INVESTIGATING LEARNERS' FRACTION UNDERSTANDING: A LONGITUDINAL STUDY IN UPPER ELEMENTARY SCHOOL

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We longitudinally followed 201 upper elementary school learners in the crucial years of acquiring rational number understanding. Using latent transition analysis we investigated their conceptual change from an initial natural number based concept of a rational number towards a mathematically more correct one by characterizing the various intermediate states learners go through. Results showed that learners first develop an understanding of decimal numbers before they have an increased understanding of fractions. We also found that a first step in learners' rational number understanding is an increased understanding of the numerical size of rational numbers.

INTRODUCTION

There is broad agreement in the literature that a good understanding of the rational number domain is highly predictive for the learning of more advanced mathematics (e.g., Siegler, Thompson, & Schneider, 2011). It is therefore worrying that many elementary and secondary school learners and even (prospective) teachers face serious difficulties understanding rational numbers. An often reported source for the struggle with understanding rational numbers is the natural number bias, i.e., the tendency to (inappropriately) apply properties of natural numbers in rational numbers tasks (e.g., Christou & Vosniadou, 2009; Gomez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014; Obersteiner et al., 2014; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Vosniadou, 2013).

The literature reports at least three aspects of the natural number bias. The first aspect involves the numerical size of numbers. Learners often consider a fraction as two independent numbers, instead of a ratio between the numerator and denominator. This incorrect interpretation of a fraction can lead to the idea that the numerical value of a fraction increases when the numerator, denominator, or both increase, just like it is the case with natural numbers. For example, 1/8 can be judged larger than 1/6, just like 8 is larger than 6. Similarly, in the case of decimal numbers, some learners have been found to wrongly assume that, just like it is the case with natural numbers, longer decimals are larger, while shorter decimals are smaller. For example, these learners judge 0.12 larger than 0.8, just like 12 is larger than 8 (e.g., Vosniadou, 2013).

The second aspect concerns the effect of arithmetic operations. After learners did arithmetic with mostly natural numbers only in their first years of schooling, some

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learners have been found to apply the rules that hold for natural numbers also to rational numbers, also in cases where this is inappropriate. These learners assume for example that addition and multiplication will lead to a larger result, while subtraction and division will lead to a smaller result. For example, they think that 5 * 0.32 will result in an outcome larger than 5 (e.g., Christou, 2015).

The third aspect is density. Contrary to natural numbers that have a discrete structure (each natural number has a successor number; after 5 comes 6, after 6 comes 7, …), rational numbers are densely ordered (between any two rational numbers are always infinitely many other numbers). This difference in structure of both types of numbers leads to frequently found mistakes such as thinking that there are no numbers between two pseudo-successive numbers (e.g., between 6.2 and 6.3 or between 2/4 and 3/4 (e.g., Merenluoto & Lehtinen, 2004).

A lot of research on learners' transition from natural to rational numbers has been described from a conceptual change perspective. This perspective argues that since children encounter natural numbers much more frequently than rational numbers in daily life and in the first years of instruction, they form an idea of what numbers are and how they should behave based on these first experiences with and knowledge of natural numbers. So, to overcome the natural number bias, a conceptual change revising these initial natural number based understandings is required. Conceptual change is considered to be not an all or nothing issue but a gradual and time-consuming process, with qualitatively different intermediate states between the initial and the correct understanding (e.g., Vosniadou, 2013).

While the natural number bias has generated a lot of research, empirical evidence on the development of learners' understanding, i.e. their conceptual change from a natural-number-based towards a mathematically more correct concept of a rational number, is scarce. Nonetheless, it is important to investigate in detail how this development occurs. If general patterns can be found, a learner's profile at a certain measurement point can predict its further development. From an educational perspective, such profiles can help teachers to provide effective instruction that is adapted to the specific knowledge and needs of their learners (Schneider & Hardy, 2013).

THE PRESENT STUDY

In the present study, we longitudinally followed the development of rational number understanding of upper elementary school learners in the crucial years of acquiring rational number understanding. The aim of this study is to make a theoretical contribution to the research field by characterizing in detail the intermediate states of learners' conceptual change from an initial natural number based concept of rational numbers towards a mathematically more correct one and by investigating whether these intermediate states have a consistent character across students or not.

METHOD

Participants were recruited from four elementary schools and 11 classrooms in Flanders, Belgium. In total 201 learners from fourth (n = 113) and fifth grade (n = 88) participated in this study and 50.2% of the participants were boys. Data were collected following the ethical guidelines of the university.

Learners' rational number knowledge was measured three times over the course of two school years, spanning a total time of 15 months: at the beginning (= Time 1, learners were in 4th and 5th grade), and end of Spring of the first school year (= Time 2, learners were in 4th and 5th grade) and at the end of Spring in the second school year (= Time 3, learners were by that time in 5th and 6th grade). According to the Flemish curriculum, learners should have acquired all knowledge about rational numbers that is measured in our test instrument at the end of the 6th grade.

To measure learners' rational number understanding, we used the Rational Number Knowledge Test (RNKT). This test was already used and validated in previous research investigating the relation between learners' spontaneous focusing on quantitative relations and their rational number understanding (Van Hoof, Degrande, et al., 2016). Table 1 displays examples of items for all three aspects.

Density	Size	Operations
How many numbers are there between 0.74 and 0.75?	Which is the larger number? 0.36 or 0.5	Is 21 : 0.7 bigger or smaller than 21?
What is the smallest possible fraction?	Order the following numbers from small to large 4/7 2/6 5/10	2/6 + 1/3 = …

Table 1: Examples of both fraction and decimal test items from the Rational Number Knowledge Test per aspect.

ANALYSIS

Data were analyzed using latent transition analysis (LTA). LTA is a longitudinal data analysis technique designed to detect unknown groups of participants and to model change in group membership over time through transition probabilities (Nylund, 2007). In our study, the groups can be interpreted as developmental states in learners' conceptual change, characterized by a specific answer pattern. Our LTA analyses were conducted in the statistical software Mplus version 7.2. We estimated the model parameters using the maximum likelihood estimation with robust standard errors. We restricted the number and nature of the states to be the same over the three measurement points, reducing the number of parameters to be estimated and making it possible to compare the results across measurement points (Schneider & Hardy, 2012). There were no missing data.

RESULTS

We opted for the six-state solution, based on the lowest AIC and BIC values and because it is the simplest model that still allows to differentiate between (un)successful conceptual change in all three aspects of rational number understanding.

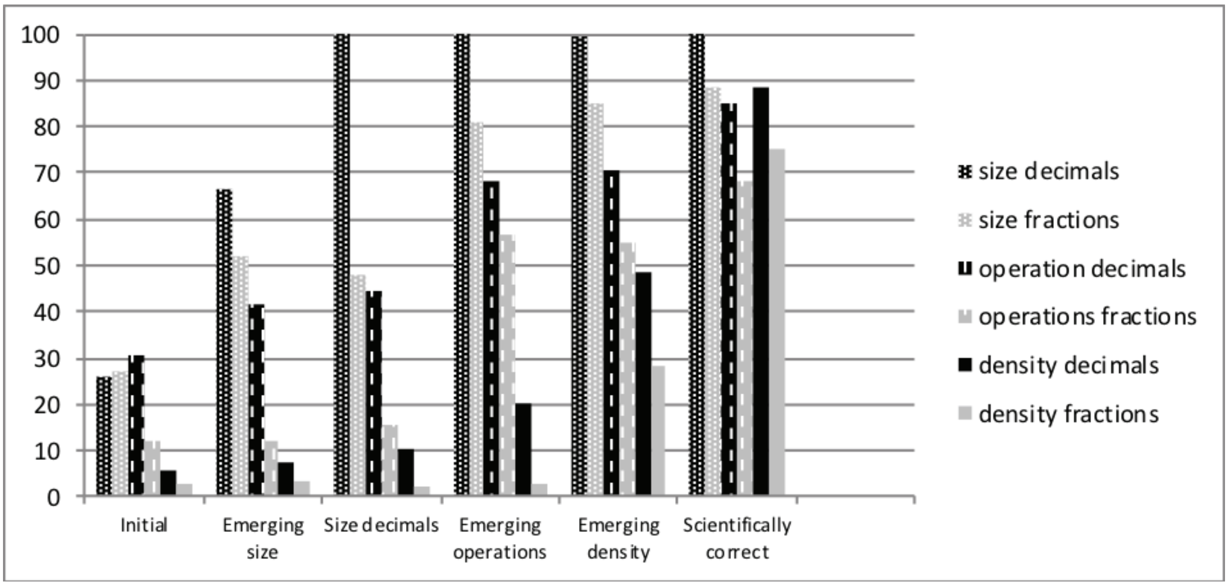

Figure 1: Accuracy levels (in %) on all aspects of the RNKT per state.

The mean accuracy scores on all subtests of the RNKT per state (see Figure 1) show, first, that learners in the '*Initial*' state are characterized by an initial natural number based understanding of rational numbers. They have a very low accuracy on all subtests, with a maximum subtest score of only 30.7% on decimal operation tasks. Second, learners in the '*Emerging size*' state have low accuracy scores on almost all subtests. Contrary to the 'Initial' state, they already have some understanding of the size of fractions (mean accuracy = 52.2%) and of decimals (mean accuracy = 66.7%). On all other subtests, they score below 50% accuracy. Third, learners in the '*Size decimals'* state are characterized by having a good understanding of the size of decimal numbers, performing almost perfectly on these items. Their mean scores on all other subtests are below 50%. Fourth, learners in the *'Emerging operations'* state have developed a good understanding of the aspect of operations. Moreover, they developed a good understanding of the size of fractions, but still have a natural number based idea of the structure of rational numbers. This is shown by their accuracy scores on decimal density tasks (mean accuracy = 20.4%), but especially on fraction density tasks (mean accuracy = 2.8%). Fifth, learners in the *'Emerging density'* state also have a good understanding of the size and operations aspect, but moreover developed already some understanding of the dense structure of rational numbers (mean accuracy decimal density tasks = 48.4% and mean accuracy fraction density tasks = 28.3%). Sixth, learners in the 'Mathematically more correct' state show a good understanding on all subtests, with a minimum subtest score of 68.1% on fraction operation tasks.

Interestingly, in every profile (except in the 'Initial' profile on size tasks) and in all three aspects of the natural number bias, learners score remarkably higher on the decimal than on the fraction tasks, indicating that understanding the size of decimals, operations with decimals, and decimals' density is easier to achieve than understanding these three aspects for the fraction counterpart.

As shown in Table 2, the number of learners in each state change over time. Both from Time 1 to Time 2 and from Time 2 to Time 3, a clear shift towards a better understanding of rational numbers is found: While half of the learners had an 'Initial' or 'Emerging size' state on Time 1, this dropped to only 13% at Time 3.

	Begin Spring Year 1			End Spring Year 1			End Spring Year 2		
	Grade4	Grade5	Total	Grade4	Grade5	Total	Grade5	Grade6	Total
Initial	42	4	46	16	2	18	7	2	9
Emerging size	40	12	52	26	13	39	13	4	17
Size decimals	22	21	43	55	23	78	9	3	12
Emerging operations	3	27	30	5	32	37	65	59	124
Emerging density	6	19	25	9	9	18	12	6	18
Mathematically more correct	0	5	5	2	9	11	7	14	21

Table 2: Number of learners in each state over time

As a second step in our LTA, we further characterized the general trend from the initial natural number based idea of a rational number ('Initial' state) to the mathematically more correct one ('Mathematically more correct' state). Therefore we had a look at the Latent Transition Probabilities (LTP) (see Table 3). Overall, the states stayed more stable from Time 1 to Time 2 compared to the stability over Time 2 to Time 3. This is not surprising given that there was less time between Time 1 and 2 than between Time 2 and 3. Further, the 'Emerging operations' state stands out as being the most stable state. Learners who are in this group at Time 1 have 89% chance of staying in this group at Time 2. In the same line, learners who have the 'Emerging operations' state at Time 2 have 94% chance of having the same state at Time 3. This suggests that once learners at the end of elementary education have developed a good understanding of the operations and size with rational numbers, they most often do not develop further and hence do not yet have a good understanding of the dense structure of rational numbers. If we take a look at the highest latent transition probabilities, as they indicate the transitions that occur most frequently, we see that from Time 1 to Time 2 learners from both the 'Initial' state and the 'Emerging size' state at Time 1 have a high chance of ending up in the 'Size decimals' at Time 2. This suggests that learners with an initial natural number based understanding of rational numbers at Time 1 first have an increased understanding of the size of decimal rational numbers. In the transition from Time 2 to Time 3, learners from both the 'Emerging size' and the 'Size decimals' state have a very high chance of ending up in the 'Emerging operations' state. This shows that learners who have an initial natural number based understanding of rational numbers, except for the size of decimal numbers, are very likely to develop an increased understanding of operations with rational numbers (both decimals and fractions) and the size of fractions in a next step, while they still have an initial natural number based understanding of the dense structure of rational numbers.

T1 \ T2	Initial	Emerging size	Size decimals	Emerging operations	Emerging density	Mathematically more correct
Initial	0.33	0.12	0.37	0.00	0.18	0.00
Emerging size	0.06	0.29	0.53	0.00	0.10	0.02
Size decimals	0.00	0.16	0.70	0.13	0.01	0.00
Emerging operations	0.00	0.00	0.00	0.89	0.04	0.07
Emerging density	0.00	0.12	0.06	0.17	0.45	0.20
Mathematically more correct	0.00	0.00	0.00	0.20	0.20	0.60
T2 \ T3	**Initial**	**Emerging size**	**Size decimals**	**Emerging operations**	**Emerging density**	**Mathematically more correct**
Initial	0.33	0.22	0.16	0.28	0.00	0.00
Emerging size	0.05	0.13	0.12	0.70	0.00	0.00
Size decimals	0.01	0.00	0.10	0.72	0.13	0.04
Emerging operations	0.00	0.00	0.00	0.94	0.00	0.06
Emerging density	0.00	0.00	0.00	0.10	0.37	0.53
Mathematically more correct	0.00	0.00	0.00	0.29	0.00	0.71

Table 3: Latent transition probabilities from Time 1 to Time 2 and from Time 2 to Time 3.

While a large group of learners who have a good understanding of operations first go through the early states of a good understanding of size, no such developmental path is found in the transition probabilities in the group of learners with (good) understanding of density. Very few learners of these qualitatively different group go through previous states. This suggests that the two states 'Emerging density' and 'Mathematically more correct' describe qualitatively different learners who understand density as opposed to the rest of the learners who do not see the dense structure of rational numbers.

CONCLUSION AND DISCUSSION

Our results add to our current theoretical understanding of the several different intermediate states going from learners' initial natural number based concept of rational numbers towards a mathematically more correct one. The finding that six different profiles can be distinguished in learners' rational number understanding shows that although all learners in our sample received similar rational number instruction, substantial individual differences could be found at every time point in learners' conceptual understanding of fractions and decimal numbers. It should be noted however that although we found several rational number understanding profiles and differences in learners' learning trajectories, we also found that the number of rational number profiles ($n = 6$) and transition paths ($n = 56$, of which only 11 were frequent) was much smaller than the number of participants in this study ($n = 201$). This indicates that learners' conceptual change from an initial to a more correct concept of rational numbers is constrained along certain patterns, and general developmental paths can be described. Based on the trends that we observed, we can characterize the development from the initial natural number based to the mathematically more correct idea of rational number as follows: First, learners develop a good understanding of the size of decimal numbers, followed by a good understanding of the size of fractions. Once learners have a good understanding of the size of rational numbers, they develop an understanding of operations with rational numbers (first decimals, then fractions). A qualitatively different group of learners also develops its understanding of the dense structure of rational numbers (first with decimals, then with fractions), without necessarily going through the profiles of good understanding of size and operations. These findings are consistent with the integrated theory of numerical development (Siegler, et al., 2011), which states that understanding the numerical sizes of fractions forms a crucial step in the understanding of fractions.

We continue with an important educational implication. From the theoretical background, we know that the process of conceptual change is gradual, time-consuming, and far from easy. Still, while instruction aimed at conceptual change in mathematics needs a lot of effort, research has shown that it can be successful under appropriate conditions. For example, curriculum designers should focus on a deep exploration and understanding of a few concepts instead of superficially covering a great amount of material (Vosniadou, 2013). The results of the present study show that a first step in learners' rational number understanding is a good understanding of the size of rational numbers. Therefore, we would suggest that instruction in the beginning explicitly focuses on the numerical size of rational numbers before introducing the more advanced content, such as operations with rational numbers.

Important to note is, finally, that the notion of natural number bias should not only be associated with its adverse effect of learners' prior knowledge on their further learning (Vamvakoussi, 2015). Using natural number knowledge acts as a facilitator too, namely in contexts that are compatible (congruent) with natural number knowledge. However, there is a need for a stronger awareness of the possible negative consequences of introducing rational numbers without an explicit attention for both the similarities and differences with natural numbers.

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