

## ON THE APPLICABILITY OF $\kappa$ -DISTRIBUTIONS

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### ABSTRACT

The standard (non-relativistic)  $\kappa$ -distribution is widely used to fit data and to describe macroscopic thermodynamical behavior, e.g. the pressure (temperature) as the second moment of the distribution function. By contrast to a Maxwellian distribution, for small relevant values  $\kappa < 2$  there exists a significant, but unphysical contribution to the pressure from unrealistic, superluminal particles with speeds exceeding the speed of light. Similar concerns exist for the entropy. We demonstrate here that by using the recently introduced regularized  $\kappa$ -distribution one can avoid such unphysical behaviour.

### 1. INTRODUCTION

There are numerous data sets of particle velocity distributions exhibiting power law tails that can be fitted well with standard  $\kappa$ -distributions (SKDs, see below), for reviews see, e.g., Pierrard & Lazar (2010), Lazar et al. (2012) or Livadiotis & McComas (2013). Limitations of the use of SKDs have been discussed recently by Lazar et al. (2016), Scherer et al. (2017), and Fichtner et al. (2018). Here we discuss a further restriction to be observed when using SKDs.

For various applications employing SKDs the associated temperature and pressure are needed (e.g., Heerikhuisen et al. 2008; Fahr et al. 2014, 2016; Kim et al. 2018), and, thus, the second-order moment of the velocity distribution. Since all moments are calculated over the entire velocity space, which in a non-relativistic treatment extends to infinity, this can lead to a non-negligible but unrealistic contribution from particles with superluminal speeds (with  $v > c$ , where  $c = 3 \times 10^5$  km/s, is the speed of light in vacuum). For instance, recently Kim et al. (2018) have outlined such an unphysical contribution of superluminal electrons in theoretical estimations of spontaneous quasi-thermal emissions. Here we show that this happens in general for the pressure of an SKD. We further demonstrate that the regularized  $\kappa$ -distribution (RKD) introduced recently by Scherer et al. (2017) allows one to avoid this problem: A proper choice of the cut-off parameter in the RKD effectively truncates the superluminal contribution, as in the case of a Maxwellian distribution. The RKD has the further advantage to be consistent with an extensive entropy (Fichtner et al. 2018), which appears not to be the case for the SKD (Silva et al. 1998).

In order to show the significance or insignificance of the contribution of superluminal particles to pressure and entropy we calculate ‘partial pressures’ and ‘partial entropies’ for isotropic distribution functions defined in section 2, i.e. we integrate to a finite speed rather than to infinity and compare with the pressure and entropy obtained from an integration over the entire velocity space. This is described in section 3. After a discussion of the results in sections 4 and 5, we discuss relevant physical systems in section 6 before drawing conclusions in section 7.

### 2. DISTRIBUTION FUNCTIONS, PRESSURE, AND ENTROPY

#### 2.1. *Distribution functions*

We discuss the following three velocity distribution functions. First, the Maxwellian

$$f_M = \frac{n_0}{\Theta^3 \sqrt{\pi^3}} \exp\left(-\frac{v^2}{\Theta^2}\right), \quad (1)$$

second, the standard  $\kappa$  distribution (SKD)

$$f_S = \frac{n_0}{\Theta^3 \sqrt{\pi^3 \kappa^3}} \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - \frac{1}{2})} \left(1 + \frac{v^2}{\Theta^2 \kappa}\right)^{-\kappa-1} \quad (2)$$

and, third, the regularized  $\kappa$  distribution (RKD)

$$f_R(\kappa, \alpha) = \frac{n_0}{\Theta^3 \sqrt{\pi^3 \kappa^3}} \mathcal{U}_{[0]} \left( 1 + \frac{v^2}{\Theta^2 \kappa} \right)^{-\kappa-1} \exp \left( -\alpha^2 \frac{v^2}{\Theta^2} \right) \quad (3)$$

$\Theta$  denotes the thermal speed in the Maxwellian, while for the SKD and RKD it is a reference speed characteristic to the Maxwellian core of the distributions (Lazar et al. 2015, 2016). For the definition of  $\mathcal{U}_{[0]}$  see below. The cut-off parameter  $\alpha$  is process-dependent (see Scherer et al. 2017) and has to be chosen accordingly.  $v$  is the particle speed and  $\kappa$  a parameter producing the power-law tail in the distribution function.

## 2.2. Pressures

The isotropic pressures are all calculated via the second-order moment of the corresponding distribution functions (in spherical coordinates)

$$P_i = A \int_0^\infty v^4 f_i dv \quad (4)$$

with the  $A$  containing all constant factors. The temperature is defined via the ideal gas law (see, e.g. Scherer et al. 2017, 2019).

The corresponding pressures  $P_M, P_S, P_R$  of the above distribution functions can be calculated analytically (Scherer et al. 2017, 2019):

$$p_M = \frac{3}{2} \Theta^2 n_0 \quad (5)$$

$$P_S(\kappa) = \frac{3}{2} \frac{\kappa}{2\kappa - \frac{3}{2}} \Theta^2 n_0 \quad (6)$$

$$p_R(\alpha, \kappa) = \frac{3}{2} n_0 \Theta^2 \kappa \mathcal{U}_{[0]} \quad (7)$$

$$(8)$$

where  $\mathcal{U}_{[n]}$  is the ratio of the Kummer- $U$  (or Tricomi) functions  $U(a, b, x)$ :

$$\mathcal{U}_{[n]}(\kappa, \alpha) = \frac{U\left(\frac{3+m}{2}, \frac{3+m}{2} - \kappa, \alpha^2 \kappa\right)}{U\left(\frac{3+n}{2}, \frac{3+n}{2} - \kappa, \alpha^2 \kappa\right)} \quad (9)$$

with the empty bracket notation

$$\mathcal{U}_{[n]}(\kappa, \alpha) = \left[ U\left(\frac{3+n}{2}, \frac{3+n}{2} - \kappa, \alpha^2 \kappa\right) \right]^{-1}$$

$$\mathcal{U}_{[]}(\kappa, \alpha) = U\left(\frac{3+m}{2}, \frac{3+m}{2} - \kappa, \alpha^2 \kappa\right)$$

$$\mathcal{U}_{[]}(\kappa, \alpha) = 1$$

and

$$\mathcal{U}_{[n]}(\kappa, \alpha) = 1$$

From the above equations  $\mathcal{U}_{[0]}$  and  $\mathcal{U}_{[2]}$  can be estimated as functions of  $\alpha$  and  $\kappa$ .

## 2.3. Entropies

A general definition of the entropy  $S$  of a gas was given originally by Boltzmann (1872) and Gibbs (1902) and for a plasma, e.g., by Balescu (1975), Balescu (1988), and Cercignani (1988):

$$S_i = -k_B \iint f_i [\ln(f_i) - 1] d^3r d^3v - k_B N \ln \left( \frac{h^3}{m^3} \right) \quad (10)$$

with the phase space distribution function  $f = f(\vec{r}, \vec{v}, t)$  of  $N$  particles species and  $h$  the Planck constant. This definition of the Gibbs entropy (sometimes called Boltzmann-Gibbs entropy) is valid for both equilibrium and non-equilibrium systems, takes into account the quantum mechanical lower limit of the phase space volume occupied by a single particle, and avoids the Gibbs paradoxon. For an evaluation of this expression for the RKD see Fichtner et al. (2018).

## 3. PARTIAL PRESSURES, ENTROPIES, AND RELATIVE RATIOS

We denote the isotropic ‘partial’ pressures with  $P'_i(w)$

$$P'_i(w) = A \int_0^w v^4 f_i dv \quad (11)$$

where the constant  $A$ , as given in Eq. 4, is not affected and  $w$  is the cut-off speed. We define the relative contribution as the ratio ( $i \in \{M, S, R\}$ )

$$R_i = \frac{P_i - P'_i(w)}{P_i} = 1 - \frac{P'_i(w)}{P_i} \quad (12)$$

that quantifies the relative pressure contribution of particles with speeds greater than  $w$ . For a physically valid model this contribution should be negligible in the limit  $w \rightarrow c$ . In the following, we use the normalized cut-off speed  $w/\Theta$ , where for physical reasons  $\Theta$  is commonly associated to the thermal speed. This immediately leads to the condition  $c/\Theta > 1$  for the Maxwellian and  $c/\Theta > \alpha^{-1}$  for the RKD. For the SKD no cut-off is defined.

In the same manner we define the partial entropies:

$$S'_i(w) = -4\pi k_B \int_0^w \left[ \int f_i [\ln(f_i) - 1] d^3r \right] v^2 dv \quad (13)$$

and the relative entropy contributions

$$r_i = \frac{\hat{S}_i - S'_i(w)}{\hat{S}_i} = 1 - \frac{S'_i(w)}{\hat{S}_i} \quad (14a)$$

$$\hat{S}_i = S_i + k_B N \ln \left( \frac{h^3}{m^3} \right) \quad (14b)$$

Note, first, that only the velocity integration is ‘partial’ and that the integration w.r.t. position still extends over the entire configuration space. Second, we have omitted the constant ‘quantum mechanical’ term involving  $h$  in order to have  $r_i$  as a direct measure of the partial contribution of the distribution function to the entropy.

## 4. THE RESULTS FOR THE PRESSURES

Figs. 1 to 3 show the calculated  $R_i$  as functions of the normalized speed  $w/\Theta$ . First, we discuss the Maxwellian distribution, which is important as it is a limit for both the SKD and the RKD, next the SKD and finally the RKD.

## 4.1. The Maxwellian

The results for a Maxwellian are well known but we discuss them here for a later comparison with those for the SKD and RKD.

The partial pressure for a Maxwellian is:

$$P'_M(\hat{w}) = \frac{3}{2} \Theta^2 n_0 \left( \operatorname{erf}(\hat{w}) - \left[ \frac{2\hat{w}}{\sqrt{\pi}} - \frac{4\hat{w}^3}{3\sqrt{\pi}} \right] e^{-\hat{w}^2} \right) \quad (15)$$

with the above formulas one finds for the relative ratio:

$$R_M = 1 - \operatorname{erf}(\hat{w}) + \frac{1}{3\sqrt{\pi}} \left( [4\hat{w}^3 + 6\hat{w}] e^{-\hat{w}^2} \right) \quad (16)$$

with  $\hat{w} = w/\Theta$ . This ratio is shown in Fig. 1. Because we have plotted  $w/\Theta$ , but not specified  $\Theta$  one can read Fig. 1 (and also Figs. 2 and 3) as follows: Assuming the normalisation speed  $\Theta$  is fixed at some value (say  $\Theta = 30$  km/s for solar wind protons or  $\Theta \approx 1000$  km/s for solar wind electrons at 1 AU), avoiding a significant pressure contribution of particles with superluminal speeds requires the relative ratio to become smaller than  $c/\Theta = 10^4$  for protons and  $c/\Theta = 3 \cdot 10^2$  for electrons.

As is obvious from Fig. 1,  $R_M$  becomes negligibly low already far below these limits. Consequently, the unphysical contribution of superluminal particles is negligible as well. Only if  $c/\Theta$  is close to one, i.e. only if the considered particle population is characterized by relativistic temperatures,  $R_M$  is significantly different from zero and the use of the non-relativistic Maxwellian is prohibited. In that case the relativistic Maxwell-Jüttner distribution must be used (Jüttner 1911; Hakim 2011).

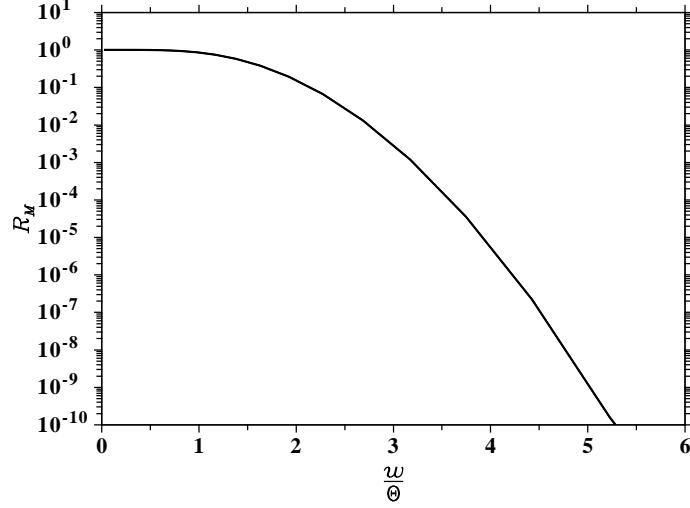


FIG. 1.— Plot of the relative ratio  $R_M$ , i.e. the relative pressure contribution of superluminal particles in case of an isotropic Maxwellian velocity distribution.

#### 4.2. The SKD

The partial pressure for the SKD is

$$P'_S(\kappa, \hat{w}) = \frac{4}{5} n_0 \Theta^2 \hat{w}^5 \frac{\Gamma(\kappa)}{\sqrt{k\pi} \Gamma(1 - \frac{1}{2})} \quad (17)$$

$$[{}_2F_{[1]}] \left( \left[ \frac{5}{2}, \kappa + 1 \right], \left[ \frac{7}{2}, \frac{\hat{w}^2}{\kappa} \right] \right)$$

where  $\Gamma$  is the Gamma function and  ${}_2F_{[1]}([a, b], [c], x)$  a hypergeometric function. Thus, the relative ratio is:

$$R_S(\hat{w}) = 1 - \frac{8}{15} \frac{\Gamma(\kappa - 1)}{\sqrt{\pi\kappa} \Gamma(\kappa - \frac{3}{2})} \quad (18)$$

$$\hat{w}^5 [{}_2F_{[1]}] \left( \left[ \frac{5}{2}, \kappa + 1 \right], \left[ \frac{7}{2}, \frac{\hat{w}^2}{\kappa} \right] \right)$$

In Fig. 2 the relative ratio for the SKD pressure is shown for different values of  $\kappa$ . The figure reveals that for, e.g., reference or thermal speed  $\Theta = 1$  km/s the superluminal particles contribute about 1% of the pressure for  $\kappa \leq 1.7$ . This contribution becomes far more significant for higher  $\Theta$ , which is more realistic for space plasmas. For example, if  $\Theta = 30$  km/s for solar wind protons and  $\Theta \approx 1000$  km/s for electrons. As above, in the latter case the regions of superluminal speeds are then  $w/\Theta > 10^4$  and  $w/\Theta > 3 \cdot 10^2$ , respectively. In these cases, the contributions of superluminal particles to the pressure is about 3.5% resp. 10% for  $\kappa = 1.7$  and even 20% resp. 40% for  $\kappa = 1.6$ . Given that such and even lower  $\kappa$ -values are discussed in the literature, the unphysical contribution of particles with speeds higher than the speed of light is often intolerably high. While one can debate how high such a ‘superluminal’ contribution to the pressure can be to be tolerated, we think that one should only use the SKD for  $\kappa$ -values for which  $R_S \leq 0.01$ . For lower values of  $\kappa$  the relativistic SKD (Xiao 2006) must be used.

#### 4.3. The RKD

Analogously to the previous subsections we can calculate the partial pressure and the relative ratio for the RKD as

$$P'_R = \frac{n_0 \Theta^2}{\sqrt{\pi^3 \kappa^3}} [U_{[0]}] \int_0^{\hat{w}} \left(1 + \frac{x^2}{\kappa}\right)^{-\kappa-1} e^{-\alpha x^2} x^4 dx \quad (19)$$

$$R_R = 1 - \frac{8}{3\sqrt{\pi\kappa^5}} [U_{[2]}] \quad (20)$$

$$\int_0^{\hat{w}} \left(1 + \frac{x^2}{\kappa}\right)^{-\kappa-1} x^4 e^{-\alpha^2 x^2} dx$$

In Fig. 3 the contribution of the relative ratio  $R_R$  is shown: While in the left panel the parameter  $\alpha$  is varied for two given  $\kappa$ -values ( $\kappa = 0.5$  and  $\kappa = 2$ ), in the right panel  $\kappa$  is varied for two given values of the cut-off parameter

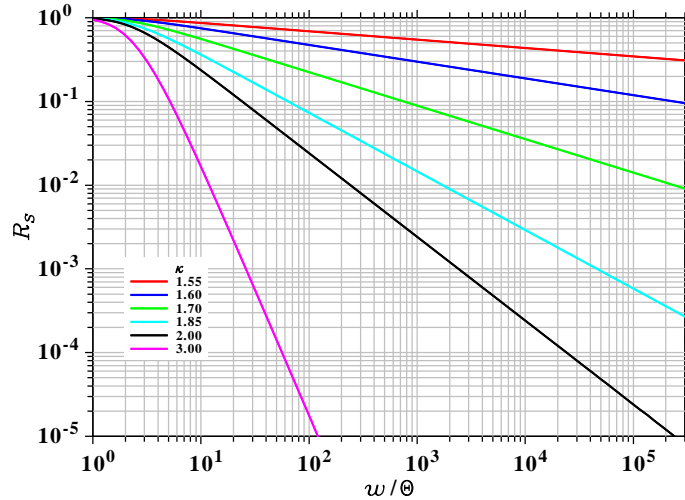


FIG. 2.— Plot of the relative ratio  $R_S$  vs. the normalized speed  $w/\Theta$  for different  $\kappa$  values labelled with different colors. The abscissa ends at  $w/\Theta = 3 \cdot 10^5$ , so that for  $\Theta = 1$  km/s or higher reference speeds one should use the gray vertical lines to determine the region of superluminal particle speeds, e. g. for  $\Theta = 1000$  km/s all values  $w/\Theta > 3 \cdot 10^2$  lie beyond the speed of light.

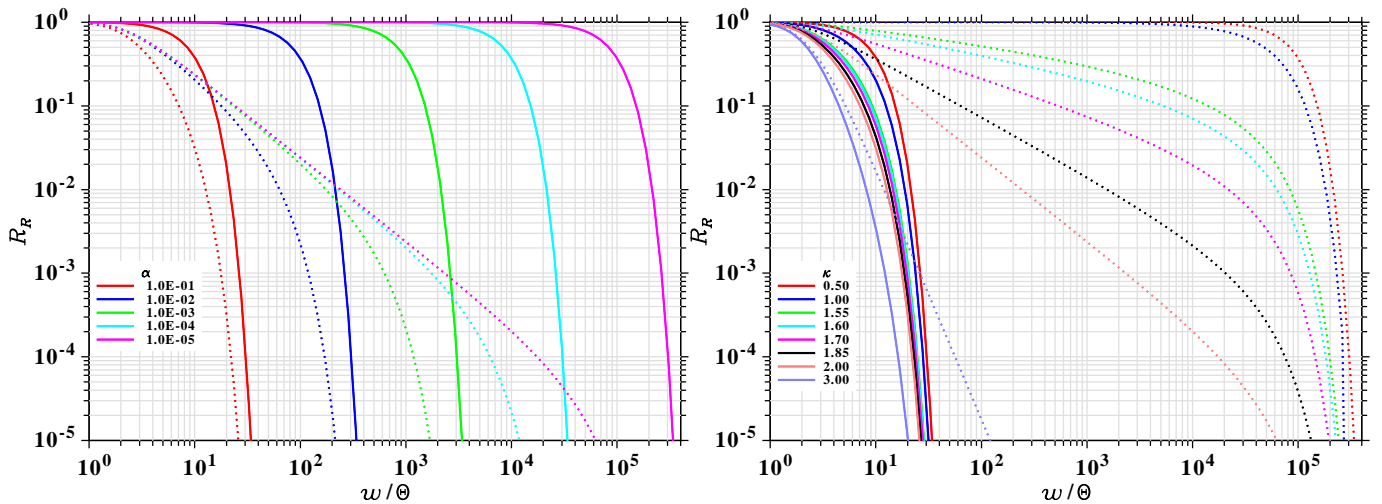


FIG. 3.— Left panel: The relative ratio  $R_R$  vs.  $w/\Theta$  for  $\log(\alpha) = -1, -2, -3, -4, -5$  (red, blue, green, cyan, and magenta, respectively). The solid lines for  $\kappa = 0.5$  and the dotted lines for  $\kappa = 2$  are the envelopes for  $0.5 < \kappa < 2$ . Right panel:  $R_R$  vs.  $w/\Theta$  for  $\kappa = 0.5, 1, 1.55, 1.6, 1.7, 1.85, 2, 3$  (red, blue, green, cyan, magenta, black, light red, and light blue, respectively). The solid lines for  $\alpha = 10^{-1}$  and the dotted lines for  $\alpha = 10^{-5}$  are the envelopes for  $10^{-5} < \alpha < 10^{-1}$ .

( $\alpha = 10^{-5}$  and  $\alpha = 10^{-1}$ ). For the RKD one has the additional requirement

$$\alpha > \frac{\Theta}{c} \quad (21)$$

in order to have an exponential cut-off at speeds lower than the speed of light  $c$  (Scherer et al. 2019). While this is not a strict condition, one should choose  $\alpha$  sufficiently high so that the cut-off occurs not too close to  $c$ .

The left panel of Fig. 3 illustrates this: for  $\alpha = 10^{-5}$  and  $\kappa = 0.5$  (solid magenta line) the cut-off is rather close to  $c$  but the pressure contribution of superluminal particles is smaller than 0.01%. The curve also reveals that high-speed particles dominate the pressure. The higher  $\alpha$  the smaller is this contribution, of course. For the higher value  $\kappa = 2$ , the relative pressure contribution is smaller and, as expected, mainly provided by particles with lower speeds, i.e.  $R_R(w/\Theta)$  has a strongly negative slope. For the above discussed values of  $\Theta = 30$  km/s and  $\Theta = 1000$  km/s appropriate choices for  $\alpha$  (see Eq. 21) are  $\alpha > 10^{-4}$  and  $\alpha > 3 \cdot 10^{-2}$ , respectively. This is illustrated explicitly in the right panel, where for the two fixed values  $\alpha = 10^{-1}$  (solid lines) and  $\alpha = 10^{-5}$  (dotted lines)  $\kappa$  is varied from 0.5 to 3. One can see that for  $\kappa < 1$  the main contribution comes from high-speed particles, while for  $\kappa > 1$  it is provided by those with relatively low speeds. In cases where  $\Theta$  or, correspondingly, the temperature reaches relativistic values one has to use a relativistic version of the RKD. Such version has, to the best of our knowledge, not yet been formulated, in general, but is only available for the ultra-relativistic case (see Treumann & Baumjohann 2018, and references therein).

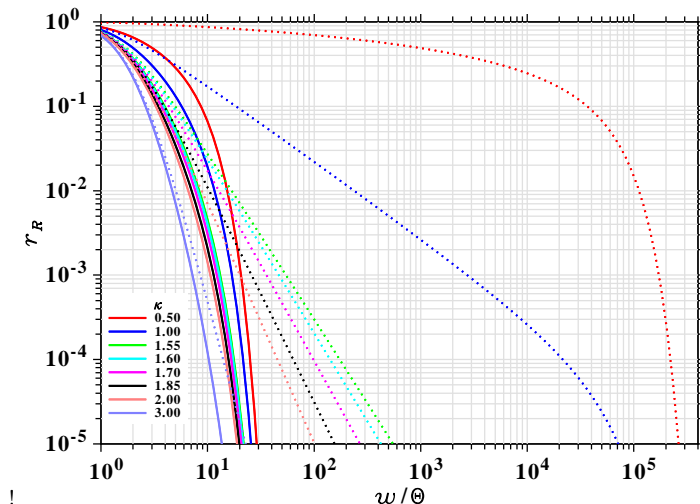


FIG. 4.— The relative ratio  $r_R$  vs.  $w/\Theta$  for  $\alpha = 10^{-1}$  (solid lines) and  $\kappa = 0.5, 1, 1.55, 1.6, 1.7, 1.85, 2, 3$  (red, blue, green, cyan, magenta, black, light red, and light blue, respectively), and for  $\alpha = 10^{-5}$  (dotted lines).

While not a moment of a distribution function, the entropy is another thermodynamic quantity characterizing the state of a given physical system. In order to have such state correctly described within a non-relativistic treatment one again can not allow superluminal particles to have any significant contribution. As before, we quantify this contribution on the basis of the relative ratios defined in section 3.

Rather than discussing all three distributions in detail again, as is done in the previous section for the partial pressure, we illustrate the findings with the RKD that, of course, contains the Maxwellian in the limit  $\kappa \rightarrow \infty$  and that the SKD in the limit  $\alpha \rightarrow 0$ . Note, however, that formula (10) does not hold for the SKD because it requires the existence of all velocity moments and, thus, it may be necessary to use the non-extensive entropy (Silva et al. 1998).

Fig. 4 shows the relative ratio  $r_R$  as a function of  $w/\Theta$  for all combinations of  $\alpha = 10^{-5}, 10^{-1}$ , and  $\kappa = 0.5, 1, 1.55, 1.6, 1.7, 1.85, 2, 3$ . The results are very similar to those for the partial pressure: First, the contributions of particles with speeds higher than the speed of light are increasingly significant with decreasing  $\kappa$ -values. Second, choosing a higher  $\alpha$ -value and, thus, exponential cut-offs at lower speeds can reduce these contributions to insignificant amounts. This similarity is expected, from the thermodynamic dependence of the entropy on the pressure.

A comparison of the results of Figs. 3 and 4 reveals that, for given  $\alpha$  and  $\kappa$ , the inequality  $r_R < R_R$  holds, i.e. that the relative contribution of superluminal particles to the pressure is higher than that to the entropy. Also this is expected in view of the different integrands of the velocity integral of the pressure moment and the entropy.

## 6. DISCUSSION

After we have demonstrated the possibility of the unphysical significance of contributions of superluminal particles to macroscopic thermodynamic quantities, we briefly mention three physical system for which a proper representation of the distribution function is essential within the framework of a non-relativistic theory.

### 6.1. Langmuir turbulence

It was recently, shown (Yoon et al. 2018) that, in contrast to the SKD, the RKD avoids on a mesoscopic (i.e. kinetic) level the infrared catastrophe. The reason is that in the long-wavelength range the contribution of the superluminal tail in case of low  $\kappa$ -values is suppressed.

### 6.2. Electrons in the interplanetary medium

The solar wind electron distribution (e.g., Lin 1998) can extend to energies beyond 100 keV, especially in the so-called superhalo (Wang et al. 2012). For these energies the electrons are on the edge to be in the relativistic range.

In solar particle events there are observed power laws (in kinetic energy) with a power-law index below 1.5 extending to even higher energies (Oka et al. 2018). The underlying distribution function is most likely consistent with an RKD, see Wang et al. (2012).

To avoid using the comparatively complicated relativistic  $\kappa$ -distribution (Xiao 2006), one can employ for both cases the non-relativistic RKD with a suitable cut-off such that any contribution of superluminal electrons is negligibly small. Moreover, cut-offs at sufficiently high velocity may prevent alterations of plasma waves dispersion relations, e.g., for Langmuir waves in Scherer et al. (2017) and, implicitly, unrealistic interpretations of plasma parameters from an indirect plasma diagnosis measuring the plasmas wave fluctuations.

### 6.3. The heliosheath

For a simulation of the heliosheath Heerikhuisen et al. (2008) and Fahr et al. (2016) were using the SKD with constant  $\kappa = 1.63$  and  $\kappa < 1.7$ , respectively, which are according to the above results critical values. While these authors do not explicitly state the value of  $\Theta$ , from the temperature plot (figure 5 of the latter authors) one can

estimate that  $\Theta \approx 100$  km/s in the region beyond the termination shock. Fig. 2 above reveals that for  $\kappa = 1.6$  and  $w/\Theta < 3 \cdot 10^3$  more than 10% of the pressure comes from superluminal particles; for  $\kappa = 1.7$  this contribution reduces slightly to about 6%. Assuming that the effect on the heliocentric distances of the termination shock, the heliopause and the bow shock (given in their table 2) is of the same order, these estimates would change accordingly. While this needs to be carefully studied again using the RKD in the complicated charge-transfer integrals solved by the former authors for the SKD, our expectation is that such correction will reduce these distances.

## 7. CONCLUSION

We have demonstrated that the use of non-relativistic standard  $\kappa$ -distributions with low  $\kappa$ -values results in unphysical contributions of particles with speeds above the speed of light to macroscopic thermodynamic quantities like pressure and entropy. The actual limiting value of  $\kappa$  depends on the thermal velocity  $\Theta$  (characteristic to the Maxwellian core of such distributions) and on the constraint adopted for the 'superluminal' contributions, e. g. 1% .

While, in principle, such a limitation exists also for the non-relativistic regularized  $\kappa$ -distribution, the latter allows to suppress the unphysical significance of superluminal particles by an appropriate choice of the cut-off parameter  $\alpha$ . This regularizing exponential cut-off makes any undesired contribution to pressure or entropy negligible, just as in the case of a Maxwellian distribution. Consequently, we confirm the finding in Scherer et al. (2017) that the SKD has its merits fitting data but can easily be inconsistent with related macroscopic quantities. Whenever there is the need of modelling the latter, the use of the RKD with an appropriate cut-off is required.

Nevertheless, a less pragmatic and more puristic approach should aim for a consistent formulation of relativistic versions of the SKD and the RKD. For the former this has been attempted by Xiao (2006), for the latter this will be done in forthcoming work.

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## APPENDIX

## SOME REMARKS CONCERNING THE CONNECTION TO OTHER STATISTICAL DISTRIBUTIONS

Although the  $\kappa$  distribution (SKD) looks quite similar to the Pareto distribution used in economics (for example Arnold 2015; Krishnamoorth 2015) or to the Schechter luminosity “function” used in astronomy (e.g. Luo et al. 2018), there are subtle differences: On the first glimpse the univariate Pareto distribution is quite close to the SKD, but because the SKD is not a univariate distribution, one has to compare to the multivariate Pareto distribution of the fourth kind. A statistical interpretation of the SKD or RKD is difficult, because their (higher-order) moments have a clear physical significance, different to those of statistical distribution functions.

The amplitude of the velocity  $|\vec{v}| = v$  depends in general on three coordinates, which reduce to one in the case of an isotropic (spherical) distribution, because  $v$  is independent on the angle variables and any integration over the solid angle results in the factor  $4\pi$ . But the moments are no longer scalars, but vectors or (higher-order) tensors (see for a more detailed discussion Scherer et al. 2019). Because the SKD is an even function of  $v$  all odd moments vanish. Only the “central moments”, if existing, survive. The central moments are obtained when introducing bulk and/or drift speed. Moreover, the probability density function already has to be integrated over  $v^2$  (from the volume element) which would correspond to the second-order moment of the univariate Pareto distribution. Thus, one should be careful comparing the Pareto distribution with the SKD or RKD.

Moreover, the SKD, and all other distribution functions in plasma physics, must be, at least, an approximate solution of the Vlasov-, Boltzmann-, Fokker-Planck- or Landau-equation based on the Liouville theorem (Balescu 1975, 1988; Cercignani 1988). At least it is known that the transport equations for cosmic ray modulation are based on a Fokker-Planck-equation, which can be solved by stochastic differential equations (see e.g. Strauss & Effenberger 2017), which are based on stochastic Wiener processes and can be extended to Levi-flights for anomalous diffusion (Fichtner et al. 2014; Stern et al. 2014).