The influence of out-of-band modes in system inversion^{☆,☆☆}

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Abstract

Model-based system inversion is a technique which allows reconstructing the forces applied to a structure and the corresponding system response using data obtained from a limited number of sensors and a dynamic model of the structure. Very often, modally reduced order models are used in system inversion, hereby reducing the number of modes to reduce the computational cost. This paper shows that the model order reduction can lead to large estimation errors in model-based system inversion. These errors are due to the disregarded quasi-static contribution of the out-of-band modes. Existing techniques for quasi-static correction cannot be straightforwardly adopted in the state-space models which are commonly applied in recursive system inversion. This paper therefore presents a novel computationally efficient approach where the quasi-static contribution of the disregarded modes is accounted for in the state-space model by means of dummy modes. The presented approach is validated using data obtained from a full-scale experiment on a footbridge. The results show that the proposed correction of the model significantly increases the estimation accuracy, as can be seen from the significant reduction in the error on the estimated forces.

Keywords: out-of-band modes, state-space-model, system inversion, dummy mode, joint input-state estimation

1. Introduction

In civil and mechanical engineering, structures are equipped with sensors to monitor their health and control their operation. Sensors are used to observe states, inputs, or parameters of the structure. In various cases, however, the quantities of interest cannot be measured directly. For example, this is the case for the wind loads acting on a wind turbine or the dynamic vehicle loads on bridges. System inversion deals with the estimation of unobserved forces and system states based on sensor data and a model of the structure.

A wide variety of system inversion algorithms for force and state/response estimation has been proposed in the literature, tackling the problem in the frequency domain or the time domain [1, 2]. Recently, much focus has gone to recursive joint input-state estimation, where the forces applied to a

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structure and the corresponding system states are simultaneously estimated. Commonly adopted techniques are the Augmented Kalman filter [3, 4], the three-step joint input-state estimation algorithm proposed in [5] and further extended in [6] and [7], and the Dual Kalman filter [8]. The Dual Kalman filter algorithm was recently extended for joint input-state-parameter estimation, allowing for simultaneous estimation of the input applied to the system, the corresponding system states, and the unknown system parameters [9].

The use of a full order finite element model for most applications results in large computation times. For this reason, modally reduced order models, which are commonly used in forward dynamic response calculations, have also been applied to speed up the calculations in system inversion [3, 7, 8, 10]. This paper shows that modal truncation can lead to large estimation errors in model-based system inversion. The errors originate from disregarding the contribution of the out-of-band modes near antiresonance and in the quasi-static frequency range. The estimation accuracy can be significantly improved by applying a quasi-static correction for the out-of band modes. Several quasi-static correction methods have been presented in the literature to account for the contribution of out-of-band modes. A method which is commonly adopted in frequency domain system identification makes use of so-called upper and lower residual corrections of the frequency response function matrix [11, 12, 13]. Its equivalent in the time domain was presented by Chopra and Gutierrez [14]. An alternative implementation is found in [15]. An in depth discussion on state-of-the-art quasi-static correction methods with special focus on their application in system inversion is found in Section 3. Since the existing methods for quasi-static correction cannot be used in the state-space models which are commonly applied in recursive system inversion, an alternative approach is presented. Dummy modes are introduced in the state-space model to account for the quasi-static contribution of disregarded modes within a specified frequency range of interest. The presented approach is verified using numerical simulations and validated using data obtained from a full-scale experiment on a footbridge. The validation involves comparing the estimated forces with the actual, measured forces. Both in the verification examples and the validation study, it is illustrated how the proposed quasi-static correction allows for an improvement in model accuracy, as can be seen from the significant reduction in the error on the estimated forces.

The outline of the paper is as follows. Section 2 investigates the influence of out-of-band modes for model-based numerical predictions and model-based system inversion. Next, Section 3 reviews currently available methods to account for out-of-band modes. Section 4 presents a novel approach to account for the quasi-static contribution of out-of-band modes. Section 5 presents a validation of the presented approach using data obtained from a measurement campaign on the Ninove footbridge [16]. Finally, in Section 6, the work is concluded.

2. The influence of out-of-band modes

This section investigates the influence of disregarding out-of-band modes on (1) model-based numerical response predictions, applied in the design of structures, and (2) model-based system inversion.

The investigation is made through numerical simulations for the case of a simply supported steel beam (Fig. 1). The beam has a length of 1 m and a rectangular cross section of width 50.8 mm and height 25.4 mm. The Young's modulus, Poisson's ratio, and material density are taken as 210 GPa, 0.3, and 7750 kg/m³, respectively. Ideal hinges are assumed at both ends of the beam, only constraining the translation of the end nodes and not their rotation.

Fig. 1: Side view (left) and front view (right) of a simply supported steel beam, indicating the force p_1 and the locations where the vertical response of the structure is calculated (a_i : acceleration i , d_i : displacement i).

A two-dimensional finite element (FE) model of the beam with 100 Euler-Bernoulli beam elements is constructed. Only bending in the vertical plane is considered. Shear deformation is not accounted for. The first four natural frequencies obtained from the beam model are 59.9 Hz, 239.6 Hz, 538.3 Hz, and 955.2 Hz. The corresponding bending mode shapes are shown in Fig. 2. A modal damping ratio of 2.5% is assumed for all modes.

Fig. 2: Mode shape along the neutral axis of the simply supported beam, for (a) mode 1, (b) mode 2, (c) mode 3, and (d) mode 4. The undeformed neutral axis is shown by a black dashed line. The markers indicate the output coordinates (black: acceleration, white: displacement). The arrow indicates the force location.

The excitation of the beam consists of a single vertical impact force p_1 , applied at a distance of 0.05 m from the right support. The force location is chosen close to the support to have a significant contribution of the higher modes. The impact force is a triangular pulse that increases linearly from zero at time $t = 0.2$ s to 100 N at $t = 0.202$ s, before decreasing linearly to zero at $t = 0.204$ s, shown in Fig. 3. In the following analysis, the focus is put on the frequency band from 0 to 1000 Hz, which contains most of the dynamic response of the beam to the impact loading.

Fig. 3: (a) Time history, (b) detail time history, and (c) frequency spectrum up to 1000 Hz of the applied impact force. The begin and end of the impact are indicated in (b) by a vertical dashed line.

2.1. The contribution of out-of-band modes in the frequency band of interest

For a linear dynamic system described in modal coordinates, the transfer function $H_{\text{dp}}(\omega) \in \mathbb{C}$ relating the Fourier transform of a single response quantity $d(t)$ to the Fourier transform of a single force $p(t)$ can be written as a sum of modal contributions:

$$H_{\text{dp}}(\omega) = (i\omega)^q \sum_{m=1}^{\infty} \frac{\phi_{\text{dm}}\phi_{\text{pm}}}{-\omega^2 + 2i\xi_m\omega_m\omega + \omega_m^2} \quad (1)$$

where i is the imaginary unit ($i^2 = -1$), $\omega \in \mathbb{R}$ is the angular frequency in rad/s, $\omega_m \in \mathbb{R}$ and $\xi_m \in \mathbb{R}$ are the natural frequency and modal damping ratio corresponding to mode m , respectively, and $\phi_{\text{dm}} \in \mathbb{R}$ and $\phi_{\text{pm}} \in \mathbb{R}$ are obtained by selecting the component of the mode shape corresponding to mode m at the response and force location, respectively. The integer q equals 0 for displacement or strain response, 1 for velocities, and 2 for accelerations. The modes are ordered by increasing natural frequency ($\omega_1 \geq \omega_2 \geq \dots$).

For many applications, only a limited number of modes (n_{m} modes) significantly contributes to the response of the structure within a specified frequency band of interest. The remaining modes ($n_{\text{dof}} - n_{\text{m}}$ modes) are generally omitted, hereby allowing for a reduction of the model order. The transfer function $H'_{\text{dp}}(\omega) \in \mathbb{C}$ obtained after truncation is given by:

$$H'_{\text{dp}}(\omega) = (i\omega)^q \sum_{m=1}^{n_{\text{m}}} \frac{\phi_{\text{dm}}\phi_{\text{pm}}}{-\omega^2 + 2i\xi_m\omega_m\omega + \omega_m^2} \quad (2)$$

Within the specified frequency band of interest, the transfer function $H'_{\text{dp}}(\omega)$ generally provides a close approximation of the transfer function $H_{\text{dp}}(\omega)$ near resonance. The largest difference between the transfer functions is obtained near antiresonance and in the quasi-static frequency range. This is illustrated in the following example.

Example 2.1. Fig. 4 compares the transfer functions $H_{\text{dp}}(\omega)$ and $H'_{\text{dp}}(\omega)$ that relate displacement d_4 to force p_1 for the simply supported steel beam that has been previously introduced in this section (Fig. 1). The transfer function $H_{\text{dp}}(\omega)$ is obtained by including all modes in the model (200 modes), while the transfer function $H'_{\text{dp}}(\omega)$ is obtained by including only the first four vertical bending modes of the beam, i.e. all modes with a natural frequency in the frequency band of interest (0 to 1000 Hz). Both transfer functions nearly coincide at resonance, where a single mode dominates the response of the system. A large difference between the transfer functions is observed near antiresonance and in the quasi-static frequency range, where higher modes that are omitted in the modal truncation significantly contribute to the system response. The approximation errors generally decrease with the number of modes n_{m} included in the modal model.

2.2. The importance of out-of-band modes in forward problems

As mentioned in Section 1, modal truncation is commonly applied for numerical predictions in structural dynamics. When the response of the structure for a given excitation is considered, e.g. for earthquake excitation or impact loading, only the first few modes corresponding to the lowest natural frequencies generally have a significant contribution. For most types of excitation, the response frequency spectrum is dominated by one or more peaks at the resonance frequencies of the structure, while the response near antiresonance is small. Since the modal truncation allows

Fig. 4: (a) Amplitude and (b) phase of transfer functions $H_{dp}(\omega)$ (200 modes, solid blue line) and $H'_{dp}(\omega)$ (4 modes, dashed red line) that relate displacement d_4 to force p_1 . The contribution of the first four modes is indicated by black dash-dotted lines. The undamped natural frequencies are indicated by vertical dashed lines, the antiresonance frequencies of the full-order model by vertical dotted lines.

for an accurate representation of the dynamic behavior at resonance (see Section 2.1), it generally leads to an accurate prediction of the response time history. This is illustrated in the following example.

Example 2.2. *The response of the simply supported steel beam (Fig. 1) to impact loading is calculated using both the full order model and the modally reduced order model, in order to investigate the influence of the out-of-band modes in the simulation. The calculation is performed in the frequency domain, using the transfer functions shown in Fig. 4 and adopting a sampling frequency of 10 kHz and time window of 2 s. Fig. 5 shows the resulting displacement d_4 obtained for both models. The response location has been chosen close to the support to emphasize the effect of the higher modes, as was the case for the force location. The frequency spectra of the displacements obtained from both models in Fig. 5c agree very well near resonance. Larger deviations are observed near antiresonance, where the out-of-band modes have a significant contribution to the response. It is observed from the displacement time history in Figs. 5a and 5b, however, that the overall error resulting from disregarding the out-of-band modes is very small. During the impact, the broad band hammer force excites the entire frequency range considered and the truncation results in visible but small deviations. In the subsequent free vibration phase, the structure vibrates at its natural frequencies and the deviations are negligible.*

The errors introduced by the modal truncation depend on (1) the force and output locations and (2) the frequency content of the applied forces. For example, for a purely harmonic force with a frequency that does not coincide with one of the natural frequencies, the simulated response can be inaccurate, as immediately derived from the error in the transfer function introduced by modal truncation (Fig. 4).

2.3. The importance of out-of-band modes in inverse problems

If the modal truncation leads to large errors in the system transfer function, as observed in Fig. 4 near antiresonance, these errors can result in large errors in the estimated forces when the

Fig. 5: (a) Time history, (b) detail time history, and (c) frequency spectrum up to 1000 Hz of the simulated displacement d_4 for 200 modes (solid blue line) and 4 modes (red dashed line). The begin and end of the impact are indicated in (b) by a vertical dashed line. The undamped natural frequencies are indicated in (c) by vertical dashed lines, the antiresonance frequencies of the full-order model by vertical dotted lines.

model is applied for system inversion. This is illustrated in the following example.

Example 2.3. *In order to investigate the influence of the out-of-band modes in an inverse calculation, the impact force $p_1(t)$ applied to the simply supported steel beam (Fig. 1) is estimated from the displacement $d_4(t)$ using a modally reduced order model. The force identification is performed in the frequency domain, applying the following equation:*

$$\hat{p}_1(\omega) = H'_{dp}{}^{-1}(\omega)d_4(\omega) \quad (3)$$

where $\hat{p}_1(\omega)$ denotes the Fourier transform of the estimated force, $H'_{dp}(\omega)$ is the transfer function obtained after modal truncation (Fig. 4, red dashed line), and $d_4(\omega)$ is the Fourier transform of displacement $d_4(t)$, obtained from a forward calculation with the full order model (Fig. 5, blue solid line). Fig. 6 compares the estimated force $\hat{p}_1$ to the actual applied impact force p_1 . The frequency spectrum of the estimated force agrees well with the one of the applied force near resonance, while large errors originating from the modal truncation are observed near antiresonance (Fig. 6c). These errors result in large estimation errors in the force time history, which are observed right after the impact as spurious harmonic force components that decay exponentially over time.

Fig. 6: (a) Time history, (b) detail time history, and (c) frequency spectrum up to 1000 Hz of the estimated force p_1 (red dashed line) and comparison to the applied force (blue solid line). The begin and end of the impact are indicated in (b) by a vertical dashed line. The undamped natural frequencies are indicated in (c) by vertical dashed lines, the antiresonance frequencies of the full-order model by vertical dotted lines.

3. State-of-the-art methods for quasi-static correction

It was demonstrated in Section 2.3 that the truncation involved with the use of modally reduced order models can lead to large estimation errors in system inversion. This section reviews state-of-the-art methods which allow accounting for out-of-band modes in the dynamic system model and verifies their possible use in system inversion.

3.1. Frequency domain approach

As previously explained in Section 2.3, frequency domain system inversion techniques make use of the system transfer functions. The method outlined in this section allows incorporating the quasi-static contribution of the out-of-band modes in the transfer function description. This allows to significantly reduce estimation errors introduced by the use of modally reduced order models in frequency domain system inversion.

For a given frequency band of interest ($\omega \leq \omega_u$), the transfer function $H_{dp}(\omega)$ in Eq. (1) can be approximated as follows:

$$H_{dp}(\omega) \approx H''_{dp}(\omega) = (i\omega)^q \left(\sum_{m=1}^{n_m} \frac{\phi_{dm}\phi_{pm}}{-\omega^2 + 2i\xi_m\omega_m\omega + \omega_m^2} + \sum_{m=n_m+1}^{n_{dof}} \frac{\phi_{dm}\phi_{pm}}{\omega_m^2} \right) \quad (4)$$

where the first sum contains the dynamic contribution of the first n_m modes and the second sum contains the quasi-static contribution of the remaining $n_{dof} - n_m$ modes. The approximation in Eq. (4) holds if $\omega \ll \omega_{n_m+1}$. The contribution of the out-of-band modes is commonly denoted in the literature as the (frequency-independent) upper residual $UR \in \mathbb{R}$ (see for example [13]):

$$UR = \sum_{m=n_m+1}^{n_{dof}} \frac{\phi_{dm}\phi_{pm}}{\omega_m^2} \quad (5)$$

It is assumed here that the lower modes of the structure are all included in the model. As such, the modal truncation consists of the elimination of higher modes only, as represented by the upper residual. The method can be readily extended to account for the lower out-of-band modes, which requires the use of a so-called lower residual (see also [13]).

The calculation of the upper residual UR from Eq. (5) requires the calculation of all n_{dof} mode shapes and natural frequencies. This is circumvented by rewriting:

$$UR = \sum_{m=1}^{n_{dof}} \frac{\phi_{dm}\phi_{pm}}{\omega_m^2} - \sum_{m=1}^{n_m} \frac{\phi_{dm}\phi_{pm}}{\omega_m^2} = \mathbf{S}_d \mathbf{K}^{-1} \mathbf{S}_p - \sum_{m=1}^{n_m} \frac{\phi_{dm}\phi_{pm}}{\omega_m^2} \quad (6)$$

where $\mathbf{K} \in \mathbb{R}^{n_{dof} \times n_{dof}}$ is the stiffness matrix obtained from the finite element (FE) model and $\mathbf{S}_p \in \mathbb{R}^{n_{dof} \times n_p}$ and $\mathbf{S}_d \in \mathbb{R}^{n_d \times n_{dof}}$ are selection matrices specifying the force and output location, respectively, where the number of forces n_p and the number of outputs n_d both equal one for the considered single-input single-output (SISO) representation. Note that $\mathbf{S}_d \mathbf{K}^{-1} \mathbf{S}_p$ represents the static deflection of the structure for a unit load.

Combination of Eqs. (4), (5), and (6) yields the following expression for the transfer function $H''_{dp}(\omega)$:

$$H''_{dp}(\omega) = (i\omega)^q \left(\sum_{m=1}^{n_m} \left(\frac{\phi_{dm}\phi_{pm}}{-\omega^2 + 2i\xi_m\omega_m\omega + \omega_m^2} - \frac{\phi_{dm}\phi_{pm}}{\omega_m^2} \right) + \mathbf{S}_d \mathbf{K}^{-1} \mathbf{S}_p \right) \quad (7)$$

Example 3.1. *This example shows the quasi-static correction of the transfer function and its ability to reduce the error introduced by modal truncation in frequency domain force identification. Fig. 7 compares the transfer functions $H_{dp}(\omega)$, $H'_{dp}(\omega)$, and $H''_{dp}(\omega)$ that relate displacement d_4 to force p_1 for the simply supported steel beam. Within the frequency band of interest (0 to 1000 Hz), the*

transfer function $H''_{dp}(\omega)$ closely approximates the full-order transfer function $H_{dp}(\omega)$, which shows that the model accuracy can be significantly improved by including the quasi-static contribution of the out-of-band modes. Some remaining errors due to the dynamic contribution of the out-of-band modes are observed near the antiresonance frequencies. Fig. 8 shows the estimated force $\hat{p}_1$ as obtained from a frequency domain system inversion based on displacement d_4 and using the corrected transfer function $H''_{dp}(\omega)$ (see also Section 2.3). The errors on the estimated force are very small compared to the case where no upper residual correction is applied (Fig. 6). This illustrates that a quasi-static correction of the model can significantly improve the results obtained from a system inversion.

Fig. 7: (a) Amplitude and (b) phase of transfer functions $H_{dp}(\omega)$ (200 modes, solid blue line), $H'_{dp}(\omega)$ (4 modes, dashed red line), and $H''_{dp}(\omega)$ (4 modes with upper residual correction, cyan dotted line) that relate displacement d_4 to force p_1 . The quasi-static correction is indicated in (a) by a horizontal black dash-dotted line. The undamped natural frequencies are indicated by vertical dashed lines, the antiresonance frequencies of the full-order model by vertical dotted lines.

Fig. 8: (a) Time history, (b) detail time history, and (c) frequency spectrum up to 1000 Hz of the estimated force p_1 obtained after upper residual correction (red dashed line) and comparison to the applied force (blue solid line). The begin and end of the impact are indicated in (b) by a vertical dashed line. The undamped natural frequencies are indicated in (c) by vertical dashed lines, the antiresonance frequencies of the full-order model by vertical dotted lines.

3.2. Time domain approaches

This section gives an overview of two methods for quasi-static correction in the time domain that are shown to be entirely equivalent to the frequency domain correction method presented

in Section 3.1. After the presentation of the methods, their application for time domain system inversion is investigated.

When the natural frequency ω_m of a mode is much higher than the upper bound ω_u of the frequency range considered in the dynamic analysis, the corresponding response is essentially static and can be determined by a static analysis rather than a full dynamic analysis (see also Section 3.1). This is the main assumption of the static correction method as described in [15], which is briefly outlined next.

Consider a displacement/strain response $d(t)$. Within the frequency range of interest, the first n_m modes have a dynamic contribution. For higher $n_{\text{dof}} - n_m$ modes, the response is essentially static. The modal contributions to the displacement/strain response are divided in two parts:

$$d(t) \approx \sum_{m=1}^{n_m} \phi_{dm} z_m(t) + \sum_{m=n_m+1}^{n_{\text{dof}}} \phi_{dm} z_m^{\text{st}}(t) \quad (8)$$

where the modal displacement $z_m(t)$ corresponding to mode m is obtained by solving the following (decoupled) equation of motion:

$$\ddot{z}_m(t) + 2\xi_m \omega_m \dot{z}_m(t) + \omega_m^2 z_m(t) = \phi_m^T \mathbf{S}_p \mathbf{p}(t) \quad (9)$$

and the static modal displacement $z_m^{\text{st}}(t)$ is obtained as:

$$z_m^{\text{st}}(t) = \frac{1}{\omega_m^2} \phi_m^T \mathbf{S}_p \mathbf{p}(t) \quad (10)$$

Note that - as opposed to the SISO transfer function description applied in the previous sections - it is assumed here that multiple ($n_p \geq 1$) forces are acting on the structure.

The evaluation of Eq. (8) requires the calculation of all n_{dof} modes. This is circumvented by rewriting:

$$\sum_{m=n_m+1}^{n_{\text{dof}}} \phi_{dm} z_m^{\text{st}}(t) = \sum_{m=1}^{n_{\text{dof}}} \phi_{dm} z_m^{\text{st}}(t) - \sum_{m=1}^{n_m} \phi_{dm} z_m^{\text{st}}(t) = \mathbf{S}_d \mathbf{K}^{-1} \mathbf{S}_p \mathbf{p}(t) - \sum_{m=1}^{n_m} \phi_{dm} z_m^{\text{st}}(t) \quad (11)$$

Combination of Eqs. (8), (10), and (11) yields:

$$d(t) \approx \sum_{m=1}^{n_m} \left(\phi_{dm} z_m(t) - \frac{\phi_{dm}}{\omega_m^2} \phi_m^T \mathbf{S}_p \mathbf{p}(t) \right) + \mathbf{S}_d \mathbf{K}^{-1} \mathbf{S}_p \mathbf{p}(t) \quad (12)$$

In case of a single force ($n_p = 1$), introducing Eq. (6) in Eq. (12) yields:

$$d(t) \approx \sum_{m=1}^{n_m} \phi_{dm} z_m(t) + \text{UR} p(t) \quad (13)$$

where UR is the so-called upper residual as introduced in Section 3.1. Applying the Fourier transform to Eq. (12) and introducing the Fourier transform of Eq. (9) directly yields the transfer function $H''_{\text{dp}}(\omega)$ in Eq. (7). This proves that the static correction in the time domain according to Eq. (12) is equivalent to the upper residual correction in the frequency domain according to Eq. (4).

As an alternative to the presented static correction method, Chopra and Gutierrez [14] presented the so-called mode acceleration superposition method to account for higher modes that only have a static contribution to the response quantity of interest. This method only differs in the implementation of the solution and provides results which are identical to those obtained by the static correction method.

If one wishes to compute velocity or acceleration using Eq. (13), the first, respectively the second order derivative of the force time history $\mathbf{p}(t)$ with respect to time comes into play. This prohibits reformulating a dynamic model with quasi-static correction in state-space form and therefore the application of the presented time domain correction methods for recursive system inversion.

4. The application of dummy modes for static correction

This section presents an alternative way of correcting for the quasi-static contribution of the out-of-band modes. The correction is based on the introduction of dummy modes, avoiding the derivatives of the force time history or, alternative, the integration of velocity or acceleration measurements to obtain displacements.

Consider the following approximation of a displacement/strain response $d(t)$:

$$d(t) \approx \sum_{m=1}^{n_m} \phi_{dm} z_m(t) + \sum_{n=1}^{n_p} \phi'_{dn} z'_n(t) \quad (14)$$

The modal displacement $z_m(t)$ corresponding to mode m is obtained by solving Eq. (9). The contribution $\phi'_{dn} z'_n(t)$ corresponds to a dummy mode which is introduced to account for the quasi-static response of the out-of-band modes to a single force component $p_n(t)$ ($n = 1, \dots, n_p$). The modal displacement $z'_n(t)$ corresponding to dummy mode n is obtained by solving the following differential equation:

$$\ddot{z}'_n(t) + 2\xi'_n \omega'_n \dot{z}'_n(t) + \omega_n'^2 z'_n(t) = \phi'_{pn} p(t) \quad (15)$$

where ω'_n and ξ'_n are respectively the natural frequency and modal damping ratio assigned to dummy mode n , which are both user defined parameters. The choice of the natural frequency ω'_n and modal damping ratio ξ'_n of the dummy modes is discussed in the remainder of this section. The mode shape components ϕ'_{pn} and ϕ'_{dn} are determined such that the static response of the structure (at 0 Hz) is correctly represented by the modally reduced order model. The following expressions are obtained:

$$\phi'_{pn} = \omega'_n \sqrt{\mathbf{S}_{pn}^T \mathbf{K}^{-1} \mathbf{S}_{pn} - \sum_{m=1}^{n_m} \frac{\mathbf{S}_{pn}^T \phi_m \phi_m^T \mathbf{S}_{pn}}{\omega_m^2}} \quad (16)$$

$$\phi'_{dn} = \frac{\omega_n'^2}{\phi'_{pn}} \left(\mathbf{S}_d \mathbf{K}^{-1} \mathbf{S}_{pn} - \sum_{m=1}^{n_m} \frac{\phi_{dm}}{\omega_m^2} \phi_m^T \mathbf{S}_{pn} \right) \quad (17)$$

where $\mathbf{S}_{pn}$ contains the n -th column of the selection matrix $\mathbf{S}_p$. Note that the expressions for the mode shape components in Eqs. (16) and (17) only require the first n_m mode shapes and natural frequencies.

In case of a single force ($n_p = 1$), applying the Fourier transform to Eq. (14) and introducing the Fourier transform of Eqs. (9) and (15) allows calculating the transfer function $H''_{dp}(\omega)$ of the approximating system obtained by introduction of the dummy mode. After extension to velocity and acceleration measurements, the following expression is obtained:

$$H''_{dp}(\omega) = (i\omega)^q \left(\sum_{m=1}^{n_m} \frac{\phi_{dm}\phi_{pm}}{-\omega^2 + 2i\xi_m\omega_m\omega + \omega_m^2} + \frac{\phi'_{d1}\phi'_{p1}}{-\omega^2 + 2i\xi'_1\omega'_1\omega + \omega'^2_1} \right) \quad (18)$$

The natural frequency ω'_n and modal damping ratio ξ'_n of every dummy mode have to be chosen such that the contribution of this mode to the transfer function $H''_{dp}(\omega)$ is nearly constant in the frequency range of interest, hereby accounting for the quasi-static contribution of the out-of-band modes. The natural frequency ω'_n of the dummy mode should be chosen larger than the upper limit ω_u of the frequency band of interest, but smaller than the Nyquist frequency ω_N adopted for the time discretization. The damping ratio ξ'_n should be chosen sufficiently small to avoid a dynamic contribution of the dummy mode within the frequency band of interest. In this way, the quasi-static contribution is nearly constant within the frequency band of interest and closely resembles the previously mentioned upper residual correction UR (Section 3). The choice of the parameters ω'_n and ξ'_n can be verified by plotting the transfer functions, as illustrated in the following example.

Example 4.1. *Fig. 9 compares the transfer functions $H_{dp}(\omega)$, $H'_{dp}(\omega)$, and $H''_{dp}(\omega)$ that relate displacement d_4 to force p_1 for the simply supported steel beam. In the presented case, the natural frequency and modal damping ratio of the dummy mode are assigned a value of 1800 Hz and 2%, respectively. For the selected parameter values, the contribution of the dummy mode to the transfer function $H''_{dp}(\omega)$ is nearly constant within the frequency band of interest (0 to 1000 Hz). As a result, the transfer function $H''_{dp}(\omega)$ closely approximates the full-order transfer function $H_{dp}(\omega)$ within this frequency range. For higher frequencies, the approximation deviates from the actual model. It is therefore concluded that a model with dummy mode correction can only be used for calculations within the predefined frequency range.*

Since every dummy mode is related to a single force component, the derivation of the expressions for the state-space system matrices is different from the standard case where modal truncation is applied without dummy mode correction (see e.g. [7]). The expressions for the system matrices in the case where dummy modes are included, are derived in the following.

In state-space modeling, the dynamic behavior of the structure is generally described by the following discrete-time combined deterministic-stochastic system equations:

$$\mathbf{x}_{[k+1]} = \mathbf{A}\mathbf{x}_{[k]} + \mathbf{B}\mathbf{p}_{[k]} + \mathbf{w}_{[k]} \quad (19)$$

$$\mathbf{d}_{[k]} = \mathbf{G}\mathbf{x}_{[k]} + \mathbf{J}\mathbf{p}_{[k]} + \mathbf{v}_{[k]} \quad (20)$$

where $\mathbf{x}_{[k]} = \mathbf{x}(k\Delta t) \in \mathbb{R}^{n_s}$ is the state vector, $\mathbf{d}_{[k]} = \mathbf{d}(k\Delta t) \in \mathbb{R}^{n_d}$ is the output vector, and $\mathbf{p}_{[k]} = \mathbf{p}(k\Delta t) \in \mathbb{R}^{n_p}$ is the input vector, for $k = 0, \dots, N-1$, with Δt the sampling time step and N the total number of samples. The number of system states is denoted by n_s , the number of outputs by n_d , and the number of inputs by n_p . $\mathbf{w}_{[k]} \in \mathbb{R}^{n_s}$ and $\mathbf{v}_{[k]} \in \mathbb{R}^{n_d}$ are the process noise vector and the measurement noise vector, respectively. The model described by Eqs. (19) and (20) is obtained by time discretization of the following continuous-time state-space description of the system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{B}_c\mathbf{p}(t) + \mathbf{w}(t) \quad (21)$$

$$\mathbf{d}(t) = \mathbf{G}_c\mathbf{x}(t) + \mathbf{J}_c\mathbf{p}(t) + \mathbf{v}(t) \quad (22)$$

Fig. 9: (a) Amplitude and (b) phase of transfer functions $H_{dp}(\omega)$ (200 modes, blue solid line), $H'_{dp}(\omega)$ (4 modes, red dashed line), and $H''_{dp}(\omega)$ (4 modes with dummy mode correction, cyan dotted line) that relate displacement d_4 to force p_1 . The contribution of the dummy mode is indicated by a black dash-dotted line. The undamped natural frequencies are indicated by vertical dashed lines, the antiresonance frequencies of the full-order model by vertical dotted lines.

In the case where n_p dummy modes are accounted for in the model, the state vector $\mathbf{x}(t) \in \mathbb{R}^{2(n_m+n_p)}$ contains the modal displacements $\mathbf{z}(t) \in \mathbb{R}^{n_m}$ and modal velocities $\dot{\mathbf{z}}(t) \in \mathbb{R}^{n_m}$ of the n_m modes included in the model, as well as the modal displacements $\mathbf{z}'(t) \in \mathbb{R}^{n_p}$ and modal velocities $\dot{\mathbf{z}}'(t) \in \mathbb{R}^{n_p}$ of the dummy modes:

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{z}'(t) \\ \dot{\mathbf{z}}(t) \\ \dot{\mathbf{z}}'(t) \end{bmatrix} \quad (23)$$

The expressions for the system matrices $\mathbf{A}_c$ and $\mathbf{B}_c$ are obtained by combination of Eqs. (9), (15), (21), and (23), and given by:

$$\mathbf{A}_c = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_m} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_p} \\ -\mathbf{\Omega}^2 & \mathbf{0} & -\mathbf{\Gamma} & \mathbf{0} \\ \mathbf{0} & -\mathbf{\Omega}'^2 & \mathbf{0} & -\mathbf{\Gamma}' \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{\Phi}^T \mathbf{S}_p \\ \mathbf{\Phi}'_p \end{bmatrix} \quad (24)$$

where $\mathbf{I}_{n_m} \in \mathbb{R}^{n_m \times n_m}$ and $\mathbf{I}_{n_p} \in \mathbb{R}^{n_p \times n_p}$ are identity matrices. $\mathbf{\Gamma} \in \mathbb{R}^{n_m \times n_m}$ is a diagonal matrix containing the terms $2\xi_m \omega_m$ on its diagonal, where ω_m and ξ_m are the natural frequency and modal damping ratio corresponding to mode m , respectively. $\mathbf{\Gamma}' \in \mathbb{R}^{n_p \times n_p}$ is a diagonal matrix containing the terms $2\xi'_n \omega'_n$ on its diagonal, with ω'_n and ξ'_n the natural frequency and modal damping ratio assigned to dummy mode n , respectively. $\mathbf{\Omega} \in \mathbb{R}^{n_m \times n_m}$ and $\mathbf{\Omega}' \in \mathbb{R}^{n_p \times n_p}$ are diagonal matrices as well, containing respectively the natural frequencies ω_m and ω'_n on the diagonal. $\mathbf{\Phi} \in \mathbb{R}^{n_{\text{dof}} \times n_m}$ is a matrix containing the mass-normalized mode shapes ϕ_m as columns. $\mathbf{S}_p \in \mathbb{R}^{n_{\text{dof}} \times n_p}$ is the selection matrix specifying the force locations, as already considered in Section 3. Finally, $\mathbf{\Phi}'_p \in \mathbb{R}^{n_p \times n_p}$ is a diagonal matrix containing the dummy mode shape components ϕ'_{pn} obtained from Eq. (16) on its diagonal.

Similarly, the expressions for the system matrices $\mathbf{G}_c$, and $\mathbf{J}_c$ are obtained by combination of the following (modified) expression for the response vector $\mathbf{d}(t)$ with Eqs. (9), (15), (22), and (23):

$$\mathbf{d}(t) = \mathbf{S}_{d,a} (\Phi \ddot{\mathbf{z}}(t) + \Phi' \dot{\mathbf{z}}'(t)) + \mathbf{S}_{d,v} (\Phi \dot{\mathbf{z}}(t) + \Phi' \dot{\mathbf{z}}'(t)) + \mathbf{S}_{d,d} (\Phi \mathbf{z}(t) + \Phi' \mathbf{z}'(t)) + \mathbf{v}(t) \quad (25)$$

where the columns of the matrix $\Phi' \in \mathbb{R}^{n_{\text{dof}} \times n_p}$ are the dummy mode shape components ϕ'_{dn} corresponding to all degrees of freedom in the model, obtained from Eq. (17). $\mathbf{S}_{d,a}$, $\mathbf{S}_{d,v}$, and $\mathbf{S}_{d,d} \in \mathbb{R}^{n_d \times n_{\text{dof}}}$ are selection matrices indicating the degrees of freedom corresponding to the acceleration, velocity, and displacement or strain measurements, respectively, with n_d the number of outputs. The following expressions for the matrices $\mathbf{G}_c$ and $\mathbf{J}_c$ are obtained:

$$\mathbf{G}_c = [\mathbf{S}_{d,d} [\Phi \ \Phi'] - \mathbf{S}_{d,a} [\Phi \Omega^2 \ \Phi' \Omega'^2] \quad \mathbf{S}_{d,v} [\Phi \ \Phi'] - \mathbf{S}_{d,a} [\Phi \Gamma \ \Phi' \Gamma']] \quad (26)$$

$$\mathbf{J}_c = [\mathbf{S}_{d,a} \Phi \Phi^T \mathbf{S}_p + \mathbf{S}_{d,a} \Phi' \Phi_p'] \quad (27)$$

The application of the dummy mode correction for recursive system inversion is verified in the following example.

Example 4.2. *The recursive joint input-state estimation algorithm presented in [7] is applied for estimating the impact force p_1 applied to the simply supported steel beam shown in Fig. 1. In the present example, a state-space model representing the dynamic behavior of the beam in the frequency range from 0 to 1000 Hz is constructed from the first four bending modes of the beam ($n_m=4$). The static contribution of the higher modes is accounted for by a single dummy mode ($n_p=1$), with a natural frequency and modal damping ratio of 1800 Hz and 2%, respectively. A sampling rate of 10 kHz is applied, using a first-order hold (FOH) assumption on the input in the time discretization of the model.*

From the conditions for instantaneous system inversion presented in [17], it is found that the output assumed for the identification of a single force should contain at least one acceleration and one displacement measurement. The system response in the considered case consists of displacement d_4 and acceleration a_1 (see Fig. 1). The response is obtained from a forward frequency domain calculation (see also Section 2.2). No measurement noise or unknown stochastic excitation is considered.

In addition to the state-space model description in Eqs. (19) and (20) and the system response $\mathbf{d}_{[k]}$, the joint input-state estimation algorithm [7] requires the covariance matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ of the noise processes $\mathbf{w}_{[k]}$ and $\mathbf{v}_{[k]}$ and an initial estimate of the state vector $\hat{\mathbf{x}}_{[0|-1]}$. The noise covariance matrices $\mathbf{Q} \in \mathbb{R}^{n_s \times n_s}$, $\mathbf{R} \in \mathbb{R}^{n_d \times n_d}$, and $\mathbf{S} \in \mathbb{R}^{n_s \times n_d}$ are defined as:

$$\mathbb{E} \left\{ \begin{pmatrix} \mathbf{w}_{[k]} \\ \mathbf{v}_{[k]} \end{pmatrix} \begin{pmatrix} \mathbf{w}_{[l]}^T & \mathbf{v}_{[l]}^T \end{pmatrix} \right\} = \begin{bmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{bmatrix} \delta_{[k-l]} \quad (28)$$

Since no additional stochastic excitation and modeling errors are present, the noise covariance matrices $\mathbf{Q}$ and $\mathbf{S}$ are chosen as zero matrices (see also [7]). Although no measurement noise is considered in the analysis, the measurement noise covariance matrix $\mathbf{R}$ cannot be zero ($\mathbf{R} > 0$). Following Maes et al. in [7], the matrix $\mathbf{R}$ is assumed to account for (fictitious) measurement noise. A standard deviation $\sigma_{M,a} = 10^{-2}$ m/s² and $\sigma_{M,d} = 10^{-6}$ m is assumed for the measurement error on the acceleration and displacement signals, respectively. The initial state vector estimate $\hat{\mathbf{x}}_{[0|-1]}$ and its corresponding error covariance matrix $\mathbf{P}_{x[0|-1]}$ are both assigned zero matrices.

Fig. 10 shows the force $\hat{p}_1$ estimated by the recursive time domain system inversion, using the model with dummy mode correction. As a reference, Fig. 11 shows the result which is obtained

without quasi-static correction for the out-of-band modes. In both cases, the estimated force is lowpass filtered with a cutoff frequency of 1000 Hz in order to retain only information within the frequency band of interest and to remove the dynamic contribution of the dummy mode at higher frequencies. Comparison of Figs. 10 and 11 shows how the introduction of the dummy mode allows to significantly improve the recursive system inversion.

Fig. 10: (a) Time history, (b) detail time history, and (c) frequency spectrum up to 1000 Hz of the estimated force $\hat{p}_1$ obtained from the joint input-state estimation algorithm **with dummy mode correction** (red dashed line) and comparison to the applied force (blue solid line). The begin and end of the impact are indicated in (b) by a vertical dashed line. The undamped natural frequencies are indicated in (c) by vertical dashed lines, the antiresonance frequencies of the full-order model by vertical dotted lines.

Fig. 11: (a) Time history, (b) detail time history, and (c) frequency spectrum up to 1000 Hz of the estimated force $\hat{p}_1$ obtained from the joint input-state estimation algorithm **without dummy mode correction** (red dashed line) and comparison to the applied force (blue solid line). The begin and end of the impact are indicated in (b) by a vertical dashed line. The undamped natural frequencies are indicated in (c) by vertical dashed lines, the antiresonance frequencies of the full-order model by vertical dotted lines.

5. Validation

This section presents a validation of the presented quasi-static correction method using data obtained from in situ experiments on a footbridge, shown in Fig. 12. The footbridge is a two-span cable-stayed steel bridge located in Ninove, Belgium, and has a main span of 36 m and a secondary span of 22.5 m [18].

In the present validation study, the joint input-state estimation algorithm that was previously applied in Section 4 is used to estimate swept sine forces applied to the bridge deck. A modally reduced order model of the footbridge is used in the estimation. The quasi-static contribution of the out-of-band modes is accounted for by dummy modes. The validation consists of comparing the estimated forces with the actual measured swept sine loads.

5.1. Measurement setup

The structure is excited by swept sine forces applied vertically to the bridge deck by pneumatic actuators developed by the Acoustics and Vibration Research Group of the Vrije Universiteit

Fig. 12: The footbridge in Ninove, Belgium.

Brussel [19]. The sensor configuration is shown in Fig. 13. The acceleration response of the footbridge has been recorded in three orthogonal directions at 12 locations on the bridge deck, using 12 wireless GeoSIG GMS-18 units. In addition, a National Instruments (NI) PXI data acquisition system has been used to record (1) the vertical acceleration at nodes 27 and 48, obtained from PCB 393B04 uniaxial accelerometers, (2) the vertical displacement of the bridge deck at nodes 27 and 40, obtained from AWLG 008M optical displacement sensors, and (3) the actuator loads applied vertically at nodes 27 and 48 using a PCB 222B and a BD 5 load cell, respectively.

Fig. 13: Sensor configuration Ninove footbridge (white circle: GMS-18 unit, black circle: uniaxial accelerometer, gray circle: optical displacement sensor, white square: load cell with pneumatic artificial muscle).

The frequency band considered in the present validation study is 0 to 20 Hz. A sampling frequency of 200 Hz and 1000 Hz is used for the data acquisition by the GMS-18 units and the NI system, respectively. The GMS-18 acceleration data and the measurement data obtained from the NI system are synchronized by maximizing the correlation between the acceleration obtained from the GMS-18 unit at node 48 and the acceleration at node 48 obtained from the cabled uniaxial accelerometer. After synchronization, the measured response and force signals obtained from the NI system are all digitally lowpass filtered by means of an eighth-order Chebyshev type I lowpass filter with a cut-off frequency of 80 Hz, in both the forward and the reverse direction to remove all phase distortion, and then re-sampled at 200 Hz. Next, all force and response signals are digitally lowpass filtered by means of a fourth order Chebyshev type I lowpass filter with a cutoff frequency of 20 Hz, respectively, in both the forward and the reverse direction. The aim of the filter is to retain only frequency components in the frequency band of interest. The acceleration signals obtained from the NI system and the GMS-18 units are additionally digitally highpass filtered by means of a fourth order Chebyshev type I lowpass filter with a cutoff frequency of 0.5 Hz and 0.1 Hz, respectively, in both the forward and the reverse direction. The aim of the filter is to remove the low frequency components from the signals that are contaminated by measurement noise. Finally, a detrend operation is applied to all acceleration, displacement, and force signals.

5.2. System model

The force identification is based on a state-space description of the system, given by equations (19) and (20). The system model used in the present analysis is based on the modes obtained from a detailed finite element (FE) model of the structure, built using the FE program ANSYS. The reader is referred to [16] for a detailed description of the FE model and the calibration of the model based on the experimental modal parameters that have been obtained through operational modal analysis (OMA).

A reduced-order discrete-time state-space model is constructed from the modal characteristics of the structure. The model includes the dynamic contribution of all bending and torsional modes of the bridge deck with a natural frequency that falls within the frequency range 0 Hz to 20 Hz (18 modes). For each of these modes, the mass-normalized mode shape of the FE model is used. The natural frequency and modal damping ratio in the state-space model are the experimentally identified values. For the remaining (higher) modes, the quasi-static contribution is accounted for by including one dummy mode for every force considered in the analysis. A natural frequency and modal damping ratio of 25 Hz and 2% are assumed for the dummy modes, respectively. The mode shape components ϕ'_{pn} and ϕ'_{dn} are calculated from Eqs. (16) and (17), where the static deflections to a unit force $\mathbf{S}_{pn}^T \mathbf{K}^{-1} \mathbf{S}_{pn}$ and $\mathbf{S}_d \mathbf{K}^{-1} \mathbf{S}_{pn}$ are obtained from a static FE analysis in ANSYS. A first-order hold assumption on the input vector $\mathbf{p}_{[k]}$ is applied in the time discretization of the state-space model, using a sampling frequency of 200 Hz.

Fig. 14 compares the transfer functions $H'_{dp}(\omega)$ (modal truncation without dummy mode correction) and $H''_{dp}(\omega)$ (with dummy mode correction) that relate the vertical displacement and acceleration at node 27 (d_{27} and a_{27} , respectively) to the force applied at node 27 (p_{27}). A significant difference between both transfer functions is observed. As for the numerical simulations in Section 2, the largest differences are observed near the antiresonance frequencies and in the quasi-static frequency range, where the out-of-band modes significantly contribute to the dynamic behavior of the structure.

Fig. 14: Amplitude (top) and phase (bottom) of transfer functions $H'_{dp}(\omega)$ (18 modes, blue solid), and $H''_{dp}(\omega)$ (18 modes with dummy mode correction, red dashed line) that relate displacement d_{27} ((a) and (c)) and acceleration a_{27} ((b) and (d)) to force p_{27} . The natural frequencies are indicated by vertical dashed lines, the antiresonance frequencies by vertical dotted lines.

5.3. Force identification

This section presents the results of the joint input-state estimation for the identification of swept sine forces applied to the bridge deck. Firstly, the case of a single swept sine force applied at node 27 is considered. The excitation frequency of the force rises from 0.5 to 10 Hz in 285 s. The excitation frequency increases in steps of 1 mHz every 30 ms. Two cycles of 285 s are considered.

The forces are estimated using a minimum data set of one displacement measurement d_{27} and one acceleration measurement a_{27} , which are both collocated with the estimated force p_{27} . The initial state estimate vector $\mathbf{x}_{[0|-1]}$ and its error covariance matrix $\mathbf{P}_{[0|-1]}$ are both assumed zero in the estimation. In comparison to the numerical simulations in Section 4, where no noise is considered, the noise processes $\mathbf{w}_{[k]}$ and $\mathbf{v}_{[k]}$ in this case account for sensor noise, modeling errors, and stochastic forces acting on the structure besides the forces $\mathbf{p}_{[k]}$ to be estimated (see also [7]). The noise covariance matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ describing the noise processes act as tuning parameters and determine the weighing of the data by the joint input-state estimation algorithm and therefore the estimation uncertainty (see also [7]).

Following [16], the stochastic forces that are acting on the structure besides the forces $\mathbf{p}_{[k]}$ are modeled by a set of (equivalent) stochastic forces $\mathbf{p}_{S[k]} \in \mathbb{R}^{n_{ps}}$. In the present case, ten equivalent stochastic loads are assumed to be acting on the bridge deck in the vertical and lateral directions (see also [16]). If modeling errors are disregarded, the following expressions are obtained for the noise processes [7]:

$$\mathbf{w}_{[k]} = \mathbf{B}_S \mathbf{p}_{S[k]} \quad (29)$$

$$\mathbf{v}_{[k]} = \mathbf{B}_S \mathbf{p}_{S[k]} + \mathbf{v}_{M[k]} \quad (30)$$

where the state-space matrices $\mathbf{B}_S \in \mathbb{R}^{n_s \times n_{ps}}$ and $\mathbf{J}_S \in \mathbb{R}^{n_d \times n_{ps}}$ relate the state vector $\mathbf{x}_{[k+1]}$ and the output vector $\mathbf{d}_{[k]}$ to the vector of stochastic forces $\mathbf{p}_{S[k]}$, respectively, and $\mathbf{v}_{M[k]}$ represents the additive sensor noise. Substituting Eqs. (29) and (30) in Eq. (28) yields:

$$\begin{bmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_S \\ \mathbf{J}_S \end{bmatrix} \mathbf{C}_{ps} \begin{bmatrix} \mathbf{B}_S^T & \mathbf{J}_S^T \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_M \end{bmatrix} \quad (31)$$

where $\mathbf{C}_{ps} \in \mathbb{R}^{n_{ps} \times n_{ps}}$ is the covariance matrix of the stochastic forces ($\mathbf{C}_{ps} = \mathbb{E}\{\mathbf{p}_{S[k]} \mathbf{p}_{S[k]}^T\}$) and $\mathbf{R}_M \in \mathbb{R}^{n_d \times n_d}$ is the sensor noise covariance matrix ($\mathbf{R}_M = \mathbb{E}\{\mathbf{v}_{M[k]} \mathbf{v}_{M[k]}^T\}$). The covariance matrix $\mathbf{C}_{ps}$ is estimated from the response of the structure under wind loading in the frequency range from 1 to 16 Hz. This procedure is illustrated in [16]. For the calculation of the sensor noise covariance matrix $\mathbf{R}_M$, a standard deviation of 1.6×10^{-4} m/s², 6.7×10^{-5} m/s², and 1.3×10^{-5} m is assumed for the error on the acceleration measurements obtained from the GMS-18 units, the uniaxial acceleration measurements, and the displacement measurements, respectively. The measurement errors are assumed uncorrelated (i.e. $\mathbf{R}_M$ is a diagonal matrix).

Fig. 15 compares the estimated force $\hat{p}_{27}$ obtained from the model with dummy mode correction to the measured swept sine force p_{27} . As a reference, Fig. 16 shows the estimated force $\hat{p}_{27}$ obtained from the model without quasi-static correction for the out-of-band modes. In both cases, the estimated force is lowpass filtered with a cutoff frequency of 20 Hz in order to retain only information within the frequency band of interest and to remove the dynamic contribution of the dummy mode at higher frequencies. Comparison of Figs. 15 and 16 shows how the introduction of a single dummy mode for quasi-static correction leads to a significant reduction of the estimation

errors. Some errors remain after dummy mode correction, which might originate from errors in the modal characteristics of the modes included in the model (natural frequencies, mode shapes, and modal damping ratios), small errors in the quasi-static correction, the dynamic contribution of the out-of-band modes, which has not been accounted for, and violation of the assumptions of proportional damping and linear structural behavior.

Fig. 15: (a) Time history and (b) averaged amplitude of the frequency spectrum of the estimated force $\hat{p}_{27}$ obtained **with dummy mode correction** (red dotted line) and comparison to the applied force (blue solid line). The natural frequencies are indicated in (b) by vertical dashed lines.

Fig. 16: (a) Time history and (b) averaged amplitude of the frequency spectrum of the estimated force $\hat{p}_{27}$ obtained **without dummy mode correction** (red dotted line) and comparison to the applied force (blue solid line). The natural frequencies are indicated in (b) by vertical dashed lines.

The force identification problem becomes more challenging when the number of forces n_p increases. A second validation therefore consists of the identification of two simultaneously applied swept sine forces at nodes 27 and 48. The excitation frequency of the force applied at node 27 rises from 0.5 Hz to 10 Hz in 285 s, whereas for node 48, the excitation frequency rises from 0.375 Hz to 7.5 Hz in the same time period. The excitation frequency of the forces applied at nodes 27 and 48 is increased in steps of 1 mHz and 0.75 mHz every 30 ms, respectively. Two cycles of 285 s are considered. In this case, the forces are estimated using a data set consisting of two displacements d_{27} and d_{40} and two accelerations a_{27} and a_{48} . The accelerations are collocated with the estimated forces p_{27} and p_{48} .

Fig. 17 compares the estimated forces $\hat{p}_{27}$ and $\hat{p}_{48}$ obtained from the model with dummy mode correction to the measured swept sine forces p_{27} and p_{48} . Fig. 18 shows the estimated forces $\hat{p}_{27}$ and $\hat{p}_{48}$ obtained without quasi-static correction for the out-of-band modes. As in the previous case, the estimated forces are lowpass filtered with a cutoff frequency of 20 Hz. Although less pronounced than in the case of a single force (Figs. 15 and 16), the introduction of the dummy modes results in a clear reduction of the estimation errors, mainly for frequencies below 10 Hz. For some frequencies, the estimation errors remain large, however, and modeling errors prohibit a proper distinction between the two independent forces.

Fig. 17: Time history (left) and averaged amplitude of the frequency spectrum (right) of the estimated forces $\hat{p}_{27}$ ((a) and (b)) and $\hat{p}_{48}$ ((c) and (d)) obtained **with dummy mode correction** (red dotted line) and comparison to the applied forces (blue solid line). The natural frequencies are indicated in (b) and (d) by vertical dashed lines.

Fig. 18: Time history (left) and averaged amplitude of the frequency spectrum (right) of the estimated forces $\hat{p}_{27}$ ((a) and (b)) and $\hat{p}_{48}$ ((c) and (d)) obtained **without dummy mode correction** (red dotted line) and comparison to the applied forces (blue solid line). The natural frequencies are indicated in (b) and (d) by vertical dashed lines.

6. Conclusions

Although modal truncation is commonly applied in model-based system inversion to reduce the computational effort, disregarding out-of-band modes can lead to large estimation errors. These estimation errors can be significantly reduced by a quasi-static correction for the out-of-band modes. Although currently available methods for quasi-static correction succeed in improving the model accuracy within the predefined frequency band of interest in a very efficient way, they cannot be used in combination with state-space modeling, commonly adopted in recursive system inversion. An alternative method for quasi-static correction is therefore proposed, where dummy modes are introduced to account for the quasi-static contribution of the out-of-band modes. The model improvement in the frequency band of interest is similar to that of currently available correction

techniques. The proposed quasi-static correction method is validated using measurements obtained from a full-scale experiment on a footbridge. The quasi-static correction results in a clear reduction of the errors in the forces obtained from a recursive system inversion. The remaining errors are presumably mostly due to model errors.

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