Citation	<p>Florian Maurin, Francesco Greco, Wim Desmet, (2016), Title: Isogeometric analysis for nonlinear planar pantographic lattice: discrete and continuum models Continuum Mechanics and Thermodynamics, Pages 1–14</p>
Archived version	<p>Author manuscript: the content is identical to the content of the published paper, but without the final typesetting by the publisher</p>
Published version	<p>10.1007/s00161-018-0641-y</p>
Journal homepage	<p>https://link.springer.com/article/10.1007%2Fs00161-018-0641-y</p>
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IR	<p>https://lirias.kuleuven.be/handle/123456789/618302</p>

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Isogeometric analysis for nonlinear planar pantographic lattice: discrete and continuum models

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Received: date / Accepted: date

Abstract Pantographic sheets are metamaterials constituted by two interconnected layers of straight fibers. One of the great features of these structures is that they are extremely elastically compliant toward large nonlinear deformations. To model pantographic lattices, Kirchhoff rods based on Euler–Bernoulli assumptions can be used. Otherwise, if the fibers are sufficiently dense, homogenization of the micro-structure results to a two-dimensional second-order gradient continuum model.

The discrete and continuum models have in common the fact that their energy terms depend on the second-order derivatives of the displacement field, such that the classical finite element method cannot be directly employed. We propose instead the use of the isogeometric analysis where the basis functions are endowed with a higher inter-element continuity, allowing the rotation-free discretization of these problems. The discrete and continuum models are compared to existing benchmarks and are found in excellent agreement, validating the proposed approach.

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Keywords pantographic lattice · isogeometric analysis · planar Kirchhoff rod · second-order gradient theory · geometrically-nonlinear deformations

1 Introduction

Pantographic sheets are constituted by two orthogonal arrays of straight fibers inter-connected by internal pivots. This metamaterial is attractive as it is extremely compliant toward large nonlinear deformations, while the behavior remains elastic [1].

To simulate the deformations of pantographic structures, different models based on different assumptions are used. The first approach consists in modeling the fibers by three-dimensional finite elements using a standard first gradient theory [2]. While no assumptions are stated on the geometry dimensions, the simulation are really expensive. In order to reduce the computational cost, the fibers can be modeled as one-dimensional planar elements, and the elastic pivots at the intersections are modeled via virtual springs. To model these beams, the Timoshenko theory [3], often referred to as the geometrically exact beam theory when large geometrical deformations are involved [4], can be used and considers membrane, bending and shear deformations. However, if the fibers are also assumed slender, the local shear deformations can be neglected resulting to Euler–Bernoulli beams, also called the planar Kirchhoff rods or the extensible elastica when large geometrical deformations are present. This model is used in [5] and will be considered as well in the present manuscript. Based on the same kinematic assumptions, Hencky-type beams replace the finite element discretizations of Kirchhoff rods by axial and rotational springs in series, and were successfully employed to describe pantographic structures [6–9]. If the fibers are sufficiently dense, an additional level of abstraction consists in considering the discrete lattice as a two-dimensional second-order gradient continuum model [1]. Based on the work of Pipkin, the fibers can also be assumed inextensible [7, 10–17] but this might not be accurate in some configurations where axial deformations are dominant [1, 17].

The common point between the Euler–Bernoulli beam theory and the extensible second-order continuum theory is their energy terms that depend on the second-order derivatives of the displacement field, such that classical finite element formulations cannot be directly used. To solve this issue, a first solution consists in introducing rotational degrees of freedom (dofs) [16] or dofs for the gradients [1]. However, this increases the computation cost, and might lead to a bad conditioning of the stiffness matrix (the displacement and rotational dofs do not share the same units, and hence they might be different by several orders of magnitude). To overcome these drawbacks, at least C^1 -continuous shape functions must be used, and this approach was successfully applied to the homogenized pantographic structure using Argyris elements [18–21]. A more efficient method in terms of accuracy per dof, robustness and computational cost is the isogeometric analysis (IGA) method [22, 23], and it is the approach developed in this manuscript.

IGA considers an isoparametric representation for which the basis functions are usually non-uniform rational B-splines (NURBS). One of the main advantages of this method is that the inter-element continuity of these basis functions can be augmented through the k -refinement procedure. This property has been successfully adopted to solve second-order weak formulations, as encountered for instance in the incompressible elasticity with stream functions [24], the Cahn–Hilliard phase-field problem [25], the Navier–Stokes equation [26], the Kirchhoff–Love shell [27] and the Kirchhoff rod [28–33].

The paper is organized as follows. In Section 2, the discrete and continuum models for pantographic lattices are presented. In Section 3, the discretization of the variational forms are developed. Numerical examples are provided in Section 4, followed by a comparison between both models.

2 Discrete and continuum models for pantographic lattices

Starting from a single beam, the energy and weak form of the beam lattice and its homogenized version are derived in this section.

2.1 Rotation-free planar Kirchhoff rod formulation

We consider a slender planar beam undergoing large nonlinear deformations and described by the Euler–Bernoulli beam theory (shear deformations are neglected). In the initial configuration, a material point of the beam at the arc-length parameter $s \in \mathbb{R}$ of the curvilinear coordinates is described by a position vector $\mathbf{r}_0 = \mathbf{r}_0\{s\} \in \mathbb{R}^2$. In the deformed configuration, this point is $\mathbf{r} = \mathbf{r}\{s\}$ and is related to the initial one through the displacement vector $\mathbf{u} = \mathbf{r} - \mathbf{r}_0$. The normal direction of this point in the deformed configuration reads

$$\overline{\mathbf{r}}_{,s} = \frac{\mathbf{r}_{,s}}{\|\mathbf{r}_{,s}\|}, \quad (1)$$

where $\|\mathbf{r}_{,s}\|$ is the Euclidean norm and the notation $\bullet_{,s}$ denotes the derivative with respect to s (Lagrange formulation). These quantities are shown in Fig. 1.

For an initially straight Euler–Bernoulli beam, the axial strain ϵ and the curvature κ read [29]:

$$\begin{aligned} \epsilon_{\{,s\}} &= \|\mathbf{r}_{,s}\| - 1, \\ \kappa_{\{,s\}} &= \frac{\mathbf{r}_{,ss}^T \Theta \mathbf{r}_{,s}}{\|\mathbf{r}_{,s}\|^2}, \end{aligned} \quad (2)$$

where $\bullet^T$ is the transpose. The curly brackets highlight the fact that the derivatives inside ϵ and κ are calculated with respect to s (this will change in the next section). Θ is a 90° rotation matrix given by:

$$\Theta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (3)$$

Fig. 1 Beam kinematics: reference (dotted lines) and current (full lines) configurations.

Fig. 2 Beam lattice in the reference (a) and current (b) configurations.

Assuming small strains, even for large displacements, the internal energy of the beam over an arbitrary length of the domain reads:

$$W_{\text{Beam}} = \int \left(\frac{EA}{2} (\epsilon_{\{,s\}})^2 + \frac{EI}{2} (\kappa_{\{,s\}})^2 \right) ds, \quad (4)$$

where E is the Young modulus, and A and I are the area and area moment of inertia of the beam, respectively. Note that in Eq. (2), $\mathbf{r}_{,ss}^T \Theta \mathbf{r}_{,s}$ is often expressed in the literature as a norm of a cross product $\|\mathbf{r}_{,s} \times \mathbf{r}_{,ss}\| = \sqrt{\|\mathbf{r}_{,s}\|^2 \|\mathbf{r}_{,ss}\|^2 - (\mathbf{r}_{,s} \cdot \mathbf{r}_{,ss})^2}$ (e.g. [1,34]) but the derivation of the stiffness matrix would be more tedious from this last form (it does not matter if the sign of these two expressions are different, as they are squared in Eq. (4)).

2.2 Discrete lattice

We consider a lattice of arbitrary length composed by two orthogonal families of beams having for main directions x and y (Fig. 2). All the beams have the same material and geometrical properties, and the distance between two consecutive parallel beams is e . The arc-length parameter s is replaced by y and x , for the first and second family, respectively.

For the Hencky-type beam lattice [6–9], there is an axial spring element between each intersection, and the curvature continuity at each intersection between two consecutive aligned elements is enforced by an additional rotational spring. In the present case, we adopt a different strategy and all the aligned segments form a single beam. Using this approach, the continuity at the intersection is naturally enforced. However, the drawback of this method is that it is more complex to model a pivot linking the displacements between two intersecting beams. Indeed, if the beams are modeled with at least $\mathcal{C}^1$ -continuous B-splines (which is the case of the present work), the control points (“the nodes”) inside the span are not interpolatory, meaning that a static condensation method cannot be used to constrain the displacements. Instead, Lagrange multipliers are used. At the intersection between the i -th beam and the j -th beam of the first and second family, respectively, the coordinate of the intersection is $\{x_i, y_j\}$, and the energy resulting from the displacement constraint is

$$W_{\text{Lagrange}}^{(ij)} = \left(\mathbf{r}^{(i)} - \mathbf{r}^{(j)} \right) \cdot \boldsymbol{\lambda}^{(ij)}, \quad (5)$$

where $\mathbf{r}^{(j)} = \mathbf{r}^{(j)}\{x_i\}$, $\mathbf{r}^{(i)} = \mathbf{r}^{(i)}\{y_j\}$, $\boldsymbol{\lambda}^{(ij)} \in \mathbb{R}^2$ is a Lagrange multipliers vector, and $(\bullet \cdot \bullet)$ denotes the scalar product. At each intersection, the pivot is not perfect and there are some rotational elastic deformations, modeled by a nonlinear spring having for energy [1]:

$$\begin{aligned} W_{\text{Spring}}^{(ij)} &= \frac{k_\theta}{2} \left| \theta^{(ij)} - \frac{\pi}{2} \right|^\gamma \\ &= \frac{k_\theta}{2} \left| \sin^{-1} \left(\overline{\mathbf{r}}_{,y}^{(i)} \cdot \overline{\mathbf{r}}_{,x}^{(j)} \right) \right|^\gamma, \end{aligned} \quad (6)$$

where k_θ and γ are two parameters that characterize the pivot. $\overline{\mathbf{r}}_{,x}^{(j)} = \overline{\mathbf{r}}_{,x}^{(j)}\{x_i\}$, $\overline{\mathbf{r}}_{,y}^{(i)} = \overline{\mathbf{r}}_{,y}^{(i)}\{y_j\}$, and $\theta^{(ij)}$ is the deformed angle between the i -th and j -th fiber, at the intersection. After summing the energy of all the beams in both directions plus the energy of all the pivots, one get the total energy of the discrete lattice:

$$\begin{aligned} W_{\text{Lattice}} &= \sum_j \int \left(\frac{EA}{2} \left(\epsilon_{\{,x\}}^{(j)} \right)^2 + \frac{EI}{2} \left(\kappa_{\{,x\}}^{(j)} \right)^2 \right) dx \\ &\quad + \sum_i \int \left(\frac{EA}{2} \left(\epsilon_{\{,y\}}^{(i)} \right)^2 + \frac{EI}{2} \left(\kappa_{\{,y\}}^{(i)} \right)^2 \right) dy \\ &\quad + \sum_i \sum_j \frac{k_\theta}{2} \left| \sin^{-1} \left(\overline{\mathbf{r}}_{,y}^{(i)} \cdot \overline{\mathbf{r}}_{,x}^{(j)} \right) \right|^\gamma + \left(\mathbf{r}^{(i)} - \mathbf{r}^{(j)} \right) \cdot \boldsymbol{\lambda}^{(ij)}. \end{aligned} \quad (7)$$

2.3 Homogenized lattice

In order to model an homogenized medium, the distance between the beams denoted e is assumed to be small, such that the sums of Eq. (7) can be replaced

by integrals, leading to [1]:

$$W_{\text{Continuum}} = \iint \left(\sum_{s=\{x,y\}} \left(\frac{K_\epsilon}{2} (\epsilon_{\{s\}})^2 + \frac{K_\kappa}{2} (\kappa_{\{s\}})^2 \right) + \frac{K_\theta}{2} |\sin^{-1}(\overline{\mathbf{r}_{,y}} \cdot \overline{\mathbf{r}_{,x}})|^\gamma \right) dx dy, \quad (8)$$

where $\mathbf{r} = \mathbf{r}\{x, y\}$, $K_\epsilon = EA/e$, $K_\kappa = EI/e$ and $K_\theta = k_\theta/e^2$. Note that in this expression, the displacement constraints between the two fiber families are naturally satisfied by the continuum medium, such that the Lagrange multipliers are not required. Now that the energy terms for the discrete and continuum models are provided, their weak forms, required in the Galerkin discretization, are derived.

2.4 Weak form

The weak formulation of the equilibrium equation is obtained from the principle of virtual work [4] and results to a system of two equations for the beam lattice:

$$\begin{aligned} \delta W_{\text{Lattice}} = & \sum_{s=\{x,y\}} \sum_{k=\begin{cases} j & \text{if } s = x \\ i & \text{if } s = y \end{cases}} \int \left(EA \epsilon_{\{s\}}^{(k)} \delta \epsilon_{\{s\}}^{(k)} + EI \kappa_{\{s\}}^{(k)} \delta \kappa_{\{s\}}^{(k)} \right) ds \\ & + \sum_i \sum_j \left(\frac{\gamma}{2} k_\theta s_\Gamma \left| \Gamma_{\{x,y\}}^{(ij)} \right|^{\gamma-1} \delta \Gamma_{\{x,y\}}^{(ij)} + \left(\delta \mathbf{v}^{(i)} - \delta \mathbf{v}^{(j)} \right)^T \boldsymbol{\lambda}^{(ij)} \right), \end{aligned} \quad (9)$$

and

$$\delta W_{\text{Lagrange}} = \sum_i \sum_j \delta \boldsymbol{\lambda}^{(ij)T} \left(\mathbf{r}^{(i)} - \mathbf{r}^{(j)} \right), \quad (10)$$

where $\delta \mathbf{v}$ and $\delta \boldsymbol{\lambda}$ are the virtual displacements and virtual Lagrange multipliers, respectively. $\Gamma_{\{x,y\}}^{(ij)} = \sin^{-1}(\overline{\mathbf{r}_{,y}}^{(i)} \cdot \overline{\mathbf{r}_{,x}}^{(j)})$ and $s_\Gamma = \text{sgn } \Gamma_{\{x,y\}}^{(ij)}$. Noting that the derivation with respect to x and y are always associated to beams having for index j and i , respectively, these last indexes are omitted next for a sake of brevity. The virtual strain components read:

$$\begin{aligned} \delta \epsilon_{\{s\}} &= \delta \mathbf{v}_{,s}^T \overline{\mathbf{r}_{,s}}, \\ \delta \kappa_{\{s\}} &= \delta \mathbf{v}_{,s}^T (\overline{\overline{\mathbf{r}_{,s}}})_{,s} + \delta \mathbf{v}_{,ss}^T \overline{\overline{\mathbf{r}_{,s}}}, \\ \delta \Gamma_{\{x,y\}} &= \frac{\delta \mathbf{v}_{,x}^T \mathbf{G}_{\{x\}}^\epsilon \overline{\mathbf{r}_{,y}} + \delta \mathbf{v}_{,y}^T \mathbf{G}_{\{y\}}^\epsilon \overline{\mathbf{r}_{,x}}}{\left(1 - (\overline{\mathbf{r}_{,x}} \cdot \overline{\mathbf{r}_{,y}})^2 \right)^{1/2}}, \end{aligned} \quad (11)$$

where

$$\overline{\mathbf{r}_{,s}} = \frac{\mathbf{r}_{,s}}{\|\mathbf{r}_{,s}\|}, \quad \overline{\overline{\mathbf{r}_{,s}}} = \frac{\Theta \mathbf{r}_{,s}}{\|\mathbf{r}_{,s}\|^2}, \quad \text{and} \quad (\overline{\overline{\mathbf{r}_{,s}}})_{,s} = \Theta \mathbf{G}_{\{s\}}^{\kappa} \mathbf{r}_{,ss}. \quad (12)$$

In order to solve the geometrically nonlinear equilibrium equation (9), the Newton–Raphson scheme discussed in Section 3.3 is used. It requires the linearization of $\delta W_{\text{Lattice}}$ and $\delta W_{\text{Lagrange}}$, and reads:

$$\begin{aligned} \delta \delta W_{\text{Lattice}} = & \sum_{s=\{x,y\}} \sum_{k=\begin{cases} j & \text{if } s=x \\ i & \text{if } s=y \end{cases}} \int \left(EA \delta \epsilon_{\{s\}}^{(k)} \delta \epsilon_{\{s\}}^{(k)} + EI \delta \kappa_{\{s\}}^{(k)} \delta \kappa_{\{s\}}^{(k)} \right. \\ & + EA \epsilon_{\{s\}}^{(k)} \delta \delta \epsilon_{\{s\}}^{(k)} + EI \kappa_{\{s\}}^{(k)} \delta \delta \kappa_{\{s\}}^{(k)} \left. \right) ds \\ & + \sum_i \sum_j \frac{\gamma}{2} k_\theta \left((\gamma - 1) \left| \Gamma_{\{x,y\}}^{(ij)} \right|^{\gamma-2} \delta \Gamma_{\{x,y\}}^{(ij)} \delta \Gamma_{\{x,y\}}^{(ij)} \right. \\ & \left. + s_\Gamma \left| \Gamma_{\{x,y\}}^{(ij)} \right|^{\gamma-1} \delta \delta \Gamma_{\{x,y\}}^{(ij)} \right) + \left(\delta \mathbf{v}^{(i)} - \delta \mathbf{v}^{(j)} \right)^T \delta \boldsymbol{\lambda}^{(ij)}, \end{aligned} \quad (13)$$

and

$$\delta \delta W_{\text{Lagrange}} = \sum_i \sum_j \delta \boldsymbol{\lambda}^{(ij)T} \left(\delta \mathbf{u}^{(i)} - \delta \mathbf{u}^{(j)} \right), \quad (14)$$

where

$$\begin{aligned} \delta \epsilon_{\{s\}} \delta \epsilon_{\{s\}} &= \delta \mathbf{v}_{,s}^T \overline{\mathbf{r}}_{,s} \overline{\mathbf{r}}_{,s}^T \delta \mathbf{u}_{,s}, \\ \delta \kappa_{\{s\}} \delta \kappa_{\{s\}} &= \delta \mathbf{v}_{,s}^T \left(\overline{\mathbf{r}}_{,s} \right)_{,s} \left(\overline{\mathbf{r}}_{,s} \right)_{,s}^T \delta \mathbf{u}_{,s} + \delta \mathbf{v}_{,ss}^T \overline{\mathbf{r}}_{,s} \left(\overline{\mathbf{r}}_{,s} \right)_{,s}^T \delta \mathbf{u}_{,s} \\ &+ \delta \mathbf{v}_{,s}^T \left(\overline{\mathbf{r}}_{,s} \right)_{,s} \overline{\mathbf{r}}_{,ss}^T \delta \mathbf{u}_{,ss} + \delta \mathbf{v}_{,ss}^T \overline{\mathbf{r}}_{,s} \overline{\mathbf{r}}_{,ss}^T \delta \mathbf{u}_{,ss}, \\ \delta \Gamma_{\{x,y\}} \delta \Gamma_{\{x,y\}} &= \delta \mathbf{v}_{,x}^T C_{xx}^\Gamma \delta \mathbf{u}_{,x} + \delta \mathbf{v}_{,x}^T C_{xy}^\Gamma \delta \mathbf{u}_{,y} + \delta \mathbf{v}_{,y}^T C_{yx}^\Gamma \delta \mathbf{u}_{,x} + \delta \mathbf{v}_{,y}^T C_{yy}^\Gamma \delta \mathbf{u}_{,y}, \\ \delta \delta \epsilon_{\{s\}} &= \delta \mathbf{v}_{,s}^T \mathbf{G}_{\{s\}}^\epsilon \delta \mathbf{u}_{,s}, \\ \delta \delta \kappa_{\{s\}} &= \delta \mathbf{v}_{,s}^T \left(\Theta \mathbf{G}_{\{s\}}^\kappa \right)_{,s} \delta \mathbf{u}_{,s} + \delta \mathbf{v}_{,ss}^T \Theta \mathbf{G}_{\{s\}}^\kappa \delta \mathbf{u}_{,s} + \delta \mathbf{v}_{,s}^T \Theta \mathbf{G}_{\{s\}}^\kappa \delta \mathbf{u}_{,ss}, \\ \delta \delta \Gamma_{\{x,y\}} &= \delta \mathbf{v}_{,x}^T \mathbf{G}_{xx}^\Gamma \delta \mathbf{u}_{,x} + \delta \mathbf{v}_{,x}^T \mathbf{G}_{xy}^\Gamma \delta \mathbf{u}_{,y} + \delta \mathbf{v}_{,y}^T \mathbf{G}_{yx}^\Gamma \delta \mathbf{u}_{,x} + \delta \mathbf{v}_{,y}^T \mathbf{G}_{yy}^\Gamma \delta \mathbf{u}_{,y}. \end{aligned} \quad (15)$$

Moreover, the different geometrical stiffness matrices G are expressed as:

$$\begin{aligned}
G_{\{,s\}}^\epsilon &= \frac{\mathbf{I} - \overline{\mathbf{r}}_{,s} \overline{\mathbf{r}}_{,s}^T}{\|\mathbf{r}_{,s}\|^2}, \\
\Theta G_{\{,s\}}^\kappa &= \frac{\Theta [\mathbf{I} - 2\overline{\mathbf{r}}_{,s} \overline{\mathbf{r}}_{,s}^T]}{\|\mathbf{r}'\|^2}, \\
\left(\Theta G_{\{,s\}}^\kappa\right)_{,s} &= -2\Theta \frac{\mathbf{r}_{,s} \mathbf{r}_{,ss}^T + \mathbf{r}_{,ss} \mathbf{r}_{,s}^T + [\mathbf{I} - 4\overline{\mathbf{r}}_{,s} \overline{\mathbf{r}}_{,s}^T] (\mathbf{r}_{,ss}^T \mathbf{r}_{,s})}{\|\mathbf{r}_{,s}\|^4}, \\
&= -2 \frac{\mathbf{r}_{,s} \mathbf{r}_{,ss}^T \Theta + \Theta^T \mathbf{r}_{,ss} \mathbf{r}_{,s}^T + [\mathbf{I} - 4\overline{\mathbf{r}}_{,s} \overline{\mathbf{r}}_{,s}^T] \mathbf{r}_{,ss}^T \Theta \mathbf{r}_{,s}}{\|\mathbf{r}_{,s}\|^4}, \quad (16) \\
C_{st}^\Gamma &= \frac{G_{\{,s\}}^\epsilon \overline{\mathbf{r}}_{\neq s} \overline{\mathbf{r}}_{\neq t}^T G_{\{,t\}}^\epsilon}{1 - (\overline{\mathbf{r}}_{,x} \cdot \overline{\mathbf{r}}_{,y})^2}, \\
G_{st}^\Gamma &= \begin{cases} \frac{(\overline{\mathbf{r}}_{,x} \cdot \overline{\mathbf{r}}_{,y}) C_{ss}^\Gamma - \frac{\overline{\mathbf{r}}_{,x} \overline{\mathbf{r}}_{,y}^T + \overline{\mathbf{r}}_{,y} \overline{\mathbf{r}}_{,x}^T + [\mathbf{I} - 3\overline{\mathbf{r}}_{,s} \overline{\mathbf{r}}_{,s}^T] (\overline{\mathbf{r}}_{,x} \cdot \overline{\mathbf{r}}_{,y})}{\|\mathbf{r}_{,s}\|^2}}{(1 - (\overline{\mathbf{r}}_{,x} \cdot \overline{\mathbf{r}}_{,y})^2)^{1/2}}, & \text{if } s = t, \\ \frac{(\overline{\mathbf{r}}_{,x} \cdot \overline{\mathbf{r}}_{,y}) C_{st}^\Gamma + G_{\{,s\}}^\epsilon G_{\{,t\}}^\epsilon}{(1 - (\overline{\mathbf{r}}_{,x} \cdot \overline{\mathbf{r}}_{,y})^2)^{1/2}}, & \text{if } s \neq t, \end{cases}
\end{aligned}$$

where $\mathbf{I}$ is a 2×2 identity matrix and the symbol $\neq s$ means that if $s = x$, the value will be y , and vice versa. Note that $G_{\{,s\}}^\epsilon$ and $\Theta G_{\{,s\}}^\kappa$ are symmetric matrices and that $(\Theta G_{\{,s\}}^\kappa)_{11} = -(\Theta G_{\{,s\}}^\kappa)_{22}$.

For the homogenized model, the same procedure is followed. The weak form reads

$$\begin{aligned}
\delta W_{\text{Continuum}} &= \iint \sum_{s=\{x,y\}} (K_\epsilon \epsilon_{\{,s\}} \delta \epsilon_{\{,s\}} + K_\kappa \kappa_{\{,s\}} \delta \kappa_{\{,s\}} \\
&\quad + \frac{\gamma}{2} K_\theta s_\Gamma |\Gamma_{\{,x,y\}}|^{\gamma-1} \delta \Gamma_{\{,x,y\}}) dx dy, \quad (17)
\end{aligned}$$

and its linearization gives

$$\begin{aligned}
\delta \delta W_{\text{Continuum}} &= \iint (K_\epsilon \delta \epsilon_{\{,s\}} \delta \epsilon_{\{,s\}} + K_\kappa \delta \kappa_{\{,s\}} \delta \kappa_{\{,s\}} \\
&\quad + K_\epsilon \epsilon_{\{,s\}} \delta \delta \epsilon_{\{,s\}} + K_\kappa \kappa_{\{,s\}} \delta \delta \kappa_{\{,s\}} \\
&\quad + \frac{\gamma}{2} K_\theta \left((\gamma - 1) |\Gamma_{\{,x,y\}}|^{\gamma-2} \delta \Gamma_{\{,x,y\}} \delta \Gamma_{\{,x,y\}} \right. \\
&\quad \left. + s_\Gamma |\Gamma_{\{,x,y\}}|^{\gamma-1} \delta \delta \Gamma_{\{,x,y\}} \right) dx dy, \quad (18)
\end{aligned}$$

where the definition of the virtual strain components are the same as the ones of the beam lattice (Eqs. (11) and (15)). Before providing the force and stiffness matrices, isogeometric analysis is introduced for univariate and bivariate B-splines.

3 Numerical discretization

3.1 Isogeometric analysis

NURBS are rational B-splines that can represent exactly conic shapes for instance. However as the considered geometries are in their initial configuration straight, B-splines are sufficient for modeling exactly the undeformed structures. Therefor, NURBS are not considered here for a sake of brevity, but an extension from B-splines to NURBS would be straightforward.

3.1.1 Univariate B-spline

A B-spline of polynomial degree p is defined by a knot vector $\Xi = \{\xi_1 \leq \dots \leq \xi_{n+p+1}\}$, with n the number of basis functions forming the B-spline. The k^{th} ($k = 1, \dots, n$) B-spline basis function $N_p^k(\xi)$ is derived from the knot-vector using the Cox-De Boor recursive formula [35]:

$$N_0^k(\xi) = \begin{cases} 1 & \text{if } \xi_k \leq \xi < \xi_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \quad (19)$$

$$N_p^k(\xi) = \frac{\xi - \xi_k}{\xi_{k+p} - \xi_k} N_{p-1}^k(\xi) + \frac{\xi_{k+p+1} - \xi}{\xi_{k+p+1} - \xi_{k+1}} N_{p-1}^{k+1}(\xi), \quad \text{for } p \geq 1.$$

In the following, we consider basis functions which are $\mathcal{C}^{p-1}$ -continuous (no repeated knot except at the extremities where the knot multiplicity is $p+1$). Moreover, as the two weak formulations involve second-order derivatives, the shape functions have to be at least $\mathcal{C}^1$ -continuous, requiring that $p \geq 2$.

A curve of degree p parametrized by the control points $\mathbf{B}_k$ is described by

$$\mathbf{r}_0(\xi) = \sum_{k=1}^n N_p^k(\xi) \mathbf{B}_k. \quad (20)$$

The Jacobian vector $\mathbf{J}$, linking the initial curvilinear coordinate s to the parametric coordinate ξ , is defined by

$$s = \int_0^\xi \|\mathbf{J}\| d\xi. \quad (21)$$

The derivatives of B-spline basis functions with respect to s reads:

$$N_{p,s}^k(\xi) = \frac{1}{\|\mathbf{J}\|} N_{p,\xi}^k(\xi),$$

$$N_{p,ss}^k(\xi) = \frac{1}{\|\mathbf{J}\|^2} \left(N_{p,\xi\xi}^k(\xi) - \frac{\mathbf{J} \cdot \mathbf{J}_{,\xi}}{\|\mathbf{J}\|^2} N_{p,\xi}^k(\xi) \right). \quad (22)$$

Using the isogeometric approach, the displacement field $\mathbf{u}(\xi)$ is approximated by the same basis functions and reads:

$$\mathbf{u}(\xi) = \sum_{k=1}^n N_p^k(\xi) \mathbf{u}_k, \quad (23)$$

where $\mathbf{u}_k$ is the displacement associated to the control point $\mathbf{B}_k$.

3.1.2 Bivariate B-spline

Given two univariate B-spline bases $N_{p_1}^{k_1}(\xi)$ ($k_1 = 1, \dots, n_1$) and $M_{p_2}^{k_2}(\eta)$ ($k_2 = 1, \dots, n_2$), respectively of order p_1 and p_2 , and associated to the two knot vectors $\Xi = \{\xi_1 \leq \dots \leq \xi_{n_1+p_1+1}\}$ and $\mathcal{H} = \{\eta_1 \leq \dots \leq \eta_{n_2+p_2+1}\}$, a bivariate B-spline basis function is defined as

$$N_{p_1 p_2}^{k_1 k_2}(\xi, \eta) = N_{p_1}^{k_1}(\xi) M_{p_2}^{k_2}(\eta). \quad (24)$$

A B-spline surface $\mathbf{r}_0(\xi, \eta)$ and its displacement are defined by

$$\mathbf{r}_0(\xi, \eta) = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} N_{p_1 p_2}^{k_1 k_2}(\xi, \eta) \mathbf{B}_{k_1 k_2} = \sum_{k=1}^n N_{p_1 p_2}^k(\xi, \eta) \mathbf{B}_k, \quad (25)$$

$$\mathbf{u}(\xi, \eta) = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} N_{p_1 p_2}^{k_1 k_2}(\xi, \eta) \mathbf{u}_{k_1 k_2} = \sum_{k=1}^n N_{p_1 p_2}^k(\xi, \eta) \mathbf{u}_k, \quad (26)$$

where $\mathbf{B}_{k_1 k_2}$ are the control points, $\mathbf{u}_{k_1 k_2}$ its associated displacement, $n = n_1 \cdot n_2$ and $k = k_1 + n_1(k_2 - 1)$ a single counter replacing k_1 and k_2 . In the following, we consider that the two B-splines have the same degree ($p_1 = p_2 = p$) and this specification is omitted for a sake of brevity ($N_{p_1 p_2}^k = N^k$). The explicit dependency on ξ and η is also disregarded. The derivatives of B-spline basis functions with respect to the axis x and y reads:

$$\begin{bmatrix} N_{,x}^k \\ N_{,y}^k \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} N_{,\xi}^k \\ N_{,\eta}^k \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix}, \quad (27)$$

where $\mathbf{J}$ is the Jacobian matrix. Moreover,

$$\begin{bmatrix} N_{,xx}^k & N_{,xy}^k \\ N_{,xy}^k & N_{,yy}^k \end{bmatrix} = \mathbf{J}^{-1} \left[\begin{bmatrix} N_{,\xi\xi}^k & N_{,\xi\eta}^k \\ N_{,\xi\eta}^k & N_{,\eta\eta}^k \end{bmatrix} - \begin{bmatrix} x_{,\xi\xi} & x_{,\xi\eta} \\ x_{,\xi\eta} & x_{,\eta\eta} \end{bmatrix} N_{,x}^k - \begin{bmatrix} y_{,\xi\xi} & y_{,\xi\eta} \\ y_{,\xi\eta} & y_{,\eta\eta} \end{bmatrix} N_{,y}^k \right] \mathbf{J}^T. \quad (28)$$

3.2 Galerkin variational formulation of the weak form

The Bubnov–Galerkin approach considers as test functions the same basis functions as the solution field:

$$\delta \mathbf{v} = \sum_{k=1}^n N^k \delta \mathbf{v}_k. \quad (29)$$

3.2.1 Discrete model

The force vector for the beam lattice is derived from the discretization of the weak form Eq. (7) and reads:

$$\begin{aligned} \mathbf{F}_{\text{Lattice}}^k = & \sum_{e=1}^{n_e} \int_{\Omega_0^e} \left(EA \epsilon_{\{s\}} \delta \epsilon_{\{s\}}^k + EI \kappa_{\{s\}} \delta \kappa_{\{s\}}^k \right) ds \\ & + \sum_i \sum_j \frac{\gamma}{2} K_\theta s_\Gamma \left| \Gamma_{\{x,y\}} \right|^{\gamma-1} \delta \Gamma_{\{x,y\}}^k, \end{aligned} \quad (30)$$

where Ω_0^e is the domain of the element e (n_e elements in total). It is not explicitly stated, but s is x or y depending on the initial orientation of the element. The discretized strain components read:

$$\begin{aligned} \delta \epsilon_{\{s\}}^k &= N_{,s}^k \bar{\mathbf{r}}_{,s}, \\ \delta \kappa_{\{s\}}^k &= N_{,s}^k \left(\bar{\mathbf{r}}_{,s} \right)_{,s} + N_{,ss}^k \bar{\mathbf{r}}_{,s}, \\ \delta \Gamma_{\{x,y\}}^k &= \frac{R_{,x}^k G_{\{x\}}^\epsilon \bar{\mathbf{r}}_{,y} + R_{,y}^k G_{\{y\}}^\epsilon \bar{\mathbf{r}}_{,x}}{\left(1 - (\bar{\mathbf{r}}_{,x} \cdot \bar{\mathbf{r}}_{,y})^2 \right)^{1/2}}. \end{aligned} \quad (31)$$

The stiffness matrix, derived from the discretization of Eq. (13) reads

$$\begin{aligned} \mathbf{K}_{\text{Lattice}}^{kl} = & \sum_{e=1}^{n_e} \int_{\Omega_0^e} \left(EA \delta \epsilon_{\{s\}}^k \delta \epsilon_{\{s\}}^l + EI \delta \kappa_{\{s\}}^k \delta \kappa_{\{s\}}^l \right) ds \\ & + EA \epsilon_{\{s\}} \delta \delta \epsilon_{\{s\}}^{kl} + EI \kappa_{\{s\}} \delta \delta \kappa_{\{s\}}^{kl} \Big|_{\{s\}} \Big) ds \\ & + \sum_i \sum_j \frac{\gamma}{2} K_\theta \left((\gamma-1) \left| \Gamma_{\{x,y\}} \right|^{\gamma-2} \delta \Gamma_{\{x,y\}}^k \delta \Gamma_{\{x,y\}}^l \right. \\ & \quad \left. + s_\Gamma \left| \Gamma_{\{x,y\}} \right|^{\gamma-1} \delta \delta \Gamma_{\{x,y\}}^{kl} \right), \end{aligned} \quad (32)$$

where:

$$\begin{aligned} \delta \epsilon_{\{s\}}^k \delta \epsilon_{\{s\}}^l &= N_{,s}^k \bar{\mathbf{r}}_{,s} \bar{\mathbf{r}}_{,s}^T N_{,s}^l, \\ \delta \kappa_{\{s\}}^k \delta \kappa_{\{s\}}^l &= N_{,s}^k \left(\bar{\mathbf{r}}_{,s} \right)_{,s} \left(\bar{\mathbf{r}}_{,s} \right)_{,s}^T N_{,s}^l + N_{,ss}^k \bar{\mathbf{r}}_{,s} \left(\bar{\mathbf{r}}_{,s} \right)_{,s}^T N_{,ss}^l \\ & \quad + N_{,ss}^k \left(\bar{\mathbf{r}}_{,s} \right)_{,s} \bar{\mathbf{r}}_{,s}^T N_{,ss}^l + N_{,ss}^k \bar{\mathbf{r}}_{,s} \bar{\mathbf{r}}_{,s}^T N_{,ss}^l, \\ \delta \Gamma_{\{x,y\}}^k \delta \Gamma_{\{x,y\}}^l &= N_{,x}^k C_{xx}^\Gamma N_{,x}^l + N_{,x}^k C_{xy}^\Gamma N_{,y}^l + N_{,y}^k C_{yx}^\Gamma N_{,x}^l + N_{,y}^k C_{yy}^\Gamma N_{,y}^l, \\ \delta \delta \epsilon_{\{s\}}^{kl} &= N_{,s}^k G_{\{s\}}^\epsilon N_{,s}^l, \\ \delta \delta \kappa_{\{s\}}^{kl} &= N_{,s}^k \left(\Theta G_{\{s\}}^\kappa \right)_{,s} N_{,s}^l + N_{,ss}^k \Theta G_{\{s\}}^\kappa N_{,s}^l + N_{,ss}^k \Theta G_{\{s\}}^\kappa N_{,ss}^l, \\ \delta \delta \Gamma_{\{x,y\}}^{kl} &= N_{,x}^k G_{xx}^\Gamma N_{,x}^l + N_{,x}^k G_{xy}^\Gamma N_{,y}^l + N_{,y}^k G_{yx}^\Gamma N_{,x}^l + N_{,y}^k G_{yy}^\Gamma N_{,y}^l. \end{aligned} \quad (33)$$

Note that the Lagrange multipliers have not been considered so far in the force and stiffness matrices. The stiffness matrix resulting from Eq. (13) or Eq. (14) reads

$$\mathbf{K}_{\text{Lagrange}}^{kl} = \sum_i \sum_j \left(N^{l(i)} - N^{l(j)} \right), \quad (34)$$

where k is in this specific case a single counter that associates a unique index (k) to the couple $\{i, j\}$. The final system of equations reads:

$$\begin{bmatrix} \mathbf{K}_{\text{Lattice}} & \mathbf{K}_{\text{Lagrange}}^T \\ \mathbf{K}_{\text{Lagrange}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\text{Lattice}} \\ \mathbf{0} \end{bmatrix}. \quad (35)$$

3.2.2 Continuum model

For the homogenized medium, the force and stiffness matrices derived from Eqs. (17) and (18) read

$$\begin{aligned} \mathbf{F}_{\text{Continuum}}^k = \sum_{e=1}^{n_e} \iint_{\Omega_0^e} & \left(K_\epsilon \epsilon_{\{,s\}} \delta \epsilon_{\{,s\}}^k + K_\kappa \kappa_{\{,s\}} \delta \kappa_{\{,s\}}^k \right. \\ & \left. + \frac{\gamma}{2} K_\theta s_\Gamma |\Gamma_{\{,x,y\}}|^{\gamma-1} \delta \Gamma_{\{,x,y\}}^k \right) dx dy, \end{aligned} \quad (36)$$

and

$$\begin{aligned} \mathbf{K}_{\text{Continuum}}^{kl} = \sum_{e=1}^{n_e} \iint_{\Omega_0^e} & \left(K_\epsilon \delta \epsilon_{\{,s\}}^k \delta \epsilon_{\{,s\}}^l + K_\kappa \delta \kappa_{\{,s\}}^k \delta \kappa_{\{,s\}}^l \right. \\ & + K_\epsilon \epsilon_{\{,s\}} \delta \delta \epsilon_{\{,s\}}^{kl} + K_\kappa \kappa_{\{,s\}} \delta \delta \kappa_{\{,s\}}^{kl} \\ & + \frac{\gamma}{2} K_\theta \left((\gamma-1) |\Gamma_{\{,x,y\}}|^{\gamma-2} \delta \Gamma_{\{,x,y\}}^k \delta \Gamma_{\{,x,y\}}^l \right. \\ & \left. \left. + s_\Gamma |\Gamma_{\{,x,y\}}|^{\gamma-1} \delta \delta \Gamma_{\{,x,y\}}^{kl} \right) \right) dx dy. \end{aligned} \quad (37)$$

These equations are integrated numerically using the full Gauss–Legendre quadrature rule.

3.3 The MIP Newton

To solve the geometrically-nonlinear structural problem, the MIP (Mixed Integration Point) Newton method is used [36,37]. It consists in relaxing the constitutive equations at each integration point, combining the advantages of a displacement formulation (reasonable computational cost) with the advantages of a mixed formulation (convergence in Newton more robust [38]). This method has been successfully used for Reissner beams [37], isogeometric solid shells [36], and isogeometric Kirchhoff rods [39]. The implementation is relatively simple, and we refer to [36,37] for its details. We emphasize that the strain components which are relaxed into the constitutive equation are $\epsilon_{\{,x\}}$, $\epsilon_{\{,y\}}$, $\kappa_{\{,x\}}$, $\kappa_{\{,y\}}$ and $|\Gamma_{\{,x,y\}}|^{\gamma-1}$ (for both the discrete and continuum models). The constitutive law is recovered at convergence.

Fig. 3 Pantographic lattice of ratio 3 : 1 and composed by 78 beams ($e = 0.0048$ m).

4 Numerical results

A pantographic lattice is subject to different solicitations reproduced from [1], which are then simulated with both the discrete and continuum models. All the simulations are performed with an in-house *MATLAB* code that makes use of some functionalities of the open source NURBS toolbox [40, 41].

4.1 Problems description

The pantographic lattice (Fig. 3) is composed by 78 beams (of different length) in total and arranged in a rectangle having a length of $L = 30\sqrt{2}e \approx 0.20$ m and a width of $H = 10\sqrt{2}e \approx 0.068$ m (ratio 3 : 1). The distance between two consecutive parallel beams is $e = 0.0048$ m and the initial angle between the two beam layers is 90° . The parameters of the pantographic lattice are identified from experiments in [1], and are the following: $K_\epsilon = 1.34 \times 10^5 \text{ Nm}^{-1}$, $K_\kappa = 1.92 \times 10^{-2} \text{ Nm}$, $K_\theta = 1.59 \times 10^2 \text{ Nm}^{-1}$, and $\gamma = 1.36$.

The four case studies investigated in [1] are reproduced (in their most deformed configurations). They consist in applying different displacement conditions to the edges AB and CD such that: (see Fig. 3, $\xi = BC/\|BC\|$, $\eta = BA/\|BA\|$)

- Test a: AB is fixed, CD is translated by the vector 0.0567ξ .
- Test b: AB is fixed, CD is translated by the vector $2H\eta$.
- Test c: AB is fixed, CD is translated by the vector $0.5H\xi$ and anticlockwise rotated by $\pi/4$ about its centre.
- Test d: AB is clockwise rotated by $\pi/3$ about A , CD is rotated anticlockwise by $\pi/3$ about the point D and translate by the vector $-3/2H\xi$.

4.2 Comparisons of the discrete and continuum models

For the test a, the deformed shapes and the different energy densities (axial, bending and shear) for the discrete and continuum models are compared in Fig. 4, and are found in excellent agreement. A more detailed comparison

Fig. 4 Test a: discrete (a,c,e,g) and continuum (b,d,f,h) IGA models colorized following the total (a,b), the axial (c,d), the bending (e,f) and shear (g,h) strain energy density (N/m).

can be found in Fig. 5 where the tension applied to the edge CD (Fig. 5a), the angle between the two fibers intersecting at the middle of the specimen (Fig. 5c), and the elongation of the fiber indicated by an arrow in Fig. 3 (Fig. 5e) are plotted for the isogeometric discrete and continuum models. In addition we also report the results presented in [1] where the continuum model is discretized with displacement and its gradient as dofs. All the results are found in agreement, even if the curves do not perfectly overlaps.

In order to explain this small discrepancy, the same specimen is investigated with three additional beam densities ($e = 0.0096$ m, $e = 0.0024$ m and $e = 0.0012$ m). For $u_{CD} = 0.0567\xi$, the convergence curves for the different quantities discussed above are shown in Fig. 5b,d,f. As expected, it is found that when the lattice density is increased, the results converge to the continuum model. These conclusions are also confirmed in Table 1 where the values for the finest meshes are reported.

Finally, the total energy for the three other tests are reported in Fig. 6, and are found again in excellent agreement, validating the models and the proposed approaches.

Fig. 5 Test a: applied tension (a,b), angle at the middle of the specimen (c,d), and elongation of the fiber indicated with an arrow in Fig. 3 (e,f) for u_{BC} from 0 to 0.0567ξ m (a,c,e). At $u_{BC} = 0.0567\xi$ m, convergence plots in terms of the square root of the number of dof, for different models, finite element discretization levels and B-spline degrees p (b,d,f). The number of dof in (b,d,f) for [1] is set to 0 as the mesh discretization level is not specified.

5 Conclusions

Pantographic lattices are structures that can support large nonlinear deformations. They can be modeled in particular by Kirchhoff (Euler-Bernoulli) beam

Table 1 Test a: for $u_{BC} = 0.0567\xi$, applied tension, angle at the middle of the specimen, and elongation of the fiber indicated with an arrow in Fig. 3 for the isogeometric Kirchhoff-rod lattice with different beam densities, the isogeometric continuum model, and the continuum model of [1]. Values are measured graphically for [1] and otherwise correspond to the finest refinement level with $p = 4$.

	Force (N)	Angle ($^\circ$)	Elongation (%)
IGA lattice ($e = 0.0096$ m)	67.1	24.1	3.51
IGA lattice ($e = 0.0048$ m)	62.0	21.9	4.88
IGA lattice ($e = 0.0024$ m)	59.6	21.6	5.04
IGA lattice ($e = 0.0012$ m)	58.8	21.4	4.75
IGA continuum ($e = 0$)	58.6	21.2	4.66
Continuum [1] ($e = 0$)	60.8	19.6	4.74

Fig. 6 Test b (a,b) c (c,d) and d (e,f): Discrete (a,c,d) and continuum (b,d,f) models colored following the total strain energy density (N/m).

lattices, or by second-order continuum models. However, these two theories possess second-order derivatives into their weak forms, such that the classical

finite element method cannot be directly employed. We propose the use of IGA based on B-splines with high-degree continuous basis functions for the finite-element approximation.

The discrete and continuum models are compared and are found in excellent agreement. Moreover, if the beam density of the lattice is increased it converges to the continuum model. This validation opens the door to more complex isogeometric simulations of pantographic lattices, involving for instance three-dimensional buckling [20, 19] or inhomogeneous fiber distributions [19, 21].

Acknowledgements The Research Fund KU Leuven is gratefully acknowledged for its support. This research was also partially supported by Flanders Make, the strategic research center for the manufacturing industry. The research of F. Maurin is funded by an Experienced Researcher grant within the European ANTARES Project, under the FP7 Marie Curie Programme (GA606817).

References

1. F. Dell’Isola, I. Giorgio, M. Pawlikowski, and N. L. Rizzi, “Large deformations of planar extensible beams and pantographic lattices: heuristic homogenization, experimental and numerical examples of equilibrium,” *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, vol. 472, no. 2185, p. 20150790, 2016.
2. I. Giorgio, “Numerical identification procedure between a micro-Cauchy model and a macro-second gradient model for planar pantographic structures,” *Zeitschrift für angewandte Mathematik und Physik*, vol. 67, p. 95, aug 2016.
3. A. Cazzani, M. Malagù, and E. Turco, “Isogeometric analysis of plane-curved beams,” *Mathematics and Mechanics of Solids*, vol. 21, no. 5, pp. 562–577, 2014.
4. P. Wriggers, *Nonlinear finite element methods*. Springer Science & Business Media, 2008.
5. F. Dell’Isola, A. Della Corte, I. Giorgio, and D. Scerrato, “Pantographic 2D sheets: Discussion of some numerical investigations and potential applications,” *International Journal of Non-Linear Mechanics*, vol. 80, pp. 200–208, apr 2016.
6. E. Turco, F. Dell’Isola, A. Cazzani, and N. L. Rizzi, “Hencky-type discrete model for pantographic structures: numerical comparison with second gradient continuum models,” *Zeitschrift für angewandte Mathematik und Physik*, vol. 67, p. 85, aug 2016.
7. L. Placidi, E. Barchiesi, E. Turco, and N. L. Rizzi, “A review on 2D models for the description of pantographic fabrics,” *Zeitschrift für angewandte Mathematik und Physik*, vol. 67, p. 121, oct 2016.
8. E. Turco, M. Golaszewski, I. Giorgio, and F. D’Annibale, “Pantographic lattices with non-orthogonal fibres: Experiments and their numerical simulations,” *Composites Part B: Engineering*, vol. 118, pp. 1–14, jun 2017.
9. E. Turco, M. Golaszewski, A. Cazzani, and N. L. Rizzi, “Large deformations induced in planar pantographic sheets by loads applied on fibers: Experimental validation of a discrete Lagrangian model,” *Mechanics Research Communications*, vol. 76, pp. 51–56, sep 2016.
10. L. Greco, I. Giorgio, and A. Battista, “In plane shear and bending for first gradient inextensible pantographic sheets: numerical study of deformed shapes and global constraint reactions,” *Mathematics and Mechanics of Solids*, vol. 22, pp. 1950–1975, oct 2017.
11. M. Cuomo, F. Dell’Isola, L. Greco, and N. L. Rizzi, “First versus second gradient energies for planar sheets with two families of inextensible fibres: Investigation on deformation boundary layers, discontinuities and geometrical instabilities,” *Composites Part B: Engineering*, vol. 115, pp. 423–448, apr 2017.

12. M. Cuomo, F. Dell’Isola, and L. Greco, “Simplified analysis of a generalized bias test for fabrics with two families of inextensible fibres,” *Zeitschrift für angewandte Mathematik und Physik*, vol. 67, p. 61, jun 2016.
13. L. Placidi, L. Greco, S. Bucci, E. Turco, and N. L. Rizzi, “A second gradient formulation for a 2D fabric sheet with inextensible fibres,” *Zeitschrift für angewandte Mathematik und Physik*, vol. 67, p. 114, oct 2016.
14. F. Dell’Isola, A. Della Corte, L. Greco, and A. Luongo, “Plane bias extension test for a continuum with two inextensible families of fibers: A variational treatment with Lagrange multipliers and a perturbation solution,” *International Journal of Solids and Structures*, vol. 81, pp. 1–12, mar 2016.
15. Y. Rahali, I. Giorgio, J. Ganghoffer, and F. Dell’Isola, “Homogenization à la Piola produces second gradient continuum models for linear pantographic lattices,” *International Journal of Engineering Science*, vol. 97, pp. 148–172, dec 2015.
16. F. Dell’Isola, M. Cuomo, L. Greco, and A. Della Corte, “Bias extension test for pantographic sheets: numerical simulations based on second gradient shear energies,” *Journal of Engineering Mathematics*, vol. 103, pp. 127–157, apr 2017.
17. F. Dell’Isola, T. Lekszycki, M. Pawlikowski, R. Grygoruk, and L. Greco, “Designing a light fabric metamaterial being highly macroscopically tough under directional extension: first experimental evidence,” *Zeitschrift für angewandte Mathematik und Physik*, vol. 66, pp. 3473–3498, dec 2015.
18. I. Giorgio, R. Grygoruk, F. Dell’Isola, and D. J. Steigmann, “Pattern formation in the three-dimensional deformations of fibered sheets,” *Mechanics Research Communications*, vol. 69, pp. 164–171, oct 2015.
19. I. Giorgio, A. Della Corte, F. Dell’Isola, and D. J. Steigmann, “Buckling modes in pantographic lattices,” *Comptes Rendus Mécanique*, vol. 344, pp. 487–501, jul 2016.
20. I. Giorgio, N. L. Rizzi, and E. Turco, “Continuum modelling of pantographic sheets for out-of-plane bifurcation and vibrational analysis,” *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, vol. 473, p. 20170636, nov 2017.
21. D. Scerrato, I. Giorgio, and N. L. Rizzi, “Three-dimensional instabilities of pantographic sheets with parabolic lattices: numerical investigations,” *Zeitschrift für angewandte Mathematik und Physik*, vol. 67, p. 53, jun 2016.
22. T. Hughes, J. Cottrell, and Y. Bazilevs, “Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement,” *Computer Methods in Applied Mechanics and Engineering*, vol. 194, no. 39, pp. 4135–4195, 2005.
23. J. A. Cottrell, T. J. R. Hughes, and Y. Bazilevs, *Isogeometric Analysis Toward Intergration of CAD and FEA*. John Wiley & Sons, 2009.
24. F. Auricchio, L. B. da Veiga, A. Buffa, C. Lovadina, A. Reali, and G. Sangalli, “A fully ”locking-free” isogeometric approach for plane linear elasticity problems: A stream function formulation,” *Computer Methods in Applied Mechanics and Engineering*, vol. 197, pp. 160–172, dec 2007.
25. H. Gómez, V. M. Calo, Y. Bazilevs, and T. J. R. Hughes, “Isogeometric analysis of the CahnHilliard phase-field model,” *Computer Methods in Applied Mechanics and Engineering*, vol. 197, pp. 4333–4352, sep 2008.
26. A. Tagliabue, L. Dedè, and A. Quarteroni, “Isogeometric Analysis and error estimates for high order partial differential equations in fluid dynamics,” *Computers & Fluids*, vol. 102, pp. 277–303, oct 2014.
27. J. Kiendl, K. U. Bletzinger, J. Linhard, and R. Wüchner, “Isogeometric shell analysis with Kirchhoff-Love elements,” *Computer Methods in Applied Mechanics and Engineering*, vol. 198, no. 49-52, pp. 3902–3914, 2009.
28. S. B. Raknes, X. Deng, Y. Bazilevs, D. J. Benson, K. M. Mathisen, and T. Kvamsdal, “Isogeometric rotation-free bending-stabilized cables: Statics, dynamics, bending strips and coupling with shells,” *Computer Methods in Applied Mechanics and Engineering*, vol. 263, pp. 127–143, aug 2013.
29. F. Maurin, L. Dedè, and A. Spadoni, “Isogeometric rotation-free analysis of planar extensible-elastica for static and dynamic applications,” *Nonlinear Dynamics*, vol. 81, pp. 77–96, mar 2015.
30. L. Greco and M. Cuomo, “B-Spline interpolation of Kirchhoff-Love space rods,” *Computer Methods in Applied Mechanics and Engineering*, vol. 256, no. 0, pp. 251–269, 2013.

31. L. Greco and M. Cuomo, "An implicit G1 multi patch B-spline interpolation for Kirchhoff-Love space rod," *Computer Methods in Applied Mechanics and Engineering*, vol. 269, no. 0, pp. 173–197, 2014.
32. A. Borković, S. Kovačević, G. Radenković, S. Milovanović, and M. Guzijan-Dilber, "Rotation-free isogeometric analysis of an arbitrarily curved plane BernoulliEuler beam," *Computer Methods in Applied Mechanics and Engineering*, feb 2018.
33. S. F. Hosseini, B. Moetakef-Imani, S. Hadidi-Moud, and B. Hassani, "The effect of parameterization on isogeometric analysis of free-form curved beams," *Acta Mechanica*, vol. 227, no. 7, pp. 1983–1998, 2016.
34. R. Zhang and H. Zhong, "Weak form quadrature element analysis of planar slender beams based on geometrically exact beam theory," *Archive of Applied Mechanics*, vol. 83, pp. 1309–1325, apr 2013.
35. L. Piegl and W. Tiller, *The NURBS Book*. Springer Science & Business Media, 1996.
36. L. Leonetti, F. Liguori, D. Magisano, and G. Garcea, "An efficient isogeometric solid-shell formulation for geometrically nonlinear analysis of elastic shells," *Computer Methods in Applied Mechanics and Engineering*, dec 2017.
37. D. Magisano, L. Leonetti, and G. Garcea, "How to improve efficiency and robustness of the Newton method in geometrically non-linear structural problem discretized via displacement-based finite elements," *Computer Methods in Applied Mechanics and Engineering*, vol. 313, pp. 986–1005, 2017.
38. D. Magisano, L. Leonetti, and G. Garcea, "Advantages of the mixed format in geometrically nonlinear analysis of beams and shells using solid finite elements," *International Journal for Numerical Methods in Engineering*, vol. 109, no. 9, pp. 1237–1262, 2017.
39. F. Maurin, F. Greco, S. Dedoncker, and W. Desmet, "Isogeometric analysis for nonlinear planar Kirchhoff rods : weighted residual formulation and collocation of the strong form," *Submitted for publication*, 2018.
40. M. Spink, D. Claxton, C. de Falco, and R. Vazquez, "The NURBS toolbox," 2017.
41. C. de Falco, A. Reali, and R. Vázquez, "GeoPDEs: A research tool for Isogeometric Analysis of PDEs," *Advances in Engineering Software*, vol. 42, pp. 1020–1034, dec 2011.