The uniqueness of Weierstrass points with semigroup $< a; b >$ and related semigroups.

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Abstract

Assume a and $b = na + r$ with $n \geq 1$ and $0 < r < a$ are relatively prime integers. In case C is a smooth curve and P is a point on C with Weierstrass semigroup equal to $< a; b >$ then C is called a $C_{a;b}$-curve. In case $r \neq a - 1$ and $b \neq a + 1$ we prove C has no other point $Q \neq P$ having Weierstrass semigroup equal to $< a; b >$, in which case we say that the Weierstrass semigroup $< a; b >$ occurs at most once. The curve $C_{a;b}$ has genus $(a - 1)(b - 1)/2$ and the result is generalized to genus $g < (a-1)(b-1)/2$. We obtain a lower bound on g (sharp in many cases) such that all Weierstrass semigroups of genus g containing $< a; b >$ occur at most once.

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1 Introduction

We write $\mathbb{N}$ to denote the semigroup of non-negative integers (in particular including 0). A subsemigroup H of $\mathbb{N}$ is called a Weierstrass semigroup of genus g if the complement $\mathbb{N} \setminus H$ is a finite set of exactly g integers. Let C be a smooth curve of genus g and let $\mathcal{O}_C$ be the sheaf of regular functions on C. Let P be a point on C and consider $\{\deg(f)_0 : f \in \mathcal{O}_C(C \setminus \{P\})\}$. This is a Weierstrass semigroup of genus g called the Weierstrass semigroup of P and denoted by $\mathrm{WS}(P)$. In case $f \in \mathcal{O}_C(C \setminus \{P\})$ is not a constant then it defines a morphism $f : C \to \mathbb{P}^1$ with $f^{-1}(\infty) = \{P\}$ and introducing multiplicities for points on fibers of the morphism one obtains a base point free linear system $g^1_{\deg(f)}$ on C containing the divisor $\deg(f)P$. Therefore the Weierstrass semigroup of P can also be described as follows

$$\mathrm{WS}(P) = \{a \in \mathbb{N} : |aP| \text{ is a base point free linear system } \} \cup \{0\} .$$

The elements of $\mathbb{N} \setminus \mathrm{WS}(P)$ are called the gaps of P (and the elements of $\mathrm{WS}(P)$ are called the non-gaps of P). For all but finitely many points of C the set of

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gaps of P is equal to $\{1;2;\cdots;g\}$. A point P is called a Weierstrass point of C in case the set of gaps of P is different from $\{1;2;\cdots;g\}$. (For a more detailed introduction see e.g. [7] Section III-5.)

For a general curve C the set of gaps of each Weierstrass point is equal to $\{1;2;\cdots;g-1;g+1\}$. The most special curves are the hyperelliptic curves, i.e. curves having a morphism $f:C\rightarrow\mathbb{P}^1$ of degree 2. In case $g\geq 2$ such morphism is unique (if it exists) and the Weierstrass points are exactly the $2g+2$ ramification points of f. In this case the set of gaps of each Weierstrass point is equal to $\{1;3;5;\cdots;2g-1\}$. Hence the Weierstrass semigroup is the subsemigroup of $\mathbb{N}$ generated by 2 and $2g+1$ (denoted by $<2;2g+1>$). It is the only Weierstrass semigroup of genus g having first non-gap equal to 2.

From this point of view the next case is to consider Weierstrass points P with first non-gap equal to three. In this case the curve C needs to have a base point free linear system g^1_3 containing $3P$, i.e. there exists a covering $f:C\rightarrow\mathbb{P}^1$ of degree 3 having P as a total ramification point. Curves having a morphism to $\mathbb{P}^1$ of degree 3 are called trigonal curves. In case $g\geq 5$ then the linear system g^1_3 is unique. However in general a g^1_3 does not need to have a total ramification point and if it has a total ramification point then in general it is unique. Therefore the situation is different from the situation of hyperelliptic curves and the linear system g^1_3 does not determine all Weierstrass points on the curve. Moreover in case there is a total ramification point P then $\mathrm{WS}(P)$ is not completely determined by g and in general not even by f. Therefore in case f has at least two total ramification points then their Weierstrass semigroups can be different.

In [3] all possibilities of combinations of Weierstrass semigroups with first non-gap equal to 3 that can occur on some fixed curve of genus $g\geq 5$ are determined. In particular in case P has Weierstrass semigroup $<3;3n+1>$ (in this case the genus of C is equal to $3n$) then there is no other point Q on C with $\mathrm{WS}(Q)=<3;3n+1>$ (and this situation occurs). It is mentioned at the introduction of [18] that this fact is proved in [10]. It seems to me that this is not explicitly mentioned in that paper. The computations in [10] to obtain Theorem 6 of that paper imply that in case C has genus $3n$ and there is a covering $f:C\rightarrow\mathbb{P}^1$ of degree 3 having $g+2$ total ramification points then exactly one of them has Weierstrass semigroup equal to $<3;3n+1>$. From [3] (and also from [10]) it follows that for all other Weierstrass semigroups H with first non-gap equal to 3 there exist curves C having at least two points with Weierstrass semigroup equal to H.

We make the following definition

Definition 1. Let H be a Weierstrass semigroup of genus g. We say that H occurs at most once in case there exists no curve C of genus g having two different Weierstrass points P and Q with $\mathrm{WS}(P)=\mathrm{WS}(Q)=H$.

There is no Weierstrass semigroup with first non-gap equal to 2 that occurs at most once (this corresponds to the hyperelliptic curves mentioned before). The Weierstrass semigroups with first non-gap equal to 3 that occur at most once are exactly the semigroups $< 3; 3n + 1 >$ with $n \geq 2$ an integer (this corresponds to the trigonal curves mentioned before). Its genus is equal to $3n$.

In [18] the author gives a lot of Weierstrass semigroups H of some genus g with first non-gap some prime number a that occur at most once. As an example this result holds for semigroups $< a; ka - 2 >$ for any integer $k \geq 2$. More generally from the arguments in [18] it follows that for a prime number a and $b = ka - r$ with $k \geq 2$ and $2 \leq r \leq a - 1$ and $r \neq a - 1$ in case $k = 2$ there are at most $r - 1$ Weierstrass points having Weierstrass semigroup equal to $< a; b >$ on a curve C of genus $g = (a-1)(b-1)/2$ (this is indeed the number of gaps in case the Weierstrass semigroup is equal to $< a; b >$). In case $r \neq 2$ this upper bound is not sharp. In particular in [5] , Theorem 1, it is proved that in case $a \geq 5$ is any odd integer then $< a; a + 2 >$ occurs at most once. This is smaller than the bound obtained in [18] in case $a \geq 5$ is a prime number.

One of the main results of this paper is the following theorem.

Theorem A. Let $a; b$ be relatively prime integers (we denote it by $(a; b) = 1$) such that $b \geq a + 2$. Assume $b = ka + r$ with $1 \leq r \leq a - 2$. The Weierstrass semigroup $< a; b >$ (having genus $(a - 1)(b - 1)/2$) occurs at most once.

In case $b = a + 1$ or $r = a - 1$ then there exist smooth curves of genus $(a-1)(b-1)/2$ having more than one Weierstrass point with Weierstrass semigroup equal to $< a; b >$. The proofs in [18] consist of two steps. Under the assumptions of [18] (amongst others a is a prime number) the linear system g^1_a is unique on the curve. Then given some fixed linear system g^1_a on the curve, the author proves the upper bound on the number of total ramification points of g^1_a having Weierstrass semigroup $< a; b >$. In case $(a; b) = 1$ and $b \neq a + 2$, the uniqueness of g^1_a in case a curve C of genus $(a-1)(b-1)/2$ has a Weierstrass point P with $\mathrm{WS}(P) =< a; b >$ follows from results from [2] (see Theorem 10.1 for the relation). However we give an independent proof inspired by [18] but using seemingly easier arguments and not using the assumption that a is a prime number. So to prove Theorem A, we only need to consider total ramification points on a fixed g^1_a. Using more complicated computations than ours, Theorem 1 is proved in [15] for the case of Galois Weierstrass points (meaning the morphism $C \to \mathbb{P}^1$ defined by $|aP|$ defines a Galois extension $\mathbb{C}(\mathbb{P}^1) \subset \mathbb{C}(C)$).

Smooth curves C having a Weierstrass point P with $\mathrm{WS}(P) =< a, b >$ in case $(a, b) = 1$ are also called $C_{a,b}$ curves. They are studied from different points of view (see e.g. [16], [6], [8], [19], [17]). In [11] and [12] the similar nodal curves are used to develop a general method to study Weierstrass points.

For lower genus cases $g < (a-1)(b-1)/2$ with $(a; b) = 1$ and $b = na+r$ with $n \geq 1$ and $1 \leq r \leq a-1$ we consider the following situation. Let C be a smooth curve of genus g and let $P \in C$ such that a is the first non-gap of P, b is the first non-gap of P that is not a multiple of a and there are no other non-gaps between na and $(n + 1)a$. We obtain sufficient conditions in terms of $\mathrm{WS}(P)$ implying the uniqueness of the linear system g^1_a (this cannot be obtained using the results from [2]). In particular in case b is much larger than a then g^1_a is unique (independent from the value of g).

We concentrate on points Q on C with $Q \neq P$ such that $aQ \in |aP|$ and we obtain the following theorem in this described situation.

Theorem B. Let $(a,b) = 1$ with $b = na + r$ for some integers $n \geq 1$ and $1 \leq r \leq a-1$. Let C be a smooth curve of genus $g > (a - 1(b - a + r)/2$ containing some point P such that its first non-gap is equal to a, its first non-gap different from a multiple of a is equal to b and P has no other non-gaps between na and $(n+1)a$. Assume $Q \in C$ with $Q \neq P$ such that $aQ \in |aP|$, then $b \notin WS(Q)$.

From this theorem for large values of b with respect to a we obtain a lot of Weierstrass semigroups that can occur at most once. Moreover we prove that in many cases this bound on the genus in Theorem B is sharp. In those cases this implies that there exists a Weierstrass semigroup H_0 of genus $g = (a-1)(b-a+r)/2$ containing $< a; b >$ and a curve C of genus g having two Weierstrass points with semigroup H_0. Moreover fixing a and b the semigroup H_0 satisfying this property is unique.

In Section 2 we mention some general results. In particular Lemma 3 will be the basic lemma for obtaining the uniqueness of the pencil g^1_a.

In Section 3 we prove the main results of this paper. It starts with a very easy Lemma 4 which is the basic observation of all our main results. Assume C; P; a and b as before. Using a particular plane model Γ of the curve then it follows that equality $\mathrm{WS}(P) = \mathrm{WS}(Q)$ in case $aQ \in |aP|$ implies Q corresponds to a particular type of singular point on Γ. In particular it follows $\mathrm{WS}(P) \neq \mathrm{WS}(Q)$ in case $(a;b) = 1$, $g = (a-1)(b-1)/2$ and $r \neq a-1$ (see Corollary 3). In case $b \neq a+1$ we also obtain uniqueness of g^1_a in that case (Proposition 1) implying Theorem A. More general we also obtain Theorem B (Corollary 4). We also give some general statements on the uniqueness of g^1_a in case $g < (a-1)(b-1)/2$ (see Proposition 2 and Corollaries 5 and 6).

Using Lemma 4 in a more detailed manner we obtain a description for $\mathrm{WS}(Q)$ for all $Q \neq P$ satisfying $aQ \in |aP|$ in case $g = (a-1)(b-1)/2$ (Theorem 1). Continuing to use such arguments we obtain a list of non-gaps $\mathrm{WS}(P)$ needs to contain in order that there exists Q satisfying $aQ \in |aP|$ with $\mathrm{WS}(P) = \mathrm{WS}(Q)$ in case $g < (a-1)(b-1)/2$ (Lemma 6). From this fact we obtain further conditions on $\mathrm{WS}(P)$ going below the genus bound of Theorem B and implying $\mathrm{WS}(P)$ occurs at most once (Corollary 8). Moreover it also implies the genus bound in Theorem B is sharp in general (Corollary 11 and Lemma 8) and it gives a complete description of the Weierstrass semigroup implying this sharpness (Corollary 10 as a corollary of Lemma 7).

In Section 4 we consider some examples. In case $a = 4$ we show that for each integer N there exists a genus bound $g(N)$ such that for $g > g(N)$ there are at least N different Weierstrass semigroups with first non-gap equal to 4 and genus g that occur at most once (remember in case $a = 3$ this is not true). Those Weierstrass semigroups are very similar to each other. Case $a = 5$ illustrates that for growing values of a we obtain more types of Weierstrass gap sequences that occur at most once. Case $a = 6$ illustrates that the use of Lemma 3 causes that making a formulation of Theorem B similar to Theorem A without assuming $aQ \in |aP|$ is not possible using the arguments of this paper. Finally in case $n = 1$ (this is in case $a < b < 2a$) the genus bound in Theorem B is too small to obtain uniqueness of g^1_a. Using a very rough but different argument we show how to obtain a result on Weierstrass semigroups that occur at most once in this case $n = 1$ (see Lemma 9). The genus bound in the statement of Lemma 9 is sharp although is is larger than the genus bound in Theorem B (in particular for the corresponding Weierstrass points P and Q the divisors aP and aQ are not linearly equivalent). This argument used in Lemma 9 cannot be applied in case $n \geq 2$.

For two positive integers a and b we write (a, b) to denote their largest common divisor. In particular $(a, b) = 1$ means a and b are mutually prime. Remember we write $< a; b >$ to denote the subsemigroup of $\mathbb{N}$ generated by a and b. For a smooth projective variety X we write ω_X to denote the canonical sheaf of X.

2 Generalities

We are going to use some models of the smooth curve C on some surfaces. We use the following terminology and facts.

Let X be a smooth surface and let D, E be two curves on X without common components. For $Q \in D \cap E$ we write $i(D.E; Q)$ to denote the intersection multiplicity of D and E at Q. We also write $(D.E)$ to denote the intersection number of D and E on X.

Let X be a smooth surface and let Γ be an irreducible curve on X. This curve Γ has some arithmetic genus $p_a(\Gamma)$ and it can be computed by the formula $2p_a(\Gamma) - 2 = \Gamma.(\Gamma + K_X)$ with K_X a canonical divisor on X. In case Γ is smooth then this arithmetic genus is equal to the genus of the smooth curve Γ.

Let Q be a point on Γ of multiplicity ν and let $p : X' \to X$ be the blowing-up of X at Q. Let E be the associated exceptional divisor on X' and let Γ' be the proper transform of Γ on X'. It is well-known that $p_a(\Gamma') = p_a(\Gamma) - \nu(\nu - 1)/2$. In case Γ' has some singular points on E one continues this process blowing-up X' at the singular points of Γ' on E (such points are called infinitesimally near points on X and infinitesimally near singular points of Γ) and so on untill one obtains a smooth surface X_1 such that for the proper transform Γ_1 of Γ on X_1 all points mapping to Q are smooth. The difference $p_a(\Gamma) - p_a(\Gamma')$ is denoted by $\delta(Q)$.

Definition 2. We say Q is a cusp on an irreducible curve $\Gamma \subset \mathbb{P}^2$ in case for the normalisation $C \to \Gamma$ there is only one point of C mapping to Q (i.e. Γ is locally analytically irreducible at Q). Let ν be the multiplicity of Γ at Q. There is a unique line T on $\mathbb{P}^2$ containing Q such that $i(T.\Gamma; Q) = \mu > \nu$. We say Q is a cusp of type $(\nu; \mu)$ on Γ.

The following lemma should be well-known.

Lemma 1. Let $\Gamma \subset \mathbb{P}^2$ be an irreducible plane curve and assume Q is a cusp of type $(\nu; \mu)$ on Γ. In case $(\nu, \mu) = 1$ then $\delta_Q = \frac{(\nu-1)(\mu-1)}{2}$.

Proof. Using blowings-up starting at Q we obtain a sequence of singular points of Γ infinitesimally near to Q of known multiplicity as follows. We make the sequence $(c_1; c_2; \cdots; c_{k+1} = 1)$ taking $c_1 = \mu$ and $c_2 = \nu$. Then $c_1 = n_2 c_2 + c_3$ with $1 \leq c_3 \leq c_2 - 1$. In case $i \geq 3$ and $c_i \neq 1$ then $c_{i-1} = n_i c_i + c_{i+1}$ with $1 \leq c_{i+1} \leq c_i - 1$. This is the Euclidean algorithm to compute (ν, μ). Since $(\nu, \mu) = 1$ one has $(c_i, c_{i+1}) = 1$ for all $1 \leq i \leq k$ and $c_{k+1} = 1$.

Then for $2 \leq i \leq k$ there are n_i singular points of multiplicity c_i on the curve Γ infinitesimally near to Q. This implies

$$\delta_Q = \sum_{i=2}^{k} n_i \frac{c_i(c_i - 1)}{2} \ .$$

For $2 \leq j \leq k$ let $\delta_j = \sum_{i=j}^{k} \frac{c_i(c_i-1)}{2}$. By means of induction we show $\delta_j = \frac{(c_{j-1}-1)(c_j-1)}{2}$. Since $\delta_Q = \delta_2$ this implies the lemma.

For $j = k$ we have $\delta_k = n_k \frac{c_k(c_k-1)}{2}$. Also $c_{k-1} = n_k c_k + 1$. This implies $\delta_k = \frac{(c_{k-1}-1)(c_k-1)}{2}$.

Assume $3 \leq j \leq k$ and $\delta_j = \frac{(c_{j-1}-1)(c_j-1)}{2}$. We have $\delta_{j-1} = n_{j-1} \frac{c_{j-1}(c_{j-1}-1)}{2} + \delta_j$. We use $c_{j-2} = n_{j-1} c_{j-1} + c_j$ hence $\delta_{j-1} = \frac{(c_{j-2}-c_j)(c_{j-1}-1)}{2} + \frac{(c_{j-1}-1)(c_j-1)}{2} = \frac{(c_{j-1}-1)(c_{j-2}-1)}{2}$. □

Let X be the surface $\mathbb{P}^1 \times \mathbb{P}^1$. For each divisor D on X there exist unique integers α and β such that D is linearly equivalent to $\alpha\left(\mathbb{P}^1 \times \{S\}\right) + \beta\left(\{S\} \times \mathbb{P}^1\right)$ for some $S \in \mathbb{P}^1$ (see e.g. [9], Chapter II, Example 6.6.1). Such curve is said to be of type $(\alpha; \beta)$ and we write $|(\alpha; \beta)|$ to denote the complete linear system of curves of type $(\alpha; \beta)$. We write $\mathcal{O}_X(\alpha; \beta)$ to denote the corresponding invertible sheaf. For an irreducible curve Γ on X there exist so-called canonically adjoint curves to Γ describing all elements of the canonical linear system on the normalisation of Γ. Although this should be well-known we include an argument for this fact.

Lemma 2. Let Γ be an irreducible curve of type $(\alpha; \beta)$ on $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let C be the normalisation of Γ. Let $|K_C|$ be the canonical linear system on C. There exists a linear subsystem of $|(\alpha - 2; \beta - 2)|$ called the linear system of canonically adjoint curves of Γ that has a natural bijective correspondence with $|K_C|$ using intersections.

Proof. Let $\pi : Y \to X$ be a sequence of blowings-up at some points (some of them might be infinitesimally near points) such that the proper transform of Γ on Y is smooth (so we identify it with C). It is well-known that $H^i(X; \mathcal{O}_X) \cong H^i(Y; \mathcal{O}_Y)$ for all $i \geq 0$ (see [9], Chapter V, Proposition 3.4). Since $H^1(X; \mathcal{O}_X) = 0$ (see [9], Chapter III, Exercise 5.6) one has $H^1(Y; \mathcal{O}_Y) = 0$. Canonical divisors on X are of type $(-2; -2)$ (see [9], Chapter II, Exercise 8.20.3). From Serre duality (we use [9], Chapter III, Corollary 7.7) it follows $H^2(X; \mathcal{O}_X) \cong H^0(X; \mathcal{O}_X(-2; -2)) = 0$ and therefore $H^2(Y; \mathcal{O}_Y) = 0$ and also $H^0(Y; \omega_Y) = 0$

and $H^1(Y;\omega_Y) = 0$. From [9], Chapter II, Proposition 8.20 we know $\omega_C \cong \omega_Y \otimes \mathcal{O}_Y(C) \otimes \mathcal{O}_C$. Tensoring the exact sequence

$$0 \to \mathcal{O}_Y(-C) \to \mathcal{O}_Y \to \mathcal{O}_C \to 0$$

with $\mathcal{O}_Y(C) \otimes \omega_Y$ gives rise to the exact sequence

$$0 \to \omega_Y \to \omega_Y \otimes \mathcal{O}_Y(C) \to \omega_C \to 0$$

Using the exact cohomology sequence we obtain an isomorphism $H^0(Y;\omega_Y \otimes \mathcal{O}_Y(C)) \to H^0(C;\omega_C)$. The images on X of elements of the complete linear system associated to $\omega_Y \otimes \mathcal{O}_Y(C)$ are the canonically adjoint curves of Γ. From the construction it follows they are contained in $|(\alpha-2;\beta-2)|$ (this follows from an explicit description of ω_Y using the blowings-up (see [9], Chapter V, proposition 3.3)) and from the proof it follows they are in bijective correspondence to effective canonical divisors on C. □

Uniqueness of a linear system g^1_a as mentioned in the introduction will be a consequence of the following lemma.

Lemma 3. Let C be a smooth curve of genus g and let P be a point of C with first non-gap equal to a. Assume C has a base point free linear system g^1_a different from $|aP|$. There there exists a divisor $e < a$ of a (it might be 1) such that each integer $\left(\frac{a}{e} - 1\right) a + ie$ with $i \in \mathbb{Z}_{\geq 1}$ is a non-gap of P.

Proof. Let $f_1 : C \to \mathbb{P}^1$ be a morphism corresponding to $|aP|$ and let $f_2 : C \to \mathbb{P}^1$ be a morphism corresponding to g^1_a. Consider the morphism $f = (f_1; f_2) : C \to \mathbb{P}^1 \times \mathbb{P}^1$ and let Γ be the image of f. Let C' be the normalisation of Γ, then f factorizes through a finite morphism $h : C \to C'$ of some degree $e < a$ dividing a (e might be equal to 1). The rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ imply base point free linear systems g_1 and g_2 on C' such that $h^{-1}(g_1) = |aP|$; $h^{-1}(g_2) = g^1_a$. In particular for $P' = h(P) \in C'$ one has $\frac{a}{e}P' \in g_1$. The canonically adjoint curves of Γ give rise to a linear subsystem of $|(\frac{a}{e}-2;\frac{a}{e}-2)|$ and they correspond bijectively with effective canonical divisors on C'

Let E be a general element of $|\frac{a}{e}P'|$. An effective canonical divisor on C' containing E corresponds to some curve γ in $|(\frac{a}{e}-2;\frac{a}{e}-2)|$ containing E. This divisor E consists of $\frac{a}{e}$ different points on some line l belonging to $|(1;0)|$. Since the intersection number $\left((\frac{a}{e}-2;\frac{a}{e}-2).(1;0)\right) = \frac{a}{e} - 2 < \frac{a}{e}$ it follows $l \subset \gamma$. This implies there is no canonically adjoint curve of Γ containing $\frac{a}{e} - 1$ general elements of $|\frac{a}{e}P'|$. Therefore no effective canonical divisor of C' contains $\frac{a}{e} - 1$ general elements of $|\frac{a}{e}P'|$. It follows $(\frac{a}{e} - 1)\frac{a}{e}P'$ is a non-special divisor on C'. This implies for each $i \in \mathbb{Z}_{\geq 1}$ the integer $(\frac{a}{e} - 1)\frac{a}{e} + i$ is a non-gap of P'. Using the inverse image under the morphism f_1 one obtains $(\frac{a}{e} - 1)a + ie$ is a non-gap of P. □

3 Proofs

This easy lemma having a trivial proof is the basic lemma for all main results in this paper.

Lemma 4. Let C be a smooth curve, $P \in C$ and $a, b \in \mathbb{Z}_{\geq 1}$ with $b = na + r$ with $r, n \in \mathbb{Z}$ satisfying $0 < r < a$ and $n \geq 1$. Assume $a, b \in \mathrm{WS}(P)$. Assume $Q \in C$ with $Q \neq P$ and $aQ \in |aP|$. Let $\mu \in \mathbb{Z}_{\geq 1}$ with $0 < \mu < a$ and assume $|bP - \mu Q|$ does not have Q as a base point. Then $an + (a - \mu) \in \mathrm{WS}(Q)$.

Proof. Since $aQ \in |aP|$ it follows that

$$D_0 := (n-1)aQ + rP + (a-\mu)Q \in |bP - \mu Q| \ .$$

Since Q is not a base point of $|bP - \mu Q|$ it follows Q is not a base point of $|bP - \mu Q + (a-r)P|$. However $D_0 + (a-r)P = (n-1)aQ + aP + (a-\mu)Q$ and again using $aQ \in |aP|$ we obtain

$$D_1 := naQ + (a-\mu)Q \in |bP - \mu Q + (a-r)P| \ .$$

This implies Q is not a base point of $|(na + (a-\mu))Q|$ hence $|(na + (a-\mu))Q|$ is base point free. This implies $na + (a-\mu)$ is a non-gap of Q. □

From now on in this paper we make the following assumptions. C is a smooth curve of genus g and P is a smooth point of C. We assume $|aP|$ is a base point free g^1_a (i.e. a is the first non-gap of P). Let $n \in \mathbb{Z}_{\geq 1}$ such that $\dim |naP| = n$ while $\dim |(n+1)aP| > n+1$. Such n exists and it is unique. This means the first non-gap b of P that is not a multiple of a is of type $b = an + r$ with $0 < r < a$.

Lemma 5. Let $Q \in C$ with $Q \neq P$ and $aQ \in |aP|$. There is a unique integer μ satisfying $0 < \mu < a$ such that $|bP - \mu Q|$ does not have Q as a base point.

Proof. Since $aQ \in |aP|$ one has $(b-a)P \in |bP - aQ|$, hence Q is not a fixed point of $|bP - aQ|$. From the definition of a and b it follows $\dim |bP - aQ| = n - 1 = \dim |bP| - 2$. Assume $|bP - Q|$ contains Q as a base point with multiplicity ν (ν can be equal to 0). Then $|bP - (\nu+1)Q|$ does not contain Q as a base point and $\dim |bP - (\nu+1)Q| = \dim |bP| - 1 = n = \dim |bP - aQ| + 1$. In particular $\nu + 1 < a$. This implies the existence of an integer μ satisfying $0 < \mu < a$ such that $|bP - \mu Q|$ does not contain Q as a base point (taking $\mu = \nu + 1$).

In case there exists an integer $\mu' \neq \mu$ with $0 < \mu' < a$ such that $|bP - \mu' Q|$ does not contain Q as a base point, then $\mu' > \mu$ and we find $\dim |bP - \mu' Q| = n - 1$ and $\dim |bP - aQ| < n - 1$, a contradiction. □

Under the assumptions of Lemma 5 it follows from Lemma 4 that $an + (a-\mu)$ is a non-gap of Q. In case $\dim |(n+1)aP| = n+2$ then there is a unique non-gap of Q between an and $a(n+1)$. So we obtain the following conclusion.

Corollary 1. Assume $\dim |(n+1)aP| = n+2$ and $Q \in C$ with $Q \neq P$ and $aQ \in |aP|$. If $\mathrm{WS}(P) = \mathrm{WS}(Q)$ then $\mu = a - r$ is the unique integer $0 < \mu < a$ such that $|bP - \mu Q|$ does not have Q as a base point.

In case the linear system $|bP|$ is simple then we can give a geometric meaning to the number μ occuring in Lemma 5 and Corollary 1 using some specific plane model $\Gamma \subset \mathbb{P}^2$ of C. As in [4] we construct a simple base point free linear system g^2_b on C as follows. Choose $D \in |bP|$ general, in particular $P \notin D$. Inside the projective space $|bP|$ take the linear span of the line $|aP| + (b-a)P$ and D (denoted by $< |aP| + (b-a)P; D >$). This linear systems g^2_b defines a morphism from C to $\mathbb{P}^2$ and the image $\Gamma \subset \mathbb{P}^2$ is a plane curve of degree b birationally equivalent to C. (We write $\phi : C \to \Gamma$ to denote the normalization.) The image $\phi(P)$ is a cusp of Γ of type $(b-a; b)$. This singularity causes that the genus of the curve C is at most $((b-1)(a-1)+1-(a,b))/2$ (see the computation in Section 1 of [4]). It should be mentioned that it is proved in [4], Section 3 that there exist such curves C for all $g \leq ((a-1)(b-1)/2 + 1 - (a,b))/2$ (see also [11], Section 3 in case $(a,b) = 1$). From now on we assume $|bP|$ is simple and $\Gamma \subset \mathbb{P}^2$ is such a plane model of C.

Assume $Q \in C$ with $Q \neq P$ and $aQ \in |aP|$. Clearly $\phi(Q) \neq \phi(P)$ (since $bP \in g^2_b$). Let L_Q be the line in $\mathbb{P}^2$ connecting $\phi(P)$ and $\phi(Q)$. Since the pencil of lines on $\mathbb{P}^2$ through $\phi(P)$ induces $|aP|$ on C (because $(b-a)P + |aP| \subset g^2_b$) it follows $i(L_Q.\Gamma; \phi(Q)) = a$.

Let μ be the multiplicity of Γ at $\phi(Q)$. We already know $\mu \leq a$. In case $\mu = a$ then it would imply $|bP - aQ|$ is base point free. Since $|bP-aQ| = |(b-a)P|$ this would contradict the meaning of the integers a and b. It follows $1 \leq \mu \leq a-1$ and $\phi(Q)$ is a cusp of type $(\mu; a)$ of Γ. The pencil of lines in $\mathbb{P}^2$ containing $\phi(Q)$ induces a base point free linear system on C contained in $|bP - \mu Q|$. Therefore the multiplicity of $\phi(Q)$ on Γ is the integer $0 < \mu < a$ mentioned in Lemma 5 and Corollary 1. Using this plane model Γ of C we obtain the following conclusion.

Corollary 2. Assume $|bP|$ is simple, $\dim |(n+1)aP| = n+2$ and $Q \in C$ with $Q \neq P$ and $aQ \in |aP|$. If $\mathrm{WS}(P) = \mathrm{WS}(Q)$ then $\phi(Q)$ is a cusp of Γ of type $(a-r; a)$.

Corollary 3. Assume $|bP|$ is simple and $Q \in C$ with $Q \neq P$ and $aQ \in |aP|$. In case $g = ((a-1)(b-1)+1-(a;b))/2$ and $r \neq a-1$ then $\mathrm{WS}(P) \neq \mathrm{WS}(Q)$.

Proof. From the condition $g = ((a-1)(b-1)+1-(a,b))/2$ it follows $\dim |(n+1)aP| = n+2$. Let Γ be a plane model of C as described before. Then all points on Γ different from $\phi(P)$ are smooth. This implies $\phi(Q)$ is a cusp of Γ of type $(1; a)$ and $na + (a-1)$ is the non-gap of Q between an and $a(n+1)$. Therefore the Weierstrass semigroups of Q and P can be equal only in case $r = a-1$. $\square$

From now on we assume $(a,b) = 1$ with $a < b$. In this case $|bP|$ is simple. Write $b = na + r$ with $n \geq 1$ and $0 < r < a$. The equation of the plane model Γ can be reduced to some canonical form (see e.g. [11] Lemma 6.2). In case $g = (a-1)(b-1)/2$ then $\phi(P)$ is the only singular point on Γ and such curves are the so-called $C_{a,b}$ curves. In this case one has $\mathrm{WS}(P) =< a, b >$.

In case $g = (a-1)(b-1)/2$ and $b = a+1$ then $C = \Gamma$ is a smooth plane curve of degree $a+1$ defined by the linear system $|bP|$ (hence P is a total inflection point of this smooth plane curve Γ). For each point Q on C the linear system $|bP - Q|$ is a base point free linear system g^1_a on C. In case Q is also a total inflection point of Γ (i.e. $bQ \in |bP|$) then also $\mathrm{WS}(Q) =< a, b >$. In that way C can have many Weierstrass points having Weierstrass semigroup equal to $< a, a+1 >$.

Now we are going to prove that in case $g = (a-1)(b-1)/2$ and $b \neq a+1$ then the linear system $|aP|$ is the unique linear system g^1_a on C without base points. This implies that any point Q on C having $\mathrm{WS}(Q) =< a, b >$ satisfies $aQ \in |aP|$.

Proposition 1. Let C be a curve of genus $g = (a-1)(b-1)/2$ with $(a, b) = 1$ and assume C has a Weierstrass point with Weierstrass semigroup $< a; b >$. In case $b \neq a+1$ then C has a unique linear system g^1_a.

Proof. Assume C has more than one linear system g^1_a. From Lemma 3 it follows that there exists a divisor e of a different from a such that for all integers $i \geq 1$ the integer $(\frac{a}{e} - 1)a + ie$ is a non-gap of P. By assumption those integers belong to $< a; b >$ hence each one of them can be written as $xa + yb$ for some non-negative integers x and y. Since $(a, b) = 1$ and e divides a it follows e divides y. Therefore for each integer $1 \leq i \leq \frac{a}{e} - 1$ there is a pair of integers $(x_i; y_i)$ with $x_i \geq 0$ and $0 < y_i < \frac{a}{e}$ such that $(\frac{a}{e} - 1)a + ie = x_i a + y_i eb$. This implies there is an integer k_i such that $y_i b = i + k_i \frac{a}{e}$. In case $y_i = y_{i'}$ then this implies $(i - i') = (k_i - k_{i'})\frac{a}{e}$ and therefore $i = i'$. This implies there exists some $1 \leq i \leq \frac{a}{e} - 1$ such that $y_i = \frac{a}{e} - 1$ and therefore

$$x_i a + (\frac{a}{e} - 1)eb = (\frac{a}{e} - 1)a + ie < \frac{a}{e}a \ .$$

In case $e \geq 2$ one has $a - e = e(\frac{a}{e} - 1) \geq 2(\frac{a}{e} - 1) \geq \frac{a}{e}$ (since $e \neq a$ one has $\frac{a}{e} \geq 2$). Since $b > a$ this implies $(a-e)b > \frac{a}{e}a$, a contradiction. In case $e = 1$ one obtains $(a-1)b < a^2$. This is a contradiction in case $b \geq a+2$. □

From Proposition 1 and Corollary 3 we obtain Theorem A from the introduction.

Proof of Theorem A. Let C be a smooth curve and $P \in C$ such that $\mathrm{WS}(P) =< a; b >$. Since the number of non-gaps of P is equal to $(a-1)(b-1)/2$ it implies $g(C) = (a-1)(b-1)/2$. Since $b \neq a+1$ it follow from Proposition 1 that C has a unique g^1_a (namely $|aP|$). So in case $Q \in C$ with $Q \neq P$ and $\mathrm{WS}(Q) =< a; b >$ then $aQ \in |aP|$. Since $r \neq a-1$ Corollary 3 implies that $\mathrm{WS}(Q) \neq \mathrm{WS}(P)$, so we obtain a contradiction. This implies the Weierstrass semigroup $< a; b >$ occurs at most once. □

Proof of Theorem B. Since $(a; b) = 1$ the linear system $|bP|$ is simple. Let $\Gamma \subset \mathbb{P}^2$ be the plane model of C constructed before and let $\phi : C \to \Gamma$ be the normalisation. We know $\phi(P)$ is a cusp of type $(b-a; a)$.

Assume $Q \in C$ with $Q \neq P$ such that $aQ \in |aP|$ and $b \in WS(Q)$. It follows from Corollary 2 that Q is a cusp of type $(a-r; a)$ on the plane model Γ. Since $(a; b) = 1$ also $(a-r; a) = 1$ and from Lemma 1 it follows $\delta_{\phi(Q)} = (a-1)(a-r-1)/2$. It follows $g(C) \leq (a-1)(b-1)/2 - (a-1(a-r-1)/2 = (a-1)(b-a+r)/2$. Since $g > (a-1)(b-a+r)/2$ we obtain a contradiction. □

Corollary 4. Assume $\dim|(n+1)aP| = n+2$ and let $Q \in C$ with $Q \neq P$ and $aQ \in |aP|$. In case $g > (a-1)(b-a+r)/2$ then $\mathrm{WS}(Q) \neq \mathrm{WS}(P)$.

In case C has an linear system g^1_a different from $|aP|$ it follows from Lemma 3 there exists a divisor e of a different from a such that $\left(\frac{a}{e}-1\right)\frac{a}{e}+e$ is a non-gap of P. As a rough estimate this implies there is a non-gap of P not being a multiple of a having value at most $\left(\frac{a}{2}\right)^2$. Because of the meaning of b this is impossible in case $b > \left(\frac{a}{2}\right)^2$. Therefore in this case Corollary 4 implies the following statement on uniqueness of Weierstrass semigroups.

Corollary 5. Assume $\dim|(n+1)aP| = n+2$, $b > \left(\frac{a}{2}\right)^2$ and $g > (a-1)(b-a+r)/2$. Then $\mathrm{WS}(P)$ occurs at most once.

The estimate used in Corollary 5 is very rough. Using some (still rough) estimates on the number of non-gaps we obtain the following condition implying uniqueness of the g^1_a in case $g < (a-1)(b-1)/2$.

Proposition 2. Assume for each divisor e of a different from a (but including 1) one has

$$g > \frac{(a-1)(b-1)}{2} - \frac{(an-ne-\frac{a}{e}+2)(an-ne-\frac{a}{e}+1)}{2ne}$$

then C has a unique base point free g^1_a.

Proof. Let x be some integer at least 1. From divisibility arguments as used in the proof of Proposition 1, in case $(a;b)=1$ the number of elements of the type $xa+ie$ for some integer $1 \leq i \leq \frac{a}{e}-1$ inside $<a;b>$ is at most $[\frac{x}{ne}]$ (here $b = na+r$ with $0 < r < a$).

Assume C has a base point free g^1_a different from $|aP|$. In case $x \geq \frac{a}{e}-1$ it follows from Lemma 3 that $xa+ie$ is a non-gap of P for each integer $1 \leq i \leq \frac{a}{e}-1$. Therefore there are at least $(\frac{a}{e}-1)-[\frac{x}{ne}] \geq (\frac{a}{e}-1)-\frac{x}{ne}$ non-gaps of P between xa and $(x+1)a$ outside $<a;b>$. Summing up over different values of x we obtain at least $\frac{(an-ne-\frac{a}{e}+2)(an-ne-\frac{a}{e}+1)}{2ne}$ non-gaps of P outside $<a;b>$. □

In case a is a prime number we only have to consider the case $e=1$ in the statement of Proposition 2. In particular we obtain the following statement concerning uniqueness of Weierstrass semigroups.

Corollary 6. Let a be a prime number and assume $b > 3a$ is an integer not divisible by a. Write $b = na+r$ with $1 \leq r \leq a-1$. Let H be a Weierstrass semigroup containing $<a;b>$ having no non-gap outside of $<a;b>$ smaller than $(n+1)a$ and having genus $g > \frac{(a-1)(b-a+r)}{2}$. Then H occurs at most once.

Proof. From Corollary 4 it follows that in case there exists a smooth curve of genus g having two different Weierstrass points P and Q with Weierstrass semigroup equal to H then aP and aQ are not linearly equivalent. In particular

C has a base point free linear system g^1_a different from $|aP|$. From Proposition 2 we know this implies

$$g \leq \frac{(a-1)(b-1)}{2} - \frac{(an-n-a+2)(an-n-a+1)}{2n} \; .$$

Since $n \geq 3$ this implies $g \leq \frac{(a-1)(b-1)}{2} - \frac{(2a-1)(2a-2)}{6}$. Since $\frac{(a-1)(a-2)}{2} < \frac{(2a-1)(a-1)}{3}$ we obtain a contradiction. □

In Section 4 we illustrate that using Lemma 3 gives rise to better uniqueness statements for the linear system g^1_a in case of explicit examples than using Proposition 2. It should be noted that the results of [11] imply that many of those Weierstrass semigroups really occur as Weierstrass semigroups of points on certain curves.

Lemma 4 can also be used to determine the Weierstrass semigroups of the points $Q \neq P$ satisfying $aQ \in |aP|$ in some situations.

Theorem 1. Let P be a point on a smooth curve of genus $g = (a-1)(b-1)/2$ (with $(a;b) = 1$) such that $\mathrm{WS}(P) =< a;b >$. Let $Q \in C$ with $Q \neq P$ and $aQ \in |aP|$. For $t \in \mathbb{Z}_{\geq 1}$ let s be the number of non-gaps e of P satisfying $ta < e < (t+1)a$. Then $(t+1)a - i$ with $1 \leq i \leq s$ are the non-gaps of Q satisfying $ta < e < (t+1)a$.

Proof. In case $s = a-1$ the theorem is trivially true, so we assume $s < a-1$. The integer s associated to t is defined by the inequalities $sb < (t+1)a$ and $(s+1)b > (t+1)a$. Define $\epsilon \in \mathbb{Z}_{\geq 0}$ such that $at < sb + \epsilon a < a(t+1)$. Hence $sb + \epsilon a$ is a non-gap e of P satisfying $at < e < a(t+1)$. Because of the genus of C it follows that Q corresponds to a smooth point on the plane model Γ of C, hence $|bP - Q|$ does not have Q as a base point.

For each integer $0 \leq i \leq s$ one has

$$i(bP - Q) + ((s-i)b + \epsilon a)\, P \in |(sb + \epsilon a)P - iQ| \; .$$

This implies $|(sb+\epsilon a)P - iQ|$ does not contain Q as a fixed point. From Lemma 4 it follows $at + (a-i)$ is a non-gap of Q for $1 \leq i \leq s$. □

In [15], Lemma 2.7 the authors also obtain the statement of Theorem 1 assuming P is a Galois Weierstrass point. The statement is formulated in a diiferent way and it needs some computations to show both descriptions of the Weierstrass semigroup are the same.

Corollary 7. Assume $g = (a-1)(b-1)/2$ and $r = a-1$. Then for all $Q \in C$ with $aQ \in |aP|$ one has $\mathrm{WS}(Q) =< a;b >$.

Now we consider the case $g \leq (a-1)(b-a+r)/2$ to obtain a refinement of Corollary 4 and a generalisation of Theorem 1. In many cases it also implies sharpness of Corollary 4. Assume P as before and $Q \neq P$ such that $aQ \in |aP|$ and Q is a cusp of type $(\mu = a-r; a)$ on the plane model Γ. We assume $(a,b) = 1$, hence $(a,\mu) = 1$. For $1 \leq m \leq \mu - 1$ define the integer $n(m)$ such that $(n(m)-1)\mu < ma < n(m)\mu$. Note that $m \leq \mu - 1$ implies $n(m) < a$.

Lemma 6. *For $n(m) \leq i \leq a-1$ one has $ib - ma$ is a non-gap of Q.*

Proof. Since $|bP - \mu Q|$ does not have Q as a base point it follows $|n(m)bP - n(m)\mu Q| = |(n(m)b - ma)P - (n(m)\mu - ma)Q|$ does not have Q as a base point. One has $n(m)b \equiv -n(m)\mu \mod a$. This implies $(n(m)b - 2ma + n(m)\mu)\,P$ is linearly equivalent to $(n(m)b - 2ma + n(m)\mu)\,Q$. Since $|\,(n(m)b - ma)\,P - (n(m)\mu - ma)\,Q + (n(m)\mu - ma)\,P|$ does not have Q as a base point and contains $(n(m)b - ma)\,Q$ it follows $|(n(m)b - ma)Q|$ has no base point, hence $n(m)b - ma \in \mathrm{WS}(Q)$.

Since $b \in \mathrm{WS}(Q)$ (because of Lemma 4) it follows that for all integers $n(m) \leq i \leq a-1$ the integer $ib - ma \in \mathrm{WS}(Q)$. □

For all integers $1 \leq i < a$ one has ib is the smallest integer x in $< a; b >$ satisfying $x \equiv ib \mod a$. Therefore the integers $ib - ma$ with $n(m) \leq i \leq a-1$ do not belong to $< a; b >$. Varying $1 \leq m \leq \mu - 1$ we obtain $\sum_{m=1}^{\mu-1}(a - n(m))$ non-gaps of Q not belonging to $< a; b >$. We call them the *trivial new non-gaps* associated to a cusp of type $(\mu; a)$ on the plane model Γ.

Corollary 8. Assume $\dim |naP| = n + 2$ and let $Q \in C$ with $Q \neq P$ and $aQ \in |aP|$. In case $\mathrm{WS}(P)$ does not contain the list of trivial new non-gaps asssociated to a cusp of type $(a - r, a)$ on the plane model Γ then $\mathrm{WS}(P) \neq \mathrm{WS}(Q)$.

Combining Corollary 8 with Proposition2 we obtain the following statement.

Corollary 9. Assume $\dim |naP| = n+2$ and assume the Weierstrass semigroup of P does not contain the list of trivial non-gaps asssociated to a cusp of type $(a - r, a)$ on the plane model Γ. Assume for each divisor e of a different from a one has

$$g > \frac{(a-1)(b-1)}{2} - \frac{(an - ne - \frac{a}{e} + 2)(an - ne - \frac{a}{e} + 1)}{2}$$

then for each point $Q \in C$ with $Q \neq P$ one has $\mathrm{WS}(Q) \neq \mathrm{WS}(P)$.

In case $g = (a-1)(b - a + r)/2$ and $aQ \in |aP|$ then Lemma 6 completely determines $\mathrm{WS}(Q)$ in case Q is an cusp of Γ of type $(a - r; a)$. This follows from the calculations made in the following lemma.

Lemma 7. The number of trivial new non-gaps associated to a cusp of type (μ, a) on the plane model Γ is equal to $(a-1)(\mu-1)/2$

Proof. By definition for each $1 \leq m \leq \mu - 1$ one has $n(m) \leq a-1$ and $n(m) \geq 2$. Also for $2 \leq k \leq a-1$ there is at most one integer $1 \leq m \leq \mu - 1$ with $n(m) = k$. In case such m with $n(m) = k$ exists we define $x(k) = 1$, otherwise $x(k) = 0$. Each integer $1 \leq k \leq a-1$ gives rise to $x(k)(a-k)$ trivial new non-gaps. So the number of trivial new non-gaps can be written as $\sum_{k=2}^{a-1} x(k)(a-k)$.

For $1 \leq m \leq \mu - 1$ write $ma = (n(m) - 1)\mu + \epsilon$ with $0 < \epsilon < \mu$. Then $(\mu - m)a = (a - n(m))\mu + (\mu - \epsilon)$ implying $n(\mu - m) = a - n(m) + 1$.

In case a is even and $2 \leq k \leq \frac{a-2}{2}$ it implies $x(k) = x(a-k+1)$ while $\frac{a}{2} \leq a-k+1 \leq a-1$. So the number of trivial new non-gaps is equal to

$$\sum_{k=2}^{(a-2)/2} x(k)(a-k+a-(a-k+1)) = (a-1) \sum_{k=2}^{(a-2)/2} x(k) \ .$$

On the other hand $2\sum_{k=2}^{(a-2)/2} x(k) = \mu - 1$ and we obtain that the number of trivial new non-gaps is equal to $(a-1)(\mu-1)/2$ (note that μ is odd in case a is even since $(a,\mu) = 1$).

In case a is odd and $2 \leq k \leq \frac{a+1}{2}$ one has $\frac{a+1}{2} \leq a-k+1 \leq a-1$. In case $x\left(\frac{a+1}{2}\right) = 0$ we conclude as before. In case $x\left(\frac{a+1}{2}\right) = 1$ the number of trivial new non-gaps is equal to

$$\sum_{k=1}^{(a-1)/2} x(k)(a-1) + \binom{a-1}{2} \ .$$

On the other hand $2\sum_{k=1}^{(a-1)/2} x(k) + 1 = \mu - 1$ and again we obtain again the number of trivial new non-gaps is equal to $(a-1)(\mu-1)/2$. □

Corollary 10. Assume $\dim |naP| = n+2$ and assume $g = (a-1)(b-a+r)/2$. Let Q is a point on C with $aQ \in |aP|$ and assume Q corresponds to a cusp of type $(a-r;a)$ on the plane model Γ. Then $\mathrm{WS}(Q)$ is equal to the union S of $< a;b >$ and the set of trivial new non-gaps.

Proof. From Lemma 6 it follows that $\mathrm{WS}(Q)$ contains S. From the equality of the numbers obtained in Lemma 7 and Lemma 1 one finds that $\mathbb{N} \setminus S$ consists of exactly g elements. Therefore S is the Weierstrass semigroup of Q. □

We are now able to prove sharpness of Corollary 4 in a lot of cases.

Corollary 11. Same assumptions as in Corollary 10. Assume the curve C has a Weierstrass point $Q \neq P$ with non-gaps a and b with $aQ \in |aP|$. Then $\mathrm{WS}(P) = \mathrm{WS}(Q)$ and they are both equal to the union of $< a;b >$ and the set of trivial new non-gaps.

Proof. One can make a plane model once using P and once using Q applying Corollary 10 in both cases. □

Lemma 8. There exists a plane curve of degree b having a cusp of type $(b-a;a)$ and a cusp of type $(\mu;a)$ and no other singularities.

Proof. On $\mathbb{P}^1$ choose two different points P_0 and Q_0. Take $g^1_a =< aP_0; aQ_0 >$ and choose a general effective divisor E of degree $b-\mu$ on $\mathbb{P}^1$. Take $g^2_b =< (b-a)P_0+ < aP_0; aQ_0 >; \mu Q + E >$. This gives rise to a plane curve Γ_0 of degree b such that P_0 defines a cusp of type $(b-a;a)$ and Q_0 defines a cusp of type $(\mu;a)$. As in [4] Section 3 one can prove that all other singularities of Γ_0 are ordinary nodes. Using Tannenbaum's result as in loc. cit. those nodes can be smoothed in a family of plane curves obtaining a plane curve Γ of degree b having a cusp of type $(b-a;a)$ and a cusp of type $(\mu;a)$ and no other singularities. $\square$

The normalisation C of the plane curve obtained in Lemma 8 is a smooth curve of genus $g=(a-1)(b-\mu)/2$. The point P corresponding to the cusp of type $(b-a;a)$ has non-gaps a and b. The point Q corresponding to the cusp of type $(\mu;a)$ also has non-gaps a and b. In case $\dim|naP|=n+2$ then from Corollary 11 it follows both points have Weierstrass gap sequence equal to the union of $<a;b>$ and the set of trivial new non-gaps. In case $\dim|naP|>n+2$ then there is a non-gap of P between an and $a(n+1)$ different from b. This implies the existence of non-gaps not contained in $<a,b>$. In case the number of those new non-gaps is larger than $(a-1)(\mu-1)/2$ this gives a contradiction. In such cases Corollary 4 is sharp. In case b is sufficiently large with respect to a and some integer b' with $b<b'<(n+1)a$ is also a non-gap then the number of non-gaps not contained in $<a;b>$ is indeed larger than $(a-1)(\mu-1)/2$ and we obtain sharpness in Corollary 4.

4 Examples

Example 1. Assume C is a smooth curve of genus g and P is a Weierstrass point on C with first non-gap equal to 4. Note that all Weierstrass semigroups with first non-gap equal to 4 occur as Weierstrass semigroup of some point on some smooth curve (see [13]). Let $4n+1$ with $n\geq 2$ be a non-gap of P (in case $a=4$ this is the only possibility for b implying the existence of Weierstrass semigroups that occur at most once of the type considered in this paper).

Assume C has a base point free linear system g^1_4 different from $|4P|$. In Lemma 3 we have to consider the possibilities $e=1$ and $e=2$. In case $e=1$ then all integers at least 12 are non-gaps of P. In particular $g\leq 8$ in case $n=2$ and $g\leq 9$ in case $n\geq 3$. In case $e=2$ then for all integers $m\geq 2$, $2m$ a non-gap of P. This implies $g\leq 2n+2$. In case $4n-3$ would be a non-gap of P then $g\leq 6n-6$ and in case $4n+2$ or $4n+3$ also would be some non-gap of P then one concludes $g\leq 4n$. In particular in case $g>6n-6$ then $\dim|(n+1)4P|=n+2$ and the same conclusion holds in case $g>4n$ and $4n-3$ is a gap of P. In case $g>4n$ then also $e=1$ cannot occur in Lemma 3, therefore $|4P|$ is the unique g^1_4 in that case.

Assume $g>4n$ and $4n-3$ is a gap of P. Let $Q\in C$ with $Q\neq P$ and $4Q\in|4P|$. Then $\mathrm{WS}(Q)$ contains the trivial new non-gaps $8n-2$, $12n-5$ and $12n-1$. Therefore, if one of them is a gap of P, then $\mathrm{WS}(P)\neq\mathrm{WS}(Q)$. In case $g=6n-m$ with $0\leq m\leq 2n-1$ it implies $<4;4n+1>\cup\{4i+3:3n-m\leq i\leq 3n-1\}$ is a Weierstrass semigroup that occurs at most once. The only other type of Weierstrass semigroup occuring at most once as a corollary of the results in this paper has genus $g=6n-2$ and is equal to $<4;4n+1>\cup\{8n-2;12n-1\}$.

In case $g>6n-2$ then all Weierstrass semigroups of genus g containing $<4;4n+1>$ occur at most once. In case $g=6n-3$ then we can use Lemma 8 to conclude there is a smooth curve C of genus g having two different points P and Q with $\mathrm{WS}(P) = \mathrm{WS}(Q)$ both equal to $< 4; 4n+1 > \cup \{8n-2; 12n-1; 12n-5\}$.

In case $g \equiv 1 \mod 3$ there is exactly one Weierstrass semigroup with first non-gap equal to 3 that occurs at most once. For other values of g such Weierstrass semigroup does not exist. In case g is a large integer then g can be written as $6n - m$ with $0 \leq m \leq 2n - 1$ in many ways. This implies for all integers N there is a bound $g(N)$ such that for $g \geq g(N)$ there are at least N Weierstrass semigroups of genus g with first non-gap equal to 4 that occur at most once.

Example 2. Assume C is a smooth curve of genus g and P is a Weierstrass point on C with first non-gap equal to 5. Note that all Weierstrass semigroups with first non-gap equal to 5 occur as Weierstrass semigroup of some point on some smooth curve (see [14]). Let $b = 5n + r$ with $1 \leq r \leq 3$ and $n \geq 2$ in case $r = 1$ be another non-gap of P not divisible by 5. Assume C has a base point free g^1_5 different from $|5P|$. In Lemma 3 we only have to consider the possibility $e = 1$. In that case all integers at least 20 need to be non-gaps of P. This implies $g \leq 16$. In case $n = 3$ it implies $g \leq 15$ and in case $n = 2$ it implies $g \leq 14$. In case $b = 8$ it implies $g \leq 12$ and in case $b = 7$ it implies $g \leq 11$. In all other cases $|5P|$ is the only g^1_5 on C. In case there are two non-gaps x of P satisfying $5n < x < 5(n + 1)$ then $g \leq 6n + 3$. In case $5(n - 1) + r$ is also a non-gap of P then $g \leq 10n - 12 + 2r$. This implies $\dim |(n + 1)5P| = n + 2$ in case $g > 10n - 12 + 2r$ and also in case $g > 6n + 3$ provided $5(n - 1) + r$ is a gap of P.

In case $r = 1$ and $\dim |(n + 1)5P| = n + 2$ a Weierstrass semigroups of genus g containing $< 5; 5n + 1 >$ and not containing the set $\{10n - 3; 15n - 7; 15n - 2; 20n - 11; 20n - 6; 20n - 1\}$ occurs at most once unless $g \leq 14$ in case $n = 2$. Consider the case $g = 10n - 6$. Then $5n - 4$ is a gap of P. In case $n \geq 3$ we also have $6n + 3 < 10n - 6$, hence $\dim |5(n + 1)P| = n + 2$. We can apply Lemma 8 to obtain that all Weierstrass semigroups of genus $10n - 6$ containing $< 5; 5n + 1 >$ occur at most once except for $< 5; 5n + 1 > \cup \{10n-3; 15n-7; 15n-2; 20n-11; 20n-6; 20n-1\}$ (those are the trivial new non-gaps in this case). There do exist smooth curves of genus $10n - 6$ having two different Weierstrass points having that particular Weierstrass semigroup.

More concretely, for $n \geq 4$ and all non-negative integers m; m' satisfying $m \leq n+m'$; $2m+1 \geq m'$; $m+m' > 2n$ the Weierstrass semigroups $< 5; 5n+1 > \cup \{5i+3 : m' \leq i \leq 3n\} \cup \{5i+4 : m \leq i \leq 4n\}$ occur at most once. In particular we obtain Weierstrass semigroups having $g = 3n + m + m' - 2$ gaps. In case $m + m' < 7n - 3$ this is smaller than te genus bound $10n - 5$ coming from Theorem B. As an example, choosing $m = n + 1$ and $m' = n$ all inequalities are satisfied and the Weierstrass semigroup obtained using those values has genus 5n-1.

In case $r = 2$ and $n = 1$ then as soon as $g < 12$ it is possible that C has some g^1_5 different from $|5P|$. This is clear, if the plane model Γ used in Section 3 has one more singular point $S \neq P$ then the pencil of lines through S induces a base point free g^1_k for some $k \leq 5$ on C. However using different methods it is proved in [5] that a curve C of genus $g = 11$ has at most one Weierstrass point with Weierstrass semigroup containing $< 5; 7 >$ (see also Lemma 9).

In case $n \geq 2$ and $\dim |(n+1)5P| = n+2$ a Weierstrass semigroups of genus g containing $< 5; 5n+2 >$ and not containing the set $\{10n-1; 15n+1; 20n-2; 20n+3\}$ occurs at most once unless $g \leq 14$ in case $n = 2$. Consider the case $g = 10n+1$. Then $5n-3$ is a gap of P, otherwise $< 5; 5n-3 > \subset \mathrm{WS}(P)$ and therefore $g \leq 10n-8$. In case $n \geq 2$ we also have $6n+3 < 10n+1$, hence $\dim |5(n+1)P| = n+2$. Again we can obtain Lemma 8 to obtain sharpness of the uniqueness results. We leave it to the reader to obtain a more concrete description of the Weierstrass semigroups occuring at most once obtained in the paper.

In case $r = 3$ and $\dim |(n+1)5P| = n+2$ we have uniqueness of Weierstrass semigroups of genus g containing $< 5; 5n+3 >$ and not containing the set $\{15n+4; 20n+7\}$ unless $g \leq 12$ in case $n = 1$. Consider the case $g = 10n+4$. Then $5n-2$ is a gap of P and $6n+3 \leq 10n+4$, hence $\dim |5(n+1)P| = n+2$ for $n \geq 1$. Again we can apply Lemma 8 obtaining sharpness of the uniqueness results. A more concrete description of the Weierstrass semigroups occuring at most once is very similar to the description obtained in Example 1

Example 3. For the case $a = 6$ we only need to consider a non-gap $6n+1$ with $n \geq 2$. In case there is a non-gap between 6n+1 and 6n+6 then $g \leq 9n+1$. The bound in Corollary 4 is $g > 15n-10$. We consider the case $g = 15n-10$.

Since $9n+1 < 15n-10$ there are no non-gaps between $6n+1$ and $6n+6$ in case $g = 15n-10$. Also in case $6n-5$ would be a non-gap then $g \leq 15n-15$. This implies $\dim |(n+1)6P| = n+2$. Assume C has another g^1_6 different from $|6P|$. In Lemma 3 we need to consider the cases $e = 1$; 2 and 3. In case $e = 1$ then all integers at least 30 are non-gaps. This implies the existence of more than 10 non-gaps outside $< 6; 6n+1 >$ in case $n \geq 3$. In case $e = 2$ then all even integers at least equal to 12 are non-gaps. This implies the existence of $6n-4$ non-gaps outside $< 6; 6n+1 >$. In case $n \geq 3$ we obtain more than 10 non-gaps outside $< 6; 6n+1 >$. In case $e = 3$ then all integers divisible by 3 and at least equal to 6 are non-gaps. In case $n \geq 3$ this implies 3n-1 non-gaps outside $< 6; 6n+1 >$. In case $n \geq 4$ it implies the existence of more than 10 non-gaps outside $< 6; 6n+1 >$. Therefore only in case $n \geq 4$ the linear system g^1_6 is unique.

This example gives an illustration of the fact that $e > 1$ in Lemma 3 can impose conditions in applying the results of this paper. The following conclusion can only be made in case $n \geq 4$. For $g > 15n-10$ all Weierstrass semigroups of genus g containing $< 6; 6n+1 >$ occur at most once. In case $g = 15n-10$ the only Weierstrass semigroup of genus g containing $< 6; 6n+1 >$ and occuring more than once is equal to $< 6; 6n+1 > \cup \{12n-4; 18n-9; 18n-3; 24n-12; 24n-8; 24n-4; 30n-19; 30n-13; 30n-7; 30n-1\}$.

In case $n = 1$ in general Lemma 3 does not imply uniqueness of g^1_a in case $g \geq \frac{(a-1)(b-a+r)}{2}$. In this case we can use another argument to conclude uniqueness of Weierstrass semigroups containing $< a; a+r >$ with $1 \leq r \leq a-2$ and $(a, r) = 1$.

Lemma 9. Fix an integer $r \geq 2$. There is a bound $A(r)$ such that in case $a \geq A(r)$, $(a, r) = 1$ and $g > \frac{(a+r-1)(a-r)}{2}$ then a smooth curve C of genus g has at most one Weierstrass point whose Weierstrass semigroup contains $< a; a + r >$.

Proof. As a matter of fact, there is a genus bound $g(a; r)$ obtained in [1] , Theorem 4.3, such that in case C is a smooth curve of genus $g \geq g(a; r)$ then C has at most one simple g^2_{a+r}. This genus bound behaves like a polynomial in a with highest order term $\frac{a^2}{3}$. Since $\frac{(a-1)(b-1)}{2}$ is polynomial with highest order term $\frac{a^2}{2}$ it implies that for small $e \geq 1$ one has $\frac{(a-1)(b-1)}{2} - e \geq g(a; r)$ if $a >> 0$. In such case, if P and Q are two different Weierstrass points on C such that $\mathrm{WS}(P)$ and $\mathrm{WS}(Q)$ both contain $< a; a + r >$ then we obtain $|(a + r)P| = |(a + r)Q|$. In particular P and Q have to induce both a cusp of type $(r; a + r)$ on the same plane model Γ of C (as considered in Section 3). This implies

$$g \leq \frac{(a + r - 1)(a - r)}{2} .$$

This genus bound is polynomial with highest order term equal to $\frac{a^2}{2}$, hence for $a >> 0$ this bound is larger than $g(a; r)$. This implies that in case $g > \frac{(a+r-1)(a-r)}{2}$ and $a >> 0$ then a curve C of genus g has at most one Weierstrass point with Weierstrass semigroup containing $< a; a + r >$. □

In case $r = 2$ this is part of the arguments used in [5] and using more detailled arguments one obtains clear and good genus bounds. It should be noted that in case $a >> 0$ this bound on g is sharp. One can make use of plane rational curves having two cusps of type $(r; a + r)$ as follows. Choose P and Q different points on $\mathbb{P}^1$. Let E be a general effective divisor of degree $a - r$ on $\mathbb{P}^1$ and consider the pencil $rP+ < aP; rQ + E >$, a g^1_{a+r} on $\mathbb{P}^1$. Then take $g^2_b =< (a+r)Q; rP+ < aP; rQ+E >>$. Then using arguments as those used in [4], Section 3, one can show that there exists a plane curve of degree $a+r$ having exactly two cusps of type $(r; a+r)$ and no other singularities. The normalisation C of this curve has genus $\frac{(a+r-1)(a-1)}{2}$ and has two different Weierstrass points whose Weierstrass semigroups contain $< a; a + r >$.

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