Combined $\mathcal{H}_\infty$ linear parameter varying control design with optimal sensor/actuator selection

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Abstract—This paper presents an approach to co-design $\mathcal{H}_\infty$ dynamic output feedback controllers for linear parameter-varying (LPV) systems with optimal selection of sensors and actuators. This concurrent design problem is formulated as a Mixed Boolean Semi-Definite Programming (MBSDP) optimization problem. By means of the Big-M reformulation, the boolean decision variables related to the selection of sensors and actuators are decoupled from the SDP problem underlying the $\mathcal{H}_\infty$ controller synthesis. This reformulation allows us to solve the initial NP-hard problem efficiently using a Branch and Bound (BNB) algorithm. The concurrent design of an LPV vibration controller and optimal actuator selection for a smart composite plate serves as a case study to validate the developed approach.

I. INTRODUCTION

Industries have nowadays adopted the approach of designing optimal $\mathcal{H}_\infty$ feedback controllers to deal with multi-variable systems that display complex dynamic behavior. $\mathcal{H}_\infty$ techniques focusing on linear time-invariant (LTI) systems are easily extendable to LPV systems. They are a class of nonlinear systems, where the dynamics can be described with a linear model depending on one or several distinctive signals called scheduling parameters [1], [2]. These advanced control techniques usually generate complex control solutions if the number of sensors and actuators is large. In industries, it may be economically advantageous to use only a subset of all the possible sensors and actuators available with acceptable closed-loop performance costs.

Various controller independent strategies for sensor and actuator selection have been discussed before in [3]. [4] suggests to optimally select the components based on open-loop $\mathcal{H}_\infty/\mathcal{H}_2$ norms for dynamic structures. Another strategy that they adopted is to consider the controllability and observability of the systems. However, in order to discard the sensors and actuators, a judgement by the user is made on the degree of controllability and observability, which may not be the optimal one, when the aim is to design a controller.

The concurrent control design and optimal sensor and actuator selection is a combinatorial problem and hence NP-hard: it requires extensive search and may become computationally expensive if the number of sensors and actuators increases. For instance, a blended wing body passenger aircraft model would require to place 556 and 196 possible

sensors and actuators respectively [5]. In general, if n_s and n_a represents the possible number of sensors and actuators, respectively, then the total number of problems for distinct sets of components to be solved are $(2^{n_u} - 1)(2^{n_y} - 1) + 1$. Hence, in order to manage the interplay between the optimal selection of sensors/actuators and controller design, a computationally less expensive approach is required.

Few of the studies that consider optimal input and output (and hence sensors and actuators) selection combined with different control design approaches are elaborated in [3]. An approach of using heuristic techniques, such as genetic algorithms in order to solve the combined Linear Quadratic Regulator (LQR) controller design and optimal selection problem are tackled in [6] and [7] with an application focusing on active suspension control of a vehicle. In general, heuristic techniques have slow convergence. Another approach explores the design of static output feedback $\mathcal{H}_\infty/\mathcal{H}_2$ feedback control [8] and [9] with sensors or actuator selection. [10] solves the combined problem by taking uncertainty into account using approximations of the ℓ_0 norm in terms of (weighted) ℓ_1 norms [11].

Combined $\mathcal{H}_\infty$ controller design and sensor/actuator selection results into a Mixed Boolean Semi-Definite Programming (MBSDP) problem, since the sensor/actuator selection variables are boolean and the $\mathcal{H}_\infty$ controller synthesis can be formulated as an SDP problem. These problems are NP-hard problems and hence are difficult to solve. Some efforts have been carried out to separate the coupling of boolean and SDP variables, as done by [12] and [13], to result into convex problems. However, these methods do not concurrently perform the globally optimal sensors/actuators selection and feedback control, yielding suboptimal results.

This paper presents another approach to convert the MBSDP problem underlying a combined optimal $\mathcal{H}_\infty$ controller design and sensor/actuator selection into manageable optimization problem without sacrificing optimality. In the presented approach, Big-M reformulations [14] are used to decouple the boolean decision variables from the SDP problem. The resulting problem is then solved using the BNB algorithm from YALMIP [15]. A similar approach has been adopted in [16] for LTI systems. This paper extends this approach to LPV systems within the framework of a novel B-spline based LPV control design [17]. Here, the Big-M constraints are imposed on coefficients of the B-splines of the controller matrices yielding column sparsity of the controller input matrix that optimizes the number of sensors, while the row sparsity of the controller output matrix that optimizes the number of actuators.

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The remainder of the paper is organized as follows. Section II briefly elaborates the novel B-splines based LPV $\mathcal{H}_\infty$ dynamic output feedback control synthesis [17]. Section III joins the LPV LMI conditions with the actuators and/or sensor selection to form a MBSDP optimization problem. Here, the Big-M reformulations on the coefficient matrices of B-splines are performed to produce a convex problem to be solved by BNB. In Section IV, this combined algorithm is applied to the case of designing an active vibration controller for a smart composite plate and the results are shown along with the analyses of the controllers. Finally, Section V provides some concluding remarks.

II. $\mathcal{H}_\infty$ B-SPLINE BASED LPV CONTROLLER DESIGN

A. Preliminaries

Consider a continuous-time parameter varying generalized plant described in state space form:

$$\begin{aligned}\dot{x}(t) &= A(\rho(t))x(t) + B_w(\rho(t))w(t) + B_u(\rho(t))u(t) \\ z(t) &= C_z(\rho(t))x(t) + D_{zw}w(t) + D_{zu}u(t) \\ y(t) &= C_y(\rho(t))x(t) + D_{yw}w(t) + D_{yu}u(t),\end{aligned}\quad (1)$$

where $x \in \mathbb{R}^{n_x}$ denotes the plant state vector, $w \in \mathbb{R}^{n_w}$ is the exogenous input, $u \in \mathbb{R}^{n_u}$ is the control input, $z \in \mathbb{R}^{n_z}$ is the regulated output and $y \in \mathbb{R}^{n_y}$ is the measured output. Assume a continuous-time measurable scheduling parameters $\rho(t) \in \mathbb{R}^{n_\rho}$ that lies in some a priori known admissible compact set Δ . The aim is to design a strictly proper dynamic output feedback LPV controller of the form

$$\begin{aligned}\dot{\zeta} &= A_K(\rho(t))\zeta + B_K(\rho(t))y \\ u &= C_K(\rho(t))\zeta,\end{aligned}\quad (2)$$

where ζ represents the controller state vector.

In this paper we apply a polynomial spline based parameterization to model the scheduling parameter dependency of both (1) and (2) [17]. In this paper we restrict ourself to one scheduling parameter, that is, $\rho \in \mathbb{R}$, for brevity and compactness of notation. Multiple scheduling parameters ($n_\rho > 1$) leads to tensor product B-splines as discussed in [17]. Considering a B-spline basis function of degree n_d and a knot sequence $\lambda \in \mathbb{R}^{n_\lambda}$, a parameter dependent variable is formulated as

$$S(\rho) = \sum_i^{n_\lambda - n_d - 1} \mathcal{S}_i \mathbb{B}_{i, n_d, \lambda}(\rho) \quad (3)$$

where $\mathcal{S}_i$ are the coefficient matrices and $\mathbb{B}_{i, n_d, \lambda}(\rho)$ are the B-splines. The advantage of using B-splines as basis to model the scheduling parameter dependency of (1) and (2) lies in their unique properties of linear dependency, positivity, partition of unity and minimal support [17]. It is to be noted that the formulation (3) of parameter dependent variable corresponds to a univariate parameter for brevity. In case of multivariate parameter dependency, the B-splines are extended to tensor product B-splines [17].

Let $\mathcal{T}$ be the corresponding closed-loop transfer function from w to z . Closed-loop specifications are formulated using the $\mathcal{H}_\infty$ norm of $\mathcal{T}$.

B. LMI formulations of a full order dynamic output feedback LPV controller

The core of the combined approach is based on the LMI formulations of the full order dynamic output feedback LPV controller whose results for $\mathcal{H}_\infty$ controller synthesis, elaborated below without proofs (see [1] for proofs).

Lemma 2.1: There exists a feedback controller (2) that yields $\|\mathcal{T}\|_\infty < \gamma$ if, and only if there exists symmetric matrices $X, Y \in \mathbb{R}^{n_x}$ for which,

$$\begin{bmatrix} \Xi_1 & \star & \star & \star \\ \hat{\mathcal{A}}_{Ki} + \mathcal{A}_i^T & \Xi_2 & \star & \star \\ \mathcal{B}_{wi}^T & \Xi_3 & -\gamma I & \star \\ \Xi_4 & \mathcal{C}_{zi} & D_{zw} & -\gamma I \end{bmatrix} < 0 \quad (4)$$

and

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \quad (5)$$

with,

$$\begin{aligned}\Xi_1 &= \mathcal{A}_i X + X \mathcal{A}_i^T + \mathcal{B}_{ui} \hat{\mathcal{C}}_{Ki} + \hat{\mathcal{C}}_{Ki}^T \mathcal{B}_{ui}^T \\ \Xi_2 &= Y \mathcal{A}_i + \mathcal{A}_i^T Y + \hat{\mathcal{B}}_{Ki} \mathcal{C}_{yi} + \mathcal{C}_{yi}^T \hat{\mathcal{B}}_{Ki}^T \\ \Xi_3 &= Y \mathcal{B}_{wi} + \hat{\mathcal{B}}_{Ki} D_{yw} \\ \Xi_4 &= \mathcal{C}_{zi} X + D_{zu} \hat{\mathcal{C}}_{Ki},\end{aligned}\quad (6)$$

where a star ($\star$) denotes a symmetric term in the LMI formulation and $\mathcal{A}_i, \mathcal{B}_{wi}, \mathcal{B}_{ui}, \mathcal{C}_{zi}$ and $\mathcal{C}_{yi}$ (all related to system (1)) are the coefficient matrices of the parameter dependent model matrices $A(\rho(t)), B_w(\rho(t)), B_u(\rho(t)), C_z(\rho(t)), C_y(\rho(t))$, respectively, as seen in (3). Additionally, using (3), $\hat{\mathcal{A}}_{Ki}, \hat{\mathcal{B}}_{Ki}$ and $\hat{\mathcal{C}}_{Ki}$, are the coefficient matrices of $\hat{A}_K(\rho(t)), \hat{B}_K(\rho(t))$ and $\hat{C}_K(\rho(t))$ which are the additional variables with $\hat{A}_K(\rho(t)) \in \mathbb{R}^{n_x}, \hat{B}_K(\rho(t)) \in \mathbb{R}^{n_x \times n_y}$ and $\hat{C}_K(\rho(t)) \in \mathbb{R}^{n_u \times n_x}$.

The controller state space matrices are reconstructed as

$$\begin{aligned}C_K(\rho(t)) &= \hat{C}_K(\rho(t)) M^{-T} \\ B_K(\rho(t)) &= N^{-1} \hat{B}_K(\rho(t)) \\ A_K(\rho(t)) &= N^{-1} (\hat{A}_K(\rho(t)) - B_K(\rho(t)) D_{yu} \hat{C}_K(\rho(t))) - \\ &\quad - N B_K(\rho(t)) C_y(\rho(t)) X \\ &\quad - Y B_u(\rho(t)) C_K(\rho(t)) M^T \\ &\quad - Y A(\rho(t)) X M^{-T},\end{aligned}\quad (7)$$

where M and N are nonsingular matrices that are taken to satisfy

$$M N^T = I - X Y. \quad (8)$$

The $\mathcal{H}_\infty$ control optimization problem is formulated as

$$\begin{aligned}\text{minimize: } & \gamma \\ \text{subject to: } & (4), (5).\end{aligned}\quad (9)$$

It is to be noted that the symmetric matrices X and Y are considered parameter independent. The LMI formulations can also be reformulated with these matrices as parameter

dependent. However, by doing so, there are rational parameter dependent terms introduced in the controller reconstruction (7). In order to avoid that, an approach for LPV controller construction as mentioned in [18] can be adopted.

III. BIG-M REFORMULATION OF A MBSPD PROBLEM

A. MBSPD formulation

This section describes how the combination of designing an optimal parameter varying feedback controller (2) for a generalized plant (1) with the optimal selection of sensors and actuators is formulated as an MBSPD. It is easily observed from (7) that the non-zero columns of $\hat{B}_K(\rho)$ would indicate the number of sensors to be used by the controller. So, in order to obtain only a subset of the possible sensors, the columns of $\hat{B}_K(\rho)$ corresponding to the set of sensors to be neglected must be set equal to zero. Similarly, for achieving a subset of all possible actuators, the rows of $\hat{C}_K(\rho)$ corresponding to the set of actuators to be neglected must be equal to zero.

The two matrices Φ and χ that are responsible for the selection of sensors and actuators, respectively, are defined as

$$\Phi = \text{diag}(s_1, \dots, s_{n_y}), \quad \Psi = \text{diag}(a_1, \dots, a_{n_u}). \quad (10)$$

where $s_i, a_j \in \{0, 1\}$ are the boolean decision variables indicating whether a particular sensor or actuator, respectively, is selected or not. If $\hat{B}_{Ki}$ and $\hat{C}_{Ki}$ represents the matrix-valued coefficients of B_K and C_K respectively for the chosen B-spline parameterization, then $\hat{B}_{Ki}\Phi$ and $\Psi\hat{C}_{Ki}$, ($\forall i : i \in [1, n_\lambda - n_d - 1]$) provides the information regarding the selection of sensors and actuators, respectively.

These coupled variables are mixed boolean semi-definite variables which have a non-linear coupling. In order to get rid of this coupling, Big-M reformulations are considered on these variables which transforms boolean constraints to a set of constraints that describe the same feasible set [14]. This will result into following

$$m_B \Phi \leq \hat{B}_{Ki} \leq M_B \Phi, \quad \Psi m_C \leq \hat{C}_{Ki} \leq \Psi M_C, \quad (11)$$

where, $m_B \in \mathbb{R}_+^{n_x \times n_y}$, $M_B \in \mathbb{R}_+^{n_x \times n_y}$, $m_C \in \mathbb{R}_+^{n_u \times n_x}$ and $M_C \in \mathbb{R}_+^{n_u \times n_x}$ are matrices representing Big-M bounds [16]. If M_B and M_C are chosen sufficiently large (larger than any possible value) and m_B and m_C are chosen sufficiently small (smaller than any possible value) in $\hat{B}_{Ki}$ and $\hat{C}_{Ki}$ respectively, then this Big-M reformulation is equivalent to the original constraint. Here, it should be noted that all the inequalities are element-wise.

B. $\mathcal{H}_\infty$ controller synthesis with optimal selection of sensors and actuators

Suppose we require $\{k : 1 \leq k \leq n_y\}$ sensors and $\{l : 1 \leq l \leq n_u\}$, equality constraints are obtained as deduced from (10) and are shown as

$$\text{trace}(\Phi) = k, \quad \text{trace}(\chi) = l. \quad (12)$$

The MBSPD optimization problem of selection or placement of optimal set of k sensors and l actuators along with $\mathcal{H}_\infty$ LPV control design is formulated as

$$\begin{aligned} &\text{minimize: } \gamma \\ &\text{subject to: } (4), (5), (11), (12). \end{aligned} \quad (13)$$

Now, assume η_B as a normalized cost for sensors to be selected and η_C as a normalized cost for actuators. As a result, the total cost of implementation of sensors and actuators is given by, $\text{trace}(\eta_B \Phi)$ and $\text{trace}(\eta_C \chi)$. Here, η_B and η_C can be a scalar quantity, which would indicate that all the sensors or actuators have same costs or a diagonal matrix with distinct costs. As a results, the following multi-objective optimization problem for the optimal selection is formulated.

$$\begin{aligned} &\text{minimize: } \gamma + \text{trace}(\eta_B \Phi) + \text{trace}(\eta_C \chi) \\ &\text{subject to: } (4), (5), (11), (12). \end{aligned} \quad (14)$$

The Big-M reformulations in the MBSPD optimization problem (14) converts it into a convex problem which can be solved by a BNB algorithm to global optimality. The results in this paper are obtained by the BNB algorithm of YALMIP. The BNB algorithm works by simultaneously building and searching a binary tree of optimization problems where every child is obtained from a parent via a branching procedure [19].

IV. APPLICATION

A. Model Description

An attractive application of the proposed algorithm is active vibration control of smart composite plates. Since the passive vibration suppression of lightweight flexible structures reached a certain limit, active vibration control of a smart composite plate equipped piezoelectric transducers as actuators/sensors aroused considerable research. Numerous piezoelectric actuators and sensors can be integrated into these smart structures, but it is computationally expensive to find an optimal trade-off between the number of sensors/actuators on the structure and performance measured as vibration suppression. The proposed algorithm can concurrently select the sensors/actuators and design an optimal $\mathcal{H}_\infty$ controller.

In this paper, a free-free smart composite plate is used to validate the proposed algorithm. The composite plate is made by unidirectional carbon fiber laminates with a lay-up of $[-45^\circ \ 45^\circ \ 0^\circ \ 90^\circ]_s$ [20]. The size of the plate is $0.5m \times 0.4m$ and the thickness of the plate is $2.5mm$. Nine macro-fiber composite (MFC) transducers (actuators) of type M2814-P1 from Smart Material Co. are integrated into the plate along with nine accelerometer (sensors) and their placements on the plate are shown in Fig. 1. The size of the MFC actuators are $28mm \times 14mm$ and their thickness is $0.3mm$. The mechanical influence of the accelerometers on the dynamics of the model is neglected, since they are negligible (5 grams per unit) compared to actuators.

Fig. 1. Schematic representation of the smart composite plate ($d_1 - d_4$ are force disturbances applied to the plate.)

TABLE I
STRUCTURE AND MATERIAL PROPERTIES OF THE LAMINATED
COMPOSITE PLATE WITH INTEGRATED MFC-D33 TRANSDUCERS

Parameter	MFC – d33	laminates
mass density	5440 Kg/m^3	1500 Kg/m^3
Y_{11}	27.142 GPa	110.15 GPa
Y_{22}	14.8 GPa	7.868 GPa
G_{12}	4.1312 GPa	4.05 GPa
ν_{12}	0.2922	0.26
G_{13}	5.385 GPa	—
G_{23}	10.5 GPa	—
d_{33}	272 pm/V	—
d_{32}	-133 pm/V	—
C_p	0.30 nF/cm^2	—

The Equivalent substructure modeling approach is used to model the smart composite plate [20], and the properties of the smart plate are given in Table I. The dynamics of the smart plate can be expressed as follows:

$$[M]\ddot{q} + [C]\dot{q} + \underbrace{([K] - [\Theta C_p^{-1} \Theta^T])}_{[K^{ag}]} = \Theta u + Ld \quad (15)$$

where q is the displacement of the plate. $[M]$, $[C]$ and $[K]$ are the mass matrix, damping matrix and stiffness matrix, respectively. $[K^{ag}]$ is the augmented stiffness matrix which includes the piezoelectric induced stiffness. The capacitance of the used MFC actuators is noted as C_p . Θ is the piezoelectric coupling matrix and the input u is voltage. d is force disturbance and L is the localization matrix of d . A reduced order model including 5 vibration modes is obtained from (15) by using the Second-Order-Rational-Arnoldi technique [21]:

$$[M_r]\ddot{q} + [C_r]\dot{q} + [K_r^{ag}] = \Theta_r u + L_r d \quad (16)$$

A 10^{th} order state space model with the retained 5 vibration modes can be obtained from (16) simply as:

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & I \\ -M_r^{-1} K_r^{ag} & -M_r^{-1} C_r \end{bmatrix}}_A \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -M_r^{-1} \Theta_r \end{bmatrix}}_{B_u} u \\ &+ \underbrace{\begin{bmatrix} 0 \\ -M_r^{-1} L_r \end{bmatrix}}_{B_d} d. \end{aligned} \quad (17)$$

Acceleration output is used in the modeling so that the output y can be formulated as follows:

$$y = \begin{bmatrix} 0 & C_a \end{bmatrix} \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix}. \quad (18)$$

Substituting (17) into (18) yields

$$y = \underbrace{\begin{bmatrix} 0 & C_a \end{bmatrix} A}_{C} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & C_a \end{bmatrix} \begin{bmatrix} B_u & B_d \end{bmatrix}}_D \begin{bmatrix} u \\ d \end{bmatrix}. \quad (19)$$

Although the system as described above is LTI, an artificial LPV system is constructed by parameterizing the system matrix, A , as shown in (20).

$$A(\rho(t)) = A + \rho A \quad (20)$$

where the ρ is the artificially generated scheduling parameter assuming values in the interval $[0,0.2]$.

B. $\mathcal{H}_\infty$ LPV Control Design

The control configuration is shown in the Figure 2, where d_j represents the disturbance forces to the model. The considered design problem is a disturbance rejection problem with the loop-shaping filters W_s and W_u , such that a full-order feedback controller $K(\rho(t))$ of the form (2) is obtained to minimize the cost function as shown in (21).

$$\text{minimize: } \left\| \begin{bmatrix} W_s D \\ W_u U \end{bmatrix} \right\|_\infty. \quad (21)$$

where, D is the disturbance channel and U is the channel that represents input sensitivity.

Fig. 2. Control configuration of the active vibration of the plate

The weights W_s are selected as following static gains:

$$W_s = \text{diag}\{0.1778, 0.1, 0.1, 0.1, 0.1778, 0.1, 0.1, 0.1, 0.3162\} \quad (22)$$

to reduce the vibrations to keep it under 15, 20, 20, 20, 15, 20, 20, 20 and 10 db respectively for each performances $z_1, \dots, z_9$ and the 1^{st} order high-frequency roll-off for W_u are selected as:

$$W_u = \text{diag}\left\{\frac{3.162s}{s + 1.131e06} I_9\right\}. \quad (23)$$

Combining the system model (17) with weights (22) and (23) according to figure. 2, yields the generalized plant of the form (1), which is the used in the optimization problem (9). The control problem is formulated with objective (21) and is solved using 'LCToolbox'¹, an open source

¹https://github.com/meco-group/lc_toolbox

toolbox for matlab MATLAB [18]. This toolbox consists of routines that solve problem (9). The optimal γ achieved is 74.0648. The optimization variables, $\hat{A}_K(\rho(t))$, $\hat{B}_K(\rho(t))$ and $\hat{C}_K(\rho(t))$, are parameterized with B-splines of degree 1, and the knot sequence $[0, 0, 0.2, 0.2]$. The resulting 19th order full order controller (2) is reconstructed according to (7). This controller uses all the 9 sensor and actuator pairs.

C. Combined Controller Design and Optimal Selection

The matrices representing the Big-M reformulations ie. m_B, m_C, M_B and M_C , are generated selecting larger than the maximum values and smaller than the minimum values of the matrix-valued coefficients of the $\hat{B}_K(\rho)$ and $\hat{C}_K(\rho)$ obtained in the previous paragraph. In the application case study in this paper, the Big-M bounds are considered with a safety factor of 40 for these upper and lower bounds.

Consider the optimization problem given by (14). Here, we focus on actuator selection while choosing all sensors to be fed back to the controller. First we consider that all actuators have the same cost. An extra constraint was added that the number of actuators should be more than 1 and less than 9. The respective optimization problem can be formulated as shown in the (24). Different values for η_C results in different performances and actuator locations.

$$\begin{aligned} &\text{minimize: } \gamma + \text{trace}(\eta_C \chi) \\ &\text{subject to: } (4), (5), \\ &\quad \chi m_C \leq \hat{C}_{Ki} \leq \chi M_C, \\ &\quad 1 < \text{trace}(\chi) < 9, \end{aligned} \quad (24)$$

This optimization problem is parsed using YALMIP [15] and OptiSpline [22], [23] and solved using SDPT3 [24]. The optimal actuator locations are 2, 4, 6 and 8 (see figure 1.) for $\eta_C = 3$. The optimal γ achieved is 81.0569. The correctness of this result is verified by solving the control design problem with all possible actuator combinations which revealed that these locations were indeed optimal and the closed-loop performance cost is also exact. Hence, we achieve the global optimal solution for the selected value $\eta_C = 3$.

Similarly, the next step is to assign a cost to each actuator and take these into account. For that we selected following random set of normalized cost values, $\eta_C = \text{diag}(0.8147, 0.9058, 0.1270, 0.9134, 0.6324, 0.0975, 0.2785, 0.5469, 0.9575)$. Considering (24) with the refined η_C , the algorithm selected all actuators except actuators 5 and 9. Actuator 9 was not selected because it is the most expensive one and it neglects the 5th actuator since it is the most redundant actuator, because it was found out that in the equal actuator cost scenario, and irrespective of η_C , this actuator is removed from the set as the first one.

Figure 3. shows the comparison of open loop and closed loop frequency responses from the disturbances to the outputs for an equally spaced grid of scheduling parameter. It is easily noted that due to the selection of static gain as the weighting filter in the control design, there is a significant reduction in the magnitude at the resonance frequencies. Figure 4. shows the frequency response as seen in figure 3. from d_1 to y_1 for clarity.

Fig. 3. Comparison of the open loop and closed loop frequency responses from the disturbances d_j ($j = 1 \dots 4$) to outputs (accelerometer signals) y_k ($k = 1 \dots 9$).

Fig. 4. Comparison of the open loop and closed loop frequency responses from the disturbances d_1 to output (accelerometer signals) y_1 .

Figure 5. shows the comparison of open loop and closed loop step responses from the unit step force disturbances at d_j to the outputs y_k , respectively. In this simulation, the scheduling parameter ρ is varied sinusoidally within its allowable range to generate the open-loop and closed loop responses. On analyzing further, it is found that an average reduction of the rms value of the vibration of about 75% is achieved. Figure 6. shows the frequency response as seen in figure 5. from d_2 to y_3 for clarity.

Fig. 5. Comparison of the open loop and closed loop step responses from the disturbances d_j ($j = 1 \dots 4$) to outputs (accelerometer signals) y_k ($k = 1 \dots 9$).

Fig. 6. Comparison of the open loop and closed loop step responses from the disturbances d_2 to outputs (accelerometer signals) y_3 .

The calculation time to solve the problem with distinct actuator costs for (24) is 2.46 hours. However, when an exhaustive search is performed to solve all the 500 possible combinations, it is found that the total calculation time is 6.40 hours. Hence the proposed approach saves approximately 4 hours to achieve the global optimal solution.

V. CONCLUSIONS

This paper presents an approach of designing $\mathcal{H}_\infty$ linear parameter varying feedback controllers simultaneously with optimal selection of sensors and actuators. The combined design problem is converted into a mixed boolean semi-definite programming problem. Big-M reformulations are used in order to solve the mixed problem using an branch and bound algorithm.

The proposed approach is evaluated by means of an active vibration control problem of a composite plate. The results obtained with the approach indicate that the global optimal solution is found, i.e. global in terms of closed-loop performance and in actuator selection. The calculation time required to solve the combined problem is found to be considerably less than the calculation time required to perform a full combinatorial search.

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