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- The weighted residual formulation and collocation of the strong form are investigated.
- Reduced Gauss--Lobatto quadrature, Greville and superconvergent points are employed.
- Convergence-order estimates for fourth-order PDEs are provided based on observations.
- The MIP Newton method greatly improves the convergence robustness.

Isogeometric analysis for nonlinear planar Kirchhoff rods: weighted residual formulation and collocation of the strong form

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Abstract

High-order shape functions used in isogeometric analysis allow the direct solution to the weighted residual formulation of the strong form, opening the door to new integration schemes (e.g. reduced Gauss–Lobatto quadrature, integration at superconvergent sites) but also to collocation approaches (e.g. using Greville or superconvergent collocation points). The goal of the present work is to compare these different methods through the application to the planar Kirchhoff rod, a fourth-order rotation-free formulation used to model slender beams under large deformations. Robustness of the geometrically-nonlinear solution is improved using the Mixed Integration Point (MIP) Newton, a method that combines the advantages of displacement and mixed formulations. Based on the observations of the convergence plots, convergence order estimates are provided for each discretization method. Advantages and weaknesses in terms of robustness and computation cost are also discussed for a set of benchmarks.

Keywords: Isogeometric analysis, planar Kirchhoff rod, weighted residual form, collocation, nonlinear deformation, convergence order

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Highlights

- NURBS-based isogeometric rotation-free nonlinear planar Kirchhoff rods are tackled.
- The weighted residual formulation and collocation of the strong form are investigated.
- Reduced Gauss–Lobatto quadrature, Greville and superconvergent points are employed.
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- The MIP Newton method greatly improves the convergence robustness.

1. Introduction

Isogeometric analysis (IGA) [1] considers an isoparametric approach for which the basis functions are usually non-uniform rational B-splines (NURBS). One of the main advantages of this approach is that the continuity of the NURBS can be augmented through the k -refinement procedure, improving the accuracy and robustness of the simulations [2]. High-order continuous shape functions also allow the direct integration of the weighted residual formulation of the strong form, opening the door to new quadrature rules as well as collocation schemes.

Isogeometric collocation has been initially investigated in [3] and a cost comparison with the Galerkin method can be found in [4]. It has been applied to elastostatics and explicit dynamics [5], phase-field models [6, 7], contact problems [8–10], Timoshenko beams [11–16], Reissner–Mindlin plates and shells [17–19], and Kirchhoff–Love plates and shells [20–22].

An important question about the collocation method is how to choose the collocation points. Greville abscissae (C-GP) are often used, but the convergence is suboptimal compared to the Galerkin method. While for even-degree bases better points have not yet been found, for odd ones, an improved convergence can be obtained using Cauchy–Galerkin points, also called superconvergent points (points where the Galerkin solution is exact) [23, 24]. For a univariate odd-degree spline, there are two superconvergent points per element, which is about two times more than the number of equations required. This overdetermined system can be solved

using the least-squares approximation at superconvergent points (LS-SP) [23]. Alternatively, the same convergence order (which is optimal for second-order problems) can be obtained using a proper subset called the cluster superconvergent points (C-CSP) [24].

The superconvergent points can also be used as evaluation sites into the weighted residual formulation of the strong form [11]. Using an odd-degree bases, the convergence is found to be optimal for second-order partial differential equations (PDEs), whereas we will show that this is not the case for fourth-order ones. Another recent alternative based on the weighted residual form is its integration via an under-integrated Gauss–Lobatto quadrature rule that does not alter the optimal convergence order [25, 26]. The number of integration points is set to be the same as the full Gauss–Legendre quadrature rule, but can even be reduced by taking advantage of the inter-element higher continuity (element extremities are part of the Gauss–Lobatto points). Moreover, at Gauss–Lobatto locations, only one Gauss–Lobatto Lagrange basis function (obtained via Lagrange extraction of NURBS) is not zero, simplifying the matrix assembly.

The goal of the present work is to investigate the weighted residual formulation and the collocation of the strong form for fourth-order PDEs. Indeed, only few works on the collocation method involve fourth-order problems [20–22] and these do not take advantage of the last improvements (e.g. C-CSP). With the advent of IGA, the interest for fourth-order formulations has been renewed, for instance in the incompressible elasticity with stream functions [27], the Cahn–Hilliard phase-field problem [28], and the Navier–Stokes equations [29]. In structural mechanics, the higher continuity allows rotation-free discretizations, simplifying the formulation and its implementation. Examples are the Kirchhoff–Love shell [30] but also **the Kirchhoff rod [31–36]**.

The planar Kirchhoff rod theory (also called the planar extensible elastica) is used to model large nonlinear deformations of slender structures for which the shear strain can be neglected, such as beams or cables. It is based on the shear-free geometrically-exact beam theory [37, 38]. For the constitutive material relation, it is possible to use the Saint Venant–Kirchhoff model as in [31], characterized by a linear relation between the second Piola–Kirchhoff stress and the Green–Lagrange strain. This model results in a nonlinear relationship between the stress resultants (axial force and moment) and strain (axial strain and curvature) [39]. Conversely, it is possible to define a linear constitutive law between the Biot stress and the Biot strain, resulting in a linear material model. This constitutive law is adopted in [32] and has the advantage that closed-form solutions to some benchmarks are known [40–43]. For this reason, we also consider this model in the present manuscript.

To solve the geometrically-nonlinear deformation, the recently developed MIP (Mixed Integration Point) Newton is used, combining the advantages of a displacement formulation (a limited computational cost) with those of a mixed stress-displacement formulation (convergence in Newton more robust). While this method was originally introduced for the Galerkin discretization [44, 45], it is extended in Section 3.6 to the strong form.

The paper is organized as follows. In Section 2, the Kirchhoff rod theory is summarized and the strong form is derived. In Section 3, the discretization of the variational and the weighted residual form are developed, and the collocation strategy is elaborated. The advantages and disadvantages of the different methods are discussed in Section 4, supported by a set of various numerical examples. Conclusions follow.

2. Rotation-free planar Kirchhoff rod formulation

2.1. Strain measurement and constitutive law

In its initial configuration, a material point of the planar beam is described by a position vector $\mathbf{r}_0 \in \mathbb{R}^2$ and its normal direction $\mathbf{e}_0 \in \mathbb{R}^2$ inclined by an angle $\theta_0 \in \mathbb{R}$ with respect to the reference configuration x . In curvilinear coordinates, the arc-length parameter is $s_0 \in \mathbb{R}$. Correspondingly, $\mathbf{r}$, $\mathbf{e}$, θ , and s represent the position vector, the normal vectors, the rotation, and the curvilinear coordinate in the current configuration of the beam, respectively. These quantities are shown in Fig. 1 and are defined as:

$$\begin{aligned} \mathbf{r}_0 &= [r_{0,x}, r_{0,y}]^T, & \mathbf{r} &= [r_x, r_y]^T, \\ \mathbf{e}_0 &= [\cos \theta_0, \sin \theta_0]^T, & \mathbf{e} &= [\cos \theta, \sin \theta]^T, \end{aligned} \quad (1)$$

where $\mathbf{r}_0 = \mathbf{r}_0(s_0)$, $\mathbf{r} = \mathbf{r}(s)$, $\mathbf{e}_0 = \mathbf{e}_0(s_0)$, $\mathbf{e} = \mathbf{e}(s)$, $\theta_0 = \theta_0(s_0)$, and $\theta = \theta(s)$. The current configuration is related to the initial one through the displacement vector $\mathbf{u} = [u_x, u_y]^T$ such that $\mathbf{r} = \mathbf{r}_0 + \mathbf{u}$.

The axial strain ϵ , shear strain γ , and curvature κ proposed by Reissner [46] but extended to the global coordinates in order to allow initial deformations [47] read:

$$\begin{aligned} \epsilon &= r'_x \cos \theta + r'_y \sin \theta - (r'_{0,x} \cos \theta_0 + r'_{0,y} \sin \theta_0), \\ \gamma &= r'_y \cos \theta - r'_x \sin \theta - (r'_{0,y} \cos \theta_0 - r'_{0,x} \sin \theta_0), \\ \kappa &= \theta' - \theta'_0, \end{aligned} \quad (2)$$

Figure 1: Beam kinematics: reference (dotted lines), initial (dashed lines), and current (full lines) configurations.

where the notation $(\cdot)'$ denotes the derivative with respect to the initial curvilinear coordinate s_0 (Lagrange formulation). The Euler–Bernoulli beam model assumes that the cross section remains planar and normal to the tangent of the elastic axis of the beam, which is equivalent to assuming that the shear γ is zero. Consequently, the axial strain ϵ and the curvature κ can be recast as:

$$\begin{aligned}\epsilon &= \|\mathbf{r}'\| - \|\mathbf{r}_0'\|, \\ \kappa &= \frac{\mathbf{r}''^T \Theta \mathbf{r}'}{\|\mathbf{r}'\|^2} - \frac{\mathbf{r}_0''^T \Theta \mathbf{r}_0'}{\|\mathbf{r}_0'\|^2},\end{aligned}\quad (3)$$

where $\|\mathbf{r}'\| = \sqrt{r_x'^2 + r_y'^2}$ is the Euclidean norm and Θ is a 90° rotation matrix given by:

$$\Theta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (4)$$

Assuming small strains, even for large displacements, the Hooke's law can be used to describe the material behavior. After integrating through the thickness, the constitutive law between the force and the strain reads [38, 39, 46]:

$$\begin{Bmatrix} N_\epsilon \\ N_\kappa \end{Bmatrix} = \begin{bmatrix} EA & 0 \\ 0 & EI \end{bmatrix} \begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix}, \quad (5)$$

where E is the Young modulus, and A and I are the area and area moment of inertia of the beam, respectively; N_ϵ and N_κ indicate the internal axial force and bending moment, respectively. The constitutive law Eq. (5) is equivalent in continuum mechanics to the linear relationship between the Biot's stress and strain, and alternative constitutive laws are detailed in [39].

2.2. Weak form

The weak formulation of the equilibrium equation is obtained from the principle of virtual work and is given for a static problem by [38]:

$$\text{find } \mathbf{u} \in \mathcal{V} : \delta W(\mathbf{r}(\mathbf{u}))(\delta \mathbf{v}) = 0 \quad \forall \delta \mathbf{v} \in \mathcal{V}, \quad (6)$$

where $\delta \mathbf{v}$ is the virtual displacement vector and $\mathcal{V} \subset H^2(\Omega_0)$ ($\Omega_0 \in (0, l_0)$) a suitable subset of the Hilbert function space $H^2(\Omega_0)$ carrying the essential boundary conditions. Omitting for simplicity the explicit dependency on the displacement $\mathbf{u}$, δW reads for the planar Kirchhoff rod:

$$\delta W = \int_{\Omega_0} (N_\epsilon \delta \epsilon + N_\kappa \delta \kappa - \delta \mathbf{v}^T \mathbf{f} - m \delta \theta) ds_0 - [\delta \mathbf{v}^T \mathbf{F} + M \delta \theta]_0^{l_0}, \quad (7)$$

where $\mathbf{f} \in \mathbb{R}^2$ and $\mathbf{F} \in \mathbb{R}^2$ are the distributed and boundary force vectors, respectively, and $m \in \mathbb{R}$ and $M \in \mathbb{R}$ are the distributed and boundary moments, respectively. The virtual strain components read:

$$\begin{aligned} \delta \epsilon &= \delta \mathbf{v}'^T \mathbf{B}_\epsilon, \\ \delta \theta &= \delta \mathbf{v}'^T \mathbf{B}_\theta, \\ \delta \kappa &= \delta \theta' = \delta \mathbf{v}^T \mathbf{B}_\theta' + \delta \mathbf{v}''^T \mathbf{B}_\theta, \end{aligned} \quad (8)$$

where

$$\mathbf{B}_\epsilon = \frac{\mathbf{r}'}{\|\mathbf{r}'\|} \quad \text{and} \quad \mathbf{B}_\theta = \frac{\Theta \mathbf{r}'}{\|\mathbf{r}'\|^2}. \quad (9)$$

The equilibrium equation (7) is geometrically nonlinear and in order to solve the problem, the Newton–Raphson scheme discussed in Section 3.6 is used. It requires the linearization of δW and reads:

$$\delta \delta W = \int_{\Omega_0} (EA \delta \epsilon \delta \epsilon + EI \delta \kappa \delta \kappa + N_\epsilon \delta \delta \epsilon + N_\kappa \delta \delta \kappa - m \delta \delta \theta) ds_0 - [M \delta \delta \theta]_0^{l_0}, \quad (10)$$

where:

$$\begin{aligned} \delta \epsilon \delta \epsilon &= \delta \mathbf{v}'^T \mathbf{B}_\epsilon \mathbf{B}_\epsilon^T \delta \mathbf{u}', \\ \delta \kappa \delta \kappa &= \delta \mathbf{v}'^T \mathbf{B}_\theta' \mathbf{B}_\theta'^T \delta \mathbf{u}' + \delta \mathbf{v}''^T \mathbf{B}_\theta \mathbf{B}_\theta'^T \delta \mathbf{u}' + \delta \mathbf{v}'^T \mathbf{B}_\theta' \mathbf{B}_\theta^T \delta \mathbf{u}'' + \delta \mathbf{v}''^T \mathbf{B}_\theta \mathbf{B}_\theta^T \delta \mathbf{u}'', \\ \delta \delta \epsilon &= \delta \mathbf{v}'^T \mathbf{G}_1 \delta \mathbf{u}', \\ \delta \delta \theta &= \delta \mathbf{v}'^T \Theta \mathbf{G}_2 \delta \mathbf{u}', \\ \delta \delta \kappa &= \delta \mathbf{v}'^T \Theta \mathbf{G}_2' \delta \mathbf{u}' + \delta \mathbf{v}''^T \Theta \mathbf{G}_2 \delta \mathbf{u}' + \delta \mathbf{v}'^T \Theta \mathbf{G}_2 \delta \mathbf{u}''. \end{aligned}$$

(11)

The matrices $\mathbf{G}_1$, $\Theta\mathbf{G}_2$, and their derivatives are detailed in Appendix A.

2.3. Strong form

Assuming now that $\mathcal{V} \subset H^4(\Omega_0)$, and integrating the weak form of Eq. (7) by part twice, one gets the strong form

$$\begin{aligned} \delta W &= \int_{\Omega_0} \left(-\delta \mathbf{v}^T \mathbf{f} + \delta \mathbf{v}'^T \left(N_\epsilon \mathbf{B}_\epsilon + N_\kappa \mathbf{B}'_\theta - m \mathbf{B}_\theta \right) + \delta \mathbf{v}''^T N_\kappa \mathbf{B}_\theta \right) ds_0 \\ &\quad - \left[\delta \mathbf{v}^T \mathbf{F} + \delta \mathbf{v}'^T M \mathbf{B}_\theta \right]_0^{l_0} \\ &= \int_{\Omega_0} \delta \mathbf{v}^T \left(-N'_\epsilon \mathbf{B}_\epsilon - N_\epsilon \mathbf{B}'_\epsilon + \left(N'_\kappa + m' \right) \mathbf{B}_\theta + \left(N'_\kappa + m \right) \mathbf{B}'_\theta - \mathbf{f} \right) ds_0 \\ &\quad + \left[\delta \mathbf{v}^T \left(N_\epsilon \mathbf{B}_\epsilon - \left(N'_\kappa + m \right) \mathbf{B}_\theta - \mathbf{F} \right) + \delta \mathbf{v}'^T \left(N_\kappa - M \right) \mathbf{B}_\theta \right]_0^{l_0}. \end{aligned} \quad (12)$$

The derivative of the stress components can be found in Appendix A. Linearization of this equation gives

$$\begin{aligned} \delta \delta W &= \int_{\Omega_0} \delta \mathbf{v}^T \left(K_{01}^{\text{Int}} \delta \mathbf{u}' + K_{02}^{\text{Int}} \delta \mathbf{u}'' + K_{03}^{\text{Int}} \delta \mathbf{u}''' + K_{04}^{\text{Int}} \delta \mathbf{u}'''' \right) ds_0 \\ &\quad + \left[\delta \mathbf{v}^T \left(K_{01}^{\text{BC}} \delta \mathbf{u}' + K_{02}^{\text{BC}} \delta \mathbf{u}'' + K_{03}^{\text{BC}} \delta \mathbf{u}''' \right) \right. \\ &\quad \left. + \delta \mathbf{v}'^T \left(K_{11}^{\text{BC}} \delta \mathbf{u}' + K_{12}^{\text{BC}} \delta \mathbf{u}'' \right) \right]_0^{l_0}, \end{aligned} \quad (13)$$

where

$$\begin{aligned} K_{01}^{\text{Int}} &= -EA \left(\mathbf{B}'_\epsilon \mathbf{B}_\epsilon^T + \mathbf{B}_\epsilon \mathbf{B}'_\epsilon{}^T \right) + EI \left(\mathbf{B}_\theta \mathbf{B}_\theta''''^T + \mathbf{B}'_\theta \mathbf{B}_\theta''^T \right) \\ &\quad - N'_\epsilon \mathbf{G}_1 - N_\epsilon \mathbf{G}'_1 + \left(N'_\kappa + m \right)' \Theta \mathbf{G}_2 + \left(N'_\kappa + m \right) \Theta \mathbf{G}'_2, \\ K_{02}^{\text{Int}} &= -EAB_\epsilon \mathbf{B}_\epsilon^T + EI \left(3\mathbf{B}_\theta \mathbf{B}_\theta''^T + 2\mathbf{B}'_\theta \mathbf{B}_\theta'^T \right) - N_\epsilon \mathbf{G}_1 + \left(N'_\kappa + m \right) \Theta \mathbf{G}_2, \\ K_{03}^{\text{Int}} &= EI \left(3\mathbf{B}_\theta \mathbf{B}_\theta'^T + \mathbf{B}'_\theta \mathbf{B}_\theta^T \right), \\ K_{04}^{\text{Int}} &= -K_{03}^{\text{BC}} = K_{12}^{\text{BC}} = EI \mathbf{B}_\theta \mathbf{B}_\theta^T, \\ K_{01}^{\text{BC}} &= EAB_\epsilon \mathbf{B}_\epsilon^T - EI \mathbf{B}_\theta \mathbf{B}_\theta''^T + N_\epsilon \mathbf{G}_1 - \left(N'_\kappa + m \right) \Theta \mathbf{G}_2, \\ K_{02}^{\text{BC}} &= -2EI \mathbf{B}_\theta \mathbf{B}_\theta'^T, \\ K_{11}^{\text{BC}} &= EI \mathbf{B}_\theta \mathbf{B}_\theta'^T + \left(N_\kappa - M \right) \Theta \mathbf{G}_2. \end{aligned}$$

(14)

Before providing the weighted residual formulation of the strong form, some preliminaries on NURBS-based IGA are introduced.

3. Numerical discretization

3.1. NURBS-based IGA

We consider that the Kirchhoff rods are represented by means of NURBS [2]. The geometry mapping $\mathbf{r}_0$ possesses a p th-degree NURBS representation when there exist $n \in \mathbb{N}$ control points $\mathbf{B}_i \in \mathbb{R}^2$, weights $w_i \in \mathbb{R}$, $i = 1, \dots, n$, and a set of knots $\Xi = \{0 = \xi_1 \leq \dots \leq \xi_{n+p+1} = 1\}$ such that:

$$\mathbf{r}_0(\xi) = \sum_{i=1}^n R_i^p(\xi) \mathbf{B}_i, \quad (15)$$

where $R_i^p(\xi)$ is the NURBS basis defined at $\xi \in (0, 1)$ by:

$$R_i^p(\xi) = \frac{N_i^p(\xi) w_i}{W^p(\xi)}, \quad W^p(\xi) = \sum_{j=1}^n N_j^p(\xi) w_j, \quad (16)$$

with $N_i^p(\xi)$ the i -th B-spline basis function defined by the Cox–De Boor recursive formula [48]:

$$N_i^0(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{otherwise,} \end{cases} \quad (17)$$

$$N_i^p(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_i^{p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1}^{p-1}(\xi), \quad \text{for } p \geq 1.$$

The derivatives up to fourth order of the NURBS basis with respect to the initial curvilinear coordinate s_0 are given in Appendix B. In the isogeometric approach, the displacement field is described by the same basis functions as the geometry, such that:

$$\mathbf{r}(\xi) = \sum_{i=1}^n R_i^p(\xi) (\mathbf{B}_i + \mathbf{u}_i), \quad (18)$$

where $\mathbf{u}_i$ is the discretized displacement.

3.2. Galerkin variational formulation of the weak form (IGA-G)

The Bubnov–Galerkin approach considers as test functions the same basis functions as the solution field:

$$\mathbf{v}(\xi) = \sum_{i=1}^n R_i(\xi) \mathbf{v}_i. \quad (19)$$

The resulting force vector reads

$$\mathbf{F}_i^{(G)} = \sum_{e=1}^{n_e} \int_{\Omega_0^e} (N_e \delta \epsilon_i + N_\kappa \delta \kappa_i - \mathbf{f} R_i - m \delta \theta_i) ds_0 - [\mathbf{F} R_i + M \delta \theta_i]_0^{l_0}, \quad (20)$$

for which the explicit dependency on ξ and p are omitted for simplicity, Ω_0^e is the domain of the element e (n_e elements in total), and

$$\begin{aligned} \delta \epsilon_i &= \mathbf{B}_\epsilon R'_i, \\ \delta \theta_i &= \mathbf{B}_\theta R'_i, \\ \delta \kappa_i &= \mathbf{B}'_\theta R'_i + \mathbf{B}_\theta R''_i. \end{aligned} \quad (21)$$

The stiffness matrix is given by

$$\begin{aligned} \mathbf{K}_{ij}^{(G)} &= \sum_{e=1}^{n_e} \int_{\Omega_0^e} (EA \delta \epsilon_i \delta \epsilon_j + EI \delta \kappa_i \delta \kappa_j + N_e \delta \delta \epsilon_{ij} + N_\kappa \delta \delta \kappa_{ij} - m \delta \delta \theta_{ij}) ds_0 \\ &\quad - [M \delta \delta \theta_{ij}]_0^{l_0}, \end{aligned} \quad (22)$$

where

$$\begin{aligned} \delta \epsilon_i \delta \epsilon_j &= R'_i \mathbf{B}_\epsilon \mathbf{B}_\epsilon^T R'_j, \\ \delta \kappa_i \delta \kappa_j &= R'_i \mathbf{B}'_\theta \mathbf{B}_\theta'^T R'_j + R''_i \mathbf{B}_\theta \mathbf{B}_\theta'^T R'_j + R'_i \mathbf{B}'_\theta \mathbf{B}_\theta^T R''_j + R''_i \mathbf{B}_\theta \mathbf{B}_\theta^T R''_j, \\ \delta \delta \epsilon_{ij} &= R'_i \mathbf{G}_1 R'_j, \\ \delta \delta \theta_{ij} &= R'_i \Theta \mathbf{G}_2 R'_j, \\ \delta \delta \kappa_{ij} &= R'_i \Theta \mathbf{G}'_2 R'_j + R''_i \Theta \mathbf{G}_2 R'_j + R'_i \Theta \mathbf{G}_2 R''_j. \end{aligned} \quad (23)$$

These equations are integrated numerically using the full Gauss–Legendre quadrature rule, and we designate this method as IGA-G-FGLE.

3.3. The weighted residual formulation of the strong form (IGA-WR)

The discretization of the strong form Eq. (12) gives the following force vector:

$$\begin{aligned} \mathbf{F}_i^{(\text{WR})} = & \sum_{e=1}^{n_e} \int_{\Omega_0^e} R_i \left(\left(-N_\epsilon \mathbf{B}_\epsilon + (N'_\kappa + m) \mathbf{B}_\theta \right)' - \mathbf{f} \right) ds_0 \\ & + \left[R_i \left(N_\epsilon \mathbf{B}_\epsilon - (N'_\kappa + m) \mathbf{B}_\theta - \mathbf{F} \right) + R'_i (N_\kappa - M) \mathbf{B}_\theta \right]_0^{l_0}. \end{aligned} \quad (24)$$

Similarly, the discretization of the linearized functional Eq. (13) leads to the stiffness matrix:

$$\begin{aligned} \mathbf{K}_{ij}^{(\text{WR})} = & \sum_{e=1}^{n_e} \int_{\Omega_0^e} R_i \left(K_{01}^{\text{Int}} R'_j + K_{02}^{\text{Int}} R''_j + K_{03}^{\text{Int}} R'''_j + K_{04}^{\text{Int}} R''''_j \right) ds_0 \\ & + \left[R_i \left(K_{01}^{\text{BC}} R'_j + K_{02}^{\text{BC}} R''_j + K_{03}^{\text{BC}} R'''_j \right) + R'_i \left(K_{11}^{\text{BC}} R'_j + K_{12}^{\text{BC}} R''_j \right) \right]_0^{l_0}. \end{aligned} \quad (25)$$

3.3.1. Reduced Gauss–Lobatto quadrature rule (IGA-WR-RGLo)

In order to integrate numerically these equations, the classical Gauss–Legendre quadrature rule with $p + 1$ integration points can be used. Alternatively, it has been shown that employing $p + 1$ Gauss–Lobatto points provides almost identical results, whereas the equations are under-integrated (The Gauss–Legendre and Gauss–Lobatto quadrature rules give exact integrals of polynomials up to degree $2n_{\text{int}} - 1$ and $2n_{\text{int}} - 3$ respectively, where n_{int} is the number of integration points). As shown in [26, 49], one Gauss–Lobatto point is positioned at each extremity of the element, such that if the inter-element continuity is at least C^2 or C^4 for second or fourth-order problems, respectively, only one evaluation (instead of two) at each element intersection is required. As a consequence, the number of integration points per element is reduced to p , which is smaller than the classical full Gauss–Legendre rule ($p + 1$). Another advantage of those points is that if the NURBS are described in terms of Gauss–Lobatto Lagrange polynomials (obtained via Lagrange extraction [26, 49]), there is only one nonzero Gauss–Lobatto Lagrange shape function per Gauss–Lobatto point. It means that during the force and stiffness assembly, the loop over the elements and the one over the integration points can be combined, reducing the computational cost. We designate this method by IGA-WR-RGLo and we refer the reader to [26, 49] for the implementation details.

Note that in contrast to the Galerkin variational formulation (Eq. 22), the weighted residual formulation (Eq. 25) might not be symmetric due to numerical

integration approximations. As symmetry improves the computation efficiency, the symmetry of the stiffness matrix can be restored via the ultra-weak formulation, and following the same procedure as the one derived in [26], one gets

$$K_{ij}^{(\text{WR-RGLo Symmetric})} = \frac{K_{ij}^{(\text{WR-RGLo})} + K_{ji}^{(\text{WR-RGLo})}}{2}. \quad (26)$$

3.3.2. Integration at the superconvergent sites (IGA-WR-SP)

Finally, for odd-degree bases, it has been proposed in [50] to use the Cauchy–Galerkin points also called the superconvergent points which were initially derived for the collocation method [23, 51]. There are two superconvergent points per element for odd-degree bases (see Table 1) and the quadrature weights are set to 1. For even-degree bases, there are 3 points $\{-1, 0, 1\}$ corresponding to the 3 points Gauss–Lobatto quadrature rule, so the weights are $\{1/3, 4/3, 1/3\}$. The goal here is not to try to recover exactly the integrals, but to perform the evaluation at the sites where the residual between the Galerkin discretized solution and the exact solution is zero. For this reason, Eq. (26) cannot be used. The advantage of this method is that only 2 or 3 integration points per element are required, whatever p , and optimal convergence is recovered for second-order problems and odd-degree bases. To the best of the authors’ knowledge, these points have not been used for fourth-order problems so far, and we will show in Section 4 that with this method, the convergence is $p - 1$ for odd-degree bases which is suboptimal. We refer to this method as IGA-WR-SP. Note that for even-degree bases, as the 3 points Gauss–Lobatto quadrature rule can integrate polynomials up to $2n_{\text{int}} - 3 = 3$, the minimum convergence order is $\min\{p + 1, 2n_{\text{int}} - 3 + 1 = 4\}$.

3.4. Collocation of the strong form (IGA-C)

The idea of the collocation method is to use as test functions Dirac functions δ such that

$$\delta v(\xi) = \sum_{i=1}^n \delta(\xi - \hat{\xi}_i), \quad (27)$$

where $\hat{\xi}_i$ are collocation point coordinates defined in Section 3.4.2.

3.4.1. Collocation strategy

For a univariate NURBS, n collocation equations are required. It means that if the problem is of second-order type, $n - 2$ equations are provided from collocating

Figure 2: For each collocation point (CP), the number of empty and full circle portions indicates the number of equations collocated inside and at the contour of the domain, respectively.

the interior domain, and 1 equation collocates each extremity. For fourth-order problems, there are $n - 4$ equations inside and 2 at each boundary. In the planar Kirchhoff rod theory, a second and a fourth-order problem is coupled. To this end, we follow the same strategy as the one developed for the Kirchhoff shell in [22] and is presented in Fig. 2.

Inside the domain, the force and stiffness matrices read:

$$\begin{cases} F_k^{(\text{CInt})} = \left((-N_\epsilon \mathbf{B}_\epsilon + (N'_\kappa + m) \mathbf{B}_\theta)' - \mathbf{f} \right) \Big|_{\hat{\xi}_i}, & i = \{3, \dots, n-2\}, \\ K_{kl}^{(\text{CInt})} = (K_{01}^{\text{Int}} R'_j + K_{02}^{\text{Int}} R''_j + K_{03}^{\text{Int}} R'''_j + K_{04}^{\text{Int}} R''''_j) \Big|_{\hat{\xi}_i}, & k = 2i + \{-1, 0\}, \\ & l = 2j + \{-1, 0\}. \end{cases} \quad (28)$$

At the boundary, the sum of a prescribed shear and membrane force results in the two equations:

$$\begin{cases} F_k^{(\text{CBC})} = (N_\epsilon \mathbf{B}_\epsilon - (N'_\kappa + m) \mathbf{B}_\theta - \mathbf{F}) \Big|_{\hat{\xi}_i}, & i = \{1, n\}, \\ K_{kl}^{(\text{CBC})} = (K_{01}^{\text{BC}} R'_j + K_{02}^{\text{BC}} R''_j + K_{03}^{\text{BC}} R'''_j) \Big|_{\hat{\xi}_i}, & k = 2i + \{-1, 0\}, \\ & l = 2j + \{-1, 0\}. \end{cases} \quad (29)$$

At the second or penultimate collocation point, one pair of equations arises from a prescribed rotation or moment

$$\begin{cases} F_k^{(\text{CBC})} = (N_\kappa - M) \Big|_{\hat{\xi}_i}, & i = \{1, n\}, \\ K_{kl}^{(\text{CBC})} = (EI \delta \kappa_j) \Big|_{\hat{\xi}_i}, & \text{if } i = 1, k = 3 \text{ else } k = n-4, \\ & l = 2j + \{-1, 0\} \end{cases} \quad (30)$$

whereas the second equation comes from combining the two interior equations. One possibility is to consider only the membrane terms, after projecting the equa-

tions along the beam tangent, and reads

$$\begin{cases} F_k^{(\text{CInt})} = \left(-N'_\epsilon - (N'_\kappa + m) \frac{\mathbf{r}''^T \mathbf{B}_\theta}{\|\mathbf{r}'_0\|} - \mathbf{f}^T \mathbf{B}_\epsilon \right) \Big|_{\hat{\xi}_i}, & i = \{2, n-1\}, \\ K_{kl}^{(\text{CInt})} = \left(-\mathbf{f}^T \mathbf{G}_1 R'_j - EA \delta \epsilon'_j - EI \left(\frac{\mathbf{r}''^T}{\|\mathbf{r}'_0\|} \mathbf{B}_\theta \delta \kappa'_j \right. \right. & k = 2i, \\ \quad \left. \left. + (N'_\kappa + m) \left(\frac{\mathbf{B}_\theta^T}{\|\mathbf{r}'_0\|} R'_j + \mathbf{r}''^T \Theta \mathbf{G}_3 R'_j \right) \right) \right) \Big|_{\hat{\xi}_i}, & l = 2j + \{-1, 0\}, \end{cases} \quad (31)$$

where $\mathbf{G}_3$ is detailed in Appendix A. However, this form is not convenient, because the stiffness matrix becomes dependent on the external force, which requires specific adaptations of the algorithms. For this reason, instead of using a projection, we prefer here to average the two equations, such that

$$\begin{cases} F_k^{(\text{CBC})} = \mathbf{a} \left(N_\epsilon \mathbf{B}_\epsilon - (N'_\kappa + m) \mathbf{B}_\theta - \mathbf{F} \right) \Big|_{\hat{\xi}_i}, & i = \{2, n-1\}, \\ K_{kl}^{(\text{CBC})} = \mathbf{a} \left(K_{01}^{\text{BC}} R'_j + K_{02}^{\text{BC}} R'_j + K_{03}^{\text{BC}} R'_j \right) \Big|_{\hat{\xi}_i}, & k = 2i, \\ & l = 2j + \{-1, 0\}, \end{cases} \quad (32)$$

where $\mathbf{a} = [0.5, 0.5]^T$.

3.4.2. Choice of the collocation points

A common choice of collocation sites are Greville abscissae (C-GP) [3] defined by the mean location of $p-1$ consecutive knots:

$$\hat{\xi}_i = \frac{1}{p} \sum_{k=1}^p \xi_{i+k}; \quad i = 1, \dots, n. \quad (33)$$

However, as shown in Table 2, the convergence with these points is not optimal. For odd-degree bases, some recent works suggest instead the use of superconvergent points [23, 51], which are reported in Table 1. There are more superconvergent points than required and therefore a subset that covers every function can be used [24, 51]. For second-order problems, there are two possibilities so far: every other point (C-ASP) [51], resulting to a convergence of $\mathcal{O}(h^p)$, and clustered superconvergent points (C-CSP) [24] for which the convergence order is $\mathcal{O}(h^{p+1})$ (the optimal one). While C-ASP have been used for second and fourth-order problems, to the best of the authors' knowledge, C-CSP have only been investigated for second-order PDEs. We will show here that for fourth-order equations, the convergence is suboptimal. Nevertheless, it is the best convergence order that has been obtained so far for collocating fourth-order problems.

Table 1: Location of the superconvergent points on the element interval $[-1, 1]$ for a univariate NURBS of degree p and its first-fourth derivatives.

p	R	R'	R''	R'''	R''''
1	$-1, 1$	0	$\emptyset$	$\emptyset$	$\emptyset$
2	$-1, 0, 1$	$-1, 1$	0	$\emptyset$	$\emptyset$
3	$\pm\sqrt{225 - 30\sqrt{30}}/15$	$-1, 0, 1$	$\pm 1/\sqrt{3}$	0	$\emptyset$
4	$-1, 0, 1$	$\pm\sqrt{225 - 30\sqrt{30}}/15$	$-1, 0, 1$	$\pm 1/\sqrt{3}$	0
5	± 0.5049185675126533	$-1, 0, 1$	$\pm\sqrt{225 - 30\sqrt{30}}/15$	$-1, 0, 1$	$\pm 1/\sqrt{3}$
6	$-1, 0, 1$	± 0.5049185675126533	$-1, 0, 1$	$\pm\sqrt{225 - 30\sqrt{30}}/15$	$-1, 0, 1$
7	± 0.503221894597504	$-1, 0, 1$	± 0.5049185675126533	$-1, 0, 1$	$\pm\sqrt{225 - 30\sqrt{30}}/15$

3.5. A-priori error estimation: convergence order

In order to validate a numerical implementation, the convergence of the approximated solution with respect to the exact one under h -refinement is plotted and the convergence orders are compared to a-priori error estimates. The convergence can be analyzed for instance with respect to the L^2 ($L^2 \equiv H^0$) error norm, defined by [29]:

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} = \left(\int (\mathbf{u} - \mathbf{u}_h)^2 d\Omega \right)^{1/2}. \quad (34)$$

When considering PDEs of order $2m$ with $m \geq 1$, the error in the norm L^2 for a C^{p-1} -continuous NURBS basis can be estimated as:

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}^2 \leq Ch^\beta \|\mathbf{u}\|_{H^r(\Omega)}, \quad \mathbf{u} \in H^r(\Omega_0), \quad (35)$$

where h is the characteristic mesh size of the elements, and C a constant independent of $\mathbf{u}$ and h . The regularity r is assumed in this manuscript to be infinite, contrary to the Kirchhoff–Love plate where the regularity is governed by the shear singularity at the corners [22]. β gives a lower bound for the convergence order estimate, and is provided in Table 2 for the different discretization methods.

Note finally that although a-priori error estimates have been derived for linear problems, it has been observed that they are also verified for nonlinear problems [32, 52], justifying their use in the present manuscript.

3.6. The MIP Newton

For geometrically-nonlinear structural problems, it has been observed that the Newton method is more robust and allows larger steps for mixed (stress-displacement) formulations, compared to pure displacement formulations [53].

Table 2: For the different discretization procedures and PDE order $2m$, number of integration/evaluation points per element (n_{int}) and its a-priori error convergence order β in the norm L^2 . Only the convergence orders associated to a reference []* have been mathematically proved, the other ones are based on observations. The convergence orders associated to [0] are so far only investigated in this manuscript, to the best of the authors' knowledge. $\{-1\}$ indicates that if the inter-element continuity is at least C^{2m} , n_{int} can be reduced by one.

	n_{int}	β			
		$m = 1$		$m = 2$	
		p odd	p even	p odd	p even
IGA-G-FGLc	$p + 1$	$p + 1$ [29]*		$\min\{p + 1, 2p - 2\}$ [29]*	
IGA-WR-RGLo	$p + 1\{-1\}$	$p + 1$ [26]		$p + 1$ [0]	
IGA-WR-SP	2 (p odd), $3\{-1\}$ (p even)	$p + 1$ [50]	$\max\{3, p\}$ [0]	$p - 1$ [0]	$\max\{4, p - 2\}$ [0]
IGA-C-GP	1	$p - 1$ [3]*	p [3]*	$p - 3$ [20, 51]	$p - 2$ [20, 22, 51]
IGA-C-ASP	1	p [24, 51]	$\emptyset$	$p - 2$ [22, 51]	$\emptyset$
IGA-C-CSP	1	$p + 1$ [24]	$\emptyset$	$p - 1$ [0]	$\emptyset$

However, mixed formulations require also the interpolation of the stress components, increasing the cost of the stiffness matrix assembly. In [44], in the displacement formulation, it has been proposed to relax the constitutive equations at each integration point, such that one gets the advantage of the displacement formulation (almost no extra cost) with the advantage of the mixed formulation (better convergence with the Newton method). This new method is called the MIP (Mixed Integration Point) Newton, requires small implementation efforts, and has been successfully used for Reissner beams [44], isogeometric solid shells [45], isogeometric Koiters theory for buckling analysis of composite shells [54] and pantographic discrete and homogeneous lattices [36].

We emphasize that relaxing the constitutive equation at each integration point consists in setting the stress components of the weak form (N_ϵ and N_κ in Eq. (10)) as independent variables with respect to the displacement field during the Newton iteration only. The constitutive law is recovered at convergence. For the strong form (Eq. (13)), at the integration points inside the domain, the components N'_ϵ , N''_ϵ , N'_κ and N''_κ are independent, whereas at the boundary those are N_ϵ , N_κ and N'_κ .

4. Numerical results

In order to validate the formulation and the implementation of the planar Kirchhoff rod, a set of static extensible-elastica classical problems, for which closed-form solutions are known is investigated. All simulations in this work are performed with an in-house *MATLAB* code that makes use of some functionalities

of the open source NURBS toolbox [55, 56].

4.1. Single-patch Kirchhoff rods

In this section, the following properties are used for the Kirchhoff rods; Young's modulus $E = 200$ GPa, cross-sectional area $A = 4 \times 10^{-4} \text{ m}^2$ and area moment of inertia $I = 1.33 \times 10^{-8} \text{ m}^4$, initial length of the beam $l_0 = 1$ m, and radius (if curved) $R_0 = 2/\pi$ m. The different tests are described first, and the convergence curves are discussed next.

4.1.1. Straight beam under a sinusoidal loading

Before considering nonlinear deformations, a simple second and fourth-order problem is addressed. **The problem we propose is a straight beam pin-pin supported (at both extremities longitudinal and transversal displacements are locked).** The beam is under a sinusoidal loading such that

$$f_x = f_y = \sin \pi \frac{s_0}{l_0}. \quad (36)$$

As the deformations are assumed small and the beam is straight, the membrane and bending deformations are not coupled, resulting to an uncoupled second and fourth-order problem. The exact displacements are

$$\begin{aligned} u_x(s_0) &= -\frac{1}{EA} \left(\frac{l_0}{s_0 \pi} \right)^2 \sin \pi \frac{s_0}{l_0}, \\ u_y(s_0) &= +\frac{1}{EI} \left(\frac{l_0}{s_0 \pi} \right)^4 \sin \pi \frac{s_0}{l_0}. \end{aligned} \quad (37)$$

4.2. Buckling of a pinned-roller beam

This next example is also a straight simply-supported beam, but under an axial compression applied at its extremities (Fig. 5a). The deformation are not considered as infinitely small, and the buckling regime is investigated (Fig. 5a). As a consequence, the axial and bending deformations are now coupled. The applied load is $P = 1.4P_c$ with:

$$P_c = \frac{\pi^2 EI}{l_0^2}. \quad (38)$$

The geometry is perfect and in order to follow a stable path, an additional sinusoidal loading ($f_y = P' \sin \pi \frac{s_0}{l_0}$, $P' \ll P$) is applied until the bifurcation point is

Figure 3: Clamped-clamped straight beam under a sinusoidal axial loading: relative L^2 -error in terms of the normalized mesh size h_0 (averaged element length over the total beam length) and for different basis degrees (a-d).

passed ($P' = 0$ if $P > P_c$). An analytical closed-form solution for the displacement exists and can be found in [40, 41]. It reads

$$u_x(s_0) = -\frac{1}{\sqrt{P_i}} \sqrt{1 - P_a + 2P_a q^2} \left(\frac{2 - P_a + 2P_a q^2}{1 - P_a + 2P_a q^2} (K(\psi, m) - F(m)) \right. \\ \left. - 2(E(\psi, m) - E(m)) + \frac{2P_a q^2 \sin \psi \cos \psi \sqrt{1 - m \sin^2 \psi}}{1 - P_a + 2P_a q^2 - P_a q^2 \sin^2 \psi} \right), \quad (39)$$

$$u_y(s_0) = \frac{2}{\sqrt{P_i}} \frac{q(1 - P_a + 2P_a q^2) \sqrt{1 - P_a + P_a q^2} \cos \psi}{1 - P_a + 2P_a q^2 - P_a q^2 \sin^2 \psi},$$

Figure 4: Simply-supported straight beam under a sinusoidal transverse loading: relative L^2 -error in terms of the normalized mesh size and for different basis degrees p (a-d).

where

$$P_a = \frac{P}{EA}, \quad P_i = \frac{P}{EI} \quad \text{and} \quad m = q^2 \frac{1 + P_a q^2}{1 - P_a + 2P_a q^2}. \quad (40)$$

The constant q , independent of s_0 , is found by solving the equation

$$\frac{2F(m)}{\sqrt{1 - P_a + 2P_a q^2}} = l_0 \sqrt{P_i}, \quad (41)$$

and the angle $\psi = \psi(s_0)$ by solving

$$K(\psi, m) - F(m) = s_0 \sqrt{P_i} \sqrt{1 - P_a + 2P_a q^2}, \quad (42)$$

Figure 5: Buckling of a pinned-roller beam (a): relative L^2 -error in terms of the normalized mesh size and for different basis degrees p (b-e). Arrows indicate that the simulations does not converge for coarser meshes.

where $K(\psi, m)$ and $F(m)$ are the incomplete and the complete elliptical integral of first kind, respectively, and $E(\psi, m)$ and $E(m)$ are the incomplete and the complete elliptical integral of second kind, respectively.

4.3. Clamped arc under a transversal tip load

A clamped arc of radius R_0 under a concentrated tip load ($P = 3EI/R_0^2$) is used to test the formulation with an initial deformation (Fig. 6a). The closed-form

solution for this problem is derived in [32] and reads

$$\begin{aligned} u_x(s_0) &= \int_0^{s_0} r'_x ds_0 - r_{0,x} \\ &= \int_0^\theta \left(\cos \theta + \frac{P}{EA} \cos^2 \theta \right) \frac{ds_0}{d\theta} d\theta - R_0 \sin \theta_0, \\ u_y(s_0) &= \int_0^{s_0} r'_y ds_0 - r_{0,y} \\ &= \int_0^\theta \left(\sin \theta + \frac{P}{EA} \cos \theta \sin \theta \right) \frac{ds_0}{d\theta} d\theta - R_0(1 - \cos \theta_0), \end{aligned} \quad (43)$$

where:

$$\frac{d\theta}{ds_0} = \sqrt{\frac{1}{R_0^2} + \frac{2P}{EI} (\cos \theta_l - \cos \theta) + \frac{P^2}{EAEI} (\cos^2 \theta_l - \cos^2 \theta)}. \quad (44)$$

In the deformed configuration, the angles θ_l at the tip and θ at the original curvilinear coordinate s_0 are determined by solving the equations

$$\begin{aligned} l_0 &= \int_{\Omega_0} ds_0 = \int_0^{\theta_l} \frac{ds_0}{d\theta} d\theta, \\ s_0 &= \int_0^{s_0} ds_0 = \int_0^\theta \frac{ds_0}{d\theta} d\theta. \end{aligned} \quad (45)$$

4.4. Clamped arc under a transversal tip moment

We consider a clamped quarter circle beam subjected to a moment M applied at the tip (until a straight beam is obtained Fig. 7a). This problem is of pure-bending type ($\epsilon = 0$) and the curvature remains homogeneous along the beam. The applied moment to obtain the final configuration reads:

$$M = -\frac{EI}{R_0} = -\frac{\pi EI}{2l_0}. \quad (46)$$

Since κ is a constant independent of s_0 , the exact formulation is a second-order problem. The displacement is given by:

$$\begin{aligned} u_x(s_0) &= -\left(\frac{2l_0}{\pi} \sin \left(\frac{\pi s_0}{2l_0} \right) - s_0 \right), \\ u_y(s_0) &= -\frac{2l_0}{\pi} \cos \left(\frac{\pi s_0}{2l_0} \right). \end{aligned} \quad (47)$$

Figure 6: Clamped arc under a transversal tip load (a): relative L^2 -error in terms of the normalized mesh size and for different basis degrees p (b-e).

4.5. Discussion on the convergence plots

4.5.1. Convergence orders

We first focus on the convergence curves of the straight beam under a sinusoidal loading for which the membrane and bending problems are uncoupled, and are shown in Figs. 3 and 4, respectively. It is observed that all the convergence

Figure 7: Straight beams under a transversal tip moment (a): relative L^2 -error in terms of the normalized mesh size and for different basis degrees p (b-e).

orders are in agreement with the estimated ones reported in Table 2. For those which have not been reported yet in the literature, Table 2 is completed by the convergence orders observed in these plots (highlighted in table by the reference [0]), and the new contributions are the following:

- The convergence for IGA-WR-RGLo is optimal for fourth-order PDEs ($m = 2$), as it was already the case for the second-order ones ($m = 1$).
- The convergence for even p and IGA-WR-SP is found to be the same as the convergence of IGA-C-GP, except for the membrane ($m = 1$) and bending ($m = 2$) problems with $p = 2$ and $p = 4$, respectively where IGA-WR-SP is found to converge faster. This is explained by the fact that for even p , IGA-WR-SP corresponds to the 3 points Gauss–Lobatto quadrature rule which can integrate polynomials up to $2n_{\text{int}} - 3 = 3$ and gives a minimum convergence order of $\min\{p + 1, 2n_{\text{int}} - 3 + 1 = 4\}$.
- For odd-degree bases and fourth-order problems, both IGA-WR-SP and IGA-C-CSP converge at an order of $p - 1$, which is better than the use of IGA-C-GP ($p - 3$) or IGA-C-ASP ($p - 2$), but it is not the optimal one ($p + 1$). This is unexpected, as the optimal convergence order with this method is obtained for second-order problems, and could be the topic of future investigations.
- For all convergence orders concerning IGA-WR-SP and IGA-C, it is observed that $\beta_{m=1} = \beta_{m=2} - 2$ and a proper explanation could be also the topic of further investigations.

From these plots, we also conclude that for odd-degree bases, whatever the problem order, it is preferable to use IGA-C-CSP over IGA-C-GP or IGA-C-ASP, as the convergence is better for a same number of evaluation points. For this reason, these two methods (IGA-C-GP and IGA-C-ASP) won't be considered for the other tests.

Focusing on the two next fourth-order problems (the buckled beam Fig. 5 and the arc with a tip load Fig. 6), the convergence orders are found in agreement with the previous comments, even if the curve slopes are less marked. This can be explained by the fact that the problems are nonlinear, coupled, and more complex, such that a finer meshes is required to find smooth convergence. However, as the mesh is refined, the numerical accuracy decreases, and when the machine precision error becomes larger than the discretized solution error, noise and oscillations occur in the convergence curves, as observed for really fine meshes in the different convergence plots. **This phenomenon only appears for IGA-C and IGA-WR due to the fact that these methods require fourth-order derivatives, contrary to IGA-G that require only second-order ones. Indeed, the derivatives are computed iteratively and as a consequence, the numerical error is accumulated,**

especially when the number of element is large. This justify why IGA-G-FGL_e and IGA-WR-RGL_o overlap, except for really fine meshes. We also note that this phenomenon is not a locking phenomenon due to the fact that it is also present for the pure bending test of the straight beam under a sinusoidal loading (Fig. 4).

For the arc under a tip moment (Fig. 7), the exact curvature is constant along the length of the beam, such that the exact formulation (not the discretized one) is a second-order problem. This means that the superconvergent points should be selected accordingly. As shown in Fig. 7b-e, all the convergence curves are found to be in agreement with what is already reported in Table 2 for second-order problems, validating this example.

4.5.2. Robustness

It is important to note that all the nonlinear results presented here are performed using the MIP Newton method. Its relevance is demonstrated by the fact that none of the collocated nonlinear problems converge when less robust algorithms are used for IGA-C. However, this method does not solve all the issues, since for the buckled beam (Fig. 5), it is observed that IGA-C converges only for thin meshes. The buckling is a local phenomenon, really sensitive to imperfections, and IGA-C is not able to capture the full picture. The same comment applies to IGA-WR-SP, even if the convergence is slightly better. This could be a drawback of these methods that are not based on exact integrals.

4.5.3. Computation time

In order to investigate the computation cost, we focus on the configuration of the arc under a tip load and the convergence curves versus the computation time are plotted in Fig. 8. The main comment is that for a given error level, IGA-G-FGL_e and IGA-WR-RGL_o are much faster than IGA-WR-SP and IGA-C. Between IGA-G-FGL_e and IGA-WR-RGL_o, IGA-G-FGL_e is slightly faster (in our implementation). However, if a really high accuracy is required, IGA-WR-RGL_o reaches the machine precision before IGA-G-FGL_e, and so is less accurate. In conclusion, for the present problem, the weighted residual form and the collocation method doesn't bring any benefit in terms of computation time. These methods only become advantageous for two- and mostly for three-dimensional problems where the number of integration points increases as n_{int}^d (d being the problem dimension).

Figure 8: Clamped arc under a transversal tip load (a): relative L^2 -error in terms of the computation time t (s) and for different basis degrees p (b-e). Results obtained with an *Intel Core i7 4720HQ* and taking advantage of sparse matrices.

5. Conclusions

In the present paper, a fourth-order rotation-free formulation, the planar Kirchhoff rod, is investigated to compare several discretization techniques, namely the Galerkin method using the full Gauss–Legendre quadrature rule (IGA-G-FGL), the weighted residual form using a reduced Gauss–Lobatto quadrature rule (IGA-WR-RGL), the weighted residual form integrated at superconvergent sites (IGA-WR-SP), and the collocation method with Greville points (IGA-C-GP), alternative superconvergent points (IGA-C-ASP) and cluster superconvergent points (IGA-CSP). It is found that for IGA-C and IGA-WR-SP, the convergence order estimates for fourth-order PDEs are the ones for second-order PDEs minus two. Consequently, the use of superconvergent points (IGA-WR-SP or IGA-C-CSP)

does not provide the optimal convergence for odd-degree bases and fourth-order problems, contrary to what was observed for second-order PDEs.

While for this one-dimensional problem, the best discretization method in terms of the ratio between accuracy and computational time is IGA-G-FGL_e, for higher-dimensions, the difference in the number of evaluation sites between IGA-G-FGL_e and the other discretization methods is more important, making these alternatives attractive as already reported for second-order problems [24, 26, 50]. Taking care of the limitations addressed in this manuscript (i.e. robustness), these new discretization procedures applied to higher-dimensional fourth-order problems such as the Kirchhoff–Love shell [22] could be the topic of future research.

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Appendix A. Strain and stress derivatives

The axial strain ϵ , the curvature κ and their derivatives are given by

$$\begin{aligned}\epsilon &= \|\mathbf{r}'\| - \|\mathbf{r}'_0\|, \\ \epsilon' &= \mathbf{B}_\epsilon^T \mathbf{r}'' - \mathbf{R}_{0\epsilon} \mathbf{r}''_0, \\ \kappa &= \mathbf{B}_\theta^T \mathbf{r}'' - \mathbf{R}_{0\theta} \mathbf{r}''_0, \\ \kappa' &= \mathbf{B}_\theta'^T \mathbf{r}'' + \mathbf{B}_\theta^T \mathbf{r}''' - \left(\mathbf{R}_{0\theta}' \mathbf{r}''_0 + \mathbf{R}_{0\theta} \mathbf{r}'''_0 \right), \\ \kappa'' &= \mathbf{B}_\theta'^T \mathbf{r}''' + 2\mathbf{B}_\theta''^T \mathbf{r}'' + \mathbf{B}_\theta'^T \mathbf{r}''' - \left(\mathbf{R}_{0\theta}' \mathbf{r}'''_0 + 2\mathbf{R}_{0\theta}'' \mathbf{r}''_0 + \mathbf{R}_{0\theta}' \mathbf{r}'''_0 \right).\end{aligned}\tag{A.1}$$

The variation of these quantities reads

$$\begin{aligned}\delta\epsilon &= \delta\mathbf{v}'^T \mathbf{B}_\epsilon, \\ \delta\epsilon' &= \delta\mathbf{v}'^T \mathbf{B}_\epsilon' + \mathbf{v}''^T \mathbf{B}_\epsilon, \\ \delta\theta &= \mathbf{v}'^T \mathbf{B}_\theta, \\ \delta\kappa &= \delta\theta' = \mathbf{v}'^T \mathbf{B}_\theta' + \mathbf{v}''^T \mathbf{B}_\theta, \\ \delta\kappa' &= \mathbf{v}'^T \mathbf{B}_\theta'' + 2\mathbf{v}''^T \mathbf{B}_\theta' + \mathbf{v}'''^T \mathbf{B}_\theta, \\ \delta\kappa'' &= \mathbf{v}' + 3\mathbf{v}''^T \mathbf{B}_\theta'' + 3\mathbf{v}'''^T \mathbf{B}_\theta' + \mathbf{v}''''^T \mathbf{B}_\theta,\end{aligned}\tag{A.2}$$

where

$$\begin{aligned}
 B_\epsilon &= \frac{\mathbf{r}'}{\|\mathbf{r}'\|}, \\
 B'_\epsilon &= G_1 \mathbf{r}'', \\
 B_\theta &= \frac{\Theta \mathbf{r}'}{\|\mathbf{r}'\|^2}, \\
 B'_\theta &= \Theta G_2 \mathbf{r}'', \\
 B''_\theta &= \Theta G'_2 \mathbf{r}'' + \Theta G_2 \mathbf{r}''', \\
 B'''_\theta &= \Theta G''_2 \mathbf{r}'' + 2\Theta G'_2 \mathbf{r}''' + \Theta G_2 \mathbf{r}'''.
 \end{aligned} \tag{A.3}$$

Moreover

$$\begin{aligned}
 G_1 &= \frac{I - \mathbf{r}' \mathbf{r}'^T / \|\mathbf{r}'\|^2}{\|\mathbf{r}'\|}, \\
 G'_1 &= -\frac{\mathbf{r}' \mathbf{r}'^T + \mathbf{r}'' \mathbf{r}'^T + [I - 3\mathbf{r}' \mathbf{r}'^T / \|\mathbf{r}'\|^2] (\mathbf{r}'^T \mathbf{r}'')}{\|\mathbf{r}'\|^3}, \\
 \Theta G_2 &= \frac{\Theta [I - 2\mathbf{r}' \mathbf{r}'^T / \|\mathbf{r}'\|^2]}{\|\mathbf{r}'\|^2}, \\
 \Theta G'_2 &= -2\Theta \frac{\mathbf{r}' \mathbf{r}''^T + \mathbf{r}'' \mathbf{r}'^T + [I - 4\mathbf{r}' \mathbf{r}'^T / \|\mathbf{r}'\|^2] (\mathbf{r}''^T \mathbf{r}')}{\|\mathbf{r}'\|^4}, \\
 &= -2 \frac{\mathbf{r}' \mathbf{r}''^T \Theta + \Theta^T \mathbf{r}'' \mathbf{r}'^T + [I - 4\mathbf{r}' \mathbf{r}'^T / \|\mathbf{r}'\|^2] \mathbf{r}''^T \Theta \mathbf{r}'}{\|\mathbf{r}'\|^4}, \\
 \Theta G''_2 &= -\frac{2\Theta}{\|\mathbf{r}'\|^4} [\mathbf{r}' \mathbf{r}'''^T + \cancel{2\mathbf{r}'' \mathbf{r}''^T} + \mathbf{r}''' \mathbf{r}'^T + [I - 4\mathbf{r}' \mathbf{r}'^T / \|\mathbf{r}'\|^2] (\mathbf{r}'''^T \mathbf{r}' + \mathbf{r}''^T \mathbf{r}'')] \\
 &\quad - 4 [2\mathbf{r}' \mathbf{r}''^T + \cancel{2\mathbf{r}'' \mathbf{r}'^T} + [I - 6\mathbf{r}' \mathbf{r}'^T / \|\mathbf{r}'\|^2] (\mathbf{r}''^T \mathbf{r}')] (\mathbf{r}''^T \mathbf{r}') / \|\mathbf{r}'\|^2, \\
 \Theta G_3 &= \frac{\Theta [I - 3\mathbf{r}' \mathbf{r}'^T / \|\mathbf{r}'\|^2]}{\|\mathbf{r}'\|^3}.
 \end{aligned} \tag{A.4}$$

Note that G_1 , ΘG_2 , ΘG_3 and there derivatives are symmetric matrices. Moreover, $(\Theta G_2)_{11} = -(\Theta G_2)_{22}$. Note finally that some terms inside those matrices turn out to be zero once multiplied by $\mathbf{r}''$ (see for instance the terms crossed in $\Theta G''_2$ when the operation $\Theta G''_2 \mathbf{r}''$ is performed).

Appendix B. NURBS derivatives

The first-fourth derivatives of the NURBS basis in terms of the B-spline basis and the weights are given by:

$$\begin{aligned}\frac{dR_i}{d\xi} &= \frac{1}{W} \left(\frac{dN_i}{d\xi} w_i - R_i \frac{dW}{d\xi} \right), \\ \frac{d^2 R_i}{d\xi^2} &= \frac{1}{W} \left(\frac{d^2 N_i}{d\xi^2} w_i - 2 \frac{dR_i}{d\xi} \frac{dW}{d\xi} - R_i \frac{d^2 W}{d\xi^2} \right), \\ \frac{d^3 R_i}{d\xi^3} &= \frac{1}{W} \left(\frac{d^3 N_i}{d\xi^3} w_i - 3 \frac{d^2 R_i}{d\xi^2} \frac{dW}{d\xi} - 3 \frac{dR_i}{d\xi} \frac{d^2 W}{d\xi^2} - R_i \frac{d^3 W}{d\xi^3} \right), \\ \frac{d^4 R_i}{d\xi^4} &= \frac{1}{W} \left(\frac{d^4 N_i}{d\xi^4} w_i - 4 \frac{d^3 R_i}{d\xi^3} \frac{dW}{d\xi} - 6 \frac{d^2 R_i}{d\xi^2} \frac{d^2 W}{d\xi^2} - 4 \frac{dR_i}{d\xi} \frac{d^3 W}{d\xi^3} - R_i \frac{d^4 W}{d\xi^4} \right).\end{aligned}\tag{B.1}$$

The Jacobian, linking the initial curvilinear coordinate s_0 to the parametric coordinate ξ , is defined by

$$s_0 = \int_0^\xi \|\mathbf{J}_0\| d\xi.\tag{B.2}$$

Noting that

$$\frac{d}{ds_0} \left(\frac{1}{\|\mathbf{J}_0\|^n} \right) = -n \frac{\mathbf{J}_0^T \mathbf{J}_{0,\xi}}{\|\mathbf{J}_0\|^{n+3}},\tag{B.3}$$

the derivatives of NURBS basis functions with respect to s_0 reads:

$$\begin{aligned}R_i' &= \frac{1}{\|\mathbf{J}_0\|} \frac{dR_i}{d\xi}, \\ R_i'' &= \frac{1}{\|\mathbf{J}_0\|^2} \left(\frac{d^2 R_i}{d\xi^2} - \frac{\mathbf{J}_0^T \mathbf{J}_{0,\xi}}{\|\mathbf{J}_0\|^2} \frac{dR_i}{d\xi} \right), \\ R_i''' &= \frac{1}{\|\mathbf{J}_0\|^3} \left(\frac{d^3 R_i}{d\xi^3} - 3 \frac{\mathbf{J}_0^T \mathbf{J}_{0,\xi}}{\|\mathbf{J}_0\|^2} \frac{d^2 R_i}{d\xi^2} + \left(4 \frac{(\mathbf{J}_0^T \mathbf{J}_{0,\xi})^2}{\|\mathbf{J}_0\|^4} - \frac{(\mathbf{J}_0^T \mathbf{J}_{0,\xi})_{,\xi}}{\|\mathbf{J}_0\|^2} \right) \frac{dR_i}{d\xi} \right), \\ R_i'''' &= \frac{1}{\|\mathbf{J}_0\|^4} \left(\frac{d^4 R_i}{d\xi^4} - 6 \frac{\mathbf{J}_0^T \mathbf{J}_{0,\xi}}{\|\mathbf{J}_0\|^2} \frac{d^3 R_i}{d\xi^3} + \left(19 \frac{(\mathbf{J}_0^T \mathbf{J}_{0,\xi})^2}{\|\mathbf{J}_0\|^4} - 4 \frac{(\mathbf{J}_0^T \mathbf{J}_{0,\xi})_{,\xi}}{\|\mathbf{J}_0\|^2} \right) \frac{d^2 R_i}{d\xi^2} \right. \\ &\quad \left. + \left(-28 \frac{(\mathbf{J}_0^T \mathbf{J}_{0,\xi})^3}{\|\mathbf{J}_0\|^6} + 13 \frac{(\mathbf{J}_0^T \mathbf{J}_{0,\xi}) (\mathbf{J}_0^T \mathbf{J}_{0,\xi})_{,\xi}}{\|\mathbf{J}_0\|^4} - \frac{(\mathbf{J}_0^T \mathbf{J}_{0,\xi})_{,\xi\xi}}{\|\mathbf{J}_0\|^2} \right) \frac{dR_i}{d\xi} \right).\end{aligned}\tag{B.4}$$

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