Density filtering regularization of finite element model updating problems^{☆,☆☆}

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Abstract

Finite element (FE) model updating is often used as a non-destructive method to detect structural damage. Stiffness parameters of an FE model of the structure are calibrated based on experimental vibration data. If the desired spatial resolution is high, the problem is likely to be ill-conditioned and requires regularization. Tikhonov regularization is frequently used for FE model updating problems, but the selection of a proper regularization parameter and a good initial estimate of the stiffness parameters is difficult. This paper proposes an alternative, density-filtering-based method where the filter radius acts as regularization parameter. Since the filter radius controls the minimal length scale of the identifiable damaged zones, it has a clear physical meaning. Furthermore, an initial estimate of the stiffness parameters is only required as a starting point for the optimization algorithm whereas in the case of Tikhonov regularization it also guides the optimization. Both regularization methods are compared in a numerical case study. The density filtering regularization method is found to be more successful in identifying the damaged zone, while Tikhonov regularization sometimes fails to do so.

Keywords: finite element model updating, damage detection, density filter, regularization

1. Introduction

Structural damage detection is gaining increased attention in civil engineering. The development of structural health monitoring (SHM) systems allows engineers to monitor the structural behavior and to detect damage in an early stage [1]. Such systems are regularly installed on important, expensive, or sensitive structures such as bridges, skyscrapers or aircraft. The ultimate goal of SHM systems is to ensure the safety and serviceability of structures during their entire life span, avoiding failure of the structure and high repair costs. In some cases, SHM systems are installed on existing structures that are surpassing their life span or that are subject to loads exceeding their design values [2].

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It is generally assumed that a local stiffness reduction indicates structural damage. Therefore, the FE model of the structure is updated either by (1) updating the entire stiffness matrix at once (direct method) or by (2) introducing a stiffness parameter for each element or group of elements in the FE model (inverse method) [3]. The latter method corresponds to deterministic model updating, which is often performed using experimental modal characteristics. In this case, the updating scheme consists of the following steps. Measurement data (accelerations, strains, temperature variations...) are acquired from several sensors located on the structure. The experimental modal characteristics such as eigenfrequencies and mode shapes are extracted from the measurement data using modal analysis techniques [4, 5]. In this paper, however, the experimental modal characteristics are simulated in a forward problem. Next, the parameterized FE model of the structure is used to predict its modal characteristics. The discrepancy between the experimental and predicted modal characteristics is expressed in terms of eigenfrequency and mode shape residuals. These residuals are minimized by solving a non-linear least-squares problem using a gradient-based approach [6, 7]. The objective function and its gradient are evaluated in each iteration. After convergence, the updated stiffness parameters of the FE model are retrieved, from which damaged zones in the structure can be identified.

The inverse problem is likely to be ill-conditioned if the number of stiffness parameters is large compared to the number of residuals, or if the modal characteristics are not sensitive to the selected stiffness parameters [8]. The careful parameterization of the structural stiffness is, however, not straightforward [6, 9]. Several methods have been proposed to deal with ill-conditioning in FE model updating. First of all, the number of stiffness parameters can be limited using subset selection methods [10, 11]. An automated parameter selection procedure was proposed by Kim and Park [12]. More recently, Bartilson et al. [13] suggested a parameter clustering approach in order to reduce the number of parameters. On the other hand, the problem can be regularized by adding penalty terms to the objective function, such as the classical Tikhonov regularization [14]. This method adds a penalty term which guides the stiffness parameters towards an initial estimate of these parameters. Alternatively, the incremental parameter change could be constrained by the penalty term [15]. Either way, a trade-off between the objective function and the penalty term is made using a regularization parameter, which is often determined using the concept of L-curves or cross-validation [8, 16]. The solution of the inverse problem very much depends on this regularization parameter. Finally, Lee et al. [17] proposed a regularization technique based on material density penalization which is used in topology optimization to obtain a discrete solution.

In this paper, a novel method is proposed for the regularization of FE model updating problems with a large number of stiffness parameters (high resolution). The method employs the density filter, which is used in topology optimization to avoid numerical instabilities and to control the length scale of the optimized structure [18–21]. The density filter prevents stiffness parameters of neighboring elements from varying independently, avoiding strong local stiffness variations in the updated FE model. Major advantages of the proposed method compared to classical regularization methods are: (1) the careful selection of a reduced parameter set is avoided, (2) the filter radius, which controls the length scale, has a well defined physical meaning, and (3) the initial estimate of the stiffness parameters is solely used as a starting point for the optimization algorithm.

This paper is organized as follows. Section 1 states the problem and shortly outlines existing methods to avoid ill-conditioning in FE model updating problems. In Section 2, the methodology of FE model updating using eigenfrequency and mode shape residuals is briefly discussed and a numerical case study is introduced. Section 3 describes the Tikhonov regularization method

and illustrates its performance using the numerical case study. In Section 4, the density filtering regularization method is introduced and the numerical case study is revisited. The influence of the filter radius on the updated stiffness parameters is assessed and discussed. Section 5 compares the performance of both regularization methods. Section 6 concludes the paper.

2. Sensitivity-based FE model updating

The general assumption for damage detection using FE model updating is that structural damage leads to a stiffness reduction in the structure. The goal is to identify (and possibly quantify) the damage based on measurement data. From these measurement data a set of modal characteristics is extracted using modal analysis techniques [4].

2.1. Numerical case study

Throughout this paper, a relatively simple 2D problem is used to illustrate the density filtering regularization method and compare it with the Tikhonov regularization method. A simply supported, concrete beam with a length of 3 m, a height of 0.36 m, and a thickness of 0.3 m is damaged at mid span (light gray zone in Fig. 1). The damaged zone represents a cracked region caused by vertical loading at the center of the beam. It is assumed that the cracks lead to a local stiffness reduction of 30% in the damaged zone. This is a moderate stiffness reduction, which is even more challenging to identify. Apart from this damaged zone, the beam is undamaged (dark gray zone in Fig. 1).

The FE model of the beam consists of $n_e = 100 \times 12 = 1200$ square linear plane elements with two degrees of freedom per node and with side lengths $L_e = 0.03$ m. The total number of degrees of freedom in the FE model is $n_{\text{dof}} = 2623$. The concrete is assumed to be linear elastic and isotropic with a Young's modulus $E_0 = 30$ GPa, a Poisson's ratio $\nu = 0.25$, and a material density $\rho = 2500$ kg/m³. Furthermore, plane stress conditions are assumed.

A stiffness parameter $\theta_{M,j}$ is assigned to each element j in the FE model with the subscript “M” standing for model. This parameter determines the contribution of the element stiffness matrix $\mathbf{K}_{e,j}$ to the global stiffness matrix $\mathbf{K}(\boldsymbol{\theta}_M)$ of the structure:

$$\mathbf{K}(\boldsymbol{\theta}_M) = \sum_{j=1}^{n_e} \theta_{M,j} \mathbf{K}_{e,j}. \quad (1)$$

The global mass matrix $\mathbf{M}$ is independent of the stiffness parameters $\boldsymbol{\theta}_M$:

$$\mathbf{M} = \sum_{j=1}^{n_e} \mathbf{M}_{e,j}. \quad (2)$$

Fig. 1: Simply supported beam with undamaged (dark gray) and damaged (light gray) elements. The mesh corresponds to the actual FE mesh used in the calculations ($n_e = 1200$). The width of the damaged zone is 0.9 m (or 30 elements) and the height is 0.12 m (or 4 elements). The red line indicates the location of the observed degrees of freedom.

The element stiffness matrix $\mathbf{K}_{e,j}$ depends on an initial estimate of the Young's modulus E_0 of the undamaged material. The product $\theta_{M,j}E_0$ represents the Young's modulus of element j , where $\theta_{M,j} < 1$ indicates that the element is damaged.

2.2. Simulated modal characteristics

Once the global system matrices are constructed, the modal characteristics are obtained by solving the generalized undamped eigenvalue problem:

$$\mathbf{K}(\boldsymbol{\theta}_M)\boldsymbol{\phi}_r = \lambda_r\mathbf{M}\boldsymbol{\phi}_r, \quad (3)$$

where $\boldsymbol{\phi}_r \in \mathbb{R}^{n_{\text{dof}}}$ is the mode shape that corresponds to the eigenvalue λ_r .

In the numerical case study, the experimental modal characteristics are simulated in a forward problem. This approach allows assessing the accuracy of the updated stiffness parameters, since the true stiffness parameters $\hat{\boldsymbol{\theta}}_M$ of the damaged structure are known a priori. The stiffness parameters $\hat{\boldsymbol{\theta}}_M$ of the damaged beam (Fig. 1) are equal to $\hat{\theta}_{M,j} = 0.7$ in the damaged zone and $\hat{\theta}_{M,j} = 1$ in the undamaged zone. The eigenvalue problem in Eq. (3) is solved with $\hat{\boldsymbol{\theta}}_M$ to obtain the noise free modal characteristics $\hat{\lambda}_r$ and $\hat{\boldsymbol{\phi}}_r$ of the damaged structure. Subsequently, normally distributed noise is added as follows to model measurement error:

$$\bar{\lambda}_r = \hat{\lambda}_r + \sigma_\lambda \hat{\lambda}_r \mathcal{N}(0, 1) \quad (4)$$

$$\bar{\boldsymbol{\phi}}_r = \mathbf{L} \left(\hat{\boldsymbol{\phi}}_r + \sigma_\phi \max |\hat{\boldsymbol{\phi}}_r| \mathcal{N}(0, 1) \right). \quad (5)$$

Here, $\mathcal{N}(0, 1)$ represents a standard normal random variable and σ_λ and σ_ϕ determine the magnitude of the standard deviation on the eigenvalues and mode shape components, respectively. It is assumed that $\sigma_\lambda = 0.005$ and $\sigma_\phi = 0.05$, so that the measurement error is larger for mode shapes than for eigenvalues, which is generally the case [22, 23]. In order to perform a statistically meaningful assessment, 50 *noise patterns* were randomly generated from the standard normal distribution and saved. The selection matrix $\mathbf{L} \in \mathbb{R}^{n_{\text{obs}} \times n_{\text{dof}}}$ selects the mode shape components corresponding to the n_{obs} observed degrees of freedom. It is assumed that measurements are available along the beam edge (red line in Fig. 1) corresponding to $n_{\text{obs}} = 445$ degrees of freedom. In practical applications, Digital Image Correlation [24] could for instance be used to obtain such extensive measurement data. The experimental data set finally consists of $n_m = 10$ eigenvalues $\bar{\lambda}_r$ and mode shapes $\bar{\boldsymbol{\phi}}_r$.

2.3. Objective function

Once the experimental modal characteristics are simulated, the stiffness parameters $\theta_{M,j}$ of the FE model are updated by reducing the discrepancy between the experimental modal characteristics and the FE model predictions. The latter correspond to the predicted eigenvalues $\lambda_r(\boldsymbol{\theta}_M)$ and mode shapes $\boldsymbol{\phi}_r(\boldsymbol{\theta}_M)$ obtained by solving the eigenvalue problem in Eq. (3) with stiffness parameters $\boldsymbol{\theta}_M$. The discrepancy between the measured and predicted modal characteristics is quantified by the residue vector $\boldsymbol{\eta}(\boldsymbol{\theta}_M) = \{\dots, \eta_{\lambda,r}, \dots, \eta_{\phi,r}, \dots\} \in \mathbb{R}^{n_m + n_m n_{\text{obs}}}$, which collects eigenvalue residuals

$$\eta_{\lambda,r} = \bar{\lambda}_r - \lambda_r(\boldsymbol{\theta}_M) \quad (6)$$

and mode shape residuals

$$\eta_{\phi,r} = \gamma_r \bar{\boldsymbol{\phi}}_r - \mathbf{L}\boldsymbol{\phi}_r(\boldsymbol{\theta}_M). \quad (7)$$

The residuals are computed after pairing the experimental and predicted modes using the Modal Assurance Criterion [25] and scaling the experimental modes with a scaling factor γ_r , which is obtained from a least-squares fit of the experimental and predicted mode shapes [6]. A set of updated stiffness parameters $\boldsymbol{\theta}_M^*$ is sought that minimizes these residuals. The problem is formulated as a non-linear least-squares problem with the following objective function $F(\boldsymbol{\theta}_M)$:

$$F(\boldsymbol{\theta}_M) = \frac{1}{2} \boldsymbol{\eta}(\boldsymbol{\theta}_M)^T \mathbf{W} \boldsymbol{\eta}(\boldsymbol{\theta}_M) = \frac{1}{2} \|\mathbf{W}^{1/2} \boldsymbol{\eta}(\boldsymbol{\theta}_M)\|_2^2. \quad (8)$$

The weighting matrix $\mathbf{W}$ is diagonal with weights $1/(\sigma_\lambda^2 \bar{\lambda}_r^2)$ and $1/(\sigma_\phi^2 \max |\gamma_r \bar{\phi}_r|^2 n_{\text{obs}})$ for the eigenvalue and mode shape residuals, respectively. Furthermore, the factor $1/n_{\text{obs}}$ in the weights of the mode shape residuals ensures equal weight is given to eigenvalue and mode shape residuals, since the latter consist of n_{obs} components. The weighting thus accounts for the measurement error on the modal characteristics, but does not correspond to the statistically-optimal weighting for the given noise model in Eqs. (4) and (5) [26]. The gradient of the objective function depends on the eigenvalue and mode shape sensitivities which are computed following the method of Nelson [27].

2.4. Optimization problem

The experimental data set is split in two subsets. A subset $\mathcal{R}_{\text{cal}}$ is used for the calibration of the stiffness parameters, while the remaining modes $\mathcal{R}_{\text{val}}$ are used for the validation of the updated FE model afterwards. This procedure is generally known as holdout validation. The subsets are selected randomly resulting in $\mathcal{R}_{\text{cal}} = \{1, 3, 4, 6, 8, 9\}$ and $\mathcal{R}_{\text{val}} = \{2, 5, 7, 10\}$ for a total of $n_m = 10$ modes. For the calibration of the stiffness parameters, the objective function in Eq. (8) becomes:

$$F_{\text{cal}}(\boldsymbol{\theta}_M) = \frac{1}{2n_{\text{cal}}} \left(\sum_{r \in \mathcal{R}_{\text{cal}}} \frac{\eta_{\lambda,r}^2}{\sigma_\lambda^2 \bar{\lambda}_r^2} + \sum_{r \in \mathcal{R}_{\text{cal}}} \frac{1}{\sigma_\phi^2 \max |\gamma_r \bar{\phi}_r|^2 n_{\text{obs}}} \sum_{l=1}^{n_{\text{obs}}} [\boldsymbol{\eta}_{\phi,r}]_l^2 \right). \quad (9)$$

after dividing by the number of modes n_{cal} in $\mathcal{R}_{\text{cal}}$. The $[\cdot]_l$ notation indicates the selection of the l -th component of the vector inside the brackets. The non-linear optimization problem is formulated as:

$$\begin{aligned} \min_{\boldsymbol{\theta}_M} \quad & F_{\text{cal}}(\boldsymbol{\theta}_M) \\ \text{s.t.} \quad & 0 < \theta_{M,j} \quad \text{for } j = 1, \dots, n_e. \end{aligned}$$

The objective function in Eq. (9) is minimized using a trust region reflective algorithm which is implemented in the MATLAB function `lsqnonlin`. Convergence is reached when the following criteria are met [28]:

1. objective function residue: $|F_{\text{cal}}(\boldsymbol{\theta}_M^{(k)}) - F_{\text{cal}}(\boldsymbol{\theta}_M^{(k+1)})| \leq 10^{-6}(1 + |F_{\text{cal}}(\boldsymbol{\theta}_M^{(k)})|),$
2. step size: $\|\boldsymbol{\theta}_M^{(k)} - \boldsymbol{\theta}_M^{(k+1)}\|_2 \leq 10^{-6}(1 + \|\boldsymbol{\theta}_M^{(k)}\|_2),$
3. first-order optimality: $\|\nabla F_{\text{cal}}(\boldsymbol{\theta}_M^{(k+1)})\|_\infty \leq 10^{-6} \|\nabla F_{\text{cal}}(\boldsymbol{\theta}_M^{(0)})\|_\infty.$

The superscript (k) denotes the k -th iteration. Furthermore, the number of iterations is limited to 50. The updated stiffness parameters are indicated as $\boldsymbol{\theta}_M^*$.

The FE model, updated based on the calibration data set, is subsequently used to predict the modal characteristics of the damaged structure. The objective function in Eq. (9) is evaluated for

the experimental modal characteristics in the validation subset $\mathcal{R}_{\text{val}}$, after properly matching the predicted and measured mode shapes using the MAC criterion. This yields the validation residue $F_{\text{val}}(\boldsymbol{\theta}_{\text{M}}^*)$, which is a measure for the predictive performance of the updated model. In Sections 3 and 4, the regularization parameters are determined by selecting the minimum of the validation residue $F_{\text{val}}(\boldsymbol{\theta}_{\text{M}}^*)$ (holdout validation).

3. Tikhonov regularization

In this section, Tikhonov regularization is applied to improve the conditioning of the inverse problem [14]. If no regularization is applied, the updated stiffness parameters are unrealistic due to overfitting. The updated FE model then fails to predict the modal characteristics used for validation ($r \in \mathcal{R}_{\text{val}}$). Tikhonov regularization is frequently applied to FE model updating problems [8, 16, 29]. The method is briefly discussed and illustrated using the numerical case study of the simply supported beam.

3.1. Regularized objective function

Tikhonov regularization adds a penalty term to the objective function which guides the updated stiffness parameters $\boldsymbol{\theta}_{\text{M}}^*$ to an initial estimate $\boldsymbol{\theta}_{\text{M}}^{(0)}$ of the stiffness parameters. Assuming the Young's modulus E_0 of the undamaged material is approximately known, $\boldsymbol{\theta}_{\text{M}}^{(0)} = 1$ is a good estimate of the solution if the damaged zone is small compared to the undamaged zone. The regularized objective function becomes:

$$F_{\text{reg}}(\boldsymbol{\theta}_{\text{M}}) = F_{\text{cal}}(\boldsymbol{\theta}_{\text{M}}) + \beta^2 \|\boldsymbol{\theta}_{\text{M}} - \boldsymbol{\theta}_{\text{M}}^{(0)}\|_2^2 \quad (10)$$

where β is called the regularization parameter. For very small values of β , the problem converges to the unregularized solution as the second term in Eq. (10) vanishes. For high values of β on the other hand, the solution tends to the initial estimate $\boldsymbol{\theta}_{\text{M}}^{(0)}$. A trade-off is made by selecting a proper regularization parameter β .

3.2. Numerical case study

The regularized objective function $F_{\text{reg}}(\boldsymbol{\theta}_{\text{M}})$ in Eq. (10) is minimized for each of the 50 noise patterns and for 15 logarithmically sampled values of the regularization parameter β between 10^{-2} and 10^1 . Three initial estimates of the solution are considered: $\boldsymbol{\theta}_{\text{M}}^{(0)} = 0.9$, $\boldsymbol{\theta}_{\text{M}}^{(0)} = 1$, and $\boldsymbol{\theta}_{\text{M}}^{(0)} = 1.1$, corresponding to an underestimation, a perfect estimation and an overestimation of the initial stiffness, respectively. These estimates $\boldsymbol{\theta}_{\text{M}}^{(0)}$ are also used as the starting point for the optimization algorithm.

In order to compare the accuracy of the updated stiffness parameters $\boldsymbol{\theta}_{\text{M}}^*$, a root mean square (RMS) value θ_{RMS} is computed based on the stiffness parameters $\hat{\boldsymbol{\theta}}_{\text{M}}$ of the forward problem:

$$\theta_{\text{RMS}} = \sqrt{\frac{1}{n_e} \sum_{j=1}^{n_e} (\hat{\theta}_{\text{M},j} - \theta_{\text{M},j}^*)^2}. \quad (11)$$

Low RMS values indicate that the updated stiffness parameters $\boldsymbol{\theta}_{\text{M}}^*$ correspond well to the stiffness parameters $\hat{\boldsymbol{\theta}}_{\text{M}}$, which represent the actual damage in the beam. The solution with minimum RMS value is considered the best possible solution.

Fig. 2 shows the average RMS value θ_{RMS} (for the 50 noise patterns) as a function of the regularization parameter β for three initial estimates $\theta_{\text{M}}^{(0)}$. Regardless of the initial estimate $\theta_{\text{M}}^{(0)}$, the average RMS value θ_{RMS} is minimal when the regularization parameter β equals approximately 0.3. Furthermore, a poor initial estimate $\theta_{\text{M}}^{(0)}$ results in higher RMS values. The computation of θ_{RMS} is, however, not possible for practical applications since $\hat{\theta}_{\text{M}}$ is unknown. For practical applications, the regularization parameter β can be determined using holdout validation. The regularization parameter β is selected based on the calibration model with the minimal validation residue $F_{\text{val}}(\theta_{\text{M}}^*)$.

Fig. 3 shows the average calibration residue $F_{\text{cal}}(\theta_{\text{M}}^*)$ and the average validation residue $F_{\text{val}}(\theta_{\text{M}}^*)$ as a function of the regularization parameter β for three initial estimates $\theta_{\text{M}}^{(0)}$. The calibration and validation residues after optimization are computed by evaluating the objective function in Eq. (9) for the updated stiffness parameters θ_{M}^* , which is equivalent to omitting the regularization term in Eq. (10). The regularization parameter β in Fig. 3 is the value used in the optimization of the corresponding calibration set. The figure shows that the optimal regularization parameter β slightly depends on the initial estimate $\theta_{\text{M}}^{(0)}$. Selecting the regularization parameter β using holdout validation (Fig. 3) results in suboptimal RMS values θ_{RMS} (Fig. 2). Furthermore, increasing β reduces the variability of the validation residue $F_{\text{val}}(\theta_{\text{M}}^*)$ and increases the calibration residue $F_{\text{cal}}(\theta_{\text{M}}^*)$, since the updated stiffness parameters θ_{M}^* tend to the initial estimate $\theta_{\text{M}}^{(0)}$.

Fig. 2: Average RMS value θ_{RMS} (black solid line) as a function of the regularization parameter β for (a) $\theta_{\text{M}}^{(0)} = 0.9$, (b) $\theta_{\text{M}}^{(0)} = 1$, and (c) $\theta_{\text{M}}^{(0)} = 1.1$. The RMS value θ_{RMS} for each of the 50 noise patterns is shown in gray as an indication of uncertainty. The dotted line indicates the minimum of the average RMS value θ_{RMS} .

Fig. 3: Average calibration residue $F_{\text{cal}}(\theta_{\text{M}}^*)$ (black solid line) and average validation residue $F_{\text{val}}(\theta_{\text{M}}^*)$ (black dashed line) as a function of the regularization parameter β for (a) $\theta_{\text{M}}^{(0)} = 0.9$, (b) $\theta_{\text{M}}^{(0)} = 1$, and (c) $\theta_{\text{M}}^{(0)} = 1.1$. The calibration and validation residues for each of the 50 noise patterns are shown in gray as an indication of uncertainty. The dotted line indicates the minimum of the average validation residue $F_{\text{val}}(\theta_{\text{M}}^*)$.

Fig. 4 shows the updated stiffness parameters θ_{M}^* for 5 out of 50 randomly selected noise patterns. The initial estimate of the stiffness parameters is $\theta_{\text{M}}^{(0)} = 1$. For each noise pattern,

the regularization parameter β has been determined by selecting the minimum of the RMS value θ_{RMS} resulting in the best possible or reference solution. These solutions are shown in Fig. 4a. On the other hand, Fig. 4b shows the solutions obtained with holdout validation (selecting the calibrated model with the lowest validation residue $F_{\text{val}}(\boldsymbol{\theta}_{\text{M}}^*)$). Hence, the selected regularization parameter β depends on the noise pattern and the selection method. Using holdout validation, the updated stiffness parameters $\boldsymbol{\theta}_{\text{M}}^*$ show strong local variations and noise dependency in some cases. In other cases, the solution is completely smooth and approximately equal to $\hat{\boldsymbol{\theta}}_{\text{M}}$, caused by the selection of high values for β . However, the reference solutions corresponding to minimal RMS value θ_{RMS} are very similar and less noise dependent. This illustrates the difficulty of selecting a proper regularization parameter β using holdout validation. This is even more difficult if the initial estimate $\boldsymbol{\theta}_{\text{M}}^{(0)}$ is poor.

Fig. 4: Updated stiffness parameters $\boldsymbol{\theta}_{\text{M}}^*$ for five randomly selected noise patterns. The regularization parameter β is selected based on (a) the minimal RMS value and (b) the minimal validation residue $F_{\text{val}}(\boldsymbol{\theta}_{\text{M}}^*)$. The initial estimate of the stiffness parameters is $\boldsymbol{\theta}_{\text{M}}^{(0)} = 1$.

4. Density filtering regularization

In this section, the novel density filtering regularization method is presented and illustrated using the numerical case study. Furthermore, the size of the damaged zone is varied and the influence on the optimal filter radius r_{min} is investigated.

4.1. Density filtering

Density filtering is frequently used to regularize topology optimization problems. By assigning a material density $0 < \rho_e \leq 1$ to each element in the design domain, the optimal structural topology under certain constraints is found by solving an optimization problem [30]. Numerical instabilities lead to checkerboard and mesh dependent solutions if no filtering is applied [19, 21]. A possible solution is to filter each design variable ρ_e by averaging all design variables in a certain filter radius r_{min} to obtain the filtered densities $\tilde{\rho}_e$. Only the filtered densities $\tilde{\rho}_e$ have physical meaning [21].

For FE model updating problems, the density filter is applied to the stiffness parameters $\boldsymbol{\theta}_M$ in order to obtain the filtered stiffness parameters $\tilde{\boldsymbol{\theta}}_M$ [19, 31]:

$$\tilde{\theta}_{M,j} = \frac{1}{\sum_{k \in \mathcal{I}_j} w_j(\mathbf{x}_k)} \sum_{k \in \mathcal{I}_j} w_j(\mathbf{x}_k) \theta_{M,k}. \quad (12)$$

Here, $\mathcal{I}_j = \{k \mid \|\mathbf{x}_k - \mathbf{x}_j\|_2 \leq r_{\min}\}$ is the set of elements k within filter range from element j , and $\mathbf{x}_k$ indicates the center coordinates of element k . The weighting function $w_j(\mathbf{x}_k)$ is cone shaped (Fig. 5) and is mathematically expressed as:

$$w_j(\mathbf{x}_k) = \max \left[0, (r_{\min} - \|\mathbf{x}_k - \mathbf{x}_j\|_2) / L_e \right]. \quad (13)$$

The filter radius $r_{\min}$ controls the minimal length scale of the solution $\boldsymbol{\theta}_M^*$. It is expressed in physical units (in this case meters). Furthermore, the gradient of the objective function $F_{\text{cal}}(\tilde{\boldsymbol{\theta}}_M)$ with respect to the unfiltered stiffness parameters $\boldsymbol{\theta}_M$ is computed using the chain rule [19, 31]:

$$\frac{\partial F_{\text{cal}}(\tilde{\boldsymbol{\theta}}_M)}{\partial \theta_{M,i}} = \sum_{j \in \mathcal{I}_i} \frac{\partial F_{\text{cal}}(\tilde{\boldsymbol{\theta}}_M)}{\partial \tilde{\theta}_{M,j}} \frac{\partial \tilde{\theta}_{M,j}}{\partial \theta_{M,i}} = \sum_{j \in \mathcal{I}_i} \frac{1}{\sum_{k \in \mathcal{I}_j} w_j(\mathbf{x}_k)} w_j(\mathbf{x}_i) \frac{\partial F_{\text{cal}}(\tilde{\boldsymbol{\theta}}_M)}{\partial \tilde{\theta}_{M,j}}. \quad (14)$$

The procedure of density filtering regularization is summarized in Algorithm 1. An efficient MATLAB implementation of topology optimization problems using the density filter is provided by Andreassen et al. [31].

4.2. Numerical case study

The objective function $F_{\text{cal}}(\boldsymbol{\theta}_M)$ in Eq. (9) is minimized for the 50 noise patterns while applying density filtering regularization with a filter radius $r_{\min}$ varying between 0.06 m to 1.5 m with steps of 0.03 m, which corresponds to the element side lengths L_e . An optimal filter radius $r_{\min}$ is selected using holdout validation. The results are compared with the best possible solutions $\tilde{\boldsymbol{\theta}}_M^*$ with minimal RMS values θ_{RMS} .

Fig. 6 shows the average RMS value θ_{RMS} (for the 50 noise patterns) as a function of the filter radius $r_{\min}$ for three initial estimates $\boldsymbol{\theta}_M^{(0)}$, which are solely used as the starting point in the

Fig. 5: Cone shaped weighting function $w_j(\mathbf{x})$ with filter radius $r_{\min} = 2L_e$ and L_e the side lengths of each element. The value of the weighting function decreases from 2 at the top of the cone to 0 at its base.

Algorithm 1 Density filtering regularization

```

Initialize  $\boldsymbol{\theta}_M^{(0)}$ 
repeat
  Filter  $\boldsymbol{\theta}_M^{(k)}$  to obtain  $\tilde{\boldsymbol{\theta}}_M^{(k)}$  Eq. (12)
  Compute  $F_{\text{cal}}(\tilde{\boldsymbol{\theta}}_M^{(k)})$  Eq. (9)
  Compute the gradient  $\frac{\partial F_{\text{cal}}(\tilde{\boldsymbol{\theta}}_M^{(k)})}{\partial \tilde{\boldsymbol{\theta}}_M^{(k)}}$ 
  Apply the chain rule to obtain  $\frac{\partial F_{\text{cal}}(\tilde{\boldsymbol{\theta}}_M^{(k)})}{\partial \boldsymbol{\theta}_M^{(k)}}$  Eq. (14)
  Compute  $\boldsymbol{\theta}_M^{(k+1)}$ 
   $k \leftarrow k + 1$ 
until convergence
  
```

optimization algorithm. The calibration and validation residues after optimization are computed by evaluating the objective function in Eq. (9) for the updated, filtered stiffness parameters $\tilde{\boldsymbol{\theta}}_M^*$. The average RMS value θ_{RMS} shows a dip at rather low filter radii and then increases with increasing filter radius r_{min} . The best possible solutions $\tilde{\boldsymbol{\theta}}_M^*$ are obtained with a small filter radius r_{min} , regardless of the initial estimate $\boldsymbol{\theta}_M^{(0)}$. The magnitude of the minimal average RMS value is similar for both regularization methods (Figs. 2 and 6). Hence, both methods perform equally well when the optimal regularization parameter is selected. High RMS values are observed when the filter radius is very low. Since no upper bound is imposed on the model parameters $\boldsymbol{\theta}_M$, the stiffness parameters may reach arbitrarily high values in zones where they do not significantly affect the modal characteristics (e.g. the upper corners of the beam). The RMS value θ_{RMS} of these solutions is significantly influenced by these large deviations and is not necessarily a good measure of the overall fit between the updated stiffness parameters $\boldsymbol{\theta}_M^*$ and the true stiffness parameters $\tilde{\boldsymbol{\theta}}_M$. However, in zones where the modal characteristics are sensitive to stiffness variations, damage is very likely to be well identified. If the filter radius increases, the updated stiffness parameters become less noise dependent and this behavior is no longer observed.

Fig. 7 shows the average calibration residue $F_{\text{cal}}(\tilde{\boldsymbol{\theta}}_M^*)$ and the average validation residue $F_{\text{val}}(\tilde{\boldsymbol{\theta}}_M^*)$ as a function of the filter radius r_{min} for three initial estimates $\boldsymbol{\theta}_M^{(0)}$. Both residues increase with increasing filter radius r_{min} . A large filter radius entirely smooths out the damaged zone, which results in a constant stiffness variation over the height of the beam. The uncertainty of the validation residue $F_{\text{val}}(\tilde{\boldsymbol{\theta}}_M^*)$ is large compared to the calibration residue $F_{\text{cal}}(\tilde{\boldsymbol{\theta}}_M^*)$, as was the case for Tikhonov regularization (Fig. 3). Both the calibration residue $F_{\text{cal}}(\tilde{\boldsymbol{\theta}}_M^*)$ and the validation residue $F_{\text{val}}(\tilde{\boldsymbol{\theta}}_M^*)$ are virtually independent of the initial estimate $\boldsymbol{\theta}_M^{(0)}$. This was not the case for Tikhonov regularization (Fig. 3). This highlights a major benefit of the proposed method: an initial estimate $\boldsymbol{\theta}_M^{(0)}$ is only required as starting point of the optimization algorithm. Furthermore, the validation residue $F_{\text{val}}(\tilde{\boldsymbol{\theta}}_M^*)$ is minimal for a low filter radius r_{min} .

Fig. 8 shows the updated stiffness parameters $\tilde{\boldsymbol{\theta}}_M^*$ for 5 out of 50 noise patterns. The initial estimate of the stiffness parameters is $\boldsymbol{\theta}_M^{(0)} = 1$. The selected noise patterns correspond to those in Fig. 4 in order to make a fair comparison between both regularization methods. For each noise pattern, the regularization parameter r_{min} has been determined by selecting the minimum of the RMS value θ_{RMS} resulting in the best possible or reference solution. These solutions are shown in

Fig. 6: Average RMS value θ_{RMS} (black solid line) as a function of the filter radius r_{min} for (a) $\theta_{\text{M}}^{(0)} = 0.9$, (b) $\theta_{\text{M}}^{(0)} = 1$, and (c) $\theta_{\text{M}}^{(0)} = 1.1$. The RMS value θ_{RMS} for each of the 50 noise patterns is shown in gray as an indication of uncertainty. The dotted line indicates the minimum of the average RMS value θ_{RMS} .

Fig. 7: Average calibration residue $F_{\text{cal}}(\tilde{\theta}_{\text{M}}^*)$ (black solid line) and validation residue $F_{\text{val}}(\tilde{\theta}_{\text{M}}^*)$ (black dashed line) as a function of the filter radius r_{min} for (a) $\theta_{\text{M}}^{(0)} = 0.9$, (b) $\theta_{\text{M}}^{(0)} = 1$, and (c) $\theta_{\text{M}}^{(0)} = 1.1$. The calibration and validation residues for each of the 50 noise patterns are shown in gray as an indication of uncertainty. The dotted line indicates the minimum of the average validation residue $F_{\text{val}}(\tilde{\theta}_{\text{M}}^*)$.

Fig. 8a. On the other hand, Fig. 8b shows the solutions obtained with holdout validation. Hence, the selected regularization parameter r_{min} depends on the noise pattern and the selection method. The damaged zone is very well identified in both cases, since low filter radii are selected using holdout validation. The selected filter radius r_{min} varies between 0.06 and 0.24 m, depending on the noise pattern and selection criterion for r_{min} . It can be concluded that it is easier to select a proper filter radius r_{min} for density filtering than a proper regularization parameter β for Tikhonov regularization when using holdout validation.

4.3. Influence of the size of the damaged zone

The filter radius r_{min} defines the minimal length scale of the identifiable damaged zone. Therefore, the optimal filter radius will change depending on the size of the damaged zone. In addition to the damage case considered in the previous section, two cases are analyzed: a smaller damaged zone of 0.3 m by 0.12 m (or 10 by 4 elements) and a larger damaged zone of 1.5 m by 0.3 m (or 50 by 10 elements). The mesh and the stiffness reduction of 30% in the damaged zone remain unchanged. An initial estimate $\theta_{\text{M}}^{(0)} = 1$ is assumed.

Fig. 9 shows the average RMS value θ_{RMS} (for the 50 noise patterns) as a function of the filter radius r_{min} . The minimum of the average RMS value is obtained for a low filter radius r_{min} regardless of the size of the damaged zone. The average RMS value θ_{RMS} generally increases with increasing filter radius r_{min} after having reached its minimal value. Finally, it is observed that the average RMS value is generally higher if the size of the damaged zone increases. Due to the density filter, the transition from undamaged to damaged material is not abrupt, but smoothed with a

Fig. 8: Updated stiffness parameters $\tilde{\theta}_M^*$ for five randomly selected noise patterns. The filter radius $r_{\min}$ is selected based on (a) the minimal RMS value and (b) the minimal validation residue $F_{\text{val}}(\tilde{\theta}_M^*)$. The initial estimate of the stiffness parameters is $\theta_M^{(0)} = 1$.

certain filter radius. Typically, this transition zone (“gray zone”) spans a couple of elements. The effect of this transition zone on θ_{RMS} is larger when the damaged zone is larger.

Fig. 9: Average RMS value θ_{RMS} (black solid line) as a function of the filter radius $r_{\min}$ for (a) the smaller damaged zone, (b) the reference damaged zone, and (c) the larger damaged zone. The RMS value θ_{RMS} for each of the 50 noise patterns is shown in gray as an indication of uncertainty. The initial estimate of the stiffness parameters is $\theta_M^{(0)} = 1$. The dotted line indicates the minimum of the average RMS value θ_{RMS} .

Fig. 10 shows the average calibration residue $F_{\text{cal}}(\tilde{\theta}_M^*)$ and the average validation residue $F_{\text{val}}(\tilde{\theta}_M^*)$ as a function of the filter radius $r_{\min}$ for the smaller, reference, and larger damaged zone. The average validation residue $F_{\text{val}}(\tilde{\theta}_M^*)$ starts to increase when the filter radius $r_{\min}$ corresponds to the smallest dimension of the damaged zone (i.e. the height). For the larger damaged zone, the average validation residue $F_{\text{val}}(\tilde{\theta}_M^*)$ decreases again and reaches a new local minimum when the filter radius $r_{\min}$ equals the width of the damaged zone. This behavior is not observed for the smaller damaged zone or the reference damaged zone, since the height of the damaged zone is much lower than the beam height in these two cases: using a large filter radius would smooth out the damaged zone over the entire height of the beam. Hence, for the larger damaged zone, the optimal filter radius $r_{\min}$ can be low or high depending on the noise pattern. However, the average calibration residue $F_{\text{cal}}(\tilde{\theta}_M^*)$ increases monotonically, as is also the case for the smaller and reference damaged zone. Therefore, it can be concluded that a low filter radius $r_{\min}$ leads to more

Fig. 10: Average calibration residue $F_{\text{cal}}(\tilde{\theta}_M^*)$ (black solid line) and average validation residue $F_{\text{val}}(\tilde{\theta}_M^*)$ (black dashed line) as a function of the filter radius r_{min} for (a) the smaller damaged zone, (b) the reference damaged zone, and (c) the larger damaged zone. The calibration and validation residues for each of the 50 noise patterns are shown in gray as an indication of uncertainty. The initial estimate of the stiffness parameters is $\theta_M^{(0)} = 1$. The dotted line indicates the minimum of the average validation residue $F_{\text{val}}(\tilde{\theta}_M^*)$.

accurate damage detection than a high filter radius r_{min} .

It can be concluded that selecting a low filter radius r_{min} results in accurate damage identification for both small and large damaged zones. Because the modal characteristics are not very sensitive to small local reductions in stiffness, these are much harder to identify. The selection of a proper filter radius is therefore difficult, as can be observed by comparing Figs. 9a and 10a. For practical applications, the filter radius r_{min} is selected from the calibration residue $F_{\text{cal}}(\tilde{\theta}_M^*)$ and the validation residue $F_{\text{val}}(\tilde{\theta}_M^*)$. If the selected filter radius r_{min} is too large, small damaged zones are smoothed out and no longer identified. It is therefore preferred to impose an upper bound on the filter radius based on (1) the calibration residue $F_{\text{cal}}(\tilde{\theta}_M^*)$ and (2) the dimensions of the physical domain.

5. Comparison of both regularization methods

In this section, a statistical comparison is made between the performance of the Tikhonov regularization and the density filtering regularization method. For each of the 50 noise patterns, the regularization parameters β and r_{min} are selected based on the calibrated models with minimal validation residues $F_{\text{val}}(\theta_M^*)$ and $F_{\text{val}}(\tilde{\theta}_M^*)$, respectively. An upper bound of 0.36 m, corresponding to the beam height, is hereby imposed on the filter radius r_{min} . Subsequently, the RMS values of the 50 obtained solutions θ_M^* and $\tilde{\theta}_M^*$ are computed. From these RMS values, cumulative distribution functions (CDFs) are estimated. The CDFs indicate the probability of the RMS value θ_{RMS} being lower than a certain threshold value.

Fig. 11 shows the CDFs computed with Tikhonov and density filtering regularization for the smaller, reference, and larger damaged zone. The CDFs characterize the performance of both regularization methods when using holdout validation to determine the regularization parameter β or the filter radius r_{min} . Fig. 11a shows that both methods perform similarly when the damaged zone is small. In a small number of cases, density filtering performs better. Fig. 11b shows that the performance of both methods is also similar for the reference damaged zone. In some cases, the RMS values θ_{RMS} obtained with density filtering are higher. However, the damaged zone is still well identified (Fig. 13). Fig. 11c shows that the CDFs start deviating for the larger damaged zone, and that the RMS value θ_{RMS} for Tikhonov regularization is higher than the RMS value for density filtering regularization.

Fig. 11: Cumulative distribution function (CDF) of the RMS value θ_{RMS} corresponding to the minimal validation residue $F_{\text{val}}(\theta_M^*)$ for Tikhonov regularization (gray solid line). Cumulative distribution function (CDF) of the RMS value θ_{RMS} corresponding to the minimal validation residue of $F_{\text{val}}(\tilde{\theta}_M^*)$ for density filtering regularization (black solid line). The CDFs are computed for (a) the smaller damaged zone, (b) the reference damaged zone, and (c) the larger damaged zone. The initial estimate of the stiffness parameters is $\theta_M^{(0)} = 1$. The markers on the curves correspond to the updated stiffness parameters shown in Figs. 12 to 14. The gray and black triangle indicate the average of the 50 RMS values for Tikhonov and density filtering regularization, respectively.

Fig. 12 shows five solutions θ_M^* and $\tilde{\theta}_M^*$ computed with Tikhonov regularization and density filtering regularization, respectively, for the case of the smaller damaged zone. The optimal regularization parameter β and filter radius r_{min} were determined using holdout validation. When using Tikhonov regularization, the smaller damaged zone is not accurately identified. Three out of five times, a high regularization parameter β is selected leading to completely smooth solutions θ_M^* . Density filtering regularization, however, identifies the smaller damaged zone five out of five times. The RMS values of the shown solutions are indicated on the respective CDFs in Fig. 11a.

Fig. 13 shows five solutions θ_M^* and $\tilde{\theta}_M^*$ computed with Tikhonov regularization and density filtering regularization, respectively, for the case of the reference damaged zone. When using Tikhonov regularization, strong local stiffness variations are sometimes observed. Density filtering regularization on the other hand results in a very smooth stiffness distribution in the beam, from which the damaged zone is easily identified. Overall, the updated stiffness parameters are less noise dependent when density filtering regularization is applied. The RMS values of the shown solutions are situated on the respective CDFs in Fig. 11b.

Fig. 14 shows five solutions for the case of the larger damaged zone. When using Tikhonov regularization, the damaged zone is recognized relatively well, but the RMS values are very high due to the strong local stiffness variations. The solutions $\tilde{\theta}_M^*$ obtained with density filtering regularization are smooth and less noise dependent. The damaged zone is easily identified four out of five times.

6. Conclusion

Finite element model updating was used to detect structural damage. In order to avoid ill-conditioning, two regularization methods were discussed and illustrated using a numerical case study: Tikhonov regularization and density filtering regularization. This paper has shown that the selection of a proper regularization parameter for Tikhonov regularization using holdout validation is difficult, as it results in solutions with suboptimal RMS values. On the other hand, the selection of the filter radius r_{min} for density filtering regularization is more intuitive because it has a clear physical meaning, which is not the case for most other regularization parameters. It is

Fig. 12: (a) Updated stiffness parameters θ_M^* for Tikhonov regularization and (b) updated stiffness parameters $\tilde{\theta}_M^*$ for density filtering regularization for five randomly selected noise patterns corresponding to the smaller damaged zone. These solutions are also indicated with the corresponding markers in Fig. 11a. The initial estimate of the stiffness parameters is $\theta_M^{(0)} = 1$.

Fig. 13: (a) Updated stiffness parameters θ_M^* for Tikhonov regularization and (b) updated stiffness parameters $\tilde{\theta}_M^*$ for density filtering regularization for five randomly selected noise patterns corresponding to the reference damaged zone. These solutions are also indicated with the corresponding markers in Fig. 11b. The initial estimate of the stiffness parameters is $\theta_M^{(0)} = 1$.

concluded that a low filter radius $r_{\min}$ is generally preferred. A large filter radius $r_{\min}$ prohibits the identification of small damaged zones. Furthermore, the updated stiffness parameters θ_M^* depend strongly on the initial estimate $\theta_M^{(0)}$ of the solution when applying Tikhonov regularization, since this estimate is included in a penalty term in the objective function. This is not the case for density filtering regularization, where the initial estimate $\theta_M^{(0)}$ is solely used as a starting point for the optimization algorithm. Although the proposed density filtering regularization method has only been investigated for one numerical case study, it shows great potential for practical applications.

Fig. 14: (a) Updated stiffness parameters θ_M^* for Tikhonov regularization and (b) updated stiffness parameters $\tilde{\theta}_M^*$ for density filtering regularization for five randomly selected noise patterns corresponding to the larger damaged zone. These solutions are also indicated with the corresponding markers in Fig. 11c. The initial estimate of the stiffness parameters is $\theta_M^{(0)} = 1$.

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