Double rotational surfaces in Euclidean 4-space

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Outline

Introduction: in $\mathbb{E}^3$

- The construction of a *surface of revolution* or a *rotational surface* in Euclidean 3-space $\mathbb{E}^3$ is well-known: this surface is traced out by a planar curve that is rotated about an axis in its supporting plane.
- A generalization of this construction leads to a *twisted surface*: this surface is the trace of a planar curve that is subjected to two simultaneous rotations, possibly at different rotation speeds.

  For an overview of results on twisted surfaces see *W. Goemans and I. Van de Woestyne "Twisted surfaces with null rotation axis in Minkowski 3-space" Results in Mathematics, September 2016, Vol. 70, Issue 1, pp 81-93*, and the references therein.

Introduction: to $\mathbb{E}^4$

- The construction of surfaces of revolution was carried over to Euclidean 4-space $\mathbb{E}^4$ in the previous century by C.L.E. Moore and G. Vranceanu and studied later also by, amongst others, G. Ganchev and V. Milousheva, K. Arslan, B. Bayram, B. Bulca and G. Öztürk, U. Dursun and N.C. Turgay, D. V. Cuong.
- Now translate the construction of twisted surfaces to Euclidean 4-space since there is 'more space to twist'!
  Or to cite Prof. B. Rouxel: *"il y a plus de place !"*.
- Hence we construct a surface in $\mathbb{E}^4$ by performing two simultaneous rotations on a planar curve, possibly at different rotation speeds. The resulting surface is called a *double rotational surface*.

Definition of a double rotational surface in $\mathbb{E}^4$

Definition

A *double rotational surface* in Euclidean 4-space is a surface that is traced out by a planar curve, the *profile curve*, when it is subjected to two simultaneous rotations, possibly at different rotation speeds.

In Euclidean 4-space there exist two kinds of rotations:

- a rotation about a point,
- a rotation about a plane.

Parameterization of a double rotational surface in $\mathbb{E}^4$

Combining two of these possible rotations, up to a transformation, there exists only one parameterization for a non-trivial double rotational surface in Euclidean 4-space, namely,

$$\begin{aligned} x(s,t) = (&\cos(cs)\,[a + f(t)\cos(bs) - g(t)\sin(bs)]\,, \\ &\sin(cs)\,[a + f(t)\cos(bs) - g(t)\sin(bs)]\,, \\ &\cos(ds)\,[f(t)\sin(bs) + g(t)\cos(bs)]\,, \\ &\sin(ds)\,[f(t)\sin(bs) + g(t)\cos(bs)]) \end{aligned}$$

with $a, b, c, d \in \mathbb{R}$ and profile curve $\alpha(t) = (f(t), 0, g(t), 0)$.
Here we exclude the combinations of rotations that lead to a twisted surface in $\mathbb{E}^3$ or to (a part of) a plane.
If $b = 0$ this parameterization reduces to that of a surface of revolution in $\mathbb{E}^4$. This is excluded in the following.

Curvatures of double rotational surfaces in $\mathbb{E}^4$

Examine the condition for zero Gaussian curvature

$$K = \frac{1}{EG - F^2} \sum_{k=1}^{2} \left(c_{11}^k c_{22}^k - (c_{12}^k)^2 \right),$$

that is, the surface if *flat*, or zero mean curvature vector

$$\vec{H} = \frac{1}{2(EG - F^2)} \sum_{k=1}^{2} \left(c_{11}^k G - 2c_{12}^k F + c_{22}^k E \right) N_k,$$

that is, the surface is *minimal*.
Here are E, F, G the components of the first fundamental form, c_{ij}^k for $i, j, k \in \{1, 2\}$ the components of the second fundamental form and is $\{N_1, N_2\}$ an orthonormal frame field of the surface.

Flat double rotational surfaces in $\mathbb{E}^4$

The expression for the Gaussian curvature of a double rotational surface in $\mathbb{E}^4$ is huge. Therefore, Maple is used to do the calculations in the systematic study of when $K \equiv 0$ for these surfaces.
When excluding the surfaces of revolution, the following double rotational surfaces in $\mathbb{E}^4$ are flat:

- A cone over the curve

$$\begin{gathered}(\cos(cs)\,(\cos(bs) - p\sin(bs))\,, \sin(cs)\,(\cos(bs) - p\sin(bs))\,, \\ \cos(ds)\,(\sin(bs) + p\cos(bs))\,, \sin(ds)\,(\sin(bs) + p\cos(bs)))\end{gathered}$$

  where $p \in \mathbb{R}$.
- A flat twisted surface in $\mathbb{E}^3$ (which we exclude).

Clelia curve in $\mathbb{E}^3$

A Clelia curve in $\mathbb{E}^3$ is a spherical curve that has linear dependent coordinates when parameterized using spherical coordinates.

In our work on twisted surfaces in $\mathbb{E}^3$ a Clelia curve turned out to be the curve that determines the cones which are flat twisted surfaces.

See *W. Goemans and I. Van de Woestyne "Clelia Curves, Twisted Surfaces and Plücker's Conoid in Euclidean and Minkowski 3-space" Contemporary Mathematics 674. Recent Advances in Submanifold Geometry, A Proceedings Volume Dedicated to the Memory of Franki Dillen (1963-2013). Editors A. Carriazo, Y.M. Oh, J. Van der Veken and B. Suceavă, September 2016*, and the references therein for more properties of Clelia curves.

Clelia curve in $\mathbb{E}^4$

Definition

A *Clelia curve* in Euclidean 4-space is a curve on a hypersphere for which its coordinates, when its parameterization is expressed using hyperspherical coordinates, are linear dependent.

To see the linear dependency of the coordinates, use Hopf coordinates, so consider

$$x(\theta,\nu,\varphi) = r\,(\cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi\cos\nu, \sin\varphi\sin\nu)$$

as a parameterization of the hypersphere with radius $r \in \mathbb{R}$ in $\mathbb{E}^4$. Then rewrite the curve in the examples of flat double rotational surfaces.

Clelia curves in flat double rotational surfaces

The curve in the parameterization of flat double rotational surfaces can be rewritten to

$$\frac{1}{\cos\eta}\left(\cos(\eta+bs)\cos\left(cs+\frac{c\eta}{b}\right),\cos(\eta+bs)\sin\left(cs+\frac{c\eta}{b}\right),\right.$$
$$\left.\sin(\eta+bs)\cos\left(ds+\frac{d\eta}{b}\right),\sin(\eta+bs)\sin\left(ds+\frac{d\eta}{b}\right)\right)$$

where η is such that $\cos\eta=\frac{1}{\sqrt{1+p^2}}$ and $\sin\eta=\frac{p}{\sqrt{1+p^2}}$. From this the linear dependency of the coordinates is immediate.
Moreover, if $c=d$ then the Clelia curve is a curve on a flat torus $\mathbb{T}^2$ which lies itself on a hypersphere in $\mathbb{E}^4$.

Minimal double rotational surfaces in $\mathbb{E}^4$

Also the mean curvature vector of a double rotational surface in $\mathbb{E}^4$ is a large expression for which Maple is used to manipulate it.

Up to now I could not find any minimal double rotational surfaces in $\mathbb{E}^4$ when excluding the surfaces of revolution.

A non-existence result would be in line with the non-existence of minimal twisted surfaces in $\mathbb{E}^3$ (when excluding the surfaces of revolution).

Further research

- Complete the classification of the flat double rotational surfaces in $\mathbb{E}^4$.
- Find a classification of minimal double rotational surfaces in $\mathbb{E}^4$ or prove there exist none.
- Consider double rotational surfaces with non-zero constant Gaussian curvature or with non-zero constant mean curvature in $\mathbb{E}^4$.
- Study Clelia curves in $\mathbb{E}^4$.
- Study double rotational surfaces in other spaceforms, for instance 4-dimensional Minkowski space.

Thank you for your attention.

Questions?

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This presentation is available at `feb.kuleuven.be/wendy.goemans/downloads`.

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