Conditional earnings subsidies for low earners

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Abstract

Having low earnings is often not sufficient to be eligible for in-work tax credits. In the United Kingdom, New Zealand, and Ireland, for example, eligibility also requires that the number of hours worked is sufficiently high. Similarly, in France and Belgium the hourly wage rate must be sufficiently low. This paper provides a justification for such additional conditions. I analyze Pareto efficient redistribution from high to low ability individuals in a model where labour has several intensive margins. Besides labour hours, also labour effort—a vector of unpleasant, but productive features of labour—is an object of choice. Effort and ability determine the hourly wage rate. I find that conditional earnings subsidies for low earners are optimal: the earnings of low earners should be subsidized at the margin, but only if they earn more by working more hours at a sufficiently low wage rate.

JEL codes: H2, I3, J2

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1 Motivation

Welfare policies try to find—within budgetary limits—a good balance between two, often conflicting, goals: maintaining sufficiently high living standards and encouraging work. Over the past decades, encouraging work became a more prominent goal in many countries, witness the introduction or reinforcement of in-work tax credits (OECD, 2011). Such tax credits reduce the effective marginal tax rates and hence increase the incentives for low earners to work.

Having low earnings is necessary, but often not sufficient to be eligible for in-work tax credits. Eligibility may also require that either the number of hours worked is sufficiently high or the hourly wage rate is sufficiently low. I provide two examples of real world policies.

To get a Working Tax Credit in the United Kingdom you must have an annual income below £13100 (as a single between 25 and 59 years old and without children), but you must also work at least 30 hours a week.¹ The base payment is currently up to £1960 per year. Other tax credits with minimal conditions on the number of hours worked are the In-Work Tax Credit in New Zealand and the Working Family Payment in Ireland.²

To be eligible for a ‘Werkbonus’ (work bonus) in Belgium, your monthly gross earnings must be below €2510, but your gross wage rate must also be lower than 156% of the minimum wage.³ The bonus is currently up to €2562 per year (if you would work one year full-time as a laborer at the minimum wage). Another tax credit with a maximal condition on the wage rate was the ‘Prime Pour l’Emploi’ in France, which has been replaced by the ‘Prime d’Activité’ since 2016.⁴

One plausible justification for such additional conditions is to deter or even exclude mimickers, i.e., relatively able individuals who could become eligible by working fewer hours. Although deterring mimicking is a well-known mechanism in the optimal tax literature, it has, to the best of my

¹See the WTC webpage for the Working Tax Credit in the United Kingdom.

²See the IWTC webpage for the In-Work Tax Credit in New Zealand and the WFP webpage for the Working Family Payment in Ireland.

³See the WB webpage (in dutch, french, or german) for the ‘Werkbonus’ in Belgium.

⁴See the PPE wikipedia for the ‘Prime pour l’emploi’ in France.

knowledge, not been used to justify conditional earnings subsidies for low earners.

In the classical Mirrleesian model there simply is no room for subsidies (Mirrlees, 1971). In this model, *unidimensional agents* choose labour along the *intensive margin* and *only earnings* can be taxed by an *inequality averse welfarist* planner. These four assumptions together imply that the optimal marginal earnings tax rate is non-negative everywhere.

At least three assumptions turn out to be crucial. Indeed, negative marginal earnings tax rates can be optimal if agents are heterogeneous along multiple dimensions (Boadway et al., 2002, Fleurbaey and Maniquet, 2006, and Choné and Laroque, 2010), if labour is chosen along the extensive margin (Diamond, 1980, Saez, 2002, Choné and Laroque, 2011), or if the planner is either not inequality averse (Stiglitz, 1982, Werning, 2007) or not welfarist (Kanbur, Keen, and Tuomala, 1994, Kanbur, Pirttilä, and Tuomala, 2006, Pirttilä and Tuomala, 2004, Gerritsen, 2016, and Farhi and Gabaix, 2017).

This paper relaxes the remaining assumption. So, in this paper, unidimensional agents will choose labour along the intensive margin, but, for low earners, both hours worked and wages can be observed by an inequality averse welfarist planner.⁵

Now, if wages were observable and exogenous, then first-best lump-sum taxes on wages become feasible. However, empirical evidence shows that higher earnings taxes not only induce people to change hours worked, but also to move to jobs with lower wage rates and less disamenities (Gentry and Hubbard, 2004, Blomquist and Selin, 2010, Powell, 2010, and Powell and Shan, 2012). In other words, wages are endogenous, i.e., chosen by individuals, and must be modelled accordingly.⁶

To model wages in an endogenous way, labour is assumed to have multiple dimensions, called labour hours and labour effort. Similar to labour hours, labour effort is a vector of unpleasant, but productive features of

⁵To attribute the optimality of (conditional) earnings subsidies to the observability of wages and hours, I must relax observability only. Relaxing other assumptions on top is likely to reinforce the possibility of earnings subsidies.

⁶This feature not only makes the current paper more realistic, but also different from the optimal wage tax literature with observable and exogenous wages (see, e.g., Allingham, 1975, Wagstaff, 1975, Kesselman, 1976, Helpman and Sadka, 1978, and Blomquist, 1981).

labour. Effort—together with ability—will determine the hourly wage rate. To sum up: because effort is chosen, wages are endogenous; and because the determinants of wages—ability and effort—are not observable (by assumption), first-best taxation remains impossible.

In this model, there are several ways in which low earners can earn more, e.g., by working more hours or working at a higher wage rate (by exerting more effort). The optimal fiscal treatment of the different margins depends on the preferences of mimickers, being, higher ability individuals that pretend to have a lower ability type by choosing to earn less. As a general rule, if a certain margin is preferred relatively more by mimickers, then that margin will be discouraged, and thus taxed; and if a certain margin is preferred relatively less by mimickers, then that margin will be encouraged, and thus subsidized. Such a fiscal treatment deters mimicking, which allows to relax the self-selection constraint of the second-best tax problem, and, ultimately, to improve redistribution (see, e.g., Boadway, 2012, chapter 4).

Because the optimal fiscal treatment of the different margins crucially depends on the (mimicking) behavior of agents with different abilities, I have to study the impact of ability on labour choices. To add structure to the problem, I introduce single-crossingness conditions. However, with several margins, the number of combinations of single-crossingness conditions is potentially large. Yet, I argue that only two combinations are reasonable. These two combinations differ in one respect only, i.e., whether individuals with a higher ability type need to be compensated less (case 1) or more (case 2) in terms of extra consumption for a small increase in hours worked, *ceteris paribus*.

The main result of the paper consists of three conditional results.

First, the earnings-conditional marginal tax rate on hours worked must be negative for low earners. In other words, if two individuals have the same low earnings, then the one who works more hours (at a lower wage rate, necessarily) will pay less taxes.⁷

Second, the hours-conditional marginal tax rate on wages should be pos-

⁷Because working more hours and earning the same income is only possible by working at a lower wage rate, an earnings-conditional subsidy on hours worked is, by definition, equivalent to an earnings-conditional tax on the wage rate.

itive for low earners. So, earning more by choosing a job with a higher wage rate (and the same hours worked) will be taxed for low earners.

Third, the wage-conditional marginal tax rate on hours worked can be positive or negative for low earners, depending on whether case 1 or 2 applies. So, earning more by choosing to work more hours (at the same wage rate) will either be taxed (case 1) or subsidized (case 2) for low earners.

Four papers, two theoretical ones and two empirical ones, are closely related to the current paper and are therefore discussed in more detail.

Blomquist (1984) employs a model where unidimensional agents choose hours worked and educational effort along the intensive margin. Educational effort and ability together define wages. Although the set-up is very similar, the paper focuses on comparing the performance of linear wage taxation versus linear earnings taxation and finds that wage taxation is superior.

Beaudry, Blackorby, and Szalay (2009) analyze wage-contingent earnings taxation, i.e., taxes are a non-linear function of earnings and wages⁸. Although the focus of this paper is very similar, the underlying model is quite different. They also introduce wage observability, but consider multi-dimensional agents who allocate a fixed number of working hours over two sectors, a formal market sector and an informal household sector. They find that it is optimal to subsidize earnings below a cutoff wage rate and tax earnings above that cutoff.

Blundell and Shephard (2012) analyze the Working Families' Tax Credit (WFTC) reform in the United Kingdom. This reform increased generosity, but also introduced a minimal number of hours worked as an eligibility condition. If optimally set, the potential welfare gain from using hours-contingent tax credits for lone parents is equivalent to a increase of 2.5 % in current expenditures. Although non-trivial, the paper shows that if hours are misreported unwillingly (measurement error) or willingly (to increase utility), then these welfare gains decrease with the size of the measurement error or the utility cost of misreporting hours. These simulations stress the potential importance of misreporting, but do not say anything about

⁸It is worth mentioning that Nishimura (2004) and Tillmann (2005) study the use of hours-contingent taxation—taxes are a non-linear function of earnings and hours—, but to implement fair (i.e., envy-free) allocations.

whether misreporting occurs in reality.

Cahuc and Carcillo (2014) analyze the detaxation of overtime in France and show that manipulating normal and overtime hours occurs only among highly qualified employees whose hours are hard to verify. In France, these employees belong to teaching and the scientific professions, media professions, arts and entertainment, administrative and commercial managers of firms, engineers, professionals, foremen and supervisors.⁹ The main message is that conditional earnings subsidies should be restricted to jobs whose hours worked can be easily verified. Because earnings are typically lower in such jobs, I will assume observability for low earners only.

The remainder of the paper proceeds as follows. Section 2 introduces the model. Section 3 analyzes the utility-maximizing choices of individuals with different ability levels. Section 4 studies optimal taxation in case only earnings are observable and in case hours and wages are observable for low earners. I find that earnings subsidies are not optimal if only earnings are observable, but conditional earnings subsidies can be optimal if hours and wages are observable for low earners. Both results together show that the observability of hours and wages is crucial to justify subsidies (and not the feature of the model that labour has several intensive margins). A final section 5 concludes.

2 The model

In this section, I first introduce (in subsection 2.1) the fundamentals that drive choices: preferences, market environment, technology, ability, and taxation. As the main aim of the paper is to analyze optimal taxation if hours and wages are observable (for low earners), I also add a brief discussion on tax schemes that depend on hours and wages (subsection 2.2). Moreover, I discuss (in subsection 2.3) how to express preferences over jobs (a combination of hours and wages) instead of preferences over labour (a combination of hours and effort).

⁹Cahuc and Carcillo (2014) only look at employees, but it is reasonable to assume that the self-employed should be added to the group whose hours are hard to verify.

2.1 Preliminaries

Labour has several margins. It collects labour hours (a scalar) and labour effort (a vector), summarized as $(h, \mathbf{e})$. Labour effort consists of unpleasant, but productive features of labour, other than hours worked.

The choice of labour—i.e., hours and effort—depends on preferences, market environment, technology, ability, and taxation.

Preferences are the same for all individuals and can be represented by a twice differentiable and strictly quasi-concave utility function $u : (c, h, \mathbf{e}) \mapsto u(c, h, \mathbf{e})$ that strictly increases in consumption (net earnings) and strictly decreases in hours and effort.

Market environment and technology define how ability, denoted θ , and labour translates into gross earnings. In a perfectly competitive labour market with a production technology that is linear in ability and hours worked—strong, yet common assumptions in the optimal tax literature—gross earnings y are equal to $\theta f(\mathbf{e})h$, with f a differentiable, strictly increasing, and concave function. The product $\theta f(\mathbf{e})$ measures the hourly wage rate, which is denoted by w .

Taxation drives a wedge between gross earnings and net earnings, i.e., the tax t is equal to $y - c$. If only gross earnings are observable (as in subsections 3.2 and 4.2 later on), then the tax can only be a non-linear function of gross earnings, say, $t = T(y)$. If the components of gross earnings—wages and hours—are observable for low earners (as in subsections 3.3 and 4.3), then the tax can be a non-linear function of these components, say, $t = T(h, w)$.¹⁰ Taxes of the form $t = T(h, w)$ are unusual in the optimal tax literature and require therefore some additional explanation.

2.2 Taxation and conditional tax credits

Taxes of the form $t = T(h, w)$ reduce to a classical non-linear earnings tax if $T(h, w) = \phi(wh)$ holds everywhere. But new possibilities arise. For

¹⁰Because $y = wh$ holds by definition, each of the three pairs (h, w) , (y, w) , and (h, y) contains the same information and can form the basis of taxation. I choose hours and wages as the tax base. As mentioned before, Beaudry, Blackorby, and Szalay (2009) study wage-contingent earnings taxation, i.e., $t = T(w, y)$; and Nishimura (2004) and Tillmann (2005) study hours-contingent earnings taxation, i.e., $t = T(h, y)$.

example, a conditional tax credit, say $C(h, w)$, would change the tax scheme to $T(h, w) = \phi(wh) - C(h, w)$. To illustrate this possibility, I again use (stylized versions of) the two real world examples that I introduced before.

The Working Tax Credit (United Kingdom) is an earnings tax credit conditional on working a minimal amount of hours, say, $\underline{h}$. It is thus defined as $C(h, w) = A(wh)$ if $h \geq \underline{h}$ and $C(h, w) = 0$ otherwise, with $A > 0$ over some earnings range.

The ‘Werkbonus’ (Belgium) is defined as $C(h, w) = A(w)h$ with A linearly decreasing in the wage rate for wages below 156% of the minimum wage. In more detail, if $\underline{w}$ denotes the minimum wage, then $A(w) \approx \mu(1.56\underline{w} - w)$ if $\underline{w} \leq w \leq 1.56\underline{w}$ and $A(w) = 0$ otherwise.¹¹

Using the general form $t = T(h, w)$ again, the signs of the derivatives T'_h and T'_w define how the different margins are treated fiscally. For example, subsidizing hours worked at the margin—a ‘wage-conditional’ hour subsidy—corresponds with $T'_h(h, w) < 0$. And taxing the wage rate at the margin—an ‘hours-conditional’ wage tax—occurs if $T'_w(h, w) > 0$. Using the definition $w = y/h$ and differentiating $T(h, y/h)$ with respect to h one gets $T'_h(h, w) - T'_w(h, w)w/h$. The sign of this expression informs us about the fiscal treatment of hours for individuals who earn the same income. So, if the expression is negative, for example, then individuals who work more hours at a lower wage rate (for the same gross earnings) pay less taxes. I call a negative sign therefore an ‘earnings-conditional’ hour subsidy.¹² Table 1 summarizes the fiscal treatment of the different margins.¹³

As the stylized version of the ‘Werkbonus’ (Belgium) has a rather smooth formula over the relevant wage range, it is easy to check these derivatives. Because the (standard) marginal earnings tax rate is high for low earners in Belgium it turns out that conditional on wages, hours are taxed at the

¹¹The parameter $\mu > 0$ is roughly equal to 0.22 and 0.24 for white- and blue-collar workers.

¹²Conditional on earnings, hours and wages are inversely related. So, we could also have called an ‘earnings-conditional’ hour subsidy an ‘earnings-conditional’ wage tax.

¹³Note that these conditional marginal tax rates are linked. For example, a ‘wage-conditional’ hour subsidy combined with an ‘hour-conditional’ wage tax must, by definition, lead to an ‘earnings-conditional’ hour subsidy (because $T'_h(h, w) - T'_w(h, w)w/h$ is indeed negative under these conditions).

Table 1: Three conditional marginal taxes

margin		taxed at the margin if	subsidized at the margin if
h	$ \quad w$	$T'_h(h, w) > 0$	$T'_h(h, w) < 0$
w	$ \quad h$	$T'_w(h, w) > 0$	$T'_w(h, w) < 0$
h	$ \quad y$	$T'_h(h, w) - T'_w(h, w)w/h > 0$	$T'_h(h, w) - T'_w(h, w)w/h < 0$

margin ($T'_h(h, w) > 0$), conditional on hours, wages are taxed at the margin ($T'_w(h, w) > 0$), and conditional on earnings, hours are subsidized at the margin ($T'_h(h, w) - T'_w(h, w)w/h < 0$).¹⁴

2.3 From labour to jobs

Because taxes can be a function of hours and wages, it is more convenient to focus on jobs (a combination of hours and wages) rather than on labour (a combination of hours and effort). I briefly discuss therefore how preferences can be expressed over jobs instead of labour.

Let (h, w) denote a job. Because the same job can be done with different vectors of effort, the solution to the problem

$$\max_{\mathbf{e} \geq \mathbf{0}} u(c, h, \mathbf{e}) \text{ such that } \theta f(\mathbf{e}) \geq w, \quad (1)$$

gives us the best amount of effort, denoted $\mathbf{e}(c, h, w, \theta)$, that an individual with ability type θ chooses to do the specified job.

¹⁴For wages in the relevant range ($\underline{w} \leq w \leq 1.56\underline{w}$), taxes are

$$T(h, w) = \phi(wh) - \mu(1.56\underline{w} - w)h.$$

Given a sufficiently high (standard) marginal tax rate for low earners in Belgium (satisfying $\phi'(wh) > 0.13$ for sure) and given $\mu \approx 0.23 > 0$, marginal taxes become

$$\begin{aligned} T'_h(h, w) &= \phi'(wh)w - \mu(1.56\underline{w} - w) \\ &\approx (\phi'(wh) + 0.23)w - 0.36\underline{w} > 0, \\ T'_w(h, w) &= (\phi'(wh) + \mu)h > 0, \end{aligned}$$

and

$$T'_h(h, w) - T'_w(h, w)w/h = -1.56\mu\underline{w} < 0.$$

If I plug in the solution in the utility function, then utility becomes a function of net earnings, jobs, and ability. Proposition 1 summarizes. Unless stated otherwise, proofs can be found in the appendix.

Proposition 1. The utility function U , defined as

$$U : (c, h, w, \theta) \mapsto U(c, h, w, \theta) = u(c, h, \mathbf{e}(c, h, w, \theta)), \quad (2)$$

is strictly increasing in consumption (net earnings), strictly decreasing in hours and wages, and strictly increasing in ability.

3 Job choice

In this section, I analyze job choice. I first study the problem of job choice conditional on earnings (subsection 3.1), which will turn out to be of general relevance later on. Afterwards I look at job choice if only earnings can be taxed (subsection 3.2) and if hours and wages can be taxed (subsection 3.3). In each of these three subsections, I illustrate how single-crossingness conditions define how ability affects choices. A more detailed discussion of the different single-crossingness conditions is postponed to the next section.

3.1 Job choice conditional on earnings

Given an amount of net earnings c , what is the best job (h, w) according to an individual with ability type θ to obtain a level of gross earnings y ? The utility-maximizing job, conditional on earnings, follows from

$$\max_{h, w \geq 0} U(c, h, w, \theta) \text{ such that } wh \geq y. \quad (3)$$

The first-order condition can be written as

$$MWH(c, h, w, \theta) = -\frac{w}{h}, \quad (4)$$

where MWH is the marginal rate of substitution of wages for hours.¹⁵

Only the left-hand side of the first-order condition (4) depends on ability. So, it informs us how changes in ability affect choices. For example, if

¹⁵It tells us how much the wage (and thus effort) should decrease to compensate for a small increase in hours worked and is defined as $MWH(c, h, w, \theta) = -\frac{U'_h(c, h, w, \theta)}{U'_w(c, h, w, \theta)} < 0$.

the single-crossingness condition $MWH'_\theta < 0$ holds, then higher ability individuals must be compensated more in terms of wages for a small increase in hours worked. As a consequence, higher ability individuals would always choose less hours and a higher wage rate to reach the same gross earnings as an individual with a lower ability level.

3.2 Job choice if only earnings can be taxed

If only earnings can be taxed, then the utility-maximizing job solves

$$\max_{h,w \geq 0} U(c, h, w, \theta) \text{ such that } c \leq wh - T(wh). \quad (5)$$

This problem can be split up in two steps.

First, the solution to the earnings-conditional problem (3), say, $\hat{H}(c, y, \theta)$ and $\hat{W}(c, y, \theta)$, can be plugged in in the utility function U . As such, utility becomes indirectly expressed as a function of net earnings, gross earnings, and type. Proposition 2 summarizes.

Proposition 2. The utility function V , defined as

$$V : (c, y, \theta) \mapsto V(c, y, \theta) = U(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta), \quad (6)$$

is strictly increasing in consumption (net earnings), strictly decreasing in gross earnings, and strictly increasing in ability.

Second, problem (5) can be solved by maximizing utility, according to V , over earnings, i.e.,

$$\max_{c,y \geq 0} V(c, y, \theta) \text{ such that } c \leq y - T(y). \quad (7)$$

The first-order condition can be expressed as

$$MCY(c, y, \theta) = 1 - T'(y), \quad (8)$$

where MCY is the marginal rate of substitution of net for gross earnings.¹⁶

Equation (8) informs us again how changes in ability affect choices. The usual Spence-Mirrlees condition imposes that higher ability types need to

¹⁶It tells us how much net earnings should increase to compensate for a small increase in gross earnings and is defined as $MCY(c, y, \theta) = -\frac{V'_y(c, y, \theta)}{V'_c(c, y, \theta)} > 0$.

be compensated less in terms of net earnings for a small increase in gross earnings, i.e., $MCY'_\theta < 0$. This would imply that higher ability individuals always choose higher gross earnings.

3.3 Job choice if hours and wages can be taxed

If hours and wages can be taxed, then the utility-maximizing job solves

$$\max_{h,w \geq 0} U(c, h, w, \theta) \text{ such that } c \leq wh - T(w, h).$$

The first-order conditions can be expressed as

$$MCH(c, h, w, \theta) = w - T'_h(h, w), \quad (9)$$

$$MCW(c, h, w, \theta) = h - T'_w(h, w), \quad (10)$$

where MCH and MCW are the marginal rates of substitution of consumption for either hours or wages.¹⁷

Single-crossingness conditions can again give direction to the choices of individuals with different abilities. If, for example, the single-crossingness condition $MCW'_\theta < 0$ would hold, then higher ability types always need to be compensated less for a small increase in wages. As a consequence, they would always choose higher wages for a given number of labour hours. Similarly, if the condition $MCH'_\theta > 0$ were true, then higher ability types always need to be compensated more for a small increase in hours worked. As a consequence, they would always choose to work less hours, for a given wage rate.

4 Optimal taxation

To characterize optimal taxation in the simplest possible way, I follow Stiglitz (1982) and introduce two ability types, called low (L) and high (H). I assume that society wants to redistribute from high to low ability individuals.

Because single-crossingness conditions drive optimal tax results, I first analyze which conditions are reasonable and, related, which combinations of conditions are compatible (subsection 4.1). Afterwards, I study optimal

¹⁷We have $MCH(c, h, w, \theta) = -\frac{U'_h(c, h, w, \theta)}{U'_c(c, h, w, \theta)} > 0$ and $MCW(c, h, w, \theta) = -\frac{U'_w(c, h, w, \theta)}{U'_c(c, h, w, \theta)} > 0$.

taxation if only earnings are observable (subsection 4.2) and if also hours and wages are observable for low earners (subsection 4.3).

4.1 Single-crossingness conditions

In section 3 I introduced examples of single-crossingness conditions and showed how these conditions define how ability affects choices. As single-crossingness conditions drive optimal tax results, it is important to answer several questions. Are some conditions more reasonable than others? And how are the different conditions linked to each other and are some combinations therefore not possible?

First, I start with the Spence-Mirrlees condition.

Assumption A. The utility function V satisfies $MCY'_\theta < 0$. As a consequence, if taxes are function of earnings only, then higher ability individuals always choose higher gross earnings.

I impose this assumption to stick as close as possible to the standard optimal tax literature. It implies that redistribution from high to low ability individuals is equivalent to redistribution from the rich to the poor.

Second, concerning the sign of MWH'_θ , proposition 3 tells us that it cannot be chosen arbitrarily.

Proposition 3. There do not exist utility functions U such that $MWH'_\theta \geq 0$ hold. As a consequence, there do not exist utility functions such that higher ability individuals always choose more hours (combined with lower wages) to reach a given level of gross earnings.

Assumption B follows naturally.

Assumption B. The utility function U satisfies $MWH'_\theta < 0$. As a consequence, higher ability individuals always choose less hours (and thus a higher wage rate) to reach a given level of gross earnings.

Third, I must define the signs of MCW'_θ and MCH'_θ . Although a priori four combinations are possible, proposition 4 tells us that only two combinations are compatible (given assumptions A and B).

Proposition 4. The combination $MCH'_\theta > 0$ and $MCW'_\theta > 0$ contradicts the Spence-Mirrlees condition (assumption A). The combination $MCH'_\theta < 0$ and $MCW'_\theta > 0$ contradicts proposition 3 (and thus assumption B).

Assumption C follows, again, naturally:¹⁸

Assumption C. The utility function U satisfies $MCW'_\theta < 0$. As a consequence, if taxes are a function of hours and wages, then higher ability individuals would always choose higher wages for a given number of labour hours. In addition, the utility function U satisfies either $MCH'_\theta < 0$ (C1) or $MCH'_\theta > 0$ (C2). As a consequence, if taxes are a function of hours and wages, then higher ability individuals would always choose to work either more hours (C1) or less hours (C2) for a given wage rate.

In the next subsections, I analyze optimal taxation under assumption A (subsection 4.2) and under assumptions B and C (subsection 4.3). Two final questions remain however.

First, do there exist preferences (represented by the utility function u) and production technologies (summarized by the production function f) such that assumptions A, B, and C are satisfied? If not, then all later results would be meaningless. Fortunately, proposition 5 shows that CES-type utility and production functions work well.

Proposition 5. Let $\mathbf{e} = (e_1, e_2, \dots, e_J)$ be the J -dimensional vector of effort. Consider a utility function

$$u : (c, h, \mathbf{e}) \mapsto \frac{1}{1-\alpha} c^{1-\alpha} - \frac{1}{1+\gamma} [\eta h^{1+\beta} + (1-\eta) \left(\sum_{j=1}^J \kappa_j (e_j)^{1+\delta} \right)^{\frac{1+\beta}{1+\delta}}]^{\frac{1+\gamma}{1+\beta}},$$

with $\alpha, \beta, \gamma, \delta > 0$, $0 < \eta < 1$, and all $\kappa_j > 0$ with $\sum_{j=1}^J \kappa_j = 1$.¹⁹ Consider a production function

$$f : \mathbf{e} \mapsto \left(\sum_{j=1}^J \nu_j (e_j)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}},$$

¹⁸ Although $MCH'_\theta = 0$ is possible, I stick to strict conditions everywhere.

¹⁹ If $\alpha = 1$, then the utility-of-consumption function is defined as $\ln c$.

with $\sigma > 0$ and all $\nu_j > 0$ with $\sum_{j=1}^J \nu_j = 1$. For these specifications, assumptions A, B, and either C1 (if $\gamma > \beta$) or C2 (if $\gamma < \beta$) hold.

Second, which of the two assumptions, C1 or C2, is more likely? This is an empirical question that is hard to answer a priori. Suppose, for now, that effort is unidimensional and the utility function u is additively separable between consumption and labour. If the disutility of working more hours is higher at higher levels of effort, then assumption C1 holds. This seems likely if effort measures, e.g., how physically challenging a job is (and ability then measures physical ability): working an extra hour implies more disutility if the job is (physically) harder. If the disutility of working more hours is lower at higher levels of effort, then assumption C2 holds. This is not implausible if, e.g., effort measures how intellectually challenging a job is (and ability then measures intellectual ability): working an extra hour leads to more disutility if the job is (intellectually) less challenging (i.e., more boring). More realistically, if effort is multidimensional, including physical and intellectual effort, then their relative importance will decide whether C1 or C2 (or none of the two) holds.

4.2 Optimal taxation if only earnings can be taxed

If only earnings can be taxed, then Pareto efficient redistribution from high to low ability individuals follows from

$$\max_{c_L, y_L, c_H, y_H} V(c_L, y_L, \theta_L) \text{ subject to}$$

$$V(c_H, y_H, \theta_H) \geq \underline{v}, \quad (11)$$

$$V(c_H, y_H, \theta_H) \geq V(c_L, y_L, \theta_H), \quad (12)$$

$$n_L(y_L - c_L) + n_H(y_H - c_H) \geq R_0, \quad (13)$$

with n_i the number of individuals with ability type $i = L, H$. Constraint (11) guarantees a minimal utility level $\underline{v}$ to high type individuals. Constraint (12) is the self-selection constraint of the high type.²⁰ It restricts the set

²⁰Because pooling cannot be Pareto efficient and because I focus on redistribution from high to low ability individuals, only the self-selection constraint of the high type will be binding at the (second-best) optimum.

of allocations to those that can be implemented by a tax scheme based on gross earnings only. Constraint (13) is the budget constraint requiring that taxation raises an exogenous amount of revenues R_0 .

The above formulation of the problem hides the fact that there is an optimal choice of jobs, conditional on earnings, underlying the function V (defined in proposition 2). Nevertheless, the optimal tax problem is, in the end, identical to the one in Stiglitz (1982). Proposition 6 follows immediately.

Proposition 6. Suppose assumption A holds. If society wants to redistribute from high to low ability individuals in a Pareto efficient way, then there should be no distortion of high earners at the optimum. For low earners, the marginal earnings tax rate should be positive at the optimum.

Proposition 6 tells us that the number of intensive margins does not matter for signing the marginal earnings tax rate faced by low earners. So, how individuals change their earnings—through hours, wages, or a mixture of both—does not matter. Only the change in earnings matters in the end and is therefore a sufficient statistic for optimal earnings tax policy.²¹

4.3 Optimal taxation if hours and wages can be taxed

If the hours and wages of (only) low earners are observable for tax purposes, then Pareto efficient redistribution from high to low ability individuals follows from

$$\begin{aligned} \max_{c_L, h_L, w_L, c_H, y_H} \quad & U(c_L, h_L, w_L, \theta_L) \quad \text{subject to} \\ & V(c_H, y_H, \theta_H) \geq \underline{v}, \end{aligned} \tag{14}$$

$$V(c_H, y_H, \theta_H) \geq U(c_L, h_L, w_L, \theta_L), \tag{15}$$

$$n_L(w_L h_L - c_L) + n_H(y_H - c_H) \geq R_0. \tag{16}$$

The self-selection constraint of the high type (15) now restricts the set of allocations to those that can be implemented by a tax scheme that is based on earnings only for high earners, but on both hours and wages for low earners.²²

²¹See Chetty (2009) for an overview of the sufficient statistics literature.

²²Although not in line with the evidence reviewed in the introduction, all results would remain the same if also the hours and wages of high earners were observable.

Pareto efficient redistribution from high to low ability individuals implies, as usual, that the high type remains undistorted at the margin. The more interesting question is what happens for low earners.

Recall Table 1. The fiscal treatment of these three conditional margins for low earners depends on the preferences of mimickers—high ability individuals pretending having a low ability type—relative to low ability individuals. As a general rule, the fiscal treatment of each margin must be such that the direction that mimickers like (dislike) is discouraged (encouraged), and thus taxed (subsidized). Such a fiscal treatment will relax the self-selection constraint (15) and allows to further increase the utility of low ability individuals without hurting high ability individuals (who remain at utility level $\underline{v}$) and without changing the tax revenues (which remain at level R_0).

Proposition 7 summarizes the main result. In a nutshell: working more hours for the same earnings should be subsidized, earning more by higher wages should be taxed, and earning more by working more hours should either be taxed or subsidized.

Proposition 7. Suppose assumptions B and C hold. If society wants to redistribute from high to low ability individuals in a Pareto efficient way, then there should be no distortion for high earners at the optimum. For low earners,

- (7.1) the earnings-conditional marginal tax rate on hours should be negative, i.e., $T'_h(h_L, w_L) - T'_w(h_L, w_L)w_L/h_L < 0$, at the optimum,
- (7.2) the hours-conditional marginal tax rate on wages should be positive, i.e., $T'_w(h_L, w_L) > 0$, at the optimum, and
- (7.3) the wage-conditional marginal tax rate on hours should be positive, i.e., $T'_h(h_L, w_L) > 0$, at the optimum if assumption C1 holds and negative, i.e., $T'_h(h_L, w_L) < 0$, at the optimum if assumption C2 holds.

In words, Proposition 7 tells us that, at the low earners' margin,

- (7.1) individuals who have the same earnings, but work more hours at a lower wage rate should pay less taxes,

(7.2) individuals who work the same hours at a higher wage rate should pay more taxes, and

(7.3) individuals who work more hours at the same wage rate should either pay more taxes (C1) or less taxes (C2).

The stylized version of the ‘Werkbonus’ in Belgium, introduced and discussed in subsection 2.2, is consistent with proposition 7 (assuming C1).

I finish this section with a visualization of proposition 7. The iso-tax curve, denoted $w_{IT} : h \mapsto w_{IT}(h)$, collects all jobs that pay the same taxes as the low earner at the optimum and is thus implicitly defined as $T(h, w_{IT}(h)) = T(h_L, w_L)$. The iso-earnings curve $w_{IE} : h \mapsto w_{IE}(h)$ collects all jobs that pay the same gross earnings as the low earner at the optimum and is defined as $w_{IE}(h) = w_L h_L / h$. The slope of the iso-tax curve for low earners at the optimum is $w'_{IT}(h_L) = -T'_h(h_L, w_L) / T'_w(h_L, w_L)$ and the slope of the iso-earnings curve for low earners at the optimum is $w'_{IE}(h_L) = -w_L / h_L < 0$. Proposition 8 follows directly from proposition 7 (without proof).

Proposition 8. According to proposition 7.1, the slope of the iso-tax-curve for low earners $w'_{IT}(h_L)$ will be larger than the slope of the iso-earnings curve $w'_{IE}(h_L)$, at the optimum. In addition, according to propositions 7.2 and 7.3, the slope $w'_{IT}(h_L)$ can be either negative (if C1 holds) or positive (if C2 holds), at the optimum.

Figure 1 visualizes the main result. It depicts the iso-earnings curve (the thin line) and the iso-tax curve (the solid line) for the low earner at the optimum if assumption C1 (left-hand panel) or C2 (right-hand panel) holds. The vertical arrows correspond with choosing a different wage rate for the same hours worked, the horizontal arrows correspond with choosing different hours worked for the same wage rate, and the arrows along the iso-earnings curve indicate working different hours at different wage rates for the same gross earnings. The signs indicate whether more taxes (+) or less taxes (−) must be paid in the direction of the arrow. Be aware that Figure 1 shows a local result, which holds only in a neighbourhood of (w_L, h_L) ; moreover, Figure 1 assumes a differentiable tax scheme.

Figure 1: Conditional earnings subsidies for low earners

Because the iso-earnings curve and the iso-tax curve do not coincide, conditional earnings subsidies for low earners occur in each panel. Indeed, the earnings of low earners should be subsidized, but only in the direction of the hatched area, i.e., if their earnings increase by working more hours at a sufficiently low wage rate (lower than w_L if C1 holds and similar to w_L if C2 holds).

5 Conclusion

This paper starts from the observation that having low earnings is often not sufficient to be eligible for in-work tax credits. Indeed, several countries also impose that either hours worked are sufficiently high or that the wage rate is sufficiently low. The Working Tax Credit in the United Kingdom and the ‘Werkbonus’ in Belgium provide two typical examples.

To derive a theoretical justification for such additional conditions, I analyze Pareto efficient redistribution from high to low ability individuals in a model where labour has several intensive margins. Besides labour hours, also labour effort—a vector of unpleasant, but productive features of labour—is

an object of choice. Effort and ability—both unobserved by the planner—determine the wage rate.

Conditional earnings subsidies for low earners turn out to be optimal, i.e., the earnings of low earners should be subsidized, but only if they earn more by working more hours at a sufficiently low wage rate. These additional conditions enhance redistribution by deterring high ability individuals to mimick low ability individuals.

The optimal tax results in this paper are qualitative in nature. A more detailed quantitative analysis, which should ideally include the possibility of misreporting hours and wages, is left for future research.

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Proof of proposition 1

Proposition 1 is an application of the envelope theorem. First, because $U(c, h, w, \theta) = u(c, h, \mathbf{e}(c, h, w, \theta))$, I get

$$U'_c(c, h, w, \theta) = u'_c(c, h, \mathbf{e}(c, h, w, \theta)) + \quad (17)$$

$$\sum_{j=1}^J u'_j(c, h, \mathbf{e}(c, h, w, \theta)) e'_{j,c}(c, h, w, \theta), \quad (18)$$

$$U'_h(c, h, w, \theta) = u'_h(c, h, \mathbf{e}(c, h, w, \theta)) + \quad (19)$$

$$\sum_{j=1}^J u'_j(c, h, \mathbf{e}(c, h, w, \theta)) e'_{j,h}(c, h, w, \theta), \quad (20)$$

$$U'_w(c, h, w, \theta) = \sum_{j=1}^J u'_j(c, h, \mathbf{e}(c, h, w, \theta)) e'_{j,w}(c, h, w, \theta), \quad (21)$$

$$U'_\theta(c, h, w, \theta) = \sum_{j=1}^J u'_j(c, h, \mathbf{e}(c, h, w, \theta)) e'_{j,\theta}(c, h, w, \theta), \quad (22)$$

where I use $j = 1, 2, \dots, J$ to refer to the different dimensions of effort.

Second, using the first-order conditions of the problem

$$\max_{\mathbf{e} \geq 0} u(c, h, \mathbf{e}) \text{ such that } \theta f(\mathbf{e}) \geq w,$$

with Lagrange multiplier $\lambda > 0$, one obtains

$$u'_j(c, h, \mathbf{e}(c, h, w, \theta)) + \lambda \theta f'_j(\mathbf{e}(c, h, w, \theta)) = 0, \quad (23)$$

for $j = 1, 2, \dots, J$, and

$$\theta f(\mathbf{e}(c, h, w, \theta)) = w.$$

Taking derivatives of this last equation (w.r.t. c , h , w , and θ), I get

$$\theta \sum_{j=1}^J f'_j(\mathbf{e}(c, h, w, \theta)) \times e'_{j,c}(c, h, w, \theta) = 0, \quad (24)$$

$$\theta \sum_{j=1}^J f'_j(\mathbf{e}(c, h, w, \theta)) \times e'_{j,h}(c, h, w, \theta) = 0, \quad (25)$$

$$\theta \sum_{j=1}^J f'_j(\mathbf{e}(c, h, w, \theta)) \times e'_{j,w}(c, h, w, \theta) = 1, \quad (26)$$

$$\theta \sum_{j=1}^J f'_j(\mathbf{e}(c, h, w, \theta)) \times e'_{j,\theta}(c, h, w, \theta) = -f(\mathbf{e}(c, h, w, \theta)). \quad (27)$$

Third, using equations (23)-(27) in (17)-(22), I get

$$\begin{aligned}
U'_c(c, h, w, \theta) &= u'_c(c, h, \mathbf{e}(c, h, w, \theta)) > 0, \\
U'_h(c, h, w, \theta) &= u'_h(c, h, \mathbf{e}(c, h, w, \theta)) < 0, \\
U'_w(c, h, w, \theta) &= -\lambda < 0, \\
U'_\theta(c, h, w, \theta) &= \lambda f(\mathbf{e}(c, h, w, \theta)) > 0,
\end{aligned}$$

as required.

Proof of proposition 2

Proposition 2 is an application of the envelope theorem. First, because $V(c, y, \theta) = U(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta)$, the partial derivatives are

$$\begin{aligned}
V'_c(c, y, \theta) &= U'_c(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta) + \\
&\quad U'_h(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta) \hat{H}'_c(c, y, \theta) + \\
&\quad U'_w(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta) \hat{W}'_c(c, y, \theta), \\
V'_y(c, y, \theta) &= U'_h(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta) \hat{H}'_y(c, y, \theta) + \\
&\quad U'_w(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta) \hat{W}'_y(c, y, \theta), \\
V'_\theta(c, y, \theta) &= U'_\theta(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta) + \\
&\quad U'_h(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta) \hat{H}'_\theta(c, y, \theta) + \\
&\quad U'_w(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta) \hat{W}'_\theta(c, y, \theta).
\end{aligned}$$

Second, $\hat{H}(c, y, \theta)$ and $\hat{W}(c, y, \theta)$ must satisfy the first-order condition (4), i.e.,

$$\frac{U'_h(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta)}{U'_w(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta)} = \frac{\hat{W}(c, y, \theta)}{\hat{H}(c, y, \theta)}.$$

In addition, also $\hat{W}(c, y, \theta) \hat{H}(c, y, \theta) = y$ must hold; taking derivatives on both sides with respect to c , y , or θ implies

$$\begin{aligned}
\hat{W}'_c(c, y, \theta) \hat{H}(c, y, \theta) + \hat{W}(c, y, \theta) \hat{H}'_c(c, y, \theta) &= 0, \\
\hat{W}'_y(c, y, \theta) \hat{H}(c, y, \theta) + \hat{W}(c, y, \theta) \hat{H}'_y(c, y, \theta) &= 1, \\
\hat{W}'_\theta(c, y, \theta) \hat{H}(c, y, \theta) + \hat{W}(c, y, \theta) \hat{H}'_\theta(c, y, \theta) &= 0.
\end{aligned}$$

Combining these elements, I get

$$V'_c(c, y, \theta) = U'_c(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta) > 0, \quad (28)$$

$$V'_y(c, y, \theta) = \frac{U'_w(c, h, \hat{W}(c, y, \theta), \theta)}{\hat{H}(c, y, \theta)} < 0, \quad (29)$$

$$V'_\theta(c, y, \theta) = U'_\theta(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta) > 0, \quad (30)$$

where the signs follow from proposition 1.

Proof of proposition 3

Suppose there exists a utility function U for which $MWH'_\theta \geq 0$ holds. If true, then there exists a utility function U that implies $\hat{H}'_\theta \geq 0$ —with, recall, $\hat{H}(c, y, \theta)$ the solution to problem (3).

First, $U(c, h, w, \theta)$ is defined as $u(c, h, \mathbf{e}(c, h, w, \theta))$, where $\mathbf{e}(c, h, w, \theta)$ solves the problem

$$\max_{\mathbf{e} \geq \mathbf{0}} u(c, h, \mathbf{e}) \text{ such that } \theta f(\mathbf{e}) \geq w.$$

Rewriting the constraint as $f(\mathbf{e}) \geq w/\theta$, it follows immediately that $\mathbf{e}$ only depends on the ratio w/θ . I write $\tilde{\mathbf{e}}(c, h, w/\theta)$ and $\tilde{U}(c, h, w/\theta) = u(c, h, \tilde{\mathbf{e}}(c, h, w/\theta))$.

Second, $\hat{H}(c, y, \theta)$ and $\hat{W}(c, y, \theta)$ follow from the problem

$$\max_{h, w \geq 0} U(c, h, w, \theta) \text{ such that } wh \geq y,$$

which can be equivalently written as

$$\max_{h, w \geq 0} \tilde{U}(c, h, \frac{w}{\theta}) \text{ such that } \frac{w}{\theta} h \geq \frac{y}{\theta}.$$

Changing variable from w/θ to k , the solutions of the problem

$$\max_{h, k \geq 0} \tilde{U}(c, h, k) \text{ such that } kh \geq \frac{y}{\theta},$$

depend only on the ratio y/θ , say, $h(c, y/\theta)$ and $k(c, y/\theta)$. Moreover, one has

$$\hat{H}(c, y, \theta) = h(c, y/\theta) \text{ and } \hat{W}(c, y, \theta) = \theta k(c, y/\theta).$$

Third, to show that $\hat{H}'_\theta \geq 0$ is not possible I show that $h'_z(c, z) > 0$ must hold over some interval of z -values. Following proposition 1, the function

$\tilde{U}$ is strictly decreasing in h and $k = w/\theta$, which implies $h(c, 0) = k(c, 0) = 0$. In addition, $k(c, \frac{y}{\theta})h(c, \frac{y}{\theta}) = \frac{y}{\theta}$ must hold at the optimum, leading to $k(c, \frac{y}{\theta}) > 0$ and $h(c, \frac{y}{\theta}) > 0$ whenever $\frac{y}{\theta} > 0$. Both conditions require that $h'_z(c, z) > 0$ must hold somewhere, as required.

Proof of proposition 4

First, I show that $MCY'_\theta < 0$, $MCW'_\theta > 0$, and $MCH'_\theta > 0$ are not compatible. Using the definition of MCY , equations (28) and (29), and the definition of MCW , one gets

$$\begin{aligned} MCY(c, y, \theta) &= -\frac{U'_w(c, h, \hat{W}(c, y, \theta), \theta)}{U'_c(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta)\hat{H}(c, y, \theta)}, \\ &= \frac{MCW(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta)}{\hat{H}(c, y, \theta)}. \end{aligned}$$

Differentiating both sides with respect to θ , the sign of $MCY'_\theta(c, y, \theta)$ must coincide with the sign of

$$\begin{aligned} &MCW'_\theta(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta)\hat{H}(c, y, \theta) - \\ &MCW(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta)\hat{H}'_\theta(c, y, \theta). \end{aligned}$$

If $MCW'_\theta > 0$ and $MCH'_\theta > 0$ hold, then this expression becomes positive, because (i) $\hat{H}(c, y, \theta) \geq 0$ and $MCW(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta) > 0$ hold by definition and (ii) $\hat{H}'_\theta(c, y, \theta) < 0$ follows from $MCH'_\theta > 0$. But this is not compatible with $MCY'_\theta < 0$, as required.

Second, I show that $MCH'_\theta < 0$ and $MCW'_\theta > 0$ are not possible. To see this, note that, by definition, MWH is equal to $-MCH/MCW$. This implies that MWH'_θ has the opposite sign of $MCH'_\theta \times MCW - MCW'_\theta \times MCH$. Now, if $MCH'_\theta < 0$ and $MCW'_\theta > 0$ were true, and because $MCH > 0$ and $MCW > 0$ hold by definition, one gets $MWH'_\theta > 0$. This however contradicts proposition 3, as required.

Proof of proposition 5

Assumptions B and C relate to the function U (defined in proposition 1) and assumption A relates to the function V (defined in proposition 2).

First, to derive the function U , I first derive the optimal effort vector that solves

$$\max_{\mathbf{e} \geq \mathbf{0}} u(c, h, \mathbf{e}) \text{ such that } \theta f(\mathbf{e}) \geq w,$$

using

$$u : (c, h, \mathbf{e}) \mapsto \frac{1}{1-\alpha} c^{1-\alpha} - \frac{1}{1+\gamma} [\eta h^{1+\beta} + (1-\eta) (\sum_{j=1}^J \kappa_j (e_j)^{1+\delta})^{\frac{1+\beta}{1+\delta}}]^{\frac{1+\gamma}{1+\beta}},$$

with $\alpha, \beta, \gamma, \delta > 0$, $0 < \eta < 1$, and all $\kappa_j > 0$ with $\sum_{j=1}^J \kappa_j = 1$ as the utility function and

$$f : \mathbf{e} \mapsto (\sum_{j=1}^J \nu_j (e_j)^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}},$$

with $\sigma > 0$ and all $\nu_j > 0$ with $\sum_{j=1}^J \nu_j = 1$ as the production function.

The first-order conditions imply

$$\frac{\kappa_j}{\kappa_1} \left(\frac{e_j}{e_1} \right)^\delta = \frac{\nu_j}{\nu_1} \left(\frac{e_j}{e_1} \right)^{\frac{-1}{\sigma}},$$

for each $j \neq 1$, or equivalently,

$$e_j = \left(\frac{\nu_j \kappa_1}{\nu_1 \kappa_j} \right)^{\frac{\sigma}{1+\delta\sigma}} e_1,$$

for each j (including $j = 1$). Because $f(\mathbf{e}) = w/\theta$ must hold at the optimum, I find

$$e_1 = \left(\sum_{j=1}^J \nu_j \left(\frac{\nu_j \kappa_1}{\nu_1 \kappa_j} \right)^{\frac{\sigma-1}{1+\delta\sigma}} \right)^{\frac{\sigma}{1-\sigma}} \frac{w}{\theta},$$

and thus

$$e_j = \left(\frac{\nu_j \kappa_1}{\nu_1 \kappa_j} \right)^{\frac{\sigma}{1+\delta\sigma}} \left(\sum_{j=1}^J \nu_j \left(\frac{\nu_j \kappa_1}{\nu_1 \kappa_j} \right)^{\frac{\sigma-1}{1+\delta\sigma}} \right)^{\frac{\sigma}{1-\sigma}} \frac{w}{\theta},$$

for all j . To find the indirect utility function U , I plug in the solution $\mathbf{e}(c, h, w, \theta)$ in the direct utility function u . I get

$$\begin{aligned} U(c, h, w, \theta) &= u(c, h, \mathbf{e}(c, h, w, \theta)), \\ &= \frac{1}{1-\alpha} c^{1-\alpha} - \frac{1}{1+\gamma} \left(\eta h^{1+\beta} + (1-\eta) C(\boldsymbol{\nu}, \boldsymbol{\kappa}) \left(\frac{w}{\theta} \right)^{1+\beta} \right)^{\frac{1+\gamma}{1+\beta}}, \end{aligned}$$

with

$$C(\boldsymbol{\nu}, \boldsymbol{\kappa}) = \left(\sum_{j=1}^J (\kappa_j)^{\frac{1-\sigma}{1+\delta\sigma}} (\nu_j)^{\frac{\sigma(1+\delta)}{1+\delta\sigma}} \right)^{\frac{(1+\beta)(1+\delta\sigma)}{(1+\delta)(1-\sigma)}} > 0.$$

I can now derive the different marginal rates of substitution; I get

$$\begin{aligned} MCH(c, h, w, \theta) &= -\frac{U'_h(c, h, w, \theta)}{U'_c(c, h, w, \theta)}, \\ &= c^\alpha \left(\eta h^{1+\beta} + (1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa}) \left(\frac{w}{\theta} \right)^{1+\beta} \right)^{\frac{\gamma-\beta}{1+\beta}} \eta h^\beta, \end{aligned}$$

and

$$\begin{aligned} MCW(c, h, w, \theta) &= -\frac{U'_w(c, h, w, \theta)}{U'_c(c, h, w, \theta)}, \\ &= \frac{(\eta h^{1+\beta} + (1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa}) \left(\frac{w}{\theta} \right)^{1+\beta})^{\frac{\gamma-\beta}{1+\beta}} w^\beta \theta^{-(1+\beta)}}{(c^\alpha (1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa}))^{-1}}. \end{aligned}$$

The signs of the derivatives of these marginal rates of substitution with respect to ability define the single-crossingness conditions. Because

$$MCH'_\theta(c, h, w, \theta) = -(\gamma - \beta) \frac{(\eta h^{1+\beta} + (1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa}) \left(\frac{w}{\theta} \right)^{1+\beta})^{\frac{\gamma-\beta}{1+\beta}-1}}{\theta (c^\alpha \eta h^\beta (1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa}) \left(\frac{w}{\theta} \right)^{1+\beta})^{-1}},$$

the sign of MCH'_θ depends on the sign of $\gamma - \beta$ and can thus be negative ($\gamma > \beta$) or positive ($\gamma < \beta$). Because

$$MCW'_\theta(c, h, w, \theta) = -\frac{[(1+\gamma)(1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa}) \left(\frac{w}{\theta} \right)^{1+2\beta} + (1+\beta) \left(\frac{w}{\theta} \right)^\beta \eta h^{1+\beta}]}{\theta^2 (c^\alpha (1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa}))^{-1} (\eta h^{1+\beta} + (1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa}) \left(\frac{w}{\theta} \right)^{1+\beta})^{-\frac{\gamma-\beta}{1+\beta}}},$$

the sign of MCW'_θ is always (strictly) negative.

To derive the function V , I focus on problem (3). Using

$$\begin{aligned} MWH(c, h, w, \theta) &= -\frac{U'_h(c, h, w, \theta)}{U'_w(c, h, w, \theta)} = -\frac{MCH(c, h, w, \theta)}{MCW(c, h, w, \theta)}, \\ &= -\frac{\eta h^\beta}{(1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa}) w^\beta \theta^{-(1+\beta)}}, \end{aligned}$$

the first-order condition (4) leads to

$$h = \left(\frac{(1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa})}{\eta} \right)^{\frac{1}{1+\beta}} \frac{w}{\theta}.$$

Because $wh = y$ must hold at the optimum, I get

$$\begin{aligned}\hat{H}(c, y, \theta) &= \sqrt{\left(\frac{(1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa})}{\eta}\right)^{\frac{1}{1+\beta}} \frac{y}{\theta}}, \\ \hat{W}(c, y, \theta) &= \theta \sqrt{\left(\frac{(1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa})}{\eta}\right)^{-\frac{1}{1+\beta}} \frac{y}{\theta}}.\end{aligned}$$

Plugged in in the utility function U , I find

$$\begin{aligned}V(c, y, \theta) &= U(c, \hat{H}(c, y, \theta), \hat{W}(c, y, \theta), \theta) = \frac{1}{1-\alpha} c^{1-\alpha} - \\ &\frac{1}{1+\gamma} \left(\eta \left(\frac{(1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa})}{\eta} \right)^{\frac{1}{2}} + (1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa}) \left(\frac{(1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa})}{\eta} \right)^{-\frac{1}{2}} \right)^{\frac{1+\gamma}{1+\beta}} \left(\frac{y}{\theta} \right)^{\frac{1+\gamma}{2}}.\end{aligned}$$

The marginal rate of substitution is

$$\begin{aligned}MCY(c, y, \theta) &= -\frac{U'_y(c, h, w, \theta)}{U'_c(c, h, w, \theta)} \\ &= -\frac{\left(\eta \left(\frac{(1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa})}{\eta} \right)^{\frac{1}{2}} + (1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa}) \left(\frac{(1-\eta)C(\boldsymbol{\nu}, \boldsymbol{\kappa})}{\eta} \right)^{-\frac{1}{2}} \right)^{\frac{1+\gamma}{1+\beta}}}{2c^{-\alpha} y^{\frac{1-\gamma}{2}}} \theta^{-\frac{1+\gamma}{2}},\end{aligned}$$

which makes clear that the sign of $MCY'_\theta(c, y, \theta)$ depends on the sign of $-\frac{1+\gamma}{2}$, which is indeed negative.

Proof of proposition 6

The proof is essentially the same as in Stiglitz (1982). I add a proof for this paper to be self-contained. Consider the program

$$\max_{c_L, y_L, c_H, y_H} V(c_L, y_L, \theta_L) \quad \text{subject to}$$

$$V(c_H, y_H, \theta_H) - \underline{v} \geq 0, \tag{\alpha}$$

$$V(c_H, y_H, \theta_H) - V(c_L, y_L, \theta_H) \geq 0, \tag{\beta}$$

$$n_L(y_L - c_L) + n_H(y_H - c_H) - R_0 \geq 0, \tag{\gamma}$$

and let α , β , and γ be the Lagrange multipliers of the corresponding constraints. The first-order conditions (w.r.t. c_L, y_L, c_H, y_H) are

$$V'_c(c_L, y_L, \theta_L) - \beta V'_c(c_L, y_L, \theta_H) - \gamma n_L = 0, \quad (31)$$

$$V'_y(c_L, y_L, \theta_L) - \beta V'_y(c_L, y_L, \theta_H) + \gamma n_L = 0, \quad (32)$$

$$(\alpha + \beta) V'_c(c_H, y_H, \theta_H) - \gamma n_H = 0, \quad (33)$$

$$(\alpha + \beta) V'_y(c_H, y_H, \theta_H) + \gamma n_H = 0. \quad (34)$$

Combining the first-order conditions (33) and (34), and using the definition of MCY , one obtains

$$MCY(c_H, y_H, \theta_H) = 1,$$

and thus, using equation (8), I find $T'(y_H) = 0$ at the optimum.

Combining the first-order conditions (31) and (32), one obtains

$$\frac{V'_y(c_L, y_L, \theta_L)}{V'_c(c_L, y_L, \theta_L)} = \frac{\beta V'_y(c_L, y_L, \theta_H) - \gamma n_L}{\beta V'_c(c_L, y_L, \theta_H) + \gamma n_L},$$

which can also be written as

$$\frac{V'_y(c_L, y_L, \theta_L)}{V'_c(c_L, y_L, \theta_L)} + \frac{\gamma n_L}{\beta V'_c(c_L, y_L, \theta_H)} \left(\frac{V'_y(c_L, y_L, \theta_L)}{V'_c(c_L, y_L, \theta_L)} + 1 \right) = \frac{V'_y(c_L, y_L, \theta_H)}{V'_c(c_L, y_L, \theta_H)}. \quad (35)$$

Using equation (8) and the definition of MCY , I get²³

$$T'(y_L) = -\frac{\beta V'_c(c_L, y_L, \theta_H)}{\gamma n_L} [MCY(c_L, y_L, \theta_H) - MCY(c_L, y_L, \theta_L)].$$

Because the term between squared brackets is strictly negative by assumption A ($MCY'_\theta < 0$), the marginal tax rate $T'(y_L)$ must be strictly positive at the optimum.

²³For the low earners, the marginal tax rate is not well-defined at the optimum. The reason is that the self-selection constraint imposes a kink in the tax scheme at y_L . I (always) use the left-derivative (here and later). This is without loss of generality for *signing* the marginal tax rates. To see this here, the left-derivative is defined as $T'_l(y_L) = 1 - MCY(c_L, y_L, \theta_L)$ and the right-derivative as $T'_r(y_L) = 1 - MCY(c_L, y_L, \theta_H)$. Adding 1 to both sides of equation (35) and using equation (8), one can rewrite the obtained expression as

$$\underbrace{\left(1 + \frac{\gamma n_L}{\beta V'_c(c_L, y_L, \theta_H)} \right)}_{>0} T'_l(y_L) = T'_r(y_L),$$

which shows that the signs of both derivatives are the same, as required.

Proof of proposition 7

Consider the program

$$\begin{aligned} \max_{c_L, h_L, w_L, c_H, y_H} \quad & U(c_L, h_L, w_L, \theta_L) \quad \text{subject to} \\ & V(c_H, y_H, \theta_H) - \underline{v} \geq 0, \end{aligned} \quad (\alpha)$$

$$V(c_H, y_H, \theta_H) - U(c_L, h_L, w_L, \theta_H) \geq 0, \quad (\beta)$$

$$n_L(w_L h_L - c_L) + n_H(y_H - c_H) - R_0 \geq 0. \quad (\gamma)$$

Let α , β , and γ be the Lagrange multipliers of the corresponding constraints.

The first-order conditions w.r.t. c_H and y_H are again equations (33) and (34) and lead therefore to the same result: no distortion at the top. The first-order conditions w.r.t. c_L , h_L , and w_L are

$$U'_c(c_L, h_L, w_L, \theta_L) - \beta U'_c(c_L, h_L, w_L, \theta_H) - \gamma n_L = 0, \quad (36)$$

$$U'_h(c_L, h_L, w_L, \theta_L) - \beta U'_h(c_L, h_L, w_L, \theta_H) + \gamma n_L w_L = 0, \quad (37)$$

$$U'_w(c_L, h_L, w_L, \theta_L) - \beta U'_w(c_L, h_L, w_L, \theta_H) + \gamma n_L h_L = 0. \quad (38)$$

First, using equations (36) and (38), I get

$$\frac{U'_w(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)} = \frac{\beta U'_w(c_L, h_L, w_L, \theta_H) - \gamma n_L h_L}{\beta U'_c(c_L, h_L, w_L, \theta_H) + \gamma n_L}.$$

This expression can be written as

$$\frac{U'_w(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)} = \frac{\frac{U'_w(c_L, h_L, w_L, \theta_H)}{U'_c(c_L, h_L, w_L, \theta_H)} - \frac{\gamma n_L h_L}{\beta U'_c(c_L, h_L, w_L, \theta_H)}}{1 + \frac{\gamma n_L}{\beta U'_c(c_L, h_L, w_L, \theta_H)}},$$

leading to

$$\begin{aligned} \frac{U'_w(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)} + \left(\frac{U'_w(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)} + h_L \right) \frac{\gamma n_L}{\beta U'_c(c_L, h_L, w_L, \theta_H)} \\ = \frac{U'_w(c_L, h_L, w_L, \theta_H)}{U'_c(c_L, h_L, w_L, \theta_H)}. \end{aligned}$$

Using equation (10) and the definition of MCW , I obtain

$$\begin{aligned} T'_w(h_L, w_L) = - \frac{\beta U'_c(c_L, h_L, w_L, \theta_H)}{\gamma n_L} \times \\ [MCW(c_L, h_L, w_L, \theta_H) - MCW(c_L, h_L, w_L, \theta_L)]. \end{aligned}$$

Because $MCW'_\theta < 0$ holds (assumption C), the term between brackets is negative and thus $T'_w(h_L, w_L) > 0$ (hence, proposition 7.2).

Second, using equations (36) and (37), I get

$$\frac{U'_h(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)} = \frac{\beta U'_h(c_L, h_L, w_L, \theta_H) - \gamma n_L w_L}{\beta U'_c(c_L, h_L, w_L, \theta_H) + \gamma n_L}.$$

Proceeding in the same way as before, one can rewrite this expression as

$$\begin{aligned} \frac{U'_h(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)} + \left(\frac{U'_h(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)} + w_L \right) \frac{\gamma n_L}{\beta U'_c(c_L, h_L, w_L, \theta_H)} \\ = \frac{U'_h(c_L, h_L, w_L, \theta_H)}{U'_c(c_L, h_L, w_L, \theta_H)}. \end{aligned}$$

Now, using equation (9) and the definition of MCH , I obtain

$$\begin{aligned} T'_h(h_L, w_L) = - \frac{\beta U'_c(c_L, h_L, w_L, \theta_H)}{\gamma n_L} \times \\ [MCH(c_L, h_L, w_L, \theta_H) - MCH(c_L, h_L, w_L, \theta_L)]. \end{aligned}$$

The sign of $T'_h(h_L, w_L)$ is thus inversely related to the sign of MCH'_θ (hence, proposition 7.3).

Third, using equations (37) and (38) one obtains

$$\frac{U'_h(c_L, h_L, w_L, \theta_L)}{U'_w(c_L, h_L, w_L, \theta_L)} = \frac{\beta U'_h(c_L, h_L, w_L, \theta_H) - \gamma n_L w_L}{\beta U'_w(c_L, h_L, w_L, \theta_H) - \gamma n_L h_L}.$$

Proceeding in the same way as before, one gets after some manipulation

$$\begin{aligned} \frac{U'_h(c_L, h_L, w_L, \theta_L)}{U'_w(c_L, h_L, w_L, \theta_L)} - K \left(\frac{U'_h(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)} - \frac{U'_w(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)} \frac{w_L}{h_L} \right) \quad (39) \\ = \frac{U'_h(c_L, h_L, w_L, \theta_H)}{U'_w(c_L, h_L, w_L, \theta_H)}, \end{aligned}$$

with

$$K = \frac{\gamma n_L h_L U'_c(c_L, h_L, w_L, \theta_L)}{\beta U'_w(c_L, h_L, w_L, \theta_L) U'_w(c_L, h_L, w_L, \theta_H)} > 0.$$

Combining equations (9) and (10), evaluated for the low ability type L , and using the definitions of MCH and MCW , one finds

$$T'_h(h_L, w_L) = w_L + \frac{U'_h(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)}, \quad (40)$$

$$T'_w(h_L, w_L) = h_L + \frac{U'_w(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)}, \quad (41)$$

and thus

$$T'_h(h_L, w_L) - T'_w(h_L, w_L) \frac{w_L}{h_L} = \frac{U'_h(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)} - \frac{U'_w(c_L, h_L, w_L, \theta_L)}{U'_c(c_L, h_L, w_L, \theta_L)} \frac{w_L}{h_L}. \quad (42)$$

Using equation (42) and the definition of MWH , equation (39) can be written as

$$T'_h(h_L, w_L) - T'_w(h_L, w_L) \frac{w_L}{h_L} = \frac{1}{K} [MWH(c_L, h_L, w_L, \theta_H) - MWH(c_L, h_L, w_L, \theta_L)].$$

So, recalling Table 1, the sign of the earnings-conditional hours tax (the sign of the left-hand side) must coincide with the sign of MWH'_θ , and thus, given assumption B, it must be negative at the optimum (hence, proposition 7.1).