Hz is 353600. In order to guarantee the validity of the FE model
290 and check the accuracy of the WB model, the convergences of FE models with different element
291 numbers in per wavelength and WB models with different truncation parameters are firstly
292 investigated.

293 The considered analysis frequency is 2000Hz. For FE models, the consider element numbers in per
294 wavelength are 4, 6, 8, 10, and 12. The corresponding element numbers are 14144, 31824, 56576,
295 88400, and 127296, respectively. For WB models, the consider truncation parameters are 1, 2, 3, 4, and
296 5. The corresponding wave function numbers are 52, 100, 148, 192 and 240, respectively. The averaged
297 relative prediction error $\langle Er \rangle$ on displacement as a function of the computation time is shown in Fig.
298 4. As shown in Fig. 4, the convergence rate of the WB model is substantially higher than the FE model.
299 The computing time of the algorithm can be remarkably reduced by using WBM.

Fig. 4 Convergence curves

Then, the displacement responses of point p_1 (0.1m, 0.1m) and point p_F (0.2m, 0.15m) are investigated. The comparison of the displacement responses calculated by the WB model under $T_1=5$ and the FE model with 20 elements in per wavelength is shown in Fig. 5. As shown in Fig. 5, the predicted displacement responses in frequency domain agree well with the FEM results.

Fig. 5 Comparison of displacement frequency responses

Tab. 3 Comparison of displacement response distributions

Frequency	WBM	FEM	Difference
500Hz

310 Furtherly, the distribution of displacement response on the clamped plate and the difference between
 311 WBM results and FEM predictions are shown in Tab. 3. The considered analysis frequencies are
 312 $f=500\text{Hz}$, 1000Hz , and 2000Hz . As indicated in Tab. 3, the displacement response distributions
 313 obtained by WBM match well with the corresponding FEM predictions.

314 4.2.2. Simply supported plate

315 The critical buckling temperature for the simply supported plate $T_{cr}=31.6^\circ\text{C}$ is first calculated by
 316 FEM. The simply supported plate is exposed to a uniform thermal load with $T=30^\circ\text{C}$. FE model with
 317 20 elements in per wavelength is also used as references. The convergences of FE models and WB
 318 models are firstly investigated. The considered analysis frequency is 2000Hz . The consider element
 319 numbers in per wavelength for FE models and the consider truncation parameters for WB models are
 320 the same as those introduced in Section 4.2.1. The averaged relative prediction error $\langle Er \rangle$ on
 321 displacement as a function of the computation time is shown in Fig. 6. As shown in Fig. 6, the
 322 convergence rate of the WB model is substantially higher than the FE model.

Fig. 6 Convergence curves

325 Then, the displacement responses of point p_1 (0.1m, 0.1m) and point p_F (0.2m, 0.15m) are
 326 investigated. The comparison of the displacement responses calculated by the WB model under $T_1=5$
 327 and the FE model with 20 elements in per wavelength is shown in Fig. 7. As shown in Fig. 7, the WBM
 328 predictions agree well with the FEM results.

Fig. 7 Comparison of displacement frequency responses

Furtherly, the distribution of displacement response on the clamped plate and the difference between WBM results and FEM predictions are shown in Tab. 4. The considered analysis frequencies are $f=500\text{Hz}$, 1000Hz , and 2000Hz . As indicated in Tab. 4, the displacement response distribution obtained by WBM matches well with the corresponding FEM predictions.

Tab. 4 Comparison of displacement response distributions

Frequency	WBM	FEM	Difference
500Hz	Displacement Response (m) 0 1.3E-07 2.6E-07 3.9E-07 5.2E-07 6.5E-07	Displacement Response (m) 0 1.3E-07 2.6E-07 3.9E-07 5.2E-07 6.5E-07	Difference (m) 0 5E-09 1E-08 1.5E-08 2E-08 2.5E-08
1000Hz	Displacement Response (m) 0 1.7E-07 3.4E-07 5.1E-07 6.8E-07 8.5E-07	Displacement Response (m) 0 1.7E-07 3.4E-07 5.1E-07 6.8E-07 8.5E-07	Difference (m) 0 6E-09 1.2E-08 1.8E-08 2.4E-08 3E-08
2000Hz	Displacement Response (m) 0 6E-08 1.2E-07 1.8E-07 2.4E-07 3E-07	Displacement Response (m) 0 7.5E-08 1.5E-07 2.25E-07 3E-07	Difference (m) 0 4E-10 8E-10 1.2E-09 1.6E-09 2E-09

4.3. The influence of thermal load on bending vibration response

In this section, in order to investigate the influence of thermal effects on bending vibration response, simulation works of a clamped plate are conducted under three cases: *Case a*: only the effect of

340 temperature-dependent material properties is considered; *Case b*: only the effect of thermal stresses is
 341 involved; *Case c*: both the two effects are considered. The considered environment temperatures are
 342 $T=20^{\circ}\text{C}$, 30°C , 40°C , and 51.2°C . The frequency response of point p_1 (0.1m, 0.1m) is investigated. The
 343 displacement response of p_1 under different thermal effect and thermal load is shown in Fig. 8.

(a) *Case a*

(b) *Case b*

(c) *Case c*

Fig. 8 Displacement response of p_1 under different thermal effect and environment temperature

351 As shown in Fig. 8(a), when only the thermal effect of temperature-dependent material properties is
 352 involved, the dynamic response of the point p_1 slightly moves toward lower frequency as the
 353 environment temperature increases. This is because the reduction of elastic modulus and the bending
 354 stiffness induces softening effect on the plate. However, as the limited change of elastic modulus, the
 355 effect of environment temperature on the vibration behavior is not obvious.

When only the thermal effect of thermal stresses is taken into consideration, as indicated in Fig. 8(b), the resonant peak shifts to lower frequency with the increasing environment temperature. This is because the thermal expansion of the constrained plate leads to compressive forces and results in the softening effect on the structure. In addition, the resonant amplitude of displacement oscillates with the increase of the environment temperature.

When both the thermal effects of temperature-dependent material properties and thermal stress are considered, as indicated in Fig. 8(c), the resonant peak shifts toward lower frequency with the increasing environment temperature. This phenomenon can be explained by two reasons: the reduction of elastic modulus and the softening effect from the compressive thermal stresses. Furthermore, the resonant amplitude of displacement oscillates as the environment temperature increases.

The displacement responses shown in Fig. 8(c) are closely agreed with the results in Fig. 8(b), which means that the effect of thermal stresses plays a leading role in structural vibration behavior than the effect of temperature-dependent material properties. Due to the clamped boundary condition and the material properties, the influence of thermally induced force on the wave number is dominant. Accordingly, the displacement response is mainly affected by the thermal effect of thermal stresses.

The displacement response distributions of the clamped plate under normal temperature environment and thermal load are shown in Tab. 5. Both two thermal effects are involved in the bending vibration analysis. The considered environment temperatures are $T=20^{\circ}\text{C}$, 30°C , 40°C , and 51.2°C . The analysis frequencies are set to $f=500\text{Hz}$, 1000Hz , and 2000Hz .

Tab. 5 Comparison of displacement response distributions

As indicated in Tab. 5, the temperature of thermal load can significantly alter the level of displacement response as well as the distribution. This is because the displacement response shifts toward lower frequency as the increasing environment temperature. In addition, the maximum displacement responses of the plate at 500Hz and 2000Hz increase with the growing temperature, while an opposite variation trend can be observed at 1000Hz.

5. Conclusions

WBM is an efficient technique for predicting the mid-frequency vibration response of thin-walled structures. The main contribution of this work is to extend the efficient WBM to solve the dynamic bending problem of thin-walled rectangular plates under thermal load. Governing equations of dynamic plate bending problems considering thermal effect are first introduced. The general solution and the particular solution of the governing equation are derived. Then novel wave functions and particular solution function are selected to establish the wave-based model for the bending vibration analysis of thin-walled plates under thermal load. Numerical simulations of a clamped plate and a simply supported plate exposed to the uniform thermal load are presented. The influence of thermal load on the dynamic bending responses of a clamped plate is investigated in detail.

Simulation results indicate that WBM achieves higher convergence rate than FEM on predicting the thermal vibration response of thin-walled rectangular plates. The thermal load significantly changes the structural dynamic behaviors. When both the thermal effect of temperature-dependent material properties and thermal stresses are taken into consideration, the influence of thermal stresses on the displacement response is dominant. Furthermore, the resonant peak of displacement response moves

396 toward lower frequency when the environment temperature increases in the valid temperature range.
 397 The resonant amplitude of displacement oscillates as the load temperature increases.
 398 This research enriches the application of WBM towards the dynamic plate bending analysis. WBM
 399 can be used to solve the vibration analysis of more complex thin-walled structures under thermal load.
 400 However, since the developed WBM is based on the principle of linear superposition, the present
 401 approach is only applicable to the thermal vibrational analysis of structures under thermal buckling
 402 temperature.

403 Acknowledgments

404 This paper is supported by the National Natural Science Foundation of China (11572086, 11802059),
 405 the Jiangsu Natural Science Foundation (BK20170022, BK20170656, and BK20180062), the China
 406 Scholarship Council (201806090082), and the Scientific Research Foundation of Graduate School of
 407 Southeast University. The Research Fund KU Leuven is gratefully acknowledged for its support.

408 References

- 409 [1] Blevins, R., Bofilios, D., Holehouse, I., Hwa, V., Tratt, M., Laganelli, A., Pozefsky, P., and Pierucci, M., 2009,
 410 "Thermo-vibro-acoustic loads and fatigue of hypersonic flight vehicle structure", U.S. Air Force Research
 411 Lab. TR AFRL-RB-WP-TR-2009-3139.
- 412 [2] Garg, A., and Chalak, H., 2019, "A review on analysis of laminated composite and sandwich structures under
 413 hygrothermal conditions", *Thin-Walled Structures*, 142, pp. 205-226.
- 414 [3] Tzong, G., and Liguore, L. S., 2013, "Verification studies on hypersonic structure thermal acoustic response
 415 and life prediction methods", *Proceedings of the 54th AIAA/ASME/ASCE/AHS/ASC structures, structural
 416 dynamics and materials conference*, Boston, Massachusetts, pp. 1-14.
- 417 [4] Behnke, M. N., Sharma, A., Przekop, A., 2010, "Thermal-acoustic analysis of a metallic integrated thermal
 418 protection system structure", *Proceedings of the 51st AIAA/ASME/ASCE/AHS/ASC structures, structural
 419 dynamics and materials conference*, Orlando, Florida, pp. 12-15.
- 420 [5] Jooybar, N., Malekzadeh, P., Fiouzi, A., and Vaghefi, M., 2016, "Thermal effect on free vibration of
 421 functionally graded truncated conical shell panels", *Thin-Walled Structures*, 103, pp. 45-61.
- 422 [6] Moita, J., Araújo, A., Correia, V., Mota Soares, C., and Herskovits, J., 2018, "Buckling and nonlinear
 423 response of functionally graded plates under thermo-mechanical loading", *Composite Structures*, 202, pp.
 424 719-730.
- 425 [7] Li, Y., Mulani, S., Fei, Q., Wu, S. and Zhang, P., 2017, "Vibro-acoustic analysis under stationary and
 426 non-stationary random excitations with KLE/FEM/BEM", *Aerospace Science and Technology*, 66,
 427 pp.203-215.
- 428 [8] Kandasamy, R., Dimitri, R., and Tornabene, F., 2016, "Numerical study on the free vibration and thermal
 429 buckling behavior of moderately thick functionally graded structures in thermal environments", *Composite
 430 Structures*, 157, pp. 207-221.
- 431 [9] Zheng, S., Chen, D., and Wang, H., 2019, "Size dependent nonlinear free vibration of axially functionally

- graded tapered microbeams using finite element method", *Thin-Walled Structures*, 139, pp. 46-52.
- [10] Jeyaraj, P., Ganesan, N., Padmanabhan, C., 2008, "Vibration and acoustic response of an isotropic plate in a thermal environment", *Journal of Vibration and Acoustics*, 130 (5), pp. 301-306.
- [11] Geng, Q., and Li, Y., 2014, "Solutions of dynamic and acoustic responses of a clamped rectangular plate in thermal environments", *Journal of Vibration and Control*, 22(6), pp. 1593-1603.
- [12] Geng, Q., Li, H., and Li, Y., 2014, "Dynamic and acoustic response of a clamped rectangular plate in thermal environments: Experiment and numerical simulation", *The Journal of the Acoustical Society of America*, 135(5), pp. 2674-2682.
- [13] Sharma, N., Mahapatra, T., and Panda, S., 2018, "Thermoacoustic behavior of laminated composite curved panels using higher-order finite-boundary element model", *International Journal of Applied Mechanics*, 10(02), p. 1850017.
- [14] Wang, L., and Chen, T., 2019, "Structural intensity analysis of the cantilevered plate under thermal load", *Thin-Walled Structures*, 139, pp. 209-218.
- [15] Bui, T., Vo, D., Zhang, C., and Nguyen, D., 2014, "A consecutive-interpolation quadrilateral element (CQ4): Formulation and applications", *Finite Elements in Analysis and Design*, 84, pp. 14-31.
- [16] Bui, T., Nguyen, D., Zhang, X., Hirose, S., and Batra, R., 2016, "Analysis of 2-dimensional transient problems for linear elastic and piezoelectric structures using the consecutive-interpolation quadrilateral element (CQ4)", *European Journal of Mechanics - A/Solids*, 58, pp. 112-130.
- [17] Xu, X., Lee, H., Wang, Y., and Lu, C., 2004, "The energy flow analysis in stiffened plates of marine structures", *Thin-Walled Structures*, 42(7), pp. 979-994.
- [18] Zhang, P., Fei, Q., Li, Y., Wu, S., and Chen, Q., 2018, "Modal energy analysis for mechanical systems excited by spatially correlated loads", *Mechanical Systems and Signal Processing*, 111, pp. 362-375.
- [19] Chen, Q., Fei, Q., Wu, S., Li, Y. and Yang, X. 2018. An efficient transient analysis method for time-varying structures based on statistical energy analysis. *Mechanics Research Communications*, 91, pp.93-99.
- [20] Chen, Q., Fei, Q., Li, Y., Wu, S. and Yang, X. 2018. Prediction of the transient energy response for complex vibro-acoustic systems. *Journal of Mechanical Science and Technology*, 33(2), pp. 495-504.
- [21] Lyon, R. H., Dejong, R. G., 1995, *Theory and Application of Statistical Energy Analysis*, Butterworth-Heinemann, London.
- [22] Chen, Q., Fei, Q., Li, Y., Wu, S. and Zhang, P. 2019. Prediction of statistical energy analysis parameters in thermal environment. *Journal of spacecraft and rockets*, 56(3), pp.687-694.
- [23] Deckers, E., Atak, O., Coox, L., D'Amico, R., Devriendt, H., Jonckheere, S., Koo, K., Pluymers, B., Vandepitte, D., and Desmet, W., 2014, "The wave based method: An overview of 15 years of research", *Wave Motion*, 51(4), pp. 550-565.
- [24] Rouch, P., and Ladevèze, P., 2003, "The variational theory of complex rays: a predictive tool for medium-frequency vibrations", *Computer Methods in Applied Mechanics and Engineering*, 192(28-30), pp. 3301-3315.
- [25] Zhang, P., Fei, Q., Wu, S., Li, Y., 2016, "A dimensionless quotient for determining coupling strength in modal energy analysis," *Journal of Vibration and Acoustics*, 138, p. 061014.
- [26] Chappell, D., Löchel, D., Søndergaard, N., and Tanner, G., 2014, "Dynamical energy analysis on mesh grids: A new tool for describing the vibro-acoustic response of complex mechanical structures", *Wave Motion*, 51(4), pp. 589-597.
- [27] Desmet, W., 1998, "A wave based prediction technique for coupled vibro-acoustic analysis," Ph.D. thesis, K.U., Leuven, Belgium
- [28] Vanmaele, C., Vandepitte, D., and Desmet, W., 2007, "An efficient wave based prediction technique for plate

- bending vibrations", *Computer Methods in Applied Mechanics and Engineering*, 196(33-34), pp. 3178-3189.
- [29] Vanmaele, C., Vandepitte, D., and Desmet, W., 2009, "An efficient wave based prediction technique for dynamic plate bending problems with corner stress singularities", *Computer Methods in Applied Mechanics and Engineering*, 198(30-32), pp. 2227-2245.
- [30] Jonckheere, S., Vandepitte, D., and Desmet, W., 2015, "A Wave Based approach for the dynamic bending analysis of Kirchhoff plates under distributed deterministic and random excitation", *Computers & Structures*, 156, pp. 42-57.
- [31] Klanner, M., and Ellermann, K., 2015, "Wave based method for the steady-state vibrations of thick plates", *Journal of Sound and Vibration*, 345, pp. 146-161.
- [32] Chen, M., Xie, K., Xu, K., and Yu, P., 2015, "Wave based method for free and forced vibration analysis of cylindrical shells with discontinuity in thickness", *Journal of Vibration and Acoustics*, 137(5), p. 051004.
- [33] Wei, J., Chen, M., Hou, G., Xie, K., and Deng, N., 2013, "Wave based method for free vibration analysis of cylindrical shells with nonuniform stiffener distribution", *Journal of Vibration and Acoustics*, 135(6), p. 061011.
- [34] Boley, B.A. and Weiner, J.H., 1960, *Theory of Thermal Stresses*, John Wiley & Sons, Inc
- [35] Richard, B.H., and Eslami, M.R., 2008, *Thermal Stresses Advanced Theory and Applications*. Springer, Netherlands.
- [36] Junger, M. C., and Feit, D., 1986, *Sound, structures, and their interaction*, MIT Press, Cambridge.