TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9947(XX)0000-0

COMPUTING GRADED BETTI TABLES OF TORIC SURFACES

WOUTER CASTRYCK, FILIP COOLS, JEROEN DEMEYER,
AND ALEXANDER LEMMENS

Abstract. We present various facts on the graded Betti table of a projectively embedded toric surface, expressed in terms of the combinatorics of its defining lattice polygon. These facts include explicit formulas for a number of entries, as well as a lower bound on the length of the quadratic strand that we conjecture to be sharp (and prove to be so in several special cases). We also present an algorithm for determining the graded Betti table of a given toric surface by explicitly computing its Koszul cohomology, and report on an implementation in SageMath. It works well for ambient projective spaces of dimension up to roughly 25, depending on the concrete combinatorics, although the current implementation runs in finite characteristic only. As a main application we obtain the graded Betti table of the Veronese surface $\nu_6(\mathbb{P}^2) \subseteq \mathbb{P}^{27}$ in characteristic 40 009. This allows us to formulate precise conjectures predicting what certain entries look like in the case of an arbitrary Veronese surface $\nu_d(\mathbb{P}^2)$.

1. Introduction

Let k be a field of characteristic 0 and let $\Delta \subseteq \mathbb{R}^2$ be a lattice polygon, by which we mean the convex hull of a finite number of points of the standard lattice $\mathbb{Z}^2$. We write $\Delta^{(1)}$ for the convex hull of the lattice points in the interior of Δ. Assume that Δ is two-dimensional, write $N_\Delta = |\Delta \cap \mathbb{Z}^2|$, and let $S_\Delta = k[X_{i,j} \,|\, (i,j) \in \Delta \cap \mathbb{Z}^2]$, so that $\mathbb{P}^{N_\Delta - 1} = \operatorname{Proj} S_\Delta$. The toric surface over k associated with Δ is the Zariski closure of the image of

$$\varphi_\Delta : (k^*)^2 \hookrightarrow \mathbb{P}^{N_\Delta - 1} : (a,b) \mapsto (a^i b^j)_{(i,j)\in\Delta\cap\mathbb{Z}^2}.$$

We denote it by X_Δ and its ideal by I_Δ. It was proved by Koelman [31] that I_Δ is generated by quadratic and cubic binomials, where quadrics suffice if and only if $|\partial\Delta \cap \mathbb{Z}^2| > 3$. Here $\partial\Delta$ denotes the boundary of Δ.

Our object of interest is the graded Betti table of X_Δ, which gathers the exponents appearing in a minimal free resolution

$$\cdots \to \bigoplus_{q\geq 2} S_\Delta(-q)^{\beta_{2,q}} \to \bigoplus_{q\geq 1} S_\Delta(-q)^{\beta_{1,q}} \to \bigoplus_{q\geq 0} S_\Delta(-q)^{\beta_{0,q}} \to S_\Delta/I_\Delta \to 0$$

of the homogeneous coordinate ring of X_Δ as a graded S_Δ-module, obtained by taking syzygies. Traditionally one writes $\beta_{p,p+q}$ in the pth column and the qth row. Alternatively and often more conveniently, the Betti numbers $\beta_{p,p+q}$ are the dimensions of the Koszul cohomology spaces $K_{p,q}(X_\Delta, \mathcal{O}(1))$, which will be described in detail in Section 2.2. The graded Betti table of X_Δ depends on the unimodular equivalence class of Δ only (see Section 2.1 for a precise definition of unimodular

2010 *Mathematics Subject Classification.* 14M25, 13D02, 52B20.

©XXXX American Mathematical Society

equivalence, which will be denoted by $\cong$) and is known[1] by specialists to be of the form

(1.1)

	0	1	2	3	$\dots$	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	$\dots$	0	0
1	0	b_1	b_2	b_3	$\dots$	$b_{N_\Delta-4}$	$b_{N_\Delta-3}$
2	0	$c_{N_\Delta-3}$	$c_{N_\Delta-4}$	$c_{N_\Delta-5}$	$\dots$	c_2	c_1

,

where omitted entries are understood to be 0, and all c_ℓ's vanish if and only if $\Delta^{(1)} = \emptyset$. It is a standard observation that the antidiagonal differences $b_\ell - c_{N_\Delta-1-\ell}$ can be told from the Hilbert function of X_Δ. As explained in Section 2.3 this can be made explicit as follows: for $\ell = 1, \dots, N_\Delta - 2$ one has

$$b_\ell - c_{N_\Delta-1-\ell} = \ell \binom{N_\Delta - 1}{\ell + 1} - 2\binom{N_\Delta - 3}{\ell - 1} \operatorname{vol}(\Delta) \tag{1.2}$$

where vol denotes the Euclidean volume and it is understood that $b_{N_\Delta-2} = c_{N_\Delta-2} = 0$.

In this paper, we study how the entries of (1.1) relate to the combinatorics of Δ. The main result available in the existing literature is a formula due to Hering [29] and Schenck [44], which expresses the number of vanishing c_ℓ's in terms of the number of lattice points on the boundary of Δ; see formula (1.3) below for a precise statement. The goal of this paper is to add new such entries to this dictionary. Our methods are elementary, but rely on an interplay between algebra, combinatorics, geometry, and explicit computations in SageMath [43]. There are three families of unimodular equivalence classes which play an exceptional role throughout this paper: these are represented by the polygons $d\Sigma$, $d\Upsilon$ and Υ_d depicted in Figure 1, where d ranges over the positive integers.

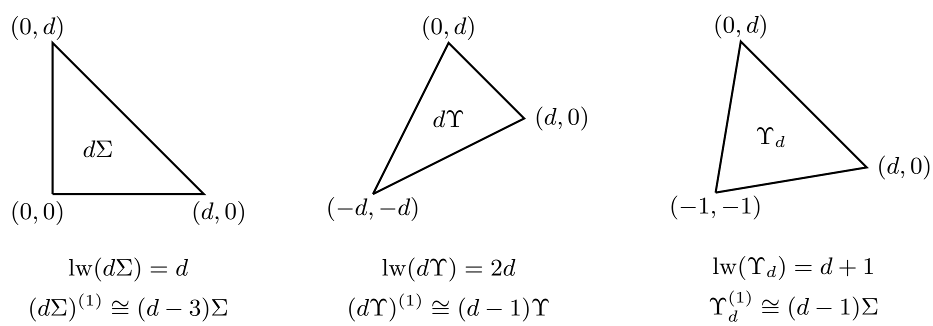


Figure 1. Three recurring families of polygons, along with some combinatorial properties. Here lw denotes the lattice width (see Section 2.1 for a definition). The formula for $(d\Sigma)^{(1)}$ assumes $d \geq 3$.

[1] The main inputs are that X_Δ is arithmetically Cohen–Macaulay [14, Ex. 9.2.8] and projectively normal [14, Cor. 2.2.13 & Thm. 2.4.1]. From this one finds that the last non-zero column has index $N_\Delta - 3$ by Auslander–Buchsbaum [18, Thm. A.2.15] and that the index of the last non-zero row equals $\operatorname{reg} X_\Delta - 1$ [18, p. 55], an explicit formula for which can be derived from [18, Cor. 4.8] or more directly from [29, Prop. IV.5]; see also [32, §0].

1.1. **Explicit determination of eight entries.** As a first contribution we present explicit formulas for eight entries of the graded Betti table:

Theorem 1.1. *On the cubic strand (i.e., the row $q = 2$) one has*

$$c_1 = N_{\Delta^{(1)}}, \qquad c_2 = \begin{cases} (N_\Delta - 3)(N_{\Delta^{(1)}} - 1) & \text{if } \Delta^{(1)} \neq \emptyset, \\ 0 & \text{if } \Delta^{(1)} = \emptyset, \end{cases}$$

$$c_{N_\Delta - 3} = \begin{cases} 0 & \text{if } |\partial\Delta \cap \mathbb{Z}^2| > 3, \\ 1 & \text{if } |\partial\Delta \cap \mathbb{Z}^2| = 3 \text{ and } \dim \Delta^{(1)} = 2, \\ N_\Delta - 3 & \text{if } |\partial\Delta \cap \mathbb{Z}^2| = 3 \text{ and } \dim \Delta^{(1)} \leq 1, \end{cases}$$

and if $\Delta \not\cong \Sigma$ then one has

$$c_3 = (N_\Delta - 4)\left((N_\Delta - 3)\operatorname{vol}(\Delta) - \tfrac{(N_\Delta - 1)(N_\Delta - 2)}{2} + B_\Delta\right)$$

where

$$B_\Delta = \begin{cases} 0 & \text{if } \dim \Delta^{(1)} = 2,\ \Delta \not\cong \Upsilon_2, \\ 1 & \text{if } \dim \Delta^{(1)} = 1 \text{ or } \Delta \cong \Upsilon_2, \\ (N_\Delta - 1)/2 & \text{if } \dim \Delta^{(1)} = 0, \\ N_\Delta - 2 & \text{if } \Delta^{(1)} = \emptyset. \end{cases}$$

On the quadratic strand (i.e., the row $q = 1$) one has

$$b_1 = \binom{N_\Delta - 1}{2} - 2\operatorname{vol}(\Delta), \qquad b_2 = 2\binom{N_\Delta - 1}{3} - 2(N_\Delta - 3)\operatorname{vol}(\Delta) + c_{N_\Delta - 3},$$

$$b_{N_\Delta - 3} = \begin{cases} 0 & \text{if } \Delta^{(1)} \neq \emptyset, \\ N_\Delta - 3 & \text{if } \Delta^{(1)} = \emptyset, \end{cases}$$

and if $\Delta \not\cong \Sigma$ then one has $b_{N_\Delta - 4} = (N_\Delta - 4)B_\Delta$.

Of course, thanks to our formula (1.2) for the antidiagonal differences, the formulas for b_1, b_2 and $b_{N_\Delta-4}$, $b_{N_\Delta-3}$ can be seen as mere consequences to those for c_1, c_2, c_3 and $c_{N_\Delta-3}$, and vice versa (modulo some rewriting using Pick's theorem). Similarly, in the range where we can apply the vanishing statements implied by Theorem 1.3 and Hering and Schenck's formula (1.3) below, we can use (1.2) to give explicit formulas for several more entries. E.g., if $1 \leq \ell \leq |\partial\Delta \cap \mathbb{Z}^2| - 2$ then b_ℓ just equals the right-hand side of (1.2).

Remark 1.2. Note that if one fixes a value of $\ell \in \mathbb{Z}_{\geq 1}$ and a two-dimensional lattice polygon Δ along with all its dilations $d\Delta$ for increasing values of d, then because $|\partial(d\Delta) \cap \mathbb{Z}^2|$ tends to infinity, eventually b_ℓ will be equal to the right-hand side of (1.2). Using Pick's theorem it is easy to rewrite this right-hand side as a polynomial expression in d for large enough values of d, thereby confirming a special case of [45, Thm. 1.1]. The same remark applies to the c_ℓ's, now using Theorem 4.1 along with Theorem 1.3 applied to $d\Sigma$, a unimodular copy of which is always contained in $d\Delta$.

1.2. **Length of the quadratic strand.** A second goal of this paper is to provide a combinatorial interpretation for the number of zeroes at the end of the quadratic strand. Unfortunately we are unable to provide a definite answer, but we present the following partial result, which involves the lattice width $\operatorname{lw}(\Delta)$ of Δ. This is a well-known combinatorial invariant whose definition will be recalled in Section 2.1.

Theorem 1.3. *If $\Delta \not\cong \Sigma, \Upsilon$ then one has*

$$\min\{\,\ell \mid b_{N_\Delta-\ell} \neq 0\,\} \leq \begin{cases} \mathrm{lw}(\Delta)+1 & \text{if } \Delta \cong d\Sigma \text{ for some } d \geq 2, \\ \mathrm{lw}(\Delta)+1 & \text{if } \Delta \cong \Upsilon_d \text{ for some } d \geq 2, \\ \mathrm{lw}(\Delta)+1 & \text{if } \Delta \cong 2\Upsilon, \\ \mathrm{lw}(\Delta)+2 & \text{in all other cases.} \end{cases}$$

Moreover:

- *If $N_\Delta \leq 32$ then the bound is met.*
- *If a certain non-exceptional lattice polygon Δ (i.e. not of the form $d\Sigma, \Upsilon_d, 2\Upsilon$) meets the bound then so does every lattice polygon containing Δ and having the same lattice width. In particular if $\mathrm{lw}(\Delta) \leq 6$ then the bound is met.*

† *If $\Delta = \Gamma^{(1)}$ for some larger lattice polygon Γ and if Green's canonical syzygy conjecture holds for smooth curves on X_Δ (known to be true if $H^0(X_\Delta, -K_{X_\Delta}) \geq 2$) then the bound is met.*

Here we note that the excluded cases $\Delta \cong \Sigma, \Upsilon$ are pathological: the Betti tables are

	0
0	1
1	0
2	0

resp.

	0	1
0	1	0
1	0	0
2	0	1

,

i.e., the entire quadratic strands are zero.

As will be explained in Section 3.1, the upper bound $\min\{\,\ell \mid b_{N_\Delta-\ell} \neq 0\,\} \leq \mathrm{lw}(\Delta)+2$ follows from the fact that our toric surface X_Δ is naturally contained in a rational normal scroll of dimension $\mathrm{lw}(\Delta)+1$, which is known to have non-zero linear syzygies up to column $p = N_\Delta - \mathrm{lw}(\Delta) - 2$ by exactness of the Eagon-Northcott complex, see e.g. [18, Cor. A2.62]. Then also X_Δ must have non-zero linear syzygies up to that point, yielding the desired bound. In the exceptional cases $d\Sigma$, Υ_d and 2Υ we can prove the sharper bound $\min\{\,\ell \mid b_{N_\Delta-\ell} \neq 0\,\} \leq \mathrm{lw}(\Delta)+1$ by following a slightly different argument, using explicit computations in Koszul cohomology, for the details of which we refer to Section 3.2.

Sharpness in the cases where $N_\Delta \leq 32$ is obtained by explicit verification, based on the data from [7] and using the algorithm described in Section 7; this covers more than half a million unimodular equivalence classes of small lattice polygons. Sharpness in the cases where $\mathrm{lw}(\Delta) \leq 6$ relies on this exhaustive verification, along with the classification of inclusion-minimal lattice polygons having a given lattice width, which is elaborated in [13]. Our third sharpness result, marked with a dagger symbol †, will not be addressed in the current paper, even though it is actually the reason why we came up with this research question in the first place. We note that to date, Green's canonical syzygy conjecture [25] for curves in toric surfaces remains open in general, but the cases where $H^0(X_\Delta, -K_{X_\Delta}) \geq 2$ are covered by recent work of Lelli-Chiesa [33], which allows one to deduce sharpness for all multiples of Υ, for all multiples of Σ, for all polygons $[0,a] \times [0,b]$ with $a, b \geq 1$, and so on. The details of this are discussed in a subsequent paper [12], which is devoted to syzygies of curves in toric surfaces; for the sake of conciseness we have chosen to keep the present document as curve-free as possible. Finally, we know of a few further sporadic cases where the bound is met, for which we refer to the Ph.D. thesis of the fourth listed author [34, p. 43]. These cases cover all polygons of the form Υ_d, allowing us to conclude that sharpness holds for each of the three (families of) exceptional lattice polygons.

Remark 1.4. Besides with Green's canonical syzygy conjecture, there is also a connection with Green and Lazarsfeld's gonality conjecture [26], a proof of which was recently found by Ein and Lazarsfeld [17]. Indeed, because S_Δ/I_Δ is Cohen-Macaulay one can view (1.1) as the graded Betti table of a generic hyperplane section of X_Δ, which is a curve of genus $N_{\Delta^{(1)}}$ that is embedded by a linear system of degree $2\operatorname{vol}(\Delta)$ and rank $N_\Delta - 2$. By [11, Cor. 6.2] the gonality of this curve equals $\operatorname{lw}(\Delta)$, unless $\Delta \cong 2\Upsilon, d\Sigma$ for some $d \geq 2$ in which case it equals $\operatorname{lw}(\Delta) - 1$. Assuming that the degree $2\operatorname{vol}(\Delta)$ is 'sufficiently large' when compared to the genus, the gonality conjecture predicts that the bound stated in Theorem 1.3 is sharp. This is apart from the cases where $\Delta \cong \Upsilon_d$ for some $d \geq 2$, where the prediction is off by 1, which just means that $2\operatorname{vol}(\Delta) = 2\operatorname{vol}(\Upsilon_d)$ is not large enough here, see [8]. But in the other cases one sees that the gonality conjecture is potentially useful for establishing further sharpness results. Unfortunately the best currently known quantifications of 'sufficiently large' are inadequate for this purpose. Indeed, the leading result is Rathmann's bound [41], which in our case reads that $2\operatorname{vol}(\Delta) \geq 4N_{\Delta^{(1)}} - 3$; here it is assumed that $N_{\Delta^{(1)}} \geq 2$. By Pick's theorem Rathmann's bound is equivalent to $|\partial\Delta \cap \mathbb{Z}^2| \geq 2N_{\Delta^{(1)}} - 1$, which by a result of Haase and Schicho [28, Lem. 11] is only possible when $\dim \Delta^{(1)} \leq 1$ or $N_\Delta \leq 19$, and in both cases we already know that sharpness holds. But in view of recent work by Farkas and Kemeny [20] it is to be expected that Rathmann's bound will be improved at some point in the future. Such an event would shed a new light on this discussion.

In fact we view the above sharpness results as evidence towards our conjecture that the upper bound stated in Theorem 1.3 is met for *all* two-dimensional lattice polygons Δ:

Conjecture 1.5. *If* $\Delta \ncong \Sigma, \Upsilon$ *then one has* $\min\{\, \ell \,|\, b_{N_\Delta - \ell} \neq 0 \,\} = \operatorname{lw}(\Delta) + 2$, *unless*

$$\Delta \cong d\Sigma \text{ for some } d \geq 2 \qquad \text{or} \qquad \Delta \cong \Upsilon_d \text{ for some } d \geq 2 \qquad \text{or} \qquad \Delta \cong 2\Upsilon$$

in which case it is $\operatorname{lw}(\Delta) + 1$.

In other words we believe that the number of zeroes at the end of the quadratic strand equals $\operatorname{lw}(\Delta) - 1$, unless Δ is of the form $d\Sigma$, Υ_d or 2Υ, in which case it equals $\operatorname{lw}(\Delta) - 2$. An informal way of reading Conjecture 1.5 is that the bound coming from the natural ambient rational normal scroll is usually sharp. This is in the philosophy of Green's $K_{p,1}$ theorem [1, Thm. 3.31] that towards the end of the resolution, 'most' linear syzygies must come from the smallest ambient variety of minimal degree, in the sense of [19].

Conjecture 1.5 can be seen as a dual statement to the formula

$$\min\{\, \ell \,|\, c_{N_\Delta - \ell} \neq 0 \,\} = |\partial\Delta \cap \mathbb{Z}^2| \tag{1.3}$$

describing the number of leading zeroes on the cubic strand; here we assume $\Delta^{(1)} \neq \emptyset$, so that the minimum is well-defined. Note that this formula can be viewed as a vast generalization of Koelman's aforementioned result on the degrees of the generators of I_Δ. A proof of (1.3) can be found in the Ph.D. thesis of Hering [29, Thm. IV.20], who built on an observation due to Schenck [44] and invoked a theorem of Gallego–Purnaprajna [23, Thm. 1.3]. Also note that Theorem 1.3 and formula (1.3) together imply that if one considers all dilations $d\Delta$ of a given two-dimensional lattice polygon Δ, then the number of a_ℓ's and b_ℓ's that vanish is in

$O(d)$. Indeed, we clearly have $|\partial(d\Delta) \cap \mathbb{Z}^2| \in O(d)$ while also $\mathrm{lw}(d\Delta) \in O(d)$ by a result of Fejes Tóth and Makai Jr. [21]. Since the length of the range $1, \ldots, N_\Delta - 3$ is in $\Theta(d^2)$, this means that asymptotically for $d \to \infty$ a proportion of 1 of the relevant Betti numbers are non-vanishing, thereby confirming a result by Ein and Lazarsfeld on asymptotic syzygies [16, Thm. A] in the case of toric surfaces.

1.3. **Algorithmic determination of the graded Betti table.** Finally, our third contribution is an algorithm for determining the graded Betti table of $X_\Delta \subseteq \mathbb{P}^{N_\Delta - 1}$ upon input of a lattice polygon Δ, by explicitly computing its Koszul cohomology. The details can be found in Section 7, but in a nutshell the ingredients are as follows. The most dramatic speed-up comes from incorporating the torus action, which decomposes the cohomology spaces into eigenspaces, one for each bidegree $(a, b) \in \mathbb{Z}^2$, all but finitely many of which are trivial. Another important speed-up comes from toric Serre duality, enabling a meet-in-the-middle approach where one fills the graded Betti table starting from the left and from the right simultaneously. A third speed-up comes from the explicit formula for the antidiagonal differences given in (1.2), thanks to which it suffices to determine half of the graded Betti table only. Moreover if $\left|\partial\Delta \cap \mathbb{Z}^2\right|$ is large (which is particularly the case for the Veronese polygons $d\Sigma$) then many of these entries come for free using Hering and Schenck's formula (1.3). A fourth theoretical ingredient is a combinatorial description of certain exact subcomplexes of the Koszul complex that can be quotiented out, resulting in smaller vector spaces, thereby making the linear algebra more manageable. Because this seems interesting in its own right, we have devoted the separate Section 6 to it. Final ingredients include sparse linear algebra, using symmetries, and working in finite characteristic. More precisely, most of the data gathered in this article, some of which can be found in Appendix A, are obtained by computing modulo 40 009, the smallest prime number larger than 40 000.

By semi-continuity the entries of the graded Betti table cannot decrease upon reduction of X_Δ modulo some prime number. Therefore working in finite characteristic is fine for proving that certain entries are zero, as is done in our partial verification of Conjecture 1.5. But entries that are found to be non-zero might a priori be too large, even though we expect this to be a very rare event (see Section 7 for an example). Therefore the non-zero entries of some of the graded Betti tables given in Appendix A are conjectural. For technical reasons our current implementation does not straightforwardly adapt to characteristic zero, but we are working on fixing this issue. Although it would come at the cost of some efficiency, this should enable us to confirm all of the data from Appendix A in characteristic zero.

In view of the wide interest in syzygies of Veronese modules [5, 15, 24, 37, 39, 40, 42], the most interesting new graded Betti table that we obtain is that of $X_{6\Sigma} \subseteq \mathbb{P}^{27}$, i.e. the image of $\mathbb{P}^2$ under the 6-uple embedding ν_6, in characteristic 40 009. Up to 5Σ this data was recently gathered (in characteristic zero) by Greco and Martino [24]. An extrapolating glance at these Betti tables naturally leads to the following conjecture:

Conjecture 1.6. *Consider the graded Betti table of the d-fold Veronese surface $X_{d\Sigma}$. If $d \geq 2$ then the last non-zero entry on the quadratic strand is*

$$b_{d(d+1)/2} = \frac{d^3(d^2-1)}{8},$$

while if $d \geq 3$ then the first non-zero entry on the cubic strand is

$$c_g = \binom{N_{(d\Sigma)^{(1)}} + 8}{9}$$

where $N_{(d\Sigma)^{(1)}} = |(d\Sigma)^{(1)} \cap \mathbb{Z}^2| = (d-1)(d-2)/2.$

In fact, about a year after we submitted the current paper, a generalization of the latter statement was proven by the fourth listed author [35], while the former statement has been put in the broader context of Schur functors by Bruce, Erman, Goldstein and Yang [4, Conj. 6.6].

Acknowledgements. This research was partially supported by the research project G093913N of the Research Foundation Flanders (FWO), by the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) with ERC Grant Agreement nr. 615722 MOTMELSUM, and by the Labex CEMPI (ANR-11-LABX-0007-01). The fourth author is supported by a Ph.D. fellowship of the Research Foundation Flanders (FWO). We would like to thank Marc Coppens, Milena Hering, Nicolas M. Thiéry and an anonymous referee for their helpful remarks. The computational resources (Stevin Supercomputer Infrastructure) and services used in this work were provided by the VSC (Flemish Supercomputer Center), funded by Ghent University, the Hercules Foundation and the Flemish Government — department EWI. After submitting a first version of this paper to `arXiv`, we learned that a group of researchers at the University of Wisconsin–Madison has been working independently on computing Betti tables of Veronese surfaces [4], thereby obtaining results that partially overlap with our own observations. In particular they also obtain the graded Betti table of $X_{6\Sigma} \subseteq \mathbb{P}^{27}$, although here too the result is conjectural, using linear algebra over the reals rather than mod p. We thank Juliette Bruce for getting in touch with us on this, and for her valuable feedback.

2. Preliminaries and first facts

2.1. **Combinatorial notions.** If Δ and Δ' are lattice polygons, we say that they are *unimodularly equivalent* (denoted by $\Delta \cong \Delta'$) if they are obtained from one another using a transformation from the affine group $\mathrm{AGL}_2(\mathbb{Z})$, that is a map of the form

$$\mathbb{R}^2 \to \mathbb{R}^2 : (x\ y) \mapsto (x\ y)A + (a\ b) \text{ with } A \in \mathrm{GL}_2(\mathbb{Z}) \text{ and } a, b \in \mathbb{Z}.$$

Unimodularly equivalent polygons yield projectively equivalent toric surfaces, which have the same graded Betti table. So we are interested in lattice polygons up to unimodular equivalence only.

The central combinatorial notion of this article is the *lattice width*. If Δ is a non-empty lattice polygon, then the lattice width of Δ, denoted $\mathrm{lw}(\Delta)$, is the minimal height d of a horizontal strip $\mathbb{R} \times [0, d]$ in which Δ can be mapped using a unimodular transformation. If $\Delta = \emptyset$ then we define $\mathrm{lw}(\Delta) = -1$. Remark that $\mathrm{lw}(\Delta) = 0$ if and only if Δ is zero- or one-dimensional. For the sake of example, the lattice widths associated with our three recurring families of lattice polygons can be found in Figure 1. In general, the lattice width can be computed recursively;

see [9, Thm. 4] or [38, Thm. 13]: if Δ is two-dimensional then

$$\mathrm{lw}(\Delta) = \begin{cases} \mathrm{lw}(\Delta^{(1)}) + 3 & \text{if } \Delta \cong d\Sigma \text{ for some } d \geq 2, \\ \mathrm{lw}(\Delta^{(1)}) + 2 & \text{if not.} \end{cases} \tag{2.1}$$

2.2. **Koszul cohomology of toric surfaces.** As is well-known, instead of using syzygies, the entries of the graded Betti table can also be defined as dimensions of Koszul cohomology spaces, which we now explicitly describe in the specific case of toric surfaces. We refer to the book by Aprodu and Nagel [1] for an introduction to Koszul cohomology, and to the books by Fulton [22] and Cox, Little and Schenck [14] for more background on toric geometry.

For a lattice polygon Δ we write V_Δ for the space of Laurent polynomials

$$\sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j}x^iy^j \ \in \ k[x^{\pm 1}, y^{\pm 1}],$$

which we view as functions on X_Δ through φ_Δ. This equals the space $H^0(X_\Delta, L_\Delta)$ of global sections of $\mathcal{O}(L_\Delta)$, where L_Δ is some very ample torus-invariant divisor on X_Δ satisfying $\mathcal{O}(L_\Delta) \cong \mathcal{O}(1)$. More generally $V_{q\Delta} = H^0(X_\Delta, qL_\Delta)$ for each $q \geq 0$.

Then the entry in the pth column and the qth row of the graded Betti table of X_Δ is the dimension of the Koszul cohomology space $K_{p,q}(X_\Delta, L_\Delta)$, defined as the cohomology in the middle of

$$\bigwedge^{p+1} H^0(X_\Delta, L_\Delta) \otimes H^0(X_\Delta, (q-1)L_\Delta) \xrightarrow{\delta} \bigwedge^{p} H^0(X_\Delta, L_\Delta) \otimes H^0(X_\Delta, qL_\Delta)$$
$$\xrightarrow{\delta'} \bigwedge^{p-1} H^0(X_\Delta, L_\Delta) \otimes H^0(X_\Delta, (q+1)L_\Delta)$$

which can be rewritten as

$$\bigwedge^{p+1} V_\Delta \otimes V_{(q-1)\Delta} \xrightarrow{\delta} \bigwedge^{p} V_\Delta \otimes V_{q\Delta} \xrightarrow{\delta'} \bigwedge^{p-1} V_\Delta \otimes V_{(q+1)\Delta}.$$

Here the coboundary maps δ and δ' are defined by

$$v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge \cdots \otimes w \ \mapsto \ \sum (-1)^s v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge \cdots \wedge \widehat{v_s} \wedge \cdots \otimes v_s w \tag{2.2}$$

where s ranges from 1 to $p+1$ resp. 1 to p, and $\widehat{v_s}$ means that v_s is being omitted. In particular one sees that b_ℓ is the dimension of the cohomology in the middle of

$$\bigwedge^{\ell+1} V_\Delta \xrightarrow{\delta} \bigwedge^{\ell} V_\Delta \otimes V_\Delta \xrightarrow{\delta'} \bigwedge^{\ell-1} V_\Delta \otimes V_{2\Delta}, \tag{2.3}$$

where we note that the left map is always injective. On the other hand c_ℓ is the dimension of the cohomology in the middle of

$$\bigwedge^{N_\Delta-1-\ell} V_\Delta \otimes V_\Delta \xrightarrow{\delta} \bigwedge^{N_\Delta-2-\ell} V_\Delta \otimes V_{2\Delta} \xrightarrow{\delta'} \bigwedge^{N_\Delta-3-\ell} V_\Delta \otimes V_{3\Delta},$$

for all $\ell = 1, \ldots, N_\Delta - 3$.

Duality. A more concise description of the c_ℓ's is obtained using Serre duality. Because the version that we will invoke requires us to work with smooth surfaces, we consider a toric resolution of singularities $X \to X_\Delta$ and let L be the pullback of L_Δ. Then L may no longer be very ample, but it remains globally generated by the same global sections V_Δ. Let K be the canonical divisor on X obtained by taking minus the sum of all torus-invariant prime divisors. By Demazure vanishing one has $H^1(X, qL) = 0$ for all $q \geq 0$, so that we can apply the duality formula from [1, Thm. 2.25], which in our case reads

$$K_{p,q}(X, L)^\vee \cong K_{N_\Delta - 3 - p, 3 - q}(X; K, L),$$

where the attribute '$; K$' denotes Koszul cohomology twisted by K (which is defined as before, except that each appearance of $\cdot \otimes H^0(X, qL)$ is replaced by $\cdot \otimes H^0(X, qL + K)$). We conclude that

$$b_\ell = \dim K_{\ell,1}(X_\Delta, L_\Delta) = \dim K_{\ell,1}(X, L) = \dim K_{N_\Delta - 3 - \ell, 2}(X; K, L),$$
$$c_\ell = \dim K_{N_\Delta - 2 - \ell, 2}(X_\Delta, L_\Delta) = \dim K_{N_\Delta - 2 - \ell, 2}(X, L) = \dim K_{\ell - 1, 1}(X; K, L),$$

again for all $\ell = 1, \ldots, N_\Delta - 3$. Using that $H^0(X, qL + K) = V_{(q\Delta)^{(1)}}$ for $q \geq 1$ and that $H^0(X, K) = 0$ we find that b_ℓ is the cohomology in the middle of

$$\bigwedge^{N_\Delta - 2 - \ell} V_\Delta \otimes V_{\Delta^{(1)}} \xrightarrow{\delta} \bigwedge^{N_\Delta - 3 - \ell} V_\Delta \otimes V_{(2\Delta)^{(1)}} \xrightarrow{\delta'} \bigwedge^{N_\Delta - 4 - \ell} V_\Delta \otimes V_{(3\Delta)^{(1)}} \tag{2.4}$$

and, more interestingly, that c_ℓ is the dimension of the kernel of

$$\bigwedge^{\ell - 1} V_\Delta \otimes V_{\Delta^{(1)}} \xrightarrow{\delta'} \bigwedge^{\ell - 2} V_\Delta \otimes V_{(2\Delta)^{(1)}}. \tag{2.5}$$

For example this gives a quick way of seeing that $c_1 = \dim \ker(V_{\Delta^{(1)}} \to 0) = N_{\Delta^{(1)}}$.

Bigrading. For $(a, b) \in \mathbb{Z}^2$ we call an element of

$$\bigwedge^p V_\Delta \otimes V_{q\Delta}$$

homogeneous of bidegree (a, b) if it is a k-linear combination of elementary tensors of the form

$$x^{i_1} y^{j_1} \wedge \cdots \wedge x^{i_p} y^{j_p} \otimes x^{i'} y^{j'}$$

satisfying $(i_1, j_1) + \cdots + (i_p, j_p) + (i', j') = (a, b)$. The coboundary morphisms δ and δ' send homogeneous elements to homogeneous elements of the same bidegree, i.e. the Koszul complex is naturally bigraded. Thus the Koszul cohomology spaces decompose as

$$K_{p,q}(X, L) = \bigoplus_{(a,b) \in \mathbb{Z}^2} K_{p,q}^{(a,b)}(X, L)$$

where in fact it suffices to let (a, b) range over $(p + q)\Delta \cap \mathbb{Z}^2$. Similarly, we have a decomposition of the twisted cohomology spaces

$$K_{p,q}(X; K, L) = \bigoplus_{(a,b) \in \mathbb{Z}^2} K_{p,q}^{(a,b)}(X; K, L)$$

where now (a, b) in fact runs over $\big(p\Delta + (q\Delta)^{(1)}\big) \cap \mathbb{Z}^2$. In particular also the b_ℓ's and the c_ℓ's, and as a matter of fact the entire graded Betti table, decompose as sums of smaller instances. We will write

$$b_{\ell,(a,b)} = \dim K_{\ell,1}^{(a,b)}(X, L), \qquad b_{\ell,(a,b)}^\vee = \dim K_{N_\Delta - 3 - \ell, 2}^{(a,b)}(X; K, L),$$
$$c_{\ell,(a,b)} = \dim K_{N_\Delta - 2 - \ell, 2}^{(a,b)}(X, L), \qquad c_{\ell,(a,b)}^\vee = \dim K_{\ell - 1, 1}^{(a,b)}(X; K, L),$$

so that

$$b_\ell = \sum_{(a,b) \in \mathbb{Z}^2} b_{\ell,(a,b)} = \sum_{(a,b) \in \mathbb{Z}^2} b_{\ell,(a,b)}^\vee \quad \text{and} \quad c_\ell = \sum_{(a,b) \in \mathbb{Z}^2} c_{\ell,(a,b)} = \sum_{(a,b) \in \mathbb{Z}^2} c_{\ell,(a,b)}^\vee.$$

Example 2.1. For $\Delta = 4\Sigma$ one can compute that $c_3 = \dim K_{2,1}(X; K, L) = 55$, which decomposes as the sum of the following numbers.

0									
0	0								
0	1	0							
0	1	1	0						
0	2	2	2	0					
0	2	3	3	2	0				
0	2	3	4	3	2	0			
0	1	2	3	3	2	1	0		
0	1	1	2	2	2	1	1	0	
0	0	0	0	0	0	0	0	0	0

Here the entry in the ath column (counting from the left) and the bth row (counting from the bottom) is the dimension $c^\vee_{3,(a,b)}$ of the degree (a, b) part. In other words we think of the above triangle as being in natural correspondence with the lattice points (a, b) inside $2\Delta + \Delta^{(1)} = (1, 1) + 9\Sigma$.

Duality versus bigrading. An interesting observation that came out of a joint discussion with Milena Hering is that duality respects the bigrading along the rule

$$K^{(a,b)}_{p,q}(X, L)^\vee \cong K^{\sigma_\Delta-(a,b)}_{N_\Delta-3-p,3-q}(X; K, L),$$

where σ_Δ denotes the sum of all lattice points in Δ. We postpone a proof to [2], but note that taking dimensions yields the formulas

$$b_{\ell,(a,b)} = b^\vee_{\ell,\sigma_\Delta-(a,b)} \quad \text{and} \quad c_{\ell,(a,b)} = c^\vee_{\ell,\sigma_\Delta-(a,b)}. \tag{2.6}$$

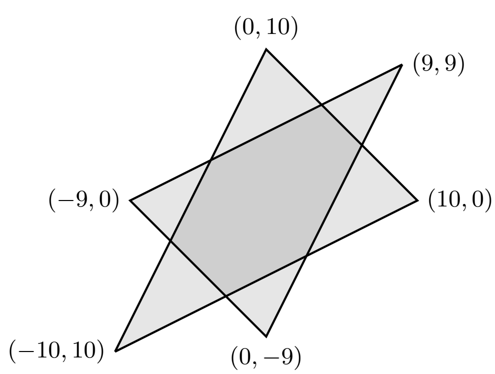
These imply that $K_{p,q}(X, L)$ is actually supported on the degrees (a, b) that are contained in

$$(p + q)\Delta \;\cap\; \left(\sigma_\Delta - (N_\Delta - 3 - p)\Delta - ((3 - q)\Delta)^{(1)}\right),$$

and similarly that $K_{p,q}(X; K, L)$ vanishes outside

$$\left(p\Delta + (q\Delta)^{(1)}\right) \cap (\sigma_\Delta - (N_\Delta - p - q)\Delta)\,.$$

The image below illustrates this for $\Delta = 2\Upsilon$, $p = 4$, $q = 1$, where $K_{p,q}(X; K, L)$ is supported on $9\Upsilon \cap (-10\Upsilon)$:



In principle this could be used to speed up our computation of the graded Betti table, because it says that certain bidegrees can be omitted. Unfortunately the vanishing happens in a range of bidegrees that is dealt with relatively easily anyway.

Therefore, the computational advantage is negligible and we will not use this in our algorithm.

2.3. **Antidiagonal differences.** We now prove the formula (1.2) from the introduction, giving a closed expression for the antidiagonal differences $b_\ell - c_{N_\Delta-1-\ell}$. The default way to proceed would be to use that the Hilbert polynomial $P_{X_\Delta}(d)$ of X_Δ is given by the Ehrhart polynomial

$$|d\Delta \cap \mathbb{Z}^2| = \operatorname{vol}(\Delta)d^2 + \frac{|\partial\Delta \cap \mathbb{Z}^2|}{2}d + 1. \tag{2.7}$$

We will give a slightly more convenient argument using Koszul cohomology.

Proof of (1.2). The proof relies on three elementary facts:

(i) Pick's theorem,

(ii) for any bounded complex of finite-dimensional vector spaces V_j one has

$$\sum_j (-1)^j \dim V_j = \sum_j (-1)^j \dim H^j,$$

where H^j is the cohomology of the complex at place j,

(iii) for all $n, k, N \geq 0$ we have $\sum_{j=0}^n (-1)^j \binom{N}{n-j}\binom{j}{k} = (-1)^k \binom{N-k-1}{n-k}$.

We compute

$$\begin{aligned}
b_\ell - c_{N_\Delta-1-\ell} &= \sum_{j=0}^{\ell+1} (-1)^{j+1} \dim K_{\ell-j+1,j}(X_\Delta, L_\Delta) \\
&\overset{\text{(ii)}}{=} \sum_{j=0}^{\ell+1} (-1)^{j+1} \dim \left(\bigwedge^{\ell+1-j} V_\Delta \otimes V_{j\Delta} \right) \\
&= \sum_{j=0}^{\ell+1} (-1)^{j+1} \binom{N_\Delta}{\ell+1-j} N_{j\Delta} \\
&\overset{\text{(i)}}{=} -\sum_{j=0}^{\ell+1} (-1)^j \binom{N_\Delta}{\ell+1-j} (j^2 \operatorname{vol}(\Delta) + \frac{j}{2} \left|\partial\Delta \cap \mathbb{Z}^2\right| + 1) \\
&\overset{\text{(i)}}{=} -\sum_{j=0}^{\ell+1} (-1)^j \binom{N_\Delta}{\ell+1-j} (j^2 \operatorname{vol}(\Delta) + j(N_\Delta - \operatorname{vol}(\Delta) - 1) + 1) \\
&= -\sum_{j=0}^{\ell+1} (-1)^j \binom{N_\Delta}{\ell+1-j} \left(2\operatorname{vol}(\Delta)\binom{j}{2} + (N_\Delta - 1)\binom{j}{1} + \binom{j}{0} \right) \\
&\overset{\text{(iii)}}{=} -2\operatorname{vol}(\Delta)\binom{N_\Delta-3}{\ell-1} + (N_\Delta-1)\binom{N_\Delta-2}{\ell} - \binom{N_\Delta-1}{\ell+1} \\
&= -2\operatorname{vol}(\Delta)\binom{N_\Delta-3}{\ell-1} + \ell\binom{N_\Delta-1}{\ell+1},
\end{aligned}$$

which equals the desired expression. □

We note the following corollary to (1.2):

Corollary 2.2. *For all ℓ one has that $b_\ell \geq c_{N_\Delta - 1 - \ell}$ if and only if*

$$\ell \leq \frac{(N_\Delta - 1)(N_\Delta - 2)}{2\operatorname{vol}(\Delta)} - 1.$$

Remark 2.3. Note that $2\operatorname{vol}(\Delta) = 2N_\Delta - |\partial\Delta \cap \mathbb{Z}^2| - 2$ by Pick's theorem. This is typically $\approx 2N_\Delta$, so the point where the c_ℓ's take over from the b_ℓ's is about halfway the Betti table. If $|\partial\Delta \cap \mathbb{Z}^2|$ is relatively large then $2\operatorname{vol}(\Delta)$ becomes smaller when compared to N_Δ, and the takeover point is shifted to the right.

2.4. **Polygons without interior.** The polygons for which $\Delta^{(1)} = \emptyset$ have the following geometric characterization:

Lemma 2.4. *The surface $X_\Delta \subseteq \mathbb{P}^{N_\Delta - 1}$ is a variety of minimal degree if and only if $\Delta^{(1)} = \emptyset$.*

Proof. By definition X_Δ has minimal degree if and only if $\deg X_\Delta = 1 + \operatorname{codim} X_\Delta$, see e.g. [19]. By the above formula (2.7) for the Hilbert polynomial this can be rewritten as

$$2\operatorname{vol}(\Delta) = N_\Delta - 2$$

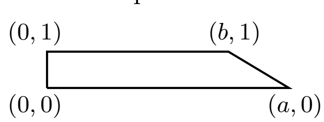
which by Pick's theorem holds if and only if $\Delta^{(1)} = \emptyset$. □

It follows that if $\Delta^{(1)} = \emptyset$ then the graded Betti table of X_Δ is of the form

(2.8)

	0	1	2	3	…	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	…	0	0
1	0	$\binom{N_\Delta - 2}{2}$	$2\binom{N_\Delta - 2}{3}$	$3\binom{N_\Delta - 2}{4}$	…	$(N_\Delta - 4)\binom{N_\Delta - 2}{N_\Delta - 3}$	$(N_\Delta - 3)\binom{N_\Delta - 2}{N_\Delta - 2}$,

because the Eagon–Northcott complex is exact in this case; see for instance [18, App. A2H]. It also follows that if $\Delta^{(1)} \neq \emptyset$ then $b_{N_\Delta - 3} = 0$; see [1, Thm. 3.31(i)]. From a combinatorial viewpoint the two-dimensional lattice polygons Δ for which $\Delta^{(1)} = \emptyset$ were classified in [30, Ch. 4]: up to unimodular equivalence they are 2Σ and the Lawrence prisms



for integers $a \geq b \geq 0$ with $a > 0$.

The respective corresponding X_Δ's are the Veronese surface in $\mathbb{P}^5$ and the rational normal surface scrolls of type (a, b). One thus sees that Theorem 1.3 and Conjecture 1.5 are true if $\Delta^{(1)} = \emptyset$.

3. Bound on the length of the quadratic strand

This section is devoted to a proof of the upper bound stated in Theorem 1.3.

3.1. **Bound through rational normal scrolls.** Let $\Delta \subseteq \mathbb{R}^2$ be a two-dimensional lattice polygon and apply a unimodular transformation in order to have $\Delta \subseteq \mathbb{R} \times [0, d]$ with $d = \operatorname{lw}(\Delta)$. For each $j = 0, \ldots, d$ consider

$$m_j = \min\{a \mid (a, j) \in \Delta \cap \mathbb{Z}^2\} \quad \text{and} \quad M_j = \max\{a \mid (a, j) \in \Delta \cap \mathbb{Z}^2\}.$$

These are well-defined, i.e., on each height j there is at least one lattice point in Δ, see for instance [11, Lem. 5.2]. Recall that X_Δ is the Zariski closure of the image of

$$\varphi_\Delta : (k^*)^2 \hookrightarrow \mathbb{P}^{N_\Delta - 1} : (\alpha, \beta) \mapsto (\alpha^{m_0}\beta^0, \alpha^{m_0+1}\beta^0, \ldots, \alpha^{M_0}\beta^0,$$

$$\alpha^{m_1}\beta^1, \alpha^{m_1+1}\beta^1, \ldots, \alpha^{M_1}\beta^1,$$
$$\vdots$$
$$\alpha^{m_d}\beta^d, \alpha^{m_d+1}\beta^d, \ldots, \alpha^{M_d}\beta^d).$$

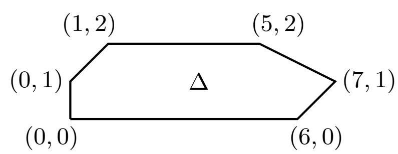
It is clear that this is contained in the Zariski closure of the image of

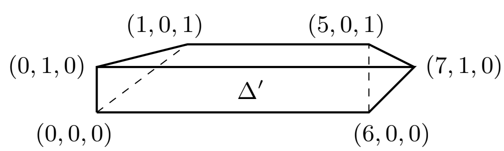
$$\begin{aligned}(k^*)^{1+d} \hookrightarrow \mathbb{P}^{N_\Delta - 1} : (\alpha, \beta_1, \ldots, \beta_d) \mapsto (&\alpha^{m_0}\beta_0, \alpha^{m_0+1}\beta_0, \ldots, \alpha^{M_0}\beta_0,\\ &\alpha^{m_1}\beta_1, \alpha^{m_1+1}\beta_1, \ldots, \alpha^{M_1}\beta_1,\\ &\vdots\\ &\alpha^{m_d}\beta_d, \alpha^{m_d+1}\beta_d, \ldots, \alpha^{M_d}\beta_d)\end{aligned}$$

where $\beta_0 = 1$. This is a $(d+1)$-dimensional rational normal scroll, spanned by rational normal curves of degrees $M_0 - m_0$, $M_1 - m_1$, …, $M_d - m_d$ (some of these degrees may be zero, in which case the 'curve' is actually a point). Its ideal is obtained from I_Δ by restricting to those binomial generators that remain valid if one forgets about the vertical structure of Δ. More precisely, we associate to Δ a lattice polytope $\Delta' \subseteq \mathbb{R}^{d+1}$ by considering for each $(a, b) \in \Delta \cap \mathbb{Z}^2$ the lattice point

$$(a, 0, 0, \ldots, 1, \ldots, 0), \qquad \text{where the 1 is in the } (b+1)\text{st place (omitted if } b = 0),$$

and taking the convex hull. For example:





Then our scroll is just the toric variety $X_{\Delta'}$ associated with Δ'; this is unambiguously defined because Δ' is normal, as is easily seen using [6, Prop. 1.2.2]. We denote its defining ideal viewed inside $I_\Delta \subseteq S_\Delta$ by $I_{\Delta'}$.

As a generalization of (2.8), it is known that a minimal free resolution of the coordinate ring $S_\Delta / I_{\Delta'}$ of a rational normal scroll is given by the Eagon–Northcott complex, from which it follows that the graded Betti table of $X_{\Delta'}$ has the following shape:

	0	1	2	3	…	$f-2$	$f-1$
0	1	0	0	0	…	0	0
1	0	$\binom{f}{2}$	$2\binom{f}{3}$	$3\binom{f}{4}$	…	$(f-2)\binom{f}{f-1}$	$(f-1)\binom{f}{f}$

where $f = \deg X_{\Delta'} = N_{\Delta'} - d - 1 = N_\Delta - d - 1$. Because all syzygies are linear, this must be a summand of the graded Betti table of X_Δ, from which it follows that

$$\min\{\, \ell \mid b_{N_\Delta - \ell} \neq 0 \,\} \leq \operatorname{lw}(\Delta) + 2. \tag{3.1}$$

Remark 3.1. The technique of proving non-vanishing results in Koszul cohomology by embedding the variety of interest in a rational normal scroll, or in a more general determinantal variety, is of course not new. E.g., this yields the 'easy' parts of Green's conjecture and the Green–Lazarsfeld gonality conjecture. We refer to [18, §8B.2] for a discussion.

3.2. **Explicit construction of non-exact cycles.** We can give an alternative proof of (3.1) by explicitly constructing non-zero elements in Koszul cohomology. From a geometric point of view this approach is less enlightening, but it allows us to prove the sharper bound $\min\{\,\ell \,|\, b_{N_\Delta-\ell} \neq 0\,\} \leq \mathrm{lw}(\Delta)+1$ in the cases $\Delta \cong d\Sigma, \Upsilon_d$ ($d \geq 2$) and $\Delta \cong 2\Upsilon$. As we will see, the sharper bound for $d\Sigma$ immediately implies the sharper bound for Υ_d.

For $\ell = 1, \ldots, N_\Delta - 3$ recall that b_ℓ is the cohomology in the middle of

$$\bigwedge^{\ell+1} V_\Delta \xrightarrow{\delta} \bigwedge^{\ell} V_\Delta \otimes V_\Delta \xrightarrow{\delta'} \bigwedge^{\ell-1} V_\Delta \otimes V_{2\Delta}.$$

It is convenient to view this as a subcomplex of

$$\bigwedge^{\ell+1} V_\Delta \otimes V_{\mathbb{Z}^2} \xrightarrow{\delta_{\mathbb{Z}^2}} \bigwedge^{\ell} V_\Delta \otimes V_{\mathbb{Z}^2} \xrightarrow{\delta'_{\mathbb{Z}^2}} \bigwedge^{\ell-1} V_\Delta \otimes V_{\mathbb{Z}^2},$$

where $V_{\mathbb{Z}^2} = k[x^{\pm 1}, y^{\pm 1}]$. In what follows we will abuse notation and describe the basis elements of V_Δ and $V_{\mathbb{Z}^2}$ using the points $(i,j) \in \mathbb{Z}^2$ rather than the monomials $x^i y^j$.

Our technique to construct an element of $\ker \delta' \setminus \operatorname{im} \delta$ will be to apply $\delta_{\mathbb{Z}^2}$ to an element of $\bigwedge^{\ell+1} V_\Delta \otimes V_{\mathbb{Z}^2}$ such that the result is in $\bigwedge^{\ell} V_\Delta \otimes V_\Delta$. This result will then automatically be contained in $\ker \delta'$, but it might land outside $\operatorname{im} \delta$. We first state an easy lemma that will be helpful in proving that certain elements are indeed not contained in $\operatorname{im} \delta$. Fix a strict total order $<$ on $\Delta \cap \mathbb{Z}^2$ and consider the bases

$$B = \{P_1 \wedge \ldots \wedge P_{\ell+1} \,|\, P_1 < \ldots < P_{\ell+1},\, P_1, \ldots, P_{\ell+1} \in \Delta \cap \mathbb{Z}^2\},$$
$$B' = \{P_1 \wedge \ldots \wedge P_{\ell} \otimes P \,|\, P_1 < \ldots < P_{\ell},\, P, P_1, \ldots, P_{\ell} \in \Delta \cap \mathbb{Z}^2\}$$

of $\bigwedge^{\ell+1} V_\Delta$ and $\bigwedge^{\ell} V_\Delta \otimes V_\Delta$, respectively.

Lemma 3.2. *If $x \in \bigwedge^{\ell+1} V_\Delta$ has n non-zero coordinates with respect to B, then $\delta(x)$ has $(\ell+1)n$ non-zero coordinates with respect to B'.*

Proof. Write $x = \sum_{i=1}^n a_i P_{i,1} \wedge \ldots \wedge P_{i,\ell+1}$, $a_i \in k \setminus \{0\}$, where the $P_{i,1} \wedge \ldots \wedge P_{i,\ell+1}$'s are distinct elements of B. Then

$$\delta(x) = \sum_{i=1}^{n} \sum_{j=1}^{\ell+1} (-1)^j a_i P_{i,1} \wedge \ldots \wedge \widehat{P_{i,j}} \wedge \ldots \wedge P_{i,\ell+1} \otimes P_{i,j}$$

Each term in this sum is $\pm a_i$ times an element of B', and the number of terms is $(\ell+1)n$, so we just have to verify that these elements of B' are mutually distinct, but that is easily done. □

Our alternative proof of the upper bound $\min\{\,\ell \,|\, b_{N_\Delta-\ell} \neq 0\,\} \leq \mathrm{lw}(\Delta)+2$ now goes as follows.

Alternative proof of (3.1). As before, we can assume that $\Delta \subseteq \mathbb{R} \times [0,d]$ with $d = \mathrm{lw}(\Delta)$. Let $\ell = N_\Delta - d - 2$ and let $P_1, \ldots, P_{\ell+1}$ be the points $(i,j) \in \Delta$ for which $i > m_j$, indexed so that $P_1 < \ldots < P_{\ell+1}$. Now consider

$$y = \delta_{\mathbb{Z}^2}(P_1 \wedge \ldots \wedge P_{\ell+1} \otimes (-1,0))$$

$$= \sum_{s=1}^{\ell+1} (-1)^s P_1 \wedge \ldots \wedge \widehat{P_s} \wedge \ldots \wedge P_{\ell+1} \otimes (P_s + (-1,0)).$$

Clearly $y \in \bigwedge^\ell V_\Delta \otimes V_\Delta$ and therefore $y \in \ker \delta'$. So it remains to show that $y \notin \operatorname{im} \delta$. Suppose $y = \delta(x)$ for some $x \in \bigwedge^{\ell+1} V_\Delta$. Since y has $\ell+1$ nonzero coordinates with respect to the basis B', by the previous lemma x has just one non-zero coordinate with respect to the basis B. Therefore we can write

$$x = aP'_1 \wedge \ldots \wedge P'_{\ell+1}, \quad a \in k \setminus \{0\}, \quad P'_1 < \ldots < P'_{\ell+1},$$

so that

$$y = \delta(x) = \sum_{s=1}^{\ell+1} a(-1)^s P'_1 \wedge \ldots \wedge \widehat{P'_s} \wedge \ldots \wedge P'_{\ell+1} \otimes P'_s.$$

Comparing both expressions for y, we deduce that $\{P_1, \ldots, P_{\ell+1}\} = \{P'_1, \ldots, P'_{\ell+1}\}$. This gives us a contradiction since the two expressions for y have a different bidegree. Summing up, we have shown that $b_{N_\Delta - d - 2} \neq 0$, from which (3.1) follows. □

The same proof technique enables us to deduce a sharper bound in the exceptional cases $d\Sigma$ ($d \geq 2$) and 2Υ.

Lemma 3.3. *If* $\Delta \cong d\Sigma$ *for some* $d \geq 2$ *then* $\min\{\, \ell \mid b_{N_\Delta - \ell} \neq 0 \,\} \leq \operatorname{lw}(\Delta) + 1$.

Proof. We can of course assume that $\Delta = d\Sigma$. Recall that $N_\Delta = (d+1)(d+2)/2$ and that $\operatorname{lw}(\Delta) = \operatorname{lw}(d\Sigma) = d$. Let $\ell = N_\Delta - d - 1 = d(d+1)/2$. Let $P_1, \ldots, P_\ell$ be the elements of $(d-1)\Sigma \cap \mathbb{Z}^2$ and define

$$\begin{aligned} y &= \delta_{\mathbb{Z}^2}\Big((d-1,1) \wedge P_1 \wedge \ldots \wedge P_\ell) \otimes (1,0) - (d,0) \wedge P_1 \wedge \ldots \wedge P_\ell \otimes (0,1) \Big) \\ &= \sum_{s=1}^{\ell} (-1)^s (d,0) \wedge P_1 \wedge \ldots \wedge \widehat{P_s} \wedge \ldots \wedge P_\ell \otimes (P_s + (0,1)) \\ &\quad - \sum_{s=1}^{\ell} (-1)^s (d-1,1) \wedge P_1 \wedge \ldots \wedge \widehat{P_s} \wedge \ldots \wedge P_\ell \otimes (P_s + (1,0)). \end{aligned}$$

As in the previous proof, since $y \in \bigwedge^\ell V_\Delta \otimes V_\Delta$ we have $y \in \ker \delta'$. The fact that $y \notin \operatorname{im} \delta$ follows from the fact that the number of nonzero coordinates with respect to B' is 2ℓ. If y were in the image, then by our lemma 2ℓ should be divisible by $\ell + 1$, hence $\ell \leq 2$. But $\ell = d(d+1)/2 \geq 3$ because $d \geq 2$: contradiction, and the lemma follows. □

Lemma 3.4. *If* $\Delta \cong 2\Upsilon$ *then* $\min\{\, \ell \mid b_{N_\Delta - \ell} \neq 0 \,\} \leq \operatorname{lw}(\Delta) + 1$.

Proof. Here we can assume $\Delta = 2\Upsilon$ and note that $N_\Delta = 10$ and $\operatorname{lw}(\Delta) = \operatorname{lw}(2\Upsilon) = 4$. With $\ell = N_\Delta - d - 1 = 5$, in exactly the same way as before we see that

$$\begin{aligned} \delta_{\mathbb{Z}^2}\Big(&(1,0) \wedge (0,1) \wedge (0,0) \wedge (-1,-1) \wedge (-1,0) \wedge (0,-1) \otimes (-1,-1) \\ &+ (1,0) \wedge (0,1) \wedge (0,0) \wedge (-1,-1) \wedge (0,-1) \wedge (-2,-2) \otimes (0,1) \\ &- (1,0) \wedge (0,1) \wedge (0,0) \wedge (-1,-1) \wedge (-1,0) \wedge (-2,-2) \otimes (1,0) \Big) \end{aligned}$$

is a non-zero cycle: it has $12 = 2(\ell+1)$ terms, so if it were in $\operatorname{im} \delta$, then any preimage should have two terms, and we leave it to the reader to verify that this again leads to a contradiction. Alternatively, the reader can just look up the graded Betti table of $X_{2\Upsilon}$ in Appendix A. □

Lemma 3.5. *If* $\Delta \cong \Upsilon_d$ *for some* $d \geq 2$ *then* $\min\{\,\ell \,|\, b_{N_\Delta - \ell} \neq 0\,\} \leq \mathrm{lw}(\Delta) + 1$.

Proof. From the combinatorics of Υ_d it is clear that if one restricts to those equations of X_{Υ_d} not involving $X_{-1,-1}$, one obtains a set of defining equations for $X_{d\Sigma}$. Thus the quadratic strand of the graded Betti table of $X_{d\Sigma}$ is a summand of the quadratic strand of the graded Betti table of X_Δ. From Lemma 3.3 we conclude that

$$\min\{\,\ell \,|\, b_{N_\Delta - \ell} \neq 0\,\} \leq \min\{\,\ell \,|\, b_{N_{d\Sigma} - \ell} \neq 0\,\} + 1 \leq \mathrm{lw}(d\Sigma) + 2 = d + 2.$$

The lemma follows from the observation that $\mathrm{lw}(\Delta) = d + 1$. □

Summarizing the results of this section, we conclude that the upper bound stated in Theorem 1.3 indeed applies.

4. Pruning off vertices without changing the lattice width

In this section we provide the theoretical ingredients needed to establish the second bullet point of Theorem 1.3, i.e., to prove that the bound is sharp for lattice polygons of lattice width at most 6.

Theorem 4.1. *Let* Δ *be a two-dimensional lattice polygon and let* $p \geq 1$*. Let* P *be a vertex of* Δ *and define* $\Delta' = \mathrm{conv}(\Delta \cap \mathbb{Z}^2 \setminus \{P\})$*, where we assume that* Δ' *is two-dimensional. If* $K_{p,1}(X_{\Delta'}, L_{\Delta'}) = 0$ *then also* $K_{p+1,1}(X_\Delta, L_\Delta) = 0$.

Proof. Consider

$$\bigwedge^{p+1} V_{\Delta'} \xrightarrow{\delta_1} \bigwedge^{p} V_{\Delta'} \otimes V_{\Delta'} \xrightarrow{\delta_2} \bigwedge^{p-1} V_{\Delta'} \otimes V_{2\Delta'}$$

and

$$\bigwedge^{p+2} V_{\Delta} \xrightarrow{\delta_3} \bigwedge^{p+1} V_{\Delta} \otimes V_{\Delta} \xrightarrow{\delta_4} \bigwedge^{p} V_{\Delta} \otimes V_{2\Delta}$$

where the δ_i's are the usual coboundary maps. Assuming that $\ker \delta_2 = \mathrm{im}\, \delta_1$ we will show that $\ker \delta_4 = \mathrm{im}\, \delta_3$. Suppose the contrary: we will find a contradiction. Let $L : \mathbb{R}^n \to \mathbb{R}$ be a linear form that maps different lattice points in Δ to different numbers, such that P attains the maximum of L on Δ. This exists because P is a vertex. For any $x \in \bigwedge^{p+1} V_\Delta \otimes V_\Delta$ define its support as the convex hull of the set of $P_{j,i}$'s occurring when expanding x in the form

$$x = \sum_i \lambda_i P_{1,i} \wedge \ldots \wedge P_{p+1,i} \otimes Q_i.$$

Here as in Section 3 we take the notational freedom to write points rather than monomials, and of course we do not write any redundant terms. Choose an $x \in \ker \delta_4 \setminus \mathrm{im}\, \delta_3$ such that the maximum that L attains on the support of x is minimal, and let $P' \in \Delta \cap \mathbb{Z}^2$ be the unique point attaining this maximum. Rearrange the above expansion as follows:

$$x = \sum_i \lambda_i P' \wedge P_{1,i} \wedge \ldots \wedge P_{p,i} \otimes Q_i + \text{ terms not containing } P' \text{ in the } \wedge \text{ part} \tag{4.1}$$

where all $P_{j,i}$'s are in Δ' and $Q_i \in \Delta$. We claim that in fact $Q_i \in \Delta'$, i.e. none of the Q_i's equals P. Indeed, otherwise when applying δ_4 the term $-\lambda_i P_{1,i} \wedge \ldots \wedge$

$P_{p,i} \otimes (P' + Q_i)$ of $\delta_4(x)$ has nothing to cancel against, contradicting that $\delta_4(x) = 0$. Let

$$y = \sum_i \lambda_i P_{1,i} \wedge \ldots \wedge P_{p,i} \otimes Q_i \in \bigwedge^p V_{\Delta'} \otimes V_{\Delta'}. \tag{4.2}$$

We have

$$0 = \delta_4(x) = -P' \wedge \delta_2(y) + \text{ terms not containing } P' \text{ in the } \wedge \text{ part.}$$

Because terms of $P' \wedge \delta_2(y)$ cannot cancel against terms without P' in the $\wedge$ part, $\delta_2(y)$ must be zero, and therefore $y \in \operatorname{im} \delta_1$ by the exactness assumption. So write $y = \delta_1(z)$ with

$$z = \sum_i \mu_i P'_{1,i} \wedge \ldots \wedge P'_{p+1,i} \in \bigwedge^{p+1} V_{\Delta'}.$$

Let P'' be the point occurring in this expression such that $L(P'')$ is maximal. Since there is no cancellation when applying δ_1 one sees that P'' is in the support of y, hence in the support of x and therefore $L(P'') < L(P')$. This means that L achieves a smaller maximum on the support of z than on the support of x. Finally, let

$$x' = x + \delta_3(P' \wedge z) = x - P' \wedge y - z \otimes P'.$$

Since $x \in \ker \delta_4 \setminus \operatorname{im} \delta_3$ we have $x' \in \ker \delta_4 \setminus \operatorname{im} \delta_3$ and by (4.1) and (4.2) one concludes that L will achieve a smaller maximum on the support of x' than on the support of x. This contradicts the choice of x. □

The theorem immediately implies the second bullet point of Theorem 1.3, except for the claim that sharpness holds for all polygons Δ with $\operatorname{lw}(\Delta) \leq 6$. In order to settle this claim, we note the following.

Lemma 4.2. *Let Δ be a two-dimensional lattice polygon, let $d = \operatorname{lw}(\Delta)$, and assume that removing an extremal lattice point makes the lattice width decrease, i.e. for every vertex $P \in \Delta$ it holds that*

$$\operatorname{lw}(\operatorname{conv}(\Delta \cap \mathbb{Z}^2 \setminus \{P\})) < d.$$

Then there exists a unimodular transformation mapping Δ into $[0, d] \times [0, d]$.

Proof. The cases where $\Delta^{(1)} \cong \emptyset$ or where $\Delta^{(1)} \cong d\Sigma$ for some $d \geq 0$ are easy to verify. In the other cases $\operatorname{lw}(\Delta^{(1)}) = \operatorname{lw}(\Delta) - 2 = d - 2$ and the lattice width directions for Δ and $\Delta^{(1)}$ are the same [38, Thm. 13]. Assume that $\Delta \subseteq \mathbb{R} \times [0, d]$, fix a vertex on height 0 and a vertex on height d, and let P be any other vertex. Then $\operatorname{lw}(\operatorname{conv}(\Delta \cap \mathbb{Z}^2 \setminus \{P\})) \leq d - 1$, where we note that a corresponding lattice width direction is necessarily non-horizontal, and that along such a direction the width of $\Delta^{(1)}$ is at most $d - 2$. But then equality must hold, and in particular it must also concern a lattice width direction for $\Delta^{(1)}$, hence it must concern a lattice width direction for Δ. We conclude that Δ has two independent lattice width directions, and the lemma follows from the remark following [11, Lem. 5.2]. □

Let us call a lattice polygon Δ as in the statement of the foregoing lemma 'minimal', and note that this attribute applies to each of the exceptional polygons $d\Sigma, \Upsilon_d, 2\Upsilon$. In order to prove sharpness for a certain non-exceptional polygon Δ it suffices to do this for any lattice polygon obtained by repeatedly pruning off vertices without changing the lattice width. Thus the proof reduces to verifying the case of a minimal lattice polygon, unless it concerns one of the exceptional cases

$d\Sigma, \Upsilon_d, 2\Upsilon$, in which case one needs to stop pruning one step earlier (otherwise this strategy has no chance of being successful).

In other words the above lemma implies that if sharpness applies to all lattice polygons Δ for which $N_\Delta \leq (d+1)^2 + 1$, then it applies to all lattice polygons Δ with $\mathrm{lw}(\Delta) \leq d$. This observation, along with our exhaustive verification in the cases where $N_\Delta \leq 32$, reported upon in Section 7, allows us to conclude that sharpness holds as soon as $\mathrm{lw}(\Delta) \leq 4$. This fact will be used in the proof of our explicit formula for $b_{N_\Delta - 4}$. But one can do better: in a spin-off paper [13] devoted to minimal polygons, the second and the fourth listed author show that if Δ is a minimal lattice polygon with $\mathrm{lw}(\Delta) \leq d$ then

$$N_\Delta \leq \max\left\{(d-1)^2 + 4, (d+1)(d+2)/2\right\}.$$

From this, using a similar reasoning, the conjecture follows for $\mathrm{lw}(\Delta) \leq 6$, as announced in the statement of Theorem 1.3.

5. Explicit formula for some entries of the graded Betti table

We are now ready to prove Theorem 1.1, thereby giving explicit formulas for eight entries of the graded Betti table of X_Δ. Six of these entries are rather straightforward. Indeed, the formulas for b_1 and c_1 follow immediately from (1.2), where in the latter case we use that $2\,\mathrm{vol}(\Delta) - N_\Delta + 2 = N_{\Delta^{(1)}}$ by Pick's theorem. The entry $c_{N_\Delta - 3}$ equals the number of cubics in a minimal set of generators of I_Δ, which was determined in [10, §2]. Together with (1.2) this then gives the formula for b_2. The formula for $b_{N_\Delta - 3}$ was discussed in Section 2.4, and the formula for c_2 again follows using (1.2) in combination with Pick's theorem.

Thus we are left with proving the formulas for c_3 and $b_{N_\Delta - 4}$, which is a considerably more difficult task. We will focus on $b_{N_\Delta - 4}$, the formula for c_3 then again follows using (1.2). Recall that the statement distinguishes between the following four cases:

$$\begin{cases} \Delta^{(1)} = \emptyset, \\ \dim \Delta^{(1)} = 0, \\ \dim \Delta^{(1)} = 1 \text{ or } \Delta \cong \Upsilon_2, \\ \dim \Delta^{(1)} = 2 \text{ and } \Delta \ncong \Upsilon_2. \end{cases}$$

We will treat these cases in the above order. The first case where $\Delta^{(1)} = \emptyset$ follows trivially from (2.8), so we can skip it. Now recall from (2.4) that $b_{N_\Delta - 4}$ is the dimension of the cohomology in the middle of

$$\bigwedge^2 V_\Delta \otimes V_{\Delta^{(1)}} \xrightarrow{\delta} V_\Delta \otimes V_{(2\Delta)^{(1)}} \xrightarrow{\delta'} V_{(3\Delta)^{(1)}}.$$

Because $K_{0,3}(X; K, L) \cong K_{N_\Delta - 3, 0}(X, L) = 0$, where we use that $\Delta \ncong \Sigma$, we have that the map δ' is surjective. In particular we obtain the formula

$$b_{N_\Delta - 4} = \dim \operatorname{coker} \delta - |(3\Delta)^{(1)} \cap \mathbb{Z}^2|.$$

Case $\dim \Delta^{(1)} = 0$. If $\dim \Delta^{(1)} = 0$ then δ is injective, so

$$b_{N_\Delta - 4} = \dim(V_\Delta \otimes V_{(2\Delta)^{(1)}}) - \dim(\bigwedge^2 V_\Delta) - |(3\Delta)^{(1)} \cap \mathbb{Z}^2| = (N_\Delta - 4)(N_\Delta - 1)/2,$$

as can be calculated using Pick's theorem, thereby yielding Theorem 1.1 in this case (alternatively, one can give an exhaustive proof by explicitly computing the graded Betti tables of the toric surfaces associated with the 16 reflexive lattice polygons).

Case $\dim \Delta^{(1)} = 1$ *or* $\Delta \cong \Upsilon_2$. The graded Betti table of X_{Υ_2} can be found in Appendix A, where one verifies that $b_{N_{\Upsilon_2}-4} = b_3 = 3$, as indeed predicted by the statement of Theorem 1.1. Therefore we can assume that $\dim \Delta^{(1)} = 1$. The polygons Δ having a one-dimensional interior were explicitly classified by Koelman [31, §4.3], but in any case it is easy to see that, using a unimodular transformation if needed, we can assume that

$$\Delta = \mathrm{conv}\{(m_1,1),(M_1,1),(m_0,0),(M_0,0),(m_{-1},-1),(M_{-1},-1)\}$$

for some $m_i \leq M_i \in \mathbb{Z}$. Here $m_0 < M_0$ can be taken such that

$$\Delta \cap (\mathbb{Z} \times \{0\}) = \{m_0, m_0+1, \ldots, M_0\} \times \{0\}.$$

Write $\Delta^{(1)} = [u,v] \times \{0\}$, then

$$(2\Delta)^{(1)} = \Delta + \Delta^{(1)} = \mathrm{conv}\{(m_i+u,i),(M_i+v,i) \,|\, i = 1,0,-1\}.$$

Now consider $V_{\mathbb{Z}} = k[x^{\pm 1}]$ and define a morphism

$$f : V_\Delta \otimes V_{(2\Delta)^{(1)}} \to k[x_{-1},x_0,x_1] \otimes V_{\mathbb{Z}}$$

by letting $(a,b) \otimes (c,d) \mapsto x_b x_d \otimes (a+c)$, where again we abusingly describe the basis elements of V_Δ, $V_{(2\Delta)^{(1)}}$ and $V_{\mathbb{Z}}$ using lattice points rather than monomials. Note that

$$f(\delta((a,b) \wedge (c,d) \otimes (e,0))) = f((a,b) \otimes (c+e,d) - (c,d) \otimes (a+e,b)) = 0,$$

so $\mathrm{im}\,\delta \subseteq \ker f$.

We claim that actually equality holds. First note that every element $\alpha \in \ker f$ decomposes into elements

$$\sum_j \lambda_j (a_j,b_j) \otimes (c_j,d_j)$$

for which $(\{b_j,d_j\}, a_j+c_j)$ is the same for all j: indeed, terms for which these are different cannot cancel out when applying f. Note that $\sum_j \lambda_j = 0$, so one can rewrite the above as a linear combination of expressions either of the form

$$\underbrace{(a,b) \otimes (c,d) - (a',b) \otimes (c',d)}_{\text{(i)}} \quad \text{or of the form} \quad \underbrace{(a,b) \otimes (c,d) - (a',d) \otimes (c',b)}_{\text{(ii)}}$$

where $a+c = a'+c'$, the points $(a,b),(a',b)$ resp. $(a,b),(a',d)$ are in Δ, and the points $(c,d),(c',d)$ resp. $(c,d),(c',b)$ are in $(2\Delta)^{(1)}$. As for case (i), these can be decomposed further as a sum (or minus a sum) of expressions of the form $(a,b) \otimes (c,d) - (a+1,b) \otimes (c-1,d)$, which can be rewritten as

$$\delta((a,b) \wedge (c-e,d) \otimes (e,0) - (a+1,b) \wedge (c-e,d) \otimes (e-1,0))$$

and therefore as an element of $\mathrm{im}\,\delta$, at least if e can be chosen in the interval $[\max(u+1, c-M_d), \min(v, c-m_d)]$. The reader can verify that this is indeed non-empty, from which the claim follows in this case. As for (ii), with e chosen from the non-empty interval $[\max(u, c'-M_b), \min(v, c'-m_b)]$ one verifies that

$$\delta((c'-e,b) \wedge (a',d) \otimes (e,0)) = (c'-e,b) \otimes (a'+e,d) - (a',d) \otimes (c',b),$$

allowing one to replace (ii) with an expression of type (i), and the claim again follows.

Summing up, we have

$$b_{N_\Delta - 4} = \dim \mathrm{im}\, f - |(3\Delta)^{(1)} \cap \mathbb{Z}^2|$$

$$= \sum_{\{i,j\} \subseteq \{-1,0,1\}} |[m_i + m_j + u, M_i + M_j + v] \cap \mathbb{Z}| - \sum_{i'=-2}^{2} \left| (3\Delta)^{(1)} \cap (\mathbb{Z} \times \{i'\}) \right| .$$

Each lattice point of $(3\Delta)^{(1)} = 2\Delta + \Delta^{(1)}$ appears in an interval on the left, and conversely. To see this it suffices to note that each lattice point of 2Δ arises as the sum of two lattice points in Δ, which is a well-known property [27]. So all terms with $i + j \neq 0$ cancel out the terms with $i' \neq 0$, and we are left with

$$|[m_1 + m_{-1} + u, M_1 + M_{-1} + v] \cap \mathbb{Z}| + |[2m_0 + u, 2M_0 + v] \cap \mathbb{Z}| \\ - \left| (3\Delta)^{(1)} \cap (\mathbb{Z} \times \{0\}) \right| .$$

Term by term this equals

$$\left(|\partial\Delta \cap \mathbb{Z}^2| + N_{\Delta^{(1)}} - 2 - \varepsilon \right) + (2(M_0 - m_0) + N_{\Delta^{(1)}}) \\ - (2(M_0 - m_0) + (2 - \varepsilon) + N_{\Delta^{(1)}})$$

where $\varepsilon := (u - m_0) + (M_0 - v) \in \{0, 1, 2\}$ denotes the cardinality of $\partial\Delta \cap (\mathbb{Z} \times \{0\})$. Because the above expression simplifies to $N_\Delta - 4$, this concludes the proof in the $\dim \Delta^{(1)} = 1$ case.

Case $\dim \Delta^{(1)} = 2$ *and* $\Delta \ncong \Upsilon_2$. Here our task amounts to proving that $b_{N_\Delta - 4} = 0$. Note that our assumptions together with (2.1) imply that $\mathrm{lw}(\Delta) \geq 3$ and even that $\mathrm{lw}(\Delta) \geq 4$ in the exceptional cases $\Delta \cong d\Sigma, \Upsilon_d, 2\Upsilon$. Therefore Theorem 1.1 arises as a consequence of Conjecture 1.5 in this case. Now recall from Section 4 that we verified Conjecture 1.5 for all polygons in the range $\mathrm{lw}(\Delta) \leq 4$. Thus we can assume that $\mathrm{lw}(\Delta) \geq 5$. But now we can reduce back to the case $\mathrm{lw}(\Delta) = 4$ by gradually removing vertices and each time applying Theorem 4.1. This works because removing a vertex reduces the lattice width by steps of at most 1, which is an easy consequence of [38, Thm. 13].

6. Quotienting the Koszul complex

We now start working towards an algorithmic determination of the graded Betti table of the toric surface $X_\Delta \subseteq \mathbb{P}^{N_\Delta - 1}$ associated with a given two-dimensional lattice polygon Δ. Essentially, the method is about reducing the dimensions of the vector spaces involved, in order to make the linear algebra more manageable. This is mainly done by incorporating bigrading and duality. However, when dealing with large polygons a further reduction is useful. In this section we show that the Koszul complex always admits certain exact subcomplexes that can be described in a combinatorial way. Quotienting out such a subcomplex does not affect the cohomology, while making the linear algebra easier, at least in theory. For reasons we don't understand our practical implementation shows that the actual gain in runtime is somewhat unpredictable: sometimes it is helpful, but other times the contrary is true. But it is worth the try, and in any case we believe that the material below is also interesting from a theoretical point of view.

We first introduce the subcomplex from an algebraic point of view, then reinterpret things combinatorially, and finally specify our discussion to the case of the Veronese surfaces $X_{d\Sigma}$. In the latter setting the idea of quotienting out such an exact subcomplex is not new: for instance it appears in the recent paper by Ein, Erman and Lazarsfeld [15, p. 2].

6.1. **An exact subcomplex.** We begin with the following lemma, which should be known to specialists, but we include a proof for the reader's convenience.

Lemma 6.1. *Let M be a graded module over $k[x_1,\dots,x_N]$ and suppose that the multiplication-by-x_N map $M \to M$ is an injection. Then the Koszul complexes*

$$\dots \to \bigwedge^{p+1} V \otimes M \to \bigwedge^{p} V \otimes M \to \bigwedge^{p-1} V \otimes M \to \dots$$

and

$$\dots \to \bigwedge^{p+1} W \otimes M/(x_N M) \to \bigwedge^{p} W \otimes M/(x_N M) \to \bigwedge^{p-1} W \otimes M/(x_N M) \to \dots$$

have the same graded cohomology. Here V and W denote the degree one parts of the polynomial rings $k[x_1,\dots,x_N]$ and $k[x_1,\dots,x_{N-1}]$, respectively.

Proof. Denote by M' the graded module $M/(x_N M)$. For every $p \geq 0$ we have a short exact sequence

$$0 \longrightarrow \left(\bigwedge^{p} W \otimes M\right) \oplus \left(\bigwedge^{p-1} W \otimes M\right) \xrightarrow{\alpha} \bigwedge^{p} V \otimes M \xrightarrow{\beta} \bigwedge^{p} W \otimes M' \longrightarrow 0,$$

by letting

$$\begin{aligned}&\alpha\,(\,v_1 \wedge \dots \wedge v_p \otimes m\,,\, w_1 \wedge \dots \wedge w_{p-1} \otimes m'\,)\\ &\qquad\qquad = v_1 \wedge \dots \wedge v_p \otimes x_N m \,+\, x_N \wedge w_1 \wedge \dots \wedge w_{p-1} \otimes m'\end{aligned}$$

and $\beta(v_1 \wedge \dots \wedge v_p \otimes m) = \pi(v_1) \wedge \dots \wedge \pi(v_p) \otimes \overline{m}$, where $\pi : V \to W$ maps x_i to itself if $i \neq N$ and to zero otherwise, and $\overline{m}$ denotes the residue class of m modulo $x_N M$. As usual if $p = 0$ then it is understood that $\bigwedge^{p-1} W \otimes M = 0$. We leave a verification of the exactness to the reader, but note that the injectivity of the multiplication-by-x_N map is important here.

On the other hand the spaces

$$C_p = \left(\bigwedge^{p} W \otimes M\right) \oplus \left(\bigwedge^{p-1} W \otimes M\right)$$

naturally form a long exact sequence $\dots \to C_2 \to C_1 \to C_0 \to 0$ along the morphisms

$$d_p : C_p \to C_{p-1} : (a,b) \mapsto (-b + \delta_p(a), -\delta_{p-1}(b))$$

where δ_p and δ_{p-1} are the usual coboundary maps, as described in (2.2). Exactness holds because if $d_p(a,b) = 0$ then $d_{p+1}(0,-a) = (a,b)$. Overall we end up with a short exact sequence of complexes:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bigwedge^{p+1} W \otimes M \oplus \bigwedge^{p} W \otimes M & \to & \bigwedge^{p+1} V \otimes M & \to & \bigwedge^{p+1} W \otimes M' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bigwedge^{p} W \otimes M \oplus \bigwedge^{p-1} W \otimes M & \to & \bigwedge^{p} V \otimes M & \to & \bigwedge^{p} W \otimes M' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

This gives a long exact sequence in (co)homology, and the result follows from the exactness of the left column. ☐

Now we explain how to exploit the above lemma for our purposes. We can apply it to the Koszul complex

$$\ldots \to \bigwedge^{p+1} V_\Delta \otimes \bigoplus_{i\geq 0} V_{i\Delta} \to \bigwedge^{p} V_\Delta \otimes \bigoplus_{i\geq 0} V_{i\Delta} \to \bigwedge^{p-1} V_\Delta \otimes \bigoplus_{i\geq 0} V_{i\Delta} \to \ldots$$

as well as to the twisted Koszul complex

$$\ldots \to \bigwedge^{p+1} V_\Delta \otimes \bigoplus_{i\geq 1} V_{(i\Delta)^{(1)}} \to \bigwedge^{p} V_\Delta \otimes \bigoplus_{i\geq 1} V_{(i\Delta)^{(1)}} \to \bigwedge^{p-1} V_\Delta \otimes \bigoplus_{i\geq 1} V_{(i\Delta)^{(1)}} \to \ldots$$

These are complexes of graded modules over the polynomial ring whose variables correspond to the lattice points of Δ. In both cases the variable corresponding to whatever point $P \in \Delta \cap \mathbb{Z}^2$ can be chosen as x_N, because multiplication by x_N will always be injective. Then the lemma yields that we can replace $V_{i\Delta}$ by $V_{(i\Delta)\setminus((i-1)\Delta+P)}$ in the first complex, and that we can replace $V_{(i\Delta)^{(1)}}$ by $V_{(i\Delta)^{(1)}\setminus(((i-1)\Delta)^{(1)}+P)}$ in the second complex. In both cases we must also replace the V_Δ's in the wedge product by $V_{\Delta\setminus\{P\}}$. Splitting these complexes into their graded pieces we conclude that $K_{p,q}(X,L)$ can be computed as the cohomology in the middle of

$$\bigwedge^{p+1} V_{\Delta\setminus\{P\}} \otimes V_{((q-1)\Delta)\setminus((q-2)\Delta+P)} \longrightarrow \bigwedge^{p} V_{\Delta\setminus\{P\}} \otimes V_{(q\Delta)\setminus((q-1)\Delta+P)} \longrightarrow \bigwedge^{p-1} V_{\Delta\setminus\{P\}} \otimes V_{((q+1)\Delta)\setminus(q\Delta+P)},$$

and that the twisted Koszul cohomology spaces $K_{p,q}(X;K,L)$ can be computed as the cohomology in the middle of

$$\bigwedge^{p+1} V_{\Delta\setminus\{P\}} \otimes V_{((q-1)\Delta)^{(1)}\setminus((q-2)\Delta+P)^{(1)}} \longrightarrow \bigwedge^{p} V_{\Delta\setminus\{P\}} \otimes V_{(q\Delta)^{(1)}\setminus((q-1)\Delta+P)^{(1)}} \longrightarrow \bigwedge^{p-1} V_{\Delta\setminus\{P\}} \otimes V_{((q+1)\Delta)^{(1)}\setminus(q\Delta+P)^{(1)}}.$$

Here for any $A \subseteq \mathbb{Z}^2$ we let $V_A \subseteq k[x^{\pm 1}, y^{\pm 1}]$ denote the space of Laurent polynomials whose support is contained in A.

Remark 6.2. The coboundary morphisms are still defined as in (2.2), with the additional rule that $x^i y^j$ is considered zero in V_A as soon as $(i,j) \notin A$.

Remark 6.3. It is important to observe that the above complexes remain naturally bigraded, and that this is compatible with the bigrading described in Section 2.2. In other words, for any $(a,b) \in \mathbb{Z}^2$, also the spaces $K_{p,q}^{(a,b)}(X,L)$ and $K_{p,q}^{(a,b)}(X;K,L)$ can be computed from the above sequences.

6.2. **Removing multiple points.** In some cases we can remove multiple points from Δ by applying Lemma 6.1 repeatedly. In algebraic terms this works if and only if these points, when viewed as elements of V_Δ, form a regular sequence for the graded module M, where M is either $\bigoplus_{i\geq 0} V_{i\Delta}$ or $\bigoplus_{i\geq 1} V_{(i\Delta)^{(1)}}$. The length of a regular sequence is bounded by the Krull dimension of M, which is equal to 3. So we can never remove more than three points. It is well-known that for graded modules over Noetherian rings any permutation of a regular sequence is again a regular sequence, so the order of removing points does not matter. Concretely, after removing the points $P_1, \ldots, P_m$ we get the complex

$$\ldots \longrightarrow \bigwedge^{p+1} V_{\Delta\setminus\{P_1,\ldots,P_m\}} \otimes \frac{M_{q-1}}{P_1 M_{q-2} + \ldots + P_m M_{q-2}}$$

$$\longrightarrow \bigwedge^p V_{\Delta \setminus \{P_1,\ldots,P_m\}} \otimes \frac{M_q}{P_1 M_{q-1} + \ldots + P_m M_{q-1}} \longrightarrow \ldots$$

where M_i denotes the degree i part of M. Here, as before, we abuse notation and identify the points $P_i \in \Delta$ with the corresponding monomials in V_Δ. So for $M = \bigoplus_{i \geq 0} V_{i\Delta}$ this gives

$$\ldots \longrightarrow \bigwedge^{p+1} V_{\Delta \setminus \{P_1,\ldots,P_m\}} \otimes V_{(q-1)\Delta \setminus ((P_1+(q-2)\Delta) \cup \ldots \cup (P_m+(q-2)\Delta))}$$
$$\longrightarrow \bigwedge^p V_{\Delta \setminus \{P_1,\ldots,P_m\}} \otimes V_{q\Delta \setminus ((P_1+(q-1)\Delta) \cup \ldots \cup (P_m+(q-1)\Delta))} \longrightarrow \ldots$$

while for $M = \bigoplus_{i \geq 1} V_{(i\Delta)^{(1)}}$ it gives

$$\ldots \longrightarrow \bigwedge^{p+1} V_{\Delta \setminus \{P_1,\ldots,P_m\}} \otimes V_{((q-1)\Delta)^{(1)} \setminus ((P_1+((q-2)\Delta)^{(1)}) \cup \ldots \cup (P_m+((q-2)\Delta)^{(1)}))}$$
$$\longrightarrow \bigwedge^p V_{\Delta \setminus \{P_1,\ldots,P_m\}} \otimes V_{q\Delta \setminus ((P_1+((q-1)\Delta)^{(1)}) \cup \ldots \cup (P_m+((q-1)\Delta)^{(1)}))} \longrightarrow \ldots$$

The question we study in this section is which sequences of points $P_1, \ldots, P_m \in \Delta \cap \mathbb{Z}^2$ are regular, where necessarily $m \leq 3$.

We first study the problem of which sequences of two points are regular. As for $M = \bigoplus_{i \geq 0} V_{i\Delta}$, if we first remove a point $P \in \Delta \cap \mathbb{Z}^2$ then we end up with M/PM, whose graded components in degree $q \geq 1$ are of the form $V_{q\Delta \setminus (P+(q-1)\Delta)}$, while the degree 0 part is just $V_{0\Delta}$. Multiplication by another point $Q \in \Delta \cap \mathbb{Z}^2$ in M/PM corresponds to

$$V_{q\Delta \setminus (P+(q-1)\Delta)} \xrightarrow{\cdot Q} V_{(q+1)\Delta \setminus (P+q\Delta)}.$$

In order for the sequence P, Q to be regular this map has to be injective for all $q \geq 1$. This means that

$$((q\Delta \setminus (P + (q-1)\Delta)) + Q) \cap (P + q\Delta) \cap \mathbb{Z}^2 = \emptyset.$$

Subtracting $P + Q$ yields

$$(q\Delta - P) \setminus ((q-1)\Delta) \cap (q\Delta - Q) \cap \mathbb{Z}^2 = \emptyset,$$

eventually leading to the criterion

$$\text{(6.1)} \quad P, Q \text{ is regular for } \bigoplus_{i \geq 0} V_{i\Delta} \quad \Leftrightarrow$$
$$\forall q \geq 1 : (q\Delta - P) \cap (q\Delta - Q) \cap \mathbb{Z}^2 \subseteq (q-1)\Delta.$$

Similarly we find

$$P, Q \text{ is regular for } \bigoplus_{i \geq 1} V_{(i\Delta)^{(1)}} \quad \Leftrightarrow$$
$$\forall q \geq 1 : (q\Delta - P)^{(1)} \cap (q\Delta - Q)^{(1)} \cap \mathbb{Z}^2 \subseteq ((q-1)\Delta)^{(1)}.$$

These criteria are strongly simplified by the equivalences *1.* $\iff$ *2.* $\iff$ *9.* of the following theorem:

Theorem 6.4. *Let Δ be a two-dimensional lattice polygon. For two distinct lattice points $P, Q \in \Delta$, the following are equivalent:*

(1) *P, Q is a regular sequence for $\bigoplus_{i \geq 0} V_{i\Delta}$.*
(2) *P, Q is a regular sequence for $\bigoplus_{i \geq 1} V_{(i\Delta)^{(1)}}$.*
(3) *$(q\Delta - P) \cap (q\Delta - Q) \subseteq (q-1)\Delta$ for some $q > 1$.*

(4) $(q\Delta - P) \cap (q\Delta - Q) \subseteq (q-1)\Delta$ *for all* $q \geq 1$.
(5) $((q\Delta)^\circ - P) \cap ((q\Delta)^\circ - Q) \subseteq ((q-1)\Delta)^\circ$ *for all* $q \geq 1$, *where* $^\circ$ *denotes the interior for the standard topology on* $\mathbb{R}^2$.
(6) $((q\Delta)^{(1)} - P) \cap ((q\Delta)^{(1)} - Q) \cap \mathbb{Z}^2 \subseteq ((q-1)\Delta)^{(1)} \cap \mathbb{Z}^2$ *for all* $q \geq 1$.
(7) $(q\Delta - P) \cap (q\Delta - Q) \cap \mathbb{Z}^2 \subseteq (q-1)\Delta \cap \mathbb{Z}^2$ *for all* $q \geq 1$.
(8) *Let* ℓ *be the line through* P *and* Q. *For both half-planes* H *bordered by* ℓ, *the polygon* $H \cap \Delta$ *is a triangle with* P *and* Q *as two vertices (this may be degenerate, in which case it is the line segment* PQ*).*
(9) Δ *is a quadrangle and* P *and* Q *are opposite vertices of this quadrangle (this may be the degenerate case where* Δ *is a triangle and* P, Q *are any pair of vertices of* Δ*).*

Proof. The equivalences *1.* $\iff$ *7.* and *2.* $\iff$ *6.* follow from the foregoing discussion.
3. $\implies$ *4.*: assume that *3.* holds for some $q > 1$. Let $q' \geq 1$, we show that it also holds for q'. Let $W \in (q'\Delta - P) \cap (q'\Delta - Q)$, we need to show that $W \in (q'-1)\Delta$.

In case $q' > q$, we define $\delta = (q-1)/(q'-1) < 1$. Now consider

$$\begin{aligned} W &\in ((q'-1)\Delta + (\Delta - P)) \cap ((q'-1)\Delta + (\Delta - Q)) \\ \delta W &\in ((q-1)\Delta + \delta(\Delta - P)) \cap ((q-1)\Delta + \delta(\Delta - Q)) \\ &\subseteq ((q-1)\Delta + (\Delta - P)) \cap ((q-1)\Delta + (\Delta - Q)) \\ &= (q\Delta - P) \cap (q\Delta - Q) \subseteq (q-1)\Delta. \end{aligned}$$

We conclude that $W \in (q'-1)\Delta$.

If $q' < q$, we find

$$\begin{aligned} W + (q-q')\Delta &\subseteq [(q'\Delta - P) \cap (q'\Delta - Q)] + (q-q')\Delta \\ &\subseteq (q'\Delta - P + (q-q')\Delta) \cap (q'\Delta - Q + (q-q')\Delta) \\ &\subseteq (q\Delta - P) \cap (q\Delta - Q) \subseteq (q-1)\Delta. \end{aligned}$$

Since $W + (q-q')\Delta \subseteq (q-1)\Delta$, it follows that $W \in (q'-1)\Delta$.

4. $\implies$ *5.*: this holds by taking interiors on both sides and using the fact that $(A \cap B)^\circ = A^\circ \cap B^\circ$.

5. $\implies$ *6.*: intersect with $\mathbb{Z}^2$ on both sides and use $\Delta^\circ \cap \mathbb{Z}^2 = \Delta^{(1)} \cap \mathbb{Z}^2$.

6. $\implies$ *7.*: let $W \in (q\Delta - P) \cap (q\Delta - Q) \cap \mathbb{Z}^2$.

$$\begin{aligned} W + \left((3\Delta)^{(1)} \cap \mathbb{Z}^2\right) &= \left(W + (3\Delta)^{(1)}\right) \cap \mathbb{Z}^2 \\ &\subseteq \left[q\Delta + (3\Delta)^{(1)} - P\right] \cap \left[q\Delta + (3\Delta)^{(1)} - Q\right] \cap \mathbb{Z}^2 \\ &\subseteq \left[((q+3)\Delta)^{(1)} - P\right] \cap \left[((q+3)\Delta)^{(1)} - Q\right] \cap \mathbb{Z}^2 \\ &\subseteq ((q+2)\Delta)^{(1)} \cap \mathbb{Z}^2. \end{aligned}$$

Since $(3\Delta)^{(1)}$ must contain a lattice point, it follows that $W \in (q-1)\Delta \cap \mathbb{Z}^2$.

7. $\implies$ *8.*: we show this by contraposition, so we assume that item *8.* is not satisfied for a half-plane H.

Let T a vertex of $H \cap \Delta$ at maximal distance from ℓ, and assume for now that this distance is positive. Let R be a vertex of $H \cap \Delta$, distinct from P, Q and T (the fact that such an R exists follows from the assumption). Without loss of generality,

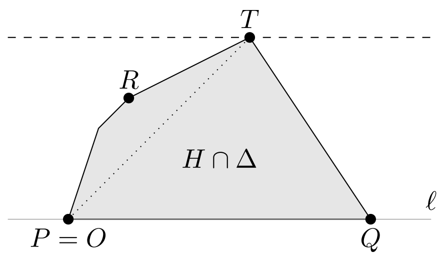


FIGURE 2. 7. $\Longrightarrow$ 8.

FIGURE 3. degenerate case (where T may be equal to Q)

we may assume that R lies in the half-plane bordered by the line PT that does not contain Q. Choose coordinates such that the origin is P.

Equip R with barycentric coordinates

$$R = \alpha T + \beta Q + \gamma P = \alpha T + \beta Q. \tag{6.2}$$

Because of the position of R, we know that $0 \leq \alpha \leq 1$ and $\beta < 0$.

Choose an integer $q > \max\{1, -\beta^{-1}\}$. Let $W = qR$. We claim that

$$W \in \big((q\Delta) \cap (q\Delta - Q) \cap \mathbb{Z}^2\big) \setminus (q-1)\Delta,$$

contradicting 7. Since R is a vertex of $H \cap \Delta$, we immediately have $W \in \big((q\Delta) \cap \mathbb{Z}^2\big) \setminus (q-1)\Delta$. It remains to show that $W \in q\Delta - Q$. Using (6.2), we have

$$\begin{aligned} W + Q = qR + Q &= qR + \beta^{-1}(R - \alpha T) \\ &= (q + \beta^{-1})R + (-\beta^{-1}\alpha)T \end{aligned}$$

This is a convex combination of $qP = O$, qR and qT because

$$q + \beta^{-1} \geq 0, \qquad -\beta^{-1}\alpha \geq 0,$$

and

$$(q + \beta^{-1}) + (-\beta^{-1}\alpha) = q + \beta^{-1}(1 - \alpha) \leq q.$$

It follows that $W + Q \in q\Delta$.

In the degenerate case where $T \in \ell$, without loss of generality one can assume that there is a vertex R such that R and Q lie on opposite sides of $P = O$. One proceeds as above with $\alpha = 0$ and $\beta < 0$.

8. $\Longrightarrow$ *9.*: this follows immediately from the geometry: Δ must be the union of two triangles on the base PQ.

9. $\Longrightarrow$ *3.*: we show this for $q = 2$. By assumption, the lattice polygon Δ is a convex quadrangle $PRQS$ (possibly degenerated into a triangle, i.e. one of R or S may coincide with P or Q). We need to show that

$$(2\Delta - P) \cap (2\Delta - Q) \subseteq \Delta$$

The left hand side is clearly contained in the cones $\widehat{RPS}$ and $\widehat{RQS}$, whose intersection is precisely our quadrangle $PRQS = \Delta$. $\square$

Now let us switch to regular sequences consisting of three points. We have the following easy fact:

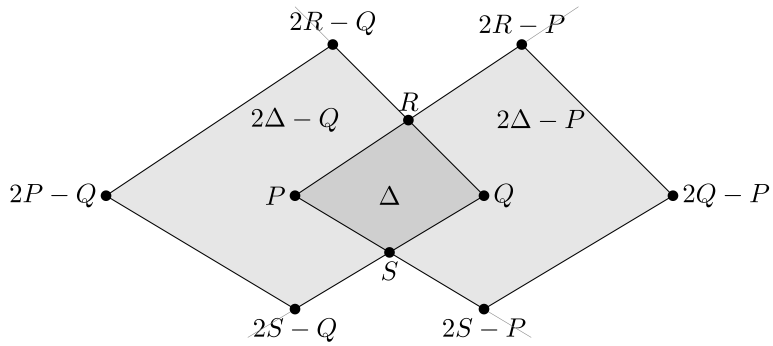


FIGURE 4. *9.* $\Longrightarrow$ *3.* with $q = 2$

Lemma 6.5. *Let $P, Q, R \in \Delta \cap \mathbb{Z}^2$ be distinct. Then P, Q, R is a regular sequence for $M = \bigoplus_{i \geq 0} V_{i\Delta}$ (resp. $M = \bigoplus_{i \geq 1} V_{(i\Delta)^{(1)}}$) if and only if*

$$P, Q, \qquad Q, R, \qquad P, R$$

are regular sequences.

Proof. It is clearly sufficient to prove the 'if' part of the claim. Assume for simplicity that $M = \bigoplus_{i \geq 0} V_{i\Delta}$, the other case is similar. Since P, Q is regular, all we have to check is that

$$V_{q\Delta \setminus ((P+(q-1)\Delta) \cup (Q+(q-1)\Delta))} \xrightarrow{\cdot R} V_{(q+1)\Delta \setminus ((P+q\Delta) \cup (Q+q\Delta))}$$

is injective, or equivalently that

$$(q\Delta \setminus ((P + (q-1)\Delta) \cup (Q + (q-1)\Delta)) + R) \cap ((P + q\Delta) \cup (Q + q\Delta)) = \emptyset.$$

This condition can be rewritten as

$$q\Delta \cap ((q\Delta + P - R) \cup (q\Delta + Q - R)) \subseteq (P + (q-1)\Delta) \cup (Q + (q-1)\Delta). \tag{6.3}$$

Since P, R is regular we know that $q\Delta \cap (q\Delta + P - R) \subseteq P + (q-1)\Delta$ by (6.1). Similarly because Q, R is regular we have $q\Delta \cap (q\Delta + Q - R) \subseteq Q + (q-1)\Delta$. Together these two inclusions imply (6.3). □

As an immediate corollary, we deduce using Theorem 6.4:

Corollary 6.6. *Let Δ be a two-dimensional lattice polygon. For three distinct lattice points $P, Q, R \in \Delta$, the following statements are equivalent:*

1. *P, Q, R is a regular sequence for $\bigoplus_{i \geq 0} V_{i\Delta}$.*
2. *P, Q, R is a regular sequence for $\bigoplus_{i \geq 1} V_{(i\Delta)^{(1)}}$.*
3. *Δ is a triangle with vertices P, Q and R.*

6.3. **Example: the case of Veronese embeddings.** Let us apply the foregoing to $\Delta = d\Sigma$ for $d \geq 2$, whose corresponding toric surface is the Veronese surface $\nu_d(\mathbb{P}^2)$ with coordinate ring

$$S_{d\Sigma} \cong k \oplus V_{d\Sigma} \oplus V_{2d\Sigma} \oplus V_{3d\Sigma} \oplus V_{4d\Sigma} \oplus V_{5d\Sigma} \oplus \ldots \tag{6.4}$$

By the foregoing corollary the sequence of points $(0, d)$, $(d, 0)$, $(0, 0)$ is regular for $S_{d\Sigma}$. When one removes these points along the above guidelines, the resulting graded module is

$$k \oplus V_{d\Sigma \setminus \{(0,d),(d,0),(0,0)\}} \oplus V_{\mathrm{conv}\{(d-1,d-1),(2,d-1),(d-1,2)\}} \oplus 0 \oplus 0 \oplus 0 \oplus \ldots$$

which can be rewritten as

$$k \oplus V_{d\Sigma\setminus\{(0,d),(d,0),(0,0)\}} \oplus V_{(d,d)-(d\Sigma)^{(1)}} \oplus 0 \oplus 0 \oplus 0 \oplus \dots \tag{6.5}$$

We recall from the end of Section 6.1 that multiplication is defined by lattice addition, with the convention that the product is zero whenever the sum falls outside the indicated range. In order to find the graded Betti table of $\nu_d(\mathbb{P}^2)$, it therefore suffices to compute the cohomology of complexes of the following type:

$$\begin{aligned}\textstyle\bigwedge^{\ell+1} V_{d\Sigma\setminus\{(0,d),(d,0),(0,0)\}} \longrightarrow \textstyle\bigwedge^{\ell} V_{d\Sigma\setminus\{(0,d),(d,0),(0,0)\}} \otimes V_{d\Sigma\setminus\{(0,d),(d,0),(0,0)\}} \\ \longrightarrow \textstyle\bigwedge^{\ell-1} V_{d\Sigma\setminus\{(0,d),(d,0),(0,0)\}} \otimes V_{(d,d)-(d\Sigma)^{(1)}}\end{aligned} \tag{6.6}$$

Indeed, the cohomology in the middle has dimension $\dim K_{\ell,1}(X,L) = b_\ell$ and the cokernel of the second morphism has dimension $\dim K_{\ell-1,2}(X,L) = c_{N_\Delta-1-\ell}$.

We can carry out the same procedure in the twisted case. The resulting graded module is

$$k \oplus V_{(d\Sigma)^{(1)}} \oplus V_{(d,d)-d\Sigma\setminus\{(0,d),(d,0),(0,0)\}} \oplus V_{\{(d,d)\}} \oplus 0 \oplus 0 \oplus \dots$$

For instance, one finds that $K^\vee_{\ell,1}(X,L) \cong K_{N_\Delta-3-\ell,2}(X;K,L)$ is the cohomology in the middle of

$$\begin{aligned}&\textstyle\bigwedge^{N_\Delta-\ell-2} V_{(d\Sigma)\setminus\{(0,d),(d,0),(0,0)\}} \otimes V_{(d\Sigma)^{(1)}} \longrightarrow \\ &\quad \textstyle\bigwedge^{N_\Delta-\ell-3} V_{d\Sigma\setminus\{(0,d),(d,0),(0,0)\}} \otimes V_{(d,d)-d\Sigma\setminus\{(0,d),(d,0),(0,0)\}} \\ &\qquad \longrightarrow \textstyle\bigwedge^{N_\Delta-\ell-4} V_{d\Sigma\setminus\{(0,d),(d,0),(0,0)\}} \otimes V_{(d,d)}.\end{aligned}$$

As a side remark, note that this complex is isomorphic to the dual of (6.6). Thus this gives a combinatorial proof of the duality formula $K^\vee_{\ell,1}(X,L) \cong K_{N_\Delta-3-\ell,2}(X;K,L)$ for Veronese surfaces.

Let us conclude with a visualization of the point removal procedure in the case where $d = 3$ (in the non-twisted setting). Figure 5 shows how the coordinate ring gradually shrinks upon removal of $(0,3)$, then of $(3,0)$, and finally of $(0,0)$. The left column shows the graded parts of the original coordinate ring (6.4) in degrees $0,1,2,3$, while the right column does the same for the eventual graded module described in (6.5).

7. Computing graded Betti numbers

7.1. **The algorithm.** To compute the entries b_ℓ and c_ℓ of the graded Betti table (1.1) of $X_\Delta \subseteq \mathbb{P}^{N_\Delta-1}$ we use the formulas (2.3) and (2.5). In other words, we determine the b_ℓ's as

$$\dim\ker\left(\textstyle\bigwedge^{\ell} V_\Delta \otimes V_\Delta \to \textstyle\bigwedge^{\ell-1} V_\Delta \otimes V_{2\Delta}\right) - \dim \textstyle\bigwedge^{\ell+1} V_\Delta,$$

while the c_ℓ's are computed as

$$\dim\ker\left(\textstyle\bigwedge^{\ell-1} V_\Delta \otimes V_{\Delta^{(1)}} \to \textstyle\bigwedge^{\ell-2} V_\Delta \otimes V_{(2\Delta)^{(1)}}\right).$$

Essentially, this requires writing down a matrix of the respective linear map and computing its rank. As explained in Section 2.2 we can consider these expressions for each bidegree (a,b) independently, and then just sum the contributions $b_{\ell,(a,b)}$

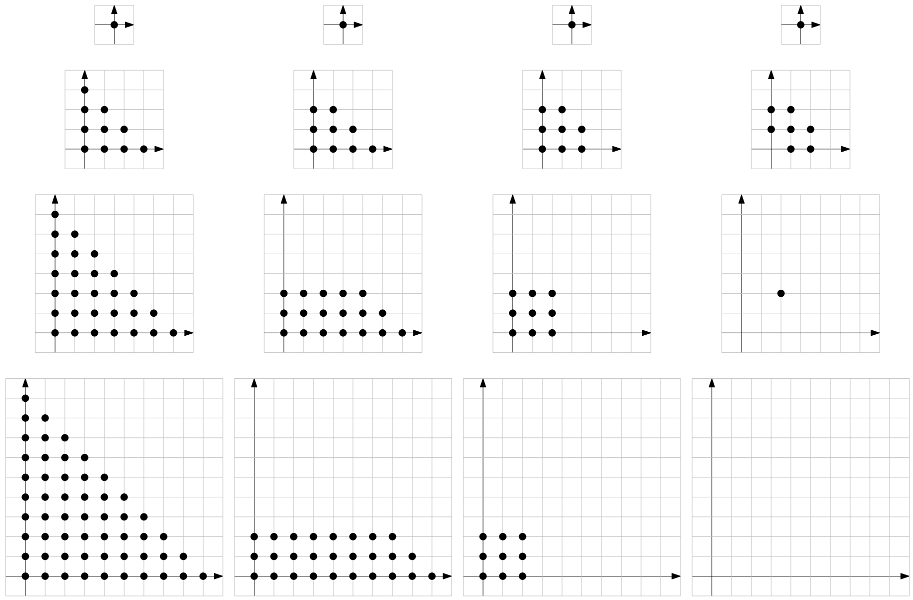


FIGURE 5. Removing three points for $\Delta = 3\Sigma$

resp. $c^\vee_{\ell,(a,b)}$. This greatly reduces the dimensions of the vector spaces and hence of the matrices that we need to deal with.

Remark 7.1. The subtracted term in the formula for b_ℓ can be made explicit:

$$\dim \bigwedge^{\ell+1} V_\Delta = \binom{N_\Delta}{\ell+1}.$$

However we prefer to compute its contribution in each bidegree separately (which is easily done, see Section 7.2), the reason being that the $b_{\ell,(a,b)}$'s are interesting in their own right; see also Remark 7.2 below.

Speed-ups. Formula (1.2) allows us to obtain $b_{N_\Delta-1-\ell}$ from c_ℓ and $c_{N_\Delta-1-\ell}$ from b_ℓ, so we only compute one of both. In practice we make an educated guess for what we think will be the easiest option, based on the dimensions of the spaces involved. Moreover, using Hering and Schenck's formula (1.3) we find that c_ℓ vanishes as soon as $\ell \geq N_\Delta + 1 - |\partial\Delta \cap \mathbb{Z}^2|$. For this reason the computation of $b_1, \ldots, b_{|\partial\Delta\cap\mathbb{Z}^2|-2}$ can be omitted, which is particularly interesting in the case of the Veronese polygons $d\Sigma$, which have many lattice points on the boundary.

Remark 7.2. From the proof of formula (1.2) given in Section 2.3 we can extract that

$$b_{\ell,(a,b)} - c_{N_\Delta-1-\ell,(a,b)} = \sum_{j=0}^{\ell+1} (-1)^{j+1} \dim \left(\bigwedge^{\ell+1-j} V_\Delta \otimes V_{j\Delta} \right)_{(a,b)}$$

for each bidegree $(a,b) \in \mathbb{Z}^2$ and each $\ell = 1, \ldots, N_\Delta - 2$. Here the subscript on the right hand side indicates that we consider the subspace of elements having bidegree (a,b). As explained in Section 7.2, we can easily compute the dimensions of the spaces on the right hand side in practice. Together with (2.6) this allows one to obtain the bigraded parts of the entire Betti table, using essentially the same method. As an illustration, bigraded versions of some of the data gathered in Appendix A have been made available on http://sage.ugent.be/www/jdemeyer/betti/.

We use the material from Section 6 to reduce the dimensions further. As soon as we are dealing with an n-gon with $n \geq 5$, then by Theorem 6.4 we can remove one lattice point only. In the case of a quadrilateral we can remove two opposite vertices. In the case of a triangle we can remove its three vertices. For simple computations we just make a random amenable choice. For larger computations it makes sense to spend a little time on optimizing the point(s) to be removed, by computing the dimensions of the resulting quotient spaces.

Remark 7.3. As we have mentioned before, from a practical point of view the effect of removing lattice points is somewhat unpredictable. In certain cases we even observed that, although the resulting matrices are of considerably lower dimension, computing the rank takes more time. We currently have no explanation for this.

Another useful optimization is to take into account symmetries of Δ, which naturally induce symmetries of multiples of Δ and $\Delta^{(1)}$. For example for b_ℓ, consider a symmetry $\psi \in \mathrm{AGL}_2(\mathbb{Z})$ of $(\ell+1)\Delta$ and let (a,b) be a bidegree. Then $b_{\ell,(a,b)} = b_{\ell,\psi(a,b)}$. The analogous remark holds for c_ℓ, using symmetries ψ of $(\ell-1)\Delta + \Delta^{(1)}$.

A final speed-up comes from computing in finite characteristic, thereby avoiding inflation of coefficients when doing rank computations. This could affect the outcome, for instance if $\Delta = \mathrm{conv}\{(0,0),(5,3),(2,5)\}$ then the graded Betti table of X_Δ in characteristic zero is

	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	36	160	315	288	45	0	0	0	0
2	0	1	9	36	129	414	441	244	72	9

while we observed that the entries 45 and 129 increase by 1 when considering X_Δ over fields of characteristic 2 or 3. We believe that this event is extremely rare; in fact the foregoing polygon is the only example of which we are currently aware, up to unimodular equivalence. Nevertheless this speed-up comes at the cost of ending up with conjectural graded Betti tables. However recall from the introduction that the graded Betti numbers can never decrease, so the zero entries are rigorous (and because of (1.2) the other entry on the corresponding antidiagonal is rigorous as well).

Writing down the matrices. The maps we need to deal with are of the form

$$\bigwedge^p V_A \otimes V_B \xrightarrow{\delta} \bigwedge^{p-1} V_A \otimes V_C, \tag{7.1}$$

where A, B and C are finite sets of lattice points and δ is as in (2.2), subject to the additional rule mentioned in Remark 6.2. For a given bidegree (a,b), as a basis of the left hand side of (7.1) we make the obvious choice

$$\{\, x^{i_1}y^{j_1} \wedge \ldots \wedge x^{i_p}y^{j_p} \otimes x^{i'}y^{j'} \mid (i',j') = (a,b) - (i_1,j_1) - \ldots - (i_p,j_p) \text{ and } \{(i_1,j_1),\ldots,(i_p,j_p)\} \subseteq A \text{ and } (i',j') \in B\},$$

where $\{(i_1, j_1), \ldots, (i_p, j_p)\}$ runs over all p-element subsets of A. In the implementation, we equip A with a total order $<$ and take subsets such that $(i_1, j_1) < \ldots < (i_p, j_p)$. We do not need to store the part $x^{i'} y^{j'}$ since that is completely determined by the rest (for a fixed bidegree). We use the analogous basis for the right hand side of (7.1). We then compute the transformation matrix corresponding to the map δ in a given bidegree, and determine its rank.

Note that the resulting matrix is very sparse: it has at most p non-zero entries in every column, while the non-zero entries are 1 or -1. Therefore we use a sparse data structure to store this matrix.

Implementation. We have implemented all this in Python and Cython, using SageMath [43] with LinBox [36] for the linear algebra. In principle the algorithm should work equally fine in characteristic zero (at the cost of some efficiency) but for technical reasons our current implementation does not support this. For the implementation details we refer to the programming code, which is made available at https://github.com/jdemeyer/toricbetti.

7.2. **Computing the dimensions of the spaces.** Given finite subsets $A, B \subseteq \mathbb{Z}^2$, computing the dimension of the space $\bigwedge^p V_A \otimes V_B$ in each bidegree can be done efficiently without explicitly constructing a basis. These dimensions determine the sizes of the matrices involved. Knowing this size allows to estimate the amount of time and memory needed to compute the rank. We use this to decide whether to compute b_ℓ or $c_{N_\Delta - 1 - \ell}$, and which point(s) we remove when applying the material from Section 6.

Namely, consider the generating function (which is actually a polynomial)

$$f_A(X, Y, T) = \prod_{(i,j) \in A} (1 + X^i Y^j T).$$

Then the coefficient of $X^a Y^b T^p$ is the dimension of the component in bidegree (a, b) of $\bigwedge^p V_A$. The generating function for $\bigwedge^p V_A \otimes V_B$ then becomes

$$f_{A,B}(X, Y, T) = \prod_{(i,j) \in A} (1 + X^i Y^j T) \cdot \sum_{(i,j) \in B} X^i Y^j.$$

If we are only interested in a fixed p, we can compute modulo T^{p+1}, throwing away all higher-order terms in T.

7.3. **Applications.** As a first application we have verified Conjecture 1.5 for all lattice polygons containing at most 32 lattice points with at least one lattice point in the interior (namely we used the list of polygons from [7] and took those polygons for which $N_\Delta \leq 32$), thereby establishing the first bullet point of Theorem 1.3. There are 583 095 such polygons; the maximal lattice width that occurs is 8. Apart from the ten exceptional polygons $3\Sigma, \ldots, 6\Sigma, \Upsilon_2, \ldots, \Upsilon_6$ and 2Υ, we verified that the entry $b_{N_\Delta - \mathrm{lw}(\Delta) - 1}$ indeed equals zero. In the exceptional cases, whose graded Betti tables are gathered in Appendix A, we found that $b_{N_\Delta - \mathrm{lw}(\Delta)}$ equals zero. This shows that the upper bound from Theorem 1.3, which was proven in Section 3, is indeed sharp for each of these lattice polygons. The computation was carried out modulo 40 009 and took 1006 CPU core-days on an Intel Xeon E5-2680 v3.

As a second application we have computed the graded Betti table of the 6-fold Veronese surface $X_{6\Sigma}$, which can be found in Appendix A. Currently the computation was done in finite characteristic only (again 40 009) and therefore some of the non-zero entries are conjectural. The computation took 12 CPU core-days on an IBM POWER8. This new data leads to the guesses stated in Conjecture 1.6, predicting certain entries of the graded Betti table of $X_{d\Sigma} = \nu_d(\mathbb{P}^2)$ for arbitrary $d \geq 2$.

- The first guess states that the last non-zero entry on the row $q = 1$ is given by $d^3(d^2-1)/8$. This is true for $d = 2, 3, 4, 5$ and has been verified in characteristic 40 009 for $d = 6, 7$. Recently, this guess has been put in the broader context of Schur functors by Bruce, Erman, Goldstein and Yang [4, Conj. 6.6].
- The second guess is about the first non-zero entry on the row $q = 2$, which we believe to be
$$\binom{N_{(d\Sigma)^{(1)}} + 8}{9}.$$
Here we have less supporting data: it is true for $d = 3, 4, 5$ and has been verified in characteristic 40 009 for $d = 6$. On the other hand our guess naturally fits within the more widely applicable formula
$$\binom{N_{\Delta^{(1)}} - 1 + \left|\{\, v \in \mathbb{Z}^2 \setminus \{(0,0)\} \,|\, \Delta^{(1)} + v \subseteq \Delta \,\}\right|}{N_{\Delta^{(1)}} - 1},$$
which we have verified for a large number of small polygons. In fact, by now we can replace the word ‘guess’ by ‘theorem’, because between the time of submission and the time of publication of this document, the fourth listed author has proven the above binomial formula in arbitrary characteristic [35] (and extended it to normal toric varieties of arbitrary dimension).

Appendix A. Some explicit graded Betti tables

This appendix contains the graded Betti tables of $X_\Delta \subseteq \mathbb{P}^{N_\Delta - 1}$ for the instances of Δ that are the most relevant to this paper. The largest of these Betti tables were computed using the algorithm described in Section 7. Because these computations were carried out modulo 40 009 the resulting tables are conjectural, except for the zero entries and the entries on the corresponding antidiagonal. The smaller Betti tables have been verified independently in characteristic zero using the Magma intrinsic [3], along the lines of [10, §2]. For the sake of clarity, we have indicated the conjectural entries by an asterisk. The question marks ‘???’ mean that the corresponding entry has not been computed.

Σ $(N_\Delta = 3)$:

	0
0	1
1	0
2	0

2Σ $(N_\Delta = 6)$:

	0	1	2	3
0	1	0	0	0
1	0	6	8	3
2	0	0	0	0

3Σ $(N_\Delta = 10)$:

	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	27	105	189	189	105	27	0
2	0	0	0	0	0	0	0	1

4Σ $(N_\Delta = 15)$:

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	75	536	1947	4488	7095	7920	6237	3344	1089	120	0	0
2	0	0	0	0	0	0	0	0	0	0	55	24	3

5Σ $(N_\Delta = 21)$:

	0	1	2	3	4	5	6	7	8	···
0	1	0	0	0	0	0	0	0	0	
1	0	165	1830	10710	41616	117300	250920	417690	548080	···
2	0	0	0	0	0	0	0	0	0	

	···	9	10	11	12	13	14	15	16	17	18
0		0	0	0	0	0	0	0	0	0	0
1	···	568854	464100	291720	134640	39780	4858	375	0	0	0
2		0	0	0	0	2002	4200	2160	595	90	6

6Σ $(N_\Delta = 28)$:

	0	1	2	3	4	5	6	7	8	···
0	1	0	0	0	0	0	0	0	0	
1	0	315	4950	41850	240120	1024650	3415500	9164925	20189400	···
2	0	0	0	0	0	0	0	0	0	

	···	9	10	11	12	13	14	···
0		0	0	0	0	0	0	
1	···	36989865	56831850	73547100	80233200	73547100	56163240	···
2		0	0	0	0	0	0	

	···	15	16	17	18	19	20	21	···
0		0	0	0	0	0	0	0	
1	···	35102025	17305200	6177545*	1256310*	160398*	17890*	945*	···
2		0	48620*	231660*	593028*	473290*	218295*	69300	

	···	22	23	24	25
0		0	0	0	0
1	···	0	0	0	0
2		15525	2376	225	10

7Σ $(N_\Delta = 36)$:

	···	26	27	28	29	30	31	32	33
0		0	0	0	0	0	0	0	0
1	···	???	53352*	2058*	0	0	0	0	0
2		27821664*	8824410*	2215136	434280	64449	6832	462	15

$\Upsilon = \Upsilon_1$ $(N_\Delta = 4)$:

	0	1
0	1	0
1	0	0
2	0	1

2Υ $(N_\Delta = 10)$:

	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	24	84	126	84	20	0	0
2	0	0	0	0	20	36	21	4

Υ_2 $(N_\Delta = 7)$:

	0	1	2	3	4
0	1	0	0	0	0
1	0	7	8	3	0
2	0	0	6	8	3

Υ_3 $(N_\Delta = 11)$:

	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	30	120	210	189	105	27	0	0
2	0	0	0	21	105	147	105	40	6

Υ_4 $(N_\Delta = 16)$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	81	598	2223	5148	7920	8172	6237	3344	1089	120	0	0	0
2	0	0	0	0	55	450	2376	4488	4950	3630	1859	612	117	10

Υ_5 $(N_\Delta = 22)$:

	0	1	2	3	4	5	6	7	8	···
0	1	0	0	0	0	0	0	0	0	
1	0	175	1995	11970	47481	135660	290820*	476385*	597415*	···
2	0	0	0	0	0	120*	1575*	9555*	52650*	

	···	9	10	11	12	13	14	15	···
0		0	0	0	0	0	0	0	
1	···	581724*	466102*	291720*	134640*	39780*	4858*	375*	···
2		172172*	291720*	338130*	291720*	194782*	102120*	39900	

	···	16	17	18	19
0		0	0	0	0
1	···	0	0	0	0
2		11305	2205	266	15

Υ_6 $(N_\Delta = 29)$:

	···	18	19	20	21	22	23	24	25	26
0		0	0	0	0	0	0	0	0	0
1	···	???	160398*	17890*	945*	0	0	0	0	0
2		16095603*	7911490*	3140445*	995280	246675	46176	6150	520	21

References

1. Marian Aprodu and Jan Nagel, *Koszul cohomology and algebraic geometry*, University Lecture Series **52**, American Mathematical Society, 125 pp. (2010)
2. Adam Boocher, Wouter Castryck, Milena Hering and Alexander Lemmens, *Torus weights of resolutions of Veronese embeddings*, article in preparation
3. Wieb Bosma, John Cannon and Catherine Playoust, *The Magma algebra system I. The user language*, Journal of Symbolic Computation **24**(3-4), pp. 235-265 (1997)
4. Juliette Bruce, Daniel Erman, Steve Goldstein and Jay Yang, *Conjectures and computations of Veronese syzygies*, Experimental Mathematics, to appear
5. Winfried Bruns, Aldo Conca and Tim Römer, *Koszul homology and syzygies of Veronese subalgebras*, Mathematische Annalen **351**(4), pp. 761-779 (2011)
6. Winfried Bruns, Joseph Gubeladze and Ngô Viêt Trung, *Normal polytopes, triangulations, and Koszul algebras*, Journal für die reine und angewandte Mathematik **485**, pp. 123-160 (1997)
7. Wouter Castryck, *Moving out the edges of a lattice polygon*, Discrete and Computational Geometry **47**, pp. 496-518 (2012)
8. Wouter Castryck, *A lower bound for the gonality conjecture*, Mathematika **63**(2), pp. 561-563 (2017)
9. Wouter Castryck and Filip Cools, *Newton polygons and curve gonalities*, Journal of Algebraic Combinatorics **35**(3), pp. 345-366 + err. pp. 367-372 (2012)

10. Wouter Castryck and Filip Cools, *A minimal set of generators for the canonical ideal of a non-degenerate curve*, Journal of the Australian Mathematical Society **98**(3), pp. 311-323 (2015)
11. Wouter Castryck and Filip Cools, *Linear pencils encoded in the Newton polygon*, International Mathematics Research Notices **2017**(10), pp. 2998-3049 (2017)
12. Wouter Castryck, Filip Cools, Jeroen Demeyer and Alexander Lemmens, *On graded Betti tables of curves on toric surfaces*, preprint
13. Filip Cools and Alexander Lemmens, *Minimal polygons with fixed lattice width*, preprint
14. David Cox, John Little and Hal Schenck, *Toric varieties*, Graduate Studies in Mathematics **124**, American Mathematical Society (2011)
15. Lawrence Ein, Daniel Erman and Robert Lazarsfeld, *A quick proof of nonvanishing for asymptotic syzygies*, Algebraic Geometry **3**(2), pp. 211-222 (2016)
16. Lawrence Ein and Robert Lazarsfeld, *Asymptotic syzygies of algebraic varieties*, Inventiones Mathematicae **190**(3), pp. 603-646 (2012)
17. Lawrence Ein and Robert Lazarsfeld, *The gonality conjecture on syzygies of algebraic curves of large degree*, Publications Mathématiques de lIHÉS **122**(1), pp. 301-313 (2015)
18. David Eisenbud, *The geometry of syzygies. A second course in commutative algebra and algebraic geometry*, Graduate Texts in Mathematics **229**, Springer-Verlag, New York (2005)
19. David Eisenbud and Joe Harris, *On varieties of minimal degree (a centennial account)*, Proceedings of Symposia in Pure Mathematics **46**(1), pp. 3-13 (1987)
20. Gavril Farkas and Michael Kemeny, *Linear syzygies of curves with prescribed gonality*, preprint
21. László Fejes Tóth and Endre Makai Jr., *On the thinnest non-separable lattice of convex plates*, Studia Scientiarum Mathematicarum Hungarica **9**, pp. 191-193 (1974)
22. William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies **131**, Princeton University Press (1993)
23. Francisco J. Gallego and Bangere P. Purnaprajna, *Some results on rational surfaces and Fano varieties*, Journal für die Reine und Angewandte Mathematik **538**, pp. 25-55 (2001)
24. Ornella Greco and Ivan Martino, *Syzygies of the Veronese modules*, Communications in Algebra **44**(9), pp. 3890-3906
25. Mark Green, *Koszul cohomology and the geometry of projective varieties*, Journal of Differential Geometry **19**, pp. 125-171 (1984)
26. Mark Green and Robert Lazarsfeld, *On the projective normality of complete linear series on an algebraic curve*, Inventiones Mathematicae **83**(1), pp. 73-90 (1986)
27. Christian Haase, Benjamin Nill, Andreas Paffenholz and Francisco Santos, *Lattice points in Minkowski sums*, Electronic Journal of Combinatorics **15**, #N11 (2008)
28. Christian Haase and Josef Schicho, *Lattice polygons and the number* $2i + 7$, American Mathematical Monthly **116**(2), pp. 151-165 (2009)
29. Milena Hering, *Syzygies of toric varieties*, Ph.D. thesis, University of Michigan (2006)
30. Robert J. Koelman, *The number of moduli of families of curves on toric surfaces*, Ph.D. thesis, Katholieke Universiteit Nijmegen (1991)
31. Robert J. Koelman, *A criterion for the ideal of a projectively embedded toric surface to be generated by quadrics*, Beiträge zur Algebra und Geometrie **34**, pp. 57-62 (1993)
32. Sijong Kwak, *Castelnuovo regularity for smooth subvarieties of dimensions* 3 *and* 4, Journal of Algebraic Geometry **7**(1), pp. 195-206 (1998)
33. Margherita Lelli-Chiesa, *Green's conjecture for curves on rational surfaces with an anticanonical pencil*, Mathematische Zeitschrift **275**(3-4), pp. 899-910 (2013)
34. Alexander Lemmens, *Combinatorial aspects of syzygies of toric varieties*, Ph.D. thesis, KU Leuven (2018)

35. Alexander Lemmens, *On the n^{th} row of the graded Betti table of an n-dimensional toric variety*, Journal of Algebraic Combinatorics, to appear
36. LinBox, Exact Linear Algebra over the Integers and Finite Fields, Version 1.3.2, http://linalg.org/ (2012)
37. Frank Loose, *On the graded Betti numbers of plane algebraic curves*, Manuscripta Mathematica **64**(4), pp. 503-514 (1989)
38. Niels Lubbes and Josef Schicho, *Lattice polygons and families of curves on rational surfaces*, Journal of Algebraic Combinatorics **34**(2), pp. 213-236 (2011)
39. Giorgio Ottaviani and Raffaela Paoletti, *Syzygies of Veronese embeddings*, Compositio Mathematica **125**(1), pp. 31-37 (2001)
40. Euisung Park, *On syzygies of Veronese embedding of arbitrary projective varieties*, Journal of Algebra **322**(1), pp. 108-121 (2009)
41. Jürgen Rathmann, *An effective bound for the gonality conjecture*, preprint
42. Elena Rubei, *A result on resolutions of Veronese embeddings*, Annali dell'Università di Ferrara Sezione VII. Scienze Matematiche **50**, pp. 151-165 (2004)
43. SageMath, the Sage Mathematics Software System, Version 7.2, http://www.sagemath.org/ (2016)
44. Hal Schenck, *Lattice polygons and Green's theorem*, Proceedings of the American Mathematical Society **132**(12), pp. 3509-3512 (2004)
45. David H. Yang, *S_n-equivariant sheaves and Koszul cohomology*, Research in the Mathematical Sciences **1**:10 (2014)

Dept. Wiskunde, KU Leuven, Celestijnenlaan 200B, Leuven, Belgium
Dept. ESAT, KU Leuven and imec, Kasteelpark Arenberg 10/2452, Leuven, Belgium
Vakgroep Wiskunde, Universiteit Gent, Krijgslaan 281, Gent, Belgium
E-mail address: wouter.castryck@esat.kuleuven.be

Dept. Wiskunde, KU Leuven, Celestijnenlaan 200B, Leuven, Belgium
E-mail address: f.cools@kuleuven.be

Vakgroep Wiskunde, Universiteit Gent, Krijgslaan 281, Gent, Belgium
E-mail address: jdemeyer@cage.ugent.be

Dept. Wiskunde, KU Leuven, Celestijnenlaan 200B, Leuven, Belgium
E-mail address: alexander.lemmens@kuleuven.be