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Du B, Zhou H, Leus R.

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# A two-stage robust model for a reliable $p$-center facility location problem

Bo Du[a], Hong Zhou[a], Roel Leus[b,*]

[a]*School of Economics and Management, Beihang University, Beijing, China*
[b]*ORSTAT, Faculty of Economics and Business, KU Leuven, Leuven, Belgium*

**Abstract**

We propose a two-stage robust model for reliable facility location when some facilities can be disrupted, for instance by a natural disaster. A reliable network is designed in a "proactive" planning phase, and when a facility is disrupted, its original clients can be reallocated to another available facility in a "reactive" phase. When demand and cost are uncertain, the initial design is also robust against the realizations (scenarios) of these data that will only be revealed post-disruption. Based on the $p$-center location model, which attempts to optimize the worst-case performance of the network, our model is concerned with the reliability for every client. Three solution methods have been implemented and tested to solve the model; we present an extensive numerical study to compare the performance of these methods. We find that, depending on the size of the instance (as given by the number of client sites and scenarios), either a *Benders dual cutting plane* method or a *column-and-constraint generation* method performs best. The effectiveness of our model is also examined in comparison with alternative facility location models.

*Keywords:* facility location, uncertainty, reliable network design, $p$-center problem, robust optimization

## 1. Introduction

The goal of facility location problems is to find appropriate locations for facilities, and to allocate them to clients. In practice, facility location is typically a strategic, long-term decision, and faces many uncertainties (for instance regarding demand, costs, facility reliability, etc.). Despite the severe uncertainties surrounding emergency situations such as a natural disaster or a terrorist attack, the pre-located facilities need to serve their clients immediately and equally (in a fair manner) in the aftermath, for humanitarian purposes. In this paper, we focus on locating facilities in an emergency logistics system, which aims to promptly deliver relief supplies or provide other emergency services (e.g., police or medical service) from facilities to every client.

In most studies, methods of stochastic optimization (SO) or robust optimization (RO) are applied to solve facility location problems under uncertainty, depending on whether probabilistic information is available or not (Laporte et al., 2015). Under many circumstances, acquiring accurate probability distributions is virtually impossible (An et al., 2014), and so in this work we apply the RO modeling approach, which is suitable for decision making with various uncertainty sets (e.g., interval, discrete set, polyhedral, box set, ellipsoidal set, etc.), each with different tractability and conservativeness (Ben-Tal et al., 2009; Bertsimas et al., 2011, 2004; Kouvelis and Yu, 1997). RO can be applied in many areas, including facility location (Lu, 2013; Opher et al., 2011). Two-stage robust optimization (2SRO) has also been used in more general facility location problems (Álvarez-Miranda et al., 2015; An et al., 2014; Atamtürk and Zhang, 2007; Caunhye et al., 2016; Gabrel et al., 2014).

When natural disasters occur, e.g. earthquakes or floods, some facilities are likely to be unavailable due to the damage of transportation infrastructure or facilities. To deal with this situation, a number of research

*Corresponding author
*Email address:* Roel.Leus@kuleuven.be (Roel Leus)

efforts have focused on defining and solving reliable facility location problems, which typically build a reliable network proactively and subsequently reassign clients reactively after facility disruption (Cui et al., 2010; Li et al., 2013; Shen et al., 2011; Snyder and Daskin, 2005). Before the occurrence of a disruptive incident, the decision maker may not be able to acquire exact information, but can only establish pre-disruption location planning by describing a number of plausible demand and cost scenarios; a post-disruption recourse decision can be made after new information is obtained. Unlike An et al. (2014) and Álvarez-Miranda et al. (2015), who both concentrate on optimizing the total performance (summed over all clients) of the supply network, we apply the $p$-center model, which minimizes the worst performance of the network: the model pursues a minimax solution, by minimizing the maximum cost between any demand point and its associated supply point. It is practical especially for situations in which service equity is more important than overall performance, such as locations for ambulances or police stations (Laporte et al., 2015).

Based on the classic uncapacitated $p$-center model, we propose a two-stage robust location model that considers both pre-disruption planning as well as post-disruption reallocation, in order to produce a reliable network design. We implement and test three solution methods, namely a *linear reformulation*, a *Benders dual cutting plane* method and a *column-and-constraint generation* method. The performance of the three methods is compared by means of a series of numerical experiments. We also compare our model with the general $p$-center model, with two-stage SO, and with two-stage RO for the $p$-median problem.

The remainder of this paper is organized as follows. In Section 2 we review the relevant literature on facility location with and without uncertainty, with a focus mainly on RO methods. As mentioned above, most research on robust facility location problems aims to minimize total cost (w.l.o.g., distance or time), but in case of emergencies that require service equity, every client site needs to be served quickly and fairly, which motivates us to study the $p$-center facility location model in the framework of 2SRO. In Section 3, we propose a two-stage robust model for a reliable $p$-center problem with consideration of pre-disruption planning and post-disruption reallocation, which minimizes the weighted sum of the maximum cost between clients and the facilities to which they are allocated in both the pre-disruption and the post-disruption stage. In Section 4 we describe three solution methods for this model, and we examine two alternative location models in Section 5. We report on a number of numerical experiments in Section 6, where we look into the performance of the solution methods and compare our model with the two alternatives. We conclude the paper in Section 7, where we also discuss some potential avenues for future research.

## 2. Literature review

Over the last decades, abundant research has been published on different facility location models, the most important ones being set covering, $p$-median, $p$-center and fixed-charge facility location models (Daskin, 2013; Laporte et al., 2015), which have different objectives and are applicable in different situations. Jia et al. (2007) use the Los Angeles area as an illustrative example, for which they propose three facility location models with different goals to suit different needs in large-scale emergencies. Instead of focusing on minimizing total cost, such as in the $p$-median or fixed-charge problem, the $p$-center problem (also called minimax location problem) optimizes the maximum distance/cost between a facility and the corresponding clients. According to Laporte et al. (2015), Huang et al. (2010), and Averbakh and Berman (1998), the $p$-center problem is concerned especially with clients "who are served the poorest," which is a fair approach in emergency situations. A detailed review of variants of the $p$-center problem and solution methods is also provided by Laporte et al. (2015). Compared to the $p$-median problem, however, the $p$-center problem appears to be harder to solve (Mladenovic et al., 2003).

Static facility location models are unsuitable in many cases, and research on facility location under uncertainty has already been conducted over the last few years. Shen et al. (2011) and Álvarez-Miranda et al. (2015) classify uncertainties in facility location problems as provider-side uncertainty, receiver-side uncertainty, and in-between uncertainty. They can also be categorized as parameter (data) uncertainty and facility disruption uncertainty. In reality, the various parameters, including cost, demand, distance, etc., can indeed be highly uncertain. A detailed review of facility location with uncertain parameters and their solution methods can be found in Laporte et al. (2015) and Snyder (2006). Facility disruption can cause serious damage to a distribution network. Based on standard facility location models, Snyder and

Daskin (2005) are the first to propose a reliable $p$-median model and a reliable uncapacitated fixed-charge location (RUFL) model to minimize the weighted sum of pre-disruption transportation cost and expected post-disruption transportation cost, where each client can be assigned to one of multiple backup facilities and all facilities have the same failure probability. Lagrangian relaxation is used for producing solutions. Based on RUFL and with site-dependent failure probabilities, Cui et al. (2010) present a mixed-integer formulation and a continuum approximation model. Shen et al. (2011) build a scenario-based stochastic programming formulation for RUFL and a single-stage nonlinear integer programming model; heuristic methods are proposed to obtain near-optimal solutions. Li et al. (2013) apply a "fortification" strategy in reliable facility location problems, which enables the decision maker to invest in a facility to make it totally reliable. For an overview of other methods for reliable facility location, we refer to the reviews by Laporte et al. (2015) and Snyder et al. (2016).

As mentioned supra, in this article we will opt for the RO framework for modelling uncertainty. Opher et al. (2011) are the first to apply RO in facility location, using two different uncertainty sets for demand in a fixed-charge location model. Lu (2013) uses interval uncertainty to represent demand and travel times, aiming to minimize the worse-case deviation of the objective for a weighted vertex $p$-center problem. Due to the two-stage nature of the facility location problem (location-allocation) and its extensions (location-transportation, location-routing, reliable facility location), two-stage stochastic optimization (2SSO) and two-stage robust optimization (2SRO) are typically applied to deal with these problems. The idea is to make partial decisions in the first stage and subsequently decide a second-stage (recourse) strategy after the realization of new information. The objective of 2SSO is to minimize the cost in the first stage plus the expected cost in the second stage; 2SRO proceeds similarly but minimizes the second-stage cost in the worst case. Another RO method to model multistage decision making is robust adaptable optimization (Bertsimas et al., 2011), or robust adjustable optimization (Ben-Tal et al., 2009), where an affinely adjustable robust counterpart is formulated as an approximation of a general robust multistage optimization problem. Atamtürk and Zhang (2007) and Gabrel et al. (2014) apply 2SRO for a location-transportation problem. Caunhye et al. (2016) propose a two-stage robust location-routing model. Contrary to the foregoing work, where all location decisions are made in the first stage, An et al. (2014) propose a 2SRO model for two-stage reliable $p$-median facility location, allowing for reallocation decisions in the second stage to compensate the disruption, and assuming that demand may change together with facility disruptions. Álvarez-Miranda et al. (2015) solve a two-stage fixed-charge location problem using a recoverable robust optimization method, in which both location and allocation decisions are made in two stages, with potential reallocation in the second stage. In general, several methods can be applied to solve 2SRO problems, including variations of Benders decomposition (Álvarez-Miranda et al., 2015; Gabrel et al., 2014; Thiele et al., 2010), column-and-constraint generation (An et al., 2014; Zeng and Zhao, 2013), and linearization to a single-stage counterpart (Caunhye et al., 2016).

## 3. Two-stage robust model for reliable $p$-center problem

In this section, we present a two-stage robust model for a reliable $p$-center problem based on the (deterministic) uncapacitated $p$-center model.

### 3.1. The deterministic problem

The $p$-center problem is to locate $p$ facilities to serve a given client set while minimizing the maximum cost between clients and the facilities to which they are allocated. The input of an instance consists of the set $I$ of client sites and the set $J$ of potential location sites. Each client $i \in I$ has a known demand $d_i$, and we also take the transportation cost (distance or time) $c_{ij}$ per unit between each client site $i \in I$ and facility site $j \in J$ as input. The demand $d_i$ from client $i$ is always served by an open facility "closest" to $i$, which means the open facility $j$ with lowest cost $c_{ij}$. The decision variables $x_{ij}$ and $y_j$ are defined as follows: $y_j = 1$ if potential facility site $j$ is open, $y_j = 0$ otherwise; $x_{ij} = 1$ if client site $i$ is assigned to facility site $j$, $x_{ij} = 0$ otherwise. A mixed-integer linear formulation of the uncapacitated $p$-center problem is then PCENTER, based on Daskin (2013):

**PCENTER**

$$\min_{\mathbf{x},\mathbf{y}} \quad L \tag{1}$$

$$\text{subject to} \quad L \geqslant \sum_{j\in J} c_{ij} d_i x_{ij} \qquad \forall i \in I \tag{2}$$

$$\sum_{j\in J} y_j = p \tag{3}$$

$$x_{ij} \leqslant y_j \qquad \forall i \in I, j \in J \tag{4}$$

$$\sum_{j\in J} x_{ij} = 1 \qquad \forall i \in I \tag{5}$$

$$x_{ij}, y_j \in \{0,1\} \qquad \forall i \in I, j \in J \tag{6}$$

The objective function (1) together with constraint set (2) minimize the maximum cost between clients and the facilities to which they are allocated. Constraint (3) requires that the total number of facilities to be opened equal $p$ (with the implicit assumption that $p \leq |J|$). Constraints (4) and (5) ensure that clients can only be assigned to an open facility, and that every single client is served. Constraints (6) require all variables to be binaries. Kariv and Hakimi (1979) prove that this $p$-center problem is NP-hard.

*3.2. A two-stage robust model*

Based on the deterministic problem formulated in PCENTER, a two-stage robust model for reliable $p$-center facility location is proposed. In this reliable $p$-center problem, initial location and allocation decisions are made before disruption based on estimated data, and a recourse decision is made with new information after disruption, which leads to cost changes and facility site perturbations. The first stage of 2SRO is the same as PCENTER, except that all $d_i$ and $c_{ij}$ take a nominal/estimated value. The second stage of 2SRO reallocates clients to surviving facilities based on updated demand and cost information. Overall, the goal of the two-stage problem is to minimize the weighted sum of the maximum cost between clients and the facilities to which they are allocated in both pre-disruption and post-disruption stage.

Contrary to the cardinality-constrained uncertainty set of An et al. (2014), in which the number of disrupted facilities is limited, we employ an uncertainty set of discrete scenarios in our study, where a scenario contains information on disrupted facility sites, modified demand of clients and updated client/facility costs. We work with a scenario set $K$, as follows: for each $k \in K$, if potential facility site $j$ is unavailable then $a_j(k) = 1$, otherwise $a_j(k) = 0$. Several facility sites may be unavailable in each scenario, and a disrupted site cannot be used again. We assume $\sum_{j\in J} a_j(k) < p$ to prevent infeasibility. In the second stage, for scenario $k$, the unit transportation cost between potential facility $j$ and client $i$ is denoted by $c^2_{ij}(k)$, and the updated demand of client $i$ is $d^2_i(k)$. Due to facility disruptions, some clients may not be served as in the initial planning. (Re)allocation decisions in the second stage are represented by decision variables $\mathbf{w}$, with $w_{ij}(k) = 1$ if client site $i$ is assigned to facility site $j$ in scenario $k$, and $w_{ij}(k) = 0$ otherwise. Following earlier work (An et al., 2014; Cui et al., 2010; Snyder and Daskin, 2005), we assume that $I = J$, which makes PCENTER a so-called vertex $p$-center problem.

Our formulation representing a two-stage robust model for the reliable $p$-center problem is given in 2SRO, with a second-stage subproblem in SP:

**2SRO**

$$
\begin{aligned}
\min_{\mathbf{x},\mathbf{y}} \quad & \alpha_1 L_1 + \alpha_2 Q(\mathbf{y}) && && (7)\\
\text{subject to} \quad & L_1 \geqslant \sum_{j\in J} c_{ij} d_i x_{ij} && \forall i \in I && (8)\\
& \sum_{j\in J} y_j = p && && (9)\\
& x_{ij} \leqslant y_j && \forall i \in I, j \in J && (10)\\
& \sum_{j\in J} x_{ij} = 1 && \forall i \in I && (11)\\
& x_{ij}, y_j \in \{0,1\} && \forall i \in I, j \in J && (12)
\end{aligned}
$$

where $Q(\mathbf{y})$ is defined in SP:

**SP**

$$
\begin{aligned}
Q(\mathbf{y}) = \max_{k\in K} \min_{\mathbf{w}} \quad & L_2(k) && && (13)\\
\text{subject to} \quad & L_2(k) \geqslant \sum_{j\in J} c_{ij}^2(k) d_i^2(k) w_{ij}(k) && \forall i \in I, k \in K && (14)\\
& w_{ij}(k) \leqslant y_j && \forall i \in I, j \in J, k \in K && (15)\\
& w_{ij}(k) \leqslant 1 - a_j(k) && \forall i \in I, j \in J, k \in K && (16)\\
& \sum_{j\in J} w_{ij}(k) = 1 && \forall i \in I, k \in K && (17)\\
& w_{ij}(k) \in \{0,1\},\ L_2(k) \geqslant 0 && \forall i \in I, j \in J, k \in K && (18)
\end{aligned}
$$

The objective function (7) consists of two terms weighted with coefficients $\alpha_1$ and $\alpha_2$, where $L_1$ is the maximum transportation cost between an opened facility and a client in the first stage, and the constraints (8) to (11) are the same as (2) to (5) in PCENTER. For the second-stage subproblem in SP, the quantity $L_2(k)$ in the objective function (13) and constraints (14) is the maximum transportation cost between an opened facility and a client under scenario $k$. Constraints (15) and (16) ensure that clients can only be served by an opened and available facility. Constraints (17) have the same function as (11). We conclude that the objective function (7) seeks to minimize the weighted sum of the maximum transportation cost between facilities and clients in the first stage and in the worst scenario of the second stage.

### *3.3. Illustrative example*

Figures 1 and 2 show the output of the reliable $p$-center problem and the deterministic $p$-center problem, respectively, for an uncapacitated six-site network ($|I| = |J| = 6$). Two facilities are to be opened under normal conditions before disruption ($p = 2$), and we assume that one of the six sites will be disabled after disruption, leading to six possible disruption scenarios ($|K| = 6$). We do not consider demand and cost uncertainty, so $c_{ij}^2(k) = c_{ij}$ and $d_i^2(k) = d_i$ for each scenario $k \in K$.

In Figures 1(a) and 2(a), the first-stage solutions of the two models (before disruption) are presented. The two models make different decisions, and the solution to the deterministic model achieves a lower maximum cost between clients and facilities. In Figure 2(a), the site at the top right only serves itself, since it is a client and open facility at the same time (remember that we assume $I = J$). The reallocation decisions in the second stage are depicted in Figures 1(b) and 2(b). For both models, the disruption affects one of the initially opened facilities, and unserved clients are reassigned to a surviving (opened) facility. We see that the solution generated by the two-stage model now has a better performance. The example shows how a sacrifice in the first-stage maximum cost can lead to a robust performance upon facility disruption.

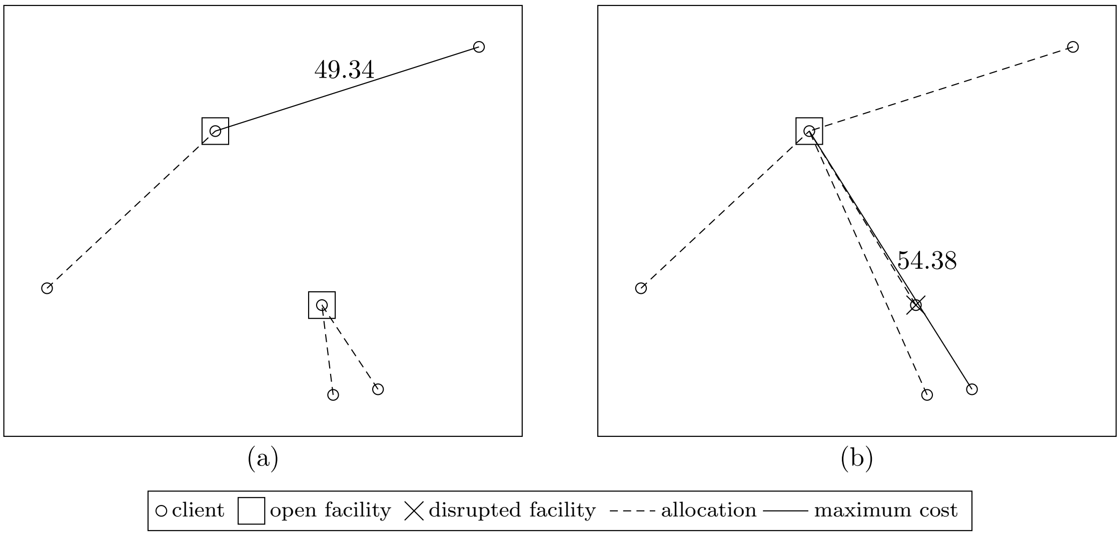


Figure 1: Solution to an example instance of the reliable $p$-center problem (a) under normal conditions, (b) for one disruption scenario

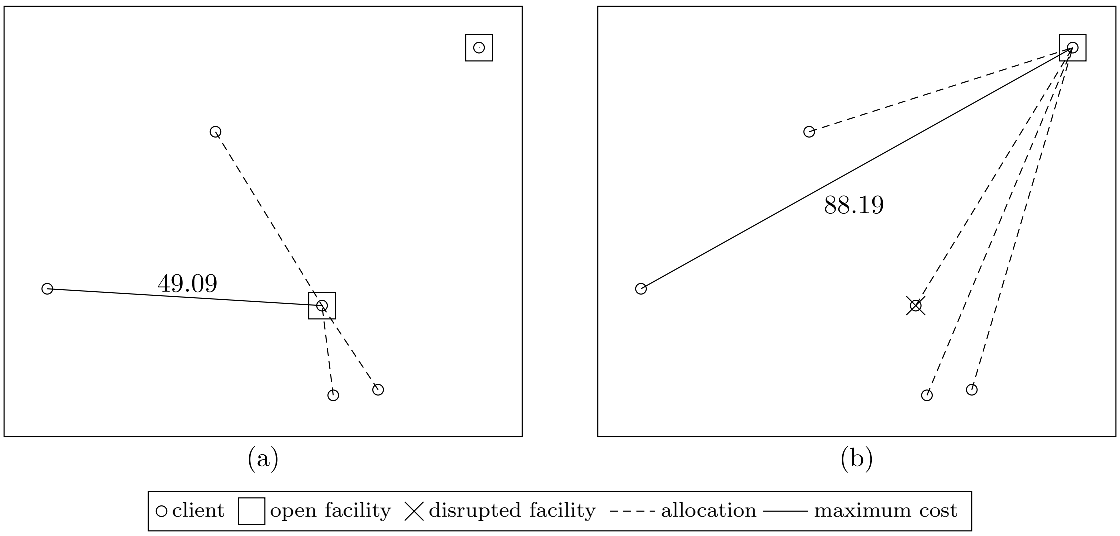


Figure 2: Solution to an example instance of the deterministic $p$-center problem (a) under normal conditions, (b) for one disruption scenario

## 4. Solution methods

We have implemented three different solution methods for the two-stage RO model 2SRO, which are described below. A report on their numerical performance will be given in Section 6.

### *4.1. Linear integer reformulation*

Caunhye et al. (2016) utilize the epigraph form of their second-stage problem to solve a two-stage RO model, converting the model into a single-stage counterpart. In 2SRO, the objective function (7) can be transformed into equation (19) together with constraints (20):

**LIP**

$$\min_{\mathbf{x},\mathbf{y}} \quad \alpha_1 L_1 + \alpha_2 L_{\max} + \epsilon \sum_{k \in K} L_2(k) \tag{19}$$

$$\text{subject to} \quad \text{constraints (8) to (12) and (14) to (18)}$$

$$L_{\max} \geqslant L_2(k) \qquad \forall k \in K \tag{20}$$

where $\epsilon$ is a small positive number, which ensures that $L_2(k)$ is minimized for every scenario $k \in K$. Then LIP can be solved by a MIP (mixed-integer programming) solver.

### *4.2. Benders dual cutting plane method*

Although the method in Section 4.1 is straightforward, it will struggle when the number $|K|$ of scenarios is too large, leading to an excessive number of variables and constraints. To deal with this issue, Benders decomposition is used, based on delayed constraint generation and the cutting plane method (Benders, 1962; Geoffrion, 1972). Following this approach, a Benders dual cutting plane (BD) method is designed to solve the two-stage RO problem by dualizing the second-stage problem (following Zeng and Zhao, 2013). BD solves a two-stage RO problem within a master-subproblem framework. With the assumption $\sum_{j \in J} a_j(k) < p$ stated earlier, the reallocation problems (second-stage subproblems) for all scenarios $k$ in SP are feasible for any given first-stage solution $(\mathbf{x}^*, \mathbf{y}^*)$. We can solve SP by constructing the dual of the inner minimization, but this requires the inner minimization to be continuous.

**Observation 1.** *In an uncapacitated discrete p-center model or a two-stage uncapacitated discrete p-center model, all allocation variables* $\mathbf{x} \in \{0,1\}^{|I|\cdot|J|}$ *can be relaxed to* $\mathbf{x} \in [0,1]^{|I|\cdot|J|}$.

Each client is always served by its "closest" facility to minimize the cost, unless more than one facility provides the same cost to a client. In the latter case, regardless of the assignment of the client to any of those facilities, the total cost remains the same. According to Observation 1, the integrality constraint on the allocation variables $\mathbf{w}$ can be relaxed.

Let $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ be the dual variables corresponding to constraints (14) to (17). For a given first-stage solution $(\mathbf{x}^*, \mathbf{y}^*)$, SP can be transformed into:

**DUAL-SP**

$$Q^D(\mathbf{y}^*) = \max_{k \in K} \max_{\mathbf{e},\mathbf{f},\mathbf{g},\mathbf{h}} \quad \sum_{i \in I} \sum_{j \in J} y_j^* f_{ij} + \sum_{i \in I} \sum_{j \in J} (1 - a_j(k)) g_{ij} + \sum_{i \in I} h_i \tag{21}$$

$$\text{subject to} \quad c_{ij}^2(k) d_i^2(k) e_i + f_{ij} + g_{ij} + h_i \leqslant 0 \qquad \forall i \in I, j \in J \tag{22}$$

$$-\sum_{i \in I} e_i \leqslant 1 \tag{23}$$

$$e_i, f_{ij}, g_{ij} \leqslant 0, \; h_i \; \textit{free} \qquad \forall i \in I, j \in J \tag{24}$$

which can be merged into a single-stage max-problem. We can now rewrite 2SRO as follows:

**MP-BD$_1$**

$$\min_{\mathbf{x},\mathbf{y}} \quad \alpha_1 L_1 + \alpha_2 \max_{k\in K} \max_{(\mathbf{e},\mathbf{f},\mathbf{g},\mathbf{h})\in S^B(k)} \left\{ \sum_{i\in I}\sum_{j\in J} y_j f_{ij} + \sum_{i\in I}\sum_{j\in J}(1-a_j(k))g_{ij} + \sum_{i\in I} h_i \right\} \tag{25}$$

$$\text{subject to} \quad \text{constraints (8) to (11)}$$

$$x_{ij} \geqslant 0, \; y_j \in \{0,1\} \qquad \forall i \in I, j \in J$$

with

$$S^B(k) = \{(\mathbf{e},\mathbf{f},\mathbf{g},\mathbf{h}) \mid \text{constraints (22) to (24) hold}\}$$

205 Define $U$ to be the set of all extreme points for the inner max-problem in (25). The optimal solution will be attained in one of the extreme points $(\mathbf{e}^u, \mathbf{f}^u, \mathbf{g}^u, \mathbf{h}^u)$, $u \in U$, and MP-BD$_1$ can be reformulated as:

**MP-BD$_2$**

$$\min_{\mathbf{x},\mathbf{y}} \quad \alpha_1 L_1 + \alpha_2 \max_{k\in K, u\in U} \left\{ \sum_{i\in I}\sum_{j\in J} y_j f^u_{ij} + \sum_{i\in I}\sum_{j\in J}(1-a_j(k))g^u_{ij} + \sum_{i\in I} h^u_i \right\} \tag{26}$$

$$\text{subject to} \quad \text{constraints (8) to (11)}$$

$$x_{ij} \geqslant 0, \; y_j \in \{0,1\} \qquad \forall i \in I, j \in J$$

This in turn can be linearized via a continuous variable $\eta_1$:

**MP-BD**

$$\min_{\mathbf{x},\mathbf{y},\eta_1} \quad \alpha_1 L_1 + \alpha_2 \eta_1 \tag{27}$$

$$\text{subject to} \quad \text{constraints (8) to (11)}$$

$$\eta_1 \geqslant \sum_{i\in I}\sum_{j\in J} y_j f^u_{ij} + \sum_{i\in I}\sum_{j\in J}(1-a_j(k))g^u_{ij} + \sum_{i\in I} h^u_i \qquad \forall k \in K, u \in U \tag{28}$$

$$x_{ij} \geqslant 0, \; y_j \in \{0,1\}, \; \eta_1 \; \textit{free} \qquad \forall i \in I, j \in J \tag{29}$$

This is the full formulation of the Benders master problem. Constraints (28) are referred to as *optimality cuts*. Rather than introduce all optimality cuts in the model immediately, the BD method includes only a subset of these constraints in an iterative fashion. Let $L = K \times U$, $\bar{L} \subseteq L$ and $|\bar{L}| = n$, with $\bar{L}$ the set of all cuts added to the master problem at iteration $n$ of the BD algorithm. The formulation of the master problem now becomes:

**MP$_1$**

$$\min_{\mathbf{x},\mathbf{y},\eta_1} \quad \alpha_1 L_1 + \alpha_2 \eta_1 \tag{30}$$

$$\text{subject to} \quad \text{constraints (8) to (11)}$$

$$\eta_1 \geqslant \sum_{i\in I}\sum_{j\in J} y_j f^l_{ij} + \sum_{i\in I}\sum_{j\in J}(1-a_j(k^l))g^l_{ij} + \sum_{i\in I} h^l_i \qquad \forall l \in \bar{L} \tag{31}$$

$$x_{ij} \geqslant 0, \; y_j \in \{0,1\}, \; \eta_1 \; \textit{free} \qquad \forall i \in I, j \in J \tag{32}$$

where $k^l$ is the scenario corresponding to cut $l$, and $f^l_{ij}$, $g^l_{ij}$ and $h^l_i$ are an optimal solution to **DUAL-SP** in cut $l$. The cutting planes in (31) iteratively tighten the objective function (30) via $\eta_1$. Algorithm 1 shows the structure of our BD method for solving 2SRO, where $\sigma$ is a small positive number that defines the stopping criterion. The objective value of MP$_1$ constitutes a lower bound $LB$, because MP$_1$ only includes part of the constraints in MP-BD. By iteratively adding cutting planes to MP$_1$, the objective value is increased until convergence. The value of $\alpha_1 L_1 + \alpha_2 Q^D_n$ then updates the upper bound, since it is the objective value of 2SRO for the solution found in iteration $n$.

**Algorithm 1:** Benders dual cutting plane method

1: Set $LB = -\infty$, $UB = \infty$, $n = 0$
2: Solve MP$_1$ and obtain an optimal solution $(\mathbf{x}^n, \mathbf{y}^n, \eta_1^n)$; set $LB$ as the optimal objective value of MP$_1$
3: Solve DUAL-SP with respect to $\mathbf{y}^n$ and derive an optimal solution $(\mathbf{e}^n, \mathbf{f}^n, \mathbf{g}^n, \mathbf{h}^n)$, the corresponding scenario $k^n$ and its optimal value $Q_n^D$. Update

$$UB = \min \left\{UB, \alpha_1 L_1 + \alpha_2 Q_n^D\right\}$$

4: If $gap = \frac{UB-LB}{LB} \leqslant \sigma$, terminate. Otherwise, generate the constraints associated with the identified scenario and add them to MP$_1$. Update $n = n+1$. Go to step 2

### *4.3. Column-and-constraint generation algorithm*

When solving a two-stage RO problem, the BD method is sometimes inefficient in dealing with real-size instances (An et al., 2014), mainly because it may take the method many iterations to reach convergence. Based on the BD method, a column-and-constraint generation (C&CG) algorithm has been developed by Zeng and Zhao (2013), which iteratively appends constraints and variables belonging to a new scenario to be included in the model. For 2SRO, the maximum iteration count for this type of C&CG algorithm is the number of scenarios $|K|$. We describe the details of this method below. With a continuous variable $\eta_2$, 2SRO can be transformed into:

**MP-CCG$_1$**

$$\min_{\mathbf{x},\mathbf{y},\eta_2} \quad \alpha_1 L_1 + \alpha_2 \eta_2 \tag{33}$$

$$\text{subject to} \quad \text{constraints (8) to (11)}$$

$$\eta_2 \geqslant \min_{\mathbf{w}\in S^C(k,\mathbf{y})} L_2(k) \qquad \forall k \in K \tag{34}$$

$$x_{ij} \geqslant 0,\ y_j \in \{0,1\},\ \eta_2\ free \qquad \forall i \in I, j \in J \tag{35}$$

with

$$S^C(k,\mathbf{y}) = \{\mathbf{w} \mid \text{constraints (14) to (17) hold, and } w_{ij}(k) \geqslant 0,\ \forall i \in I, j \in J, k \in K\}$$

Obviously, MP-CCG$_1$ is equivalent to:

**MP-CCG**

$$\min_{\mathbf{x},\mathbf{y},\mathbf{w},\eta_2} \quad \alpha_1 L_1 + \alpha_2 \eta_2 \tag{36}$$

$$\text{subject to} \quad \text{constraints (8) to (11)}$$

$$\eta_2 \geqslant L_2(k) \qquad \forall k \in K \tag{37}$$

$$w_{ij}(k) \in S^C(k,\mathbf{y}) \qquad \forall i \in I, j \in J, k \in K \tag{38}$$

$$x_{ij} \geqslant 0,\ y_j \in \{0,1\},\ \eta_2\ free \qquad \forall i \in I, j \in J \tag{39}$$

Instead of dealing with constraints and variables from all scenarios at once, C&CG includes the constraints and variables in (37) and (38) from each scenario in an iterative way. Let $\bar{K} \subseteq K$, $|\bar{K}| = n$, at iteration $n$ of the algorithm. The master problem of the C&CG method is then:

**Algorithm 2:** Column-and-constraint generation method

1: Set $LB = -\infty$, $UB = \infty$, $n = 0$

2: Solve MP$_2$ and obtain an optimal solution $(\mathbf{x}^n, \mathbf{y}^n, \eta_2^n)$; set $LB$ as the optimal objective value of MP$_2$

3: Solve SP with respect to $\mathbf{y}^n$ and identify the scenario $k^n$ and its optimal value $Q_n$. Update

$$UB = \min\{UB, \alpha_1 L_1 + \alpha_2 Q_n\}$$

4: If $gap = \frac{UB-LB}{LB} \leqslant \sigma$, terminate. Otherwise, create the variables $w_{ij}(k^n)$ and the corresponding constraints associated with scenario $k^n$ and add them to MP$_2$. Update $n = n + 1$. Go to step 2

**MP$_2$**

$$\min_{\mathbf{x},\mathbf{y},\mathbf{w},\eta_2} \quad \alpha_1 L_1 + \alpha_2 \eta_2 \tag{40}$$

subject to constraints (8) to (11)

$$\eta_2 \geqslant L_2(k) \qquad \forall k \in \bar{K} \tag{41}$$

$$w_{ij}(k) \in S^C(k, \mathbf{y}) \qquad \forall i \in I, j \in J, k \in \bar{K} \tag{42}$$

$$x_{ij} \geqslant 0,\ y_j \in \{0,1\},\ \eta_2\ free \qquad \forall i \in I, j \in J \tag{43}$$

Model MP$_2$ has a different structure than MP$_1$, and the only information MP$_2$ needs in each iteration $n$ is the new worst-case scenario $k^n$. We can solve the original subproblem simply by finding every "closest" surviving facility for every client site, rather than by solving its LP dual problem in DUAL-SP. Our implementation of the C&CG method follows the algorithmic description in Algorithm 2. Zeng and Zhao (2013) assess the computational complexity of the BD and C&CG method: BD will converge in $O(|U| \cdot |K|)$ iterations, while C&CG will converge within $O(|K|)$ iterations.

## 5. Two alternative two-stage location models

In this section, we present two alternative two-stage location models that are structurally close to 2SRO, but differ in their details. For these models we maintain the same assumptions as previously for 2SRO: within the two-stage framework, an uncertainty set consisting of a discrete set of scenarios is applied for the second-stage problem, the set of demand sites $I$ coincides with the set of facility sites $J$, and we assume $\sum_{j \in J} a_j(k) < p$ to prevent infeasibility. A numerical comparison between these two models and 2SRO is presented in Section 6.

### *5.1. Two-stage stochastic programming model for reliable p-center problem*

Contrary to stochastic optimization (SO), robust optimization (RO) is designed for situations that lack distributional information. RO is therefore applicable in a broader range of settings than SO, which is an important advantage. A possible disadvantage inherent in 2SRO resides in the fact that it protects from the worst possible scenario in the second stage, and thus might be quite conservative compared to SO, and/or produce rather "extreme" solutions that are suitable only in a limited number of "extreme" scenarios. In order to evaluate this issue, we also develop a two-stage SO model for a two-stage reliable $p$-center problem, and we compare with 2SRO. This SO model is based on PCENTER, and aims to minimize the weighted sum of the maximum cost between clients and the facilities to which they are allocated in the first stage, and the expected value of the maximum cost between clients and their assigned facilities in the second stage. The model can be formulated as follows:

**2SSO**

$$\min_{\mathbf{x},\mathbf{y}} \quad \alpha_1 L_1 + \alpha_2 \sum_{k\in K} p(k) L_2(k) \tag{44}$$

$$\text{subject to} \quad \text{constraints (8) to (11) and (14) to (17)}$$

$$x_{ij}, y_j, w_{ij}(k) \in \{0,1\},\ L_2(k) \geqslant 0 \qquad \forall i \in I, j \in J, k \in K \tag{45}$$

where $p(k)$ is the probability of scenario $k$. Model 2SSO is easier to solve than 2SRO because it is linear. Methods based on Benders decomposition and column-and-constraint generation can both be applied to solve 2SSO due to its two-stage scenario-based structure.

### *5.2. Two-stage robust model for the reliable p-median problem*

While RO focuses on the worst performance between a facility and a client, most research on facility location under uncertainty considers total cost. To validate our choice, a two-stage reliable $p$-median problem considering only transportation cost is examined, based on An et al. (2014). Under the same configuration as PCENTER, the uncapacitated $p$-median problem aims to locate $p$ facilities and minimize the total cost between clients and the facilities to which they are allocated. The (deterministic) problem can be formulated as follows:

**PMEDIAN**

$$\min_{\mathbf{x},\mathbf{y}} \quad \sum_{i\in I}\sum_{j\in J} c_{ij} d_i x_{ij} \tag{46}$$

$$\text{subject to} \quad \text{constraints (3) to (6)}$$

Based on PMEDIAN, we develop a two-stage robust model for a reliable $p$-median problem. This model minimizes the weighted sum of the total cost between clients and their facilities in the pre-disruption stage and the total cost for the worst disruption scenario, and is captured by the following formulation:

**2SRO-PMEDIAN**

$$\min_{\mathbf{x},\mathbf{y}} \quad \alpha_1 \sum_{i\in I}\sum_{j\in J} c_{ij} d_i x_{ij} + \alpha_2 \max_{k\in K} \min_{\mathbf{w}} \sum_{i\in I}\sum_{j\in J} c_{ij}^2(k) d_i^2(k) w_{ij}(k) \tag{47}$$

$$\text{subject to} \quad \text{constraints (9) to (11) and (15) to (17)} \tag{48}$$

$$x_{ij}, y_j, w_{ij}(k) \in \{0,1\} \qquad \forall i \in I, j \in J, k \in K \tag{49}$$

With its similar structure, 2SRO-PMEDIAN can also be solved with each of the three methods presented in Section 4.

## 6. Numerical study

A description of our computational setup is provided in Section 6.1. We compare the performance of the three solution methods for 2SRO in Section 6.2, using instances randomly selected from "2010 County Sorted 250" data in Daskin (2013). We also use a 25-site instance from An et al. (2014) and a 49-site instance from Daskin (2013) in Sections 6.3, 6.4 and 6.5, where we compare our two-stage robust model 2SRO for a reliable $p$-center model with the classic uncapacitated $p$-center model, with the two-stage SO model and with the two-stage robust model for reliable $p$-median.

### 6.1. Computational setup

All algorithms have been coded in Matlab 2015a; all the experiments were run on a Dell Optiplex 760 computer, with Intel Core 2 Quad Q9550 processor with 2.83 GHz clock speed and 4 GB RAM, equipped with Windows 10. GUROBI 7 was used for solving the linear formulations.

We randomly generate instances of size $|I| = 10$, 20, 30, 40, 50 and 60 from the 250-site data in "2010 County Sorted 250". For each size $|I|$, five instances are generated. In the experiments, $c_{ij}$ is the Euclidean distance between sites $i$ and $j$ based on the city coordinates, and $d_i$ is the demand in client site $i$ based on city population. For each scenario $k$, we generate $c_{ij}^2(k)$ and $d_i^2(k)$ as real numbers (uniformly) randomly chosen from the intervals $[0.5c_{ij}, 1.5c_{ij}]$ and $[0.5d_i, 1.5d_i]$, respectively. The disrupted facility sites in each scenario $k$ are chosen randomly with equal probability for each site, with $\sum_{j \in J} a_j(k) = p-1$ (for Section 6.2) or $= p/2$ (for Sections 6.3, 6.4 and 6.5).

### 6.2. Comparison of different solution methods

We compare the performance of the three solution methods LIP, BD and CC&G described in Section 4. The parameters are set as follows: the number of scenarios $|K| = 5, 10, 20, 40, 60, 80, 100$ and 200; $p$ is the integer closest to $|I|/5$, $|I|/4$ and $|I|/3$, except when $|I| = 10$, in which case the values for $p$ are 2, 3 and 4. The weights in the objective function are $\alpha_1 = 0.2, 0.5$ and 0.8, and $\alpha_1 + \alpha_2 = 1$. The time limit for each method is 1000 seconds.

Table 1 contains the computational results of the three solution methods for instances, grouped per value of $|I|$. Every cell in the table corresponds to 45 instances with different values for $p$ and $\alpha_1$. The column labeled "Time" shows the average runtime for the instances solved to guaranteed optimality; the column "Iter" indicates the average number of iterations in C&CG and BD for the solved instances; "Solved" states how many instances (out of 45) were solved; and the column "Gap (%)" shows the average gap between the upper and lower bound (expressed as a percentage) for the unsolved cases. Figure 3 displays the runtimes graphically in order to easily discern trends.

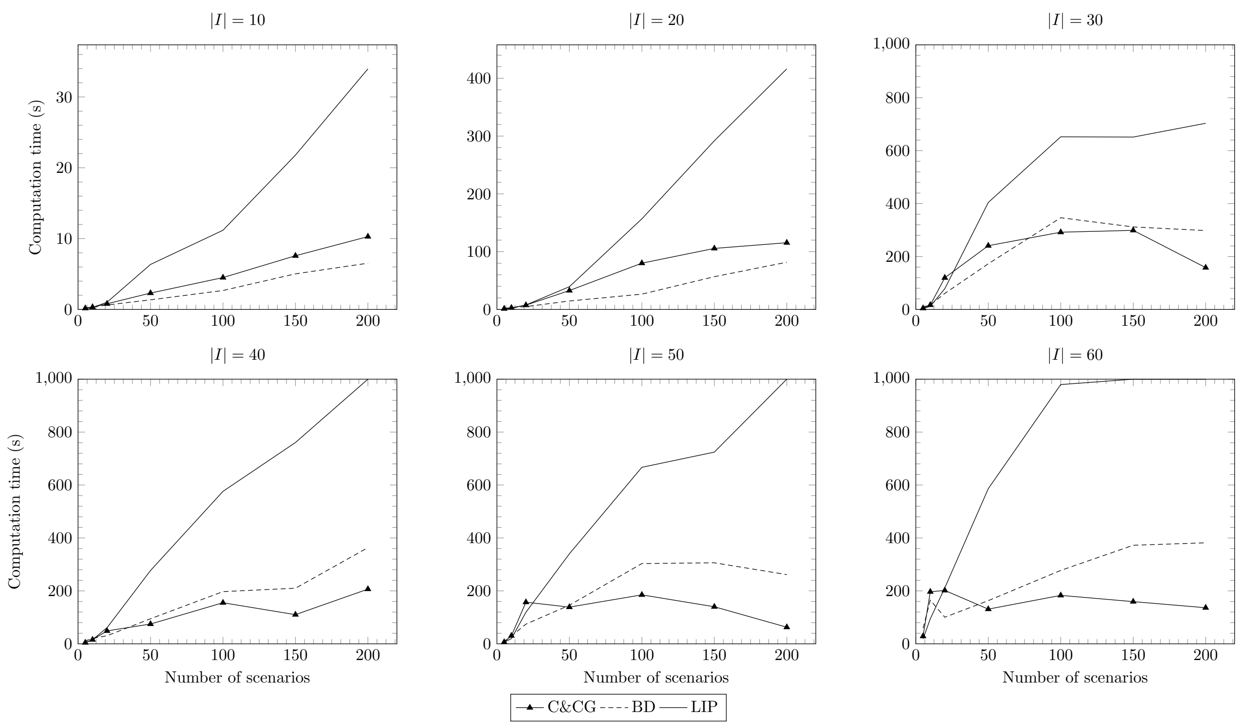


Figure 3: Plots of the evolution of the runtimes for the three solution methods of Section 4

Table 1: Computational results for the three solution methods of Section 4

| $\lvert I \rvert$ | $\lvert K \rvert$ | CC&G | | | | BD | | | | LIP | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Time | Iter | Solved | Gap (%) | Time | Iter | Solved | Gap (%) | Time | Solved | Gap (%) |
| 10 | 5 | 0.16 | 2.40 | 45 | | 0.22 | 6.20 | 45 | | 0.12 | 45 | |
| | 10 | 0.33 | 3.58 | 45 | | 0.33 | 8.07 | 45 | | 0.27 | 45 | |
| | 20 | 0.80 | 5.93 | 45 | | 0.63 | 11.76 | 45 | | 1.02 | 45 | |
| | 50 | 2.31 | 9.73 | 45 | | 1.35 | 16.53 | 45 | | 6.35 | 45 | |
| | 100 | 4.50 | 12.11 | 45 | | 2.67 | 18.51 | 45 | | 11.18 | 45 | |
| | 150 | 7.58 | 16.09 | 45 | | 5.03 | 23.96 | 45 | | 21.79 | 45 | |
| | 200 | 10.28 | 16.71 | 45 | | 6.53 | 23.51 | 45 | | 33.96 | 45 | |
| 20 | 5 | 1.27 | 2.93 | 45 | | 1.98 | 12.16 | 45 | | 0.75 | 45 | |
| | 10 | 2.83 | 4.07 | 45 | | 3.25 | 16.31 | 45 | | 2.20 | 45 | |
| | 20 | 7.45 | 5.73 | 45 | | 4.92 | 18.44 | 45 | | 7.77 | 45 | |
| | 50 | 32.73 | 8.42 | 45 | | 14.64 | 32.00 | 45 | | 39.37 | 45 | |
| | 100 | 80.09 | 12.67 | 45 | | 26.48 | 38.49 | 45 | | 156.79 | 45 | |
| | 150 | 105.60 | 14.35 | 43 | 7.50 | 56.67 | 52.47 | 45 | | 291.81 | 43 | 24.55 |
| | 200 | 115.48 | 15.90 | 39 | 11.61 | 81.71 | 55.82 | 45 | | 416.28 | 39 | 16.78 |
| 30 | 5 | 4.55 | 3.71 | 45 | | 8.39 | 20.73 | 45 | | 2.85 | 45 | |
| | 10 | 16.81 | 5.76 | 45 | | 17.41 | 33.47 | 45 | | 12.02 | 45 | |
| | 20 | 119.73 | 10.00 | 45 | | 60.50 | 60.22 | 45 | | 78.26 | 45 | |
| | 50 | 240.89 | 14.09 | 33 | 7.79 | 172.64 | 93.30 | 44 | 5.22 | 404.23 | 42 | 13.26 |
| | 100 | 291.90 | 15.31 | 16 | 21.07 | 346.58 | 131.88 | 40 | 26.04 | 652.36 | 20 | 50.13 |
| | 150 | 298.89 | 12.38 | 16 | 28.42 | 311.78 | 94.04 | 28 | 20.78 | 651.37 | 7 | 59.41 |
| | 200 | 158.11 | 14.69 | 16 | 31.04 | 298.12 | 73.81 | 21 | 26.52 | 703.47 | 6 | 55.23 |
| 40 | 5 | 4.35 | 2.38 | 45 | | 10.09 | 13.80 | 45 | | 3.92 | 45 | |
| | 10 | 15.54 | 3.76 | 45 | | 20.82 | 20.44 | 45 | | 15.63 | 45 | |
| | 20 | 48.81 | 5.16 | 44 | 6.57 | 30.52 | 23.16 | 45 | | 60.88 | 45 | |
| | 50 | 74.85 | 7.00 | 42 | 9.98 | 94.78 | 32.47 | 45 | | 277.06 | 43 | 29.67 |
| | 100 | 155.23 | 10.53 | 34 | 22.29 | 197.27 | 50.82 | 34 | 20.53 | 576.14 | 28 | 50.09 |
| | 150 | 110.26 | 8.69 | 29 | 18.74 | 210.15 | 38.97 | 33 | 29.54 | 760.65 | 12 | 39.76 |
| | 200 | 206.67 | 13.08 | 24 | 19.21 | 363.58 | 53.96 | 26 | 26.38 | 0.00 | 0 | 41.98 |
| 50 | 5 | 6.37 | 2.40 | 45 | | 8.38 | 7.73 | 45 | | 6.02 | 45 | |
| | 10 | 30.48 | 4.40 | 45 | | 29.68 | 18.69 | 45 | | 19.66 | 45 | |
| | 20 | 157.28 | 7.66 | 41 | 14.45 | 74.36 | 27.58 | 40 | 16.12 | 119.22 | 43 | 26.31 |
| | 50 | 138.74 | 8.18 | 33 | 11.49 | 145.92 | 36.75 | 36 | 22.18 | 340.15 | 39 | 30.16 |
| | 100 | 185.00 | 9.12 | 26 | 16.20 | 303.25 | 47.13 | 32 | 21.24 | 667.00 | 21 | 31.63 |
| | 150 | 140.10 | 10.12 | 25 | 14.91 | 306.17 | 36.55 | 31 | 28.27 | 724.39 | 8 | 42.31 |
| | 200 | 62.76 | 6.52 | 25 | 20.60 | 261.63 | 21.33 | 24 | 38.32 | 0.00 | 0 | 37.21 |
| 60 | 5 | 28.68 | 3.53 | 45 | | 58.92 | 20.04 | 45 | | 18.44 | 45 | |
| | 10 | 196.71 | 5.64 | 45 | | 165.04 | 25.74 | 43 | 35.69 | 94.72 | 45 | |
| | 20 | 201.81 | 7.56 | 41 | 7.03 | 100.51 | 30.93 | 40 | 10.28 | 211.31 | 44 | 5.63 |
| | 50 | 131.09 | 5.94 | 35 | 16.88 | 163.14 | 31.71 | 41 | 32.87 | 586.29 | 29 | 44.02 |
| | 100 | 183.05 | 7.96 | 25 | 14.15 | 277.44 | 31.03 | 30 | 18.03 | 979.67 | 1 | 38.36 |
| | 150 | 159.47 | 7.69 | 26 | 21.22 | 372.38 | 28.60 | 30 | 29.27 | 0.00 | 0 | 42.54 |
| | 200 | 136.38 | 8.40 | 25 | 33.87 | 381.74 | 18.68 | 22 | 41.58 | 0.00 | 0 | 50.08 |

Table 2: Comparison between the standard $p$-center model and the two-stage robust $p$-center model

| $\|I\|$ | $\alpha_1$ | Value | $p$-center | Robust $p$-center | Improvement (%) |
|---|---|---|---|---|---|
| 25 | 0.2 | $L_1$ | 1720.85 | 2307.55 | −34.09 |
| | | $L_2$ | 8803.05 | 7739.71 | 12.08 |
| | | $\alpha_1 L_1 + \alpha_2 L_2$ | 7386.61 | 6653.28 | 9.93 |
| | 0.5 | $L_1$ | 1720.85 | 1720.85 | 0.00 |
| | | $L_2$ | 7290.68 | 3399.08 | 53.38 |
| | | $\alpha_1 L_1 + \alpha_2 L_2$ | 4505.77 | 2559.97 | 43.18 |
| | 0.8 | $L_1$ | 1720.85 | 1720.85 | 0.00 |
| | | $L_2$ | 9660.37 | 9001.31 | 6.82 |
| | | $\alpha_1 L_1 + \alpha_2 L_2$ | 3308.75 | 3176.94 | 3.98 |
| 49 | 0.2 | $L_1$ | 3905.27 | 4136.25 | −5.91 |
| | | $L_2$ | 8953.10 | 5525.07 | 38.29 |
| | | $\alpha_1 L_1 + \alpha_2 L_2$ | 7943.53 | 5247.31 | 33.94 |
| | 0.5 | $L_1$ | 3905.27 | 4182.56 | −7.10 |
| | | $L_2$ | 13881.68 | 9367.47 | 32.52 |
| | | $\alpha_1 L_1 + \alpha_2 L_2$ | 8893.48 | 6775.02 | 23.82 |
| | 0.8 | $L_1$ | 3905.27 | 3998.95 | −2.40 |
| | | $L_2$ | 16171.96 | 7122.57 | 55.96 |
| | | $\alpha_1 L_1 + \alpha_2 L_2$ | 6358.61 | 4623.67 | 27.28 |

From Figure 3 and Table 1, we observe that, logically, the computational difficulty increases with the instance size $|I|$ and with the number of scenarios $|K|$. LIP is the most inefficient method in all instances: it has the highest number of unsolved instances and the largest gap for most instances. The performance of BD is clearly better than C&CG (the runtimes are lower) when $|I| = 10$ and 20 (unless for very low $|K|$). For larger instances, the comparison is less clear-cut: BD tends to solve more instances than C&CG within the runtime limit (although not consistently), but it is also displays higher runtimes and larger gaps for the remaining instances. This discrepancy is probably at least partly attributable to the different number of instances solved, but it also indicates that the performance of BD in terms of runtime and gap is more variable than C&CG. We suspect that the main cause of these differences in the behavior of the two algorithms C&CG and BD is the way in which their master problems are updated. In each iteration, all constraints and variables for one scenario are added to $MP_2$ in C&CG, while only one constraint is inserted into $MP_1$ in BD. As a result, the problem size of $MP_2$, and thus also the required computation time, grows much faster with the iteration count than for $MP_1$, but each iteration of C&CG will also significantly reduce the gap, which is less so for BD. As a result, even instances with higher $|I|$ and $|K|$ will typically be solved in only a few iterations by C&CG, while BD converges slower (needing more iterations); this obviously has a direct link with the theoretical upper bound on the iterations needed for convergence discussed at the end of Section 4.3. As the instances grow larger, the net effect seems to be beneficial for C&CG rather than BD.

### *6.3. Comparison with the deterministic p-center model*

We now compare the two-stage robust $p$-center model in 2SRO with a standard $p$-center model, as an extension of the illustration using the example in Section 3.3. We set $\alpha_1 = 0.2$, 0.5 and 0.8, $|K| = 5$, and $p = 8$. Table 2 reports the maximum cost between clients and facilities in the first stage ($L_1$) and in the worst-case scenario of the second stage ($L_2$). The column labeled "Improvement" shows the relative change from values for the $p$-center model to values for robust $p$-center. For the deterministic $p$-center model, we optimize $L_1$ in the first stage and compute $L_2$ after reassigning clients to their "closest" facility in the second stage. All the values are divided by $10^4$. The table shows that the 2SRO model consistently achieves a better solution ($L_2$) after disruption. In all cases, with no or only slight sacrifice in the first stage ($L_1$), the overall performance is clearly better.

### *6.4. Comparison with two-stage stochastic optimization*

In this section we compare 2SRO and 2SSO as two alternative models for a reliable variant of the $p$-center problem. We set $|K| = 4$, $p = 8$, $\alpha_1 = 0.2$, 0.5 and 0.8, and the probability of each scenario equals $1/|K|$.

Table 3: Comparison between 2SRO and 2SSO

| $\|I\|$ | $\alpha_1$ | Value | 2SSO | 2SRO | Improvement (%) |
|---|---|---|---|---|---|
| 25 | 0.2 | $L_1$ | 2070.44 | 2463.03 | −18.96 |
| | | $L_2$ | 7414.09 | 6655.88 | 10.23 |
| | | $L_e$ | 4091.73 | 4460.58 | −9.01 |
| | | $\alpha_1 L_1 + \alpha_2 L_2$ | 6345.36 | 5817.31 | 8.32 |
| | | $\alpha_1 L_1 + \alpha_2 L_e$ | 3687.47 | 4061.07 | −10.13 |
| | 0.5 | $L_1$ | 2226.41 | 2204.79 | 0.97 |
| | | $L_2$ | 7136.13 | 6928.13 | 2.91 |
| | | $L_e$ | 3932.15 | 4325.78 | −10.01 |
| | | $\alpha_1 L_1 + \alpha_2 L_2$ | 4681.27 | 4566.46 | 2.45 |
| | | $\alpha_1 L_1 + \alpha_2 L_e$ | 3079.28 | 3265.28 | −6.04 |
| | 0.8 | $L_1$ | 1727.40 | 1954.27 | −13.13 |
| | | $L_2$ | 10889.38 | 9330.38 | 14.32 |
| | | $L_e$ | 5513.74 | 5197.39 | 5.74 |
| | | $\alpha_1 L_1 + \alpha_2 L_2$ | 3559.80 | 3429.49 | 3.66 |
| | | $\alpha_1 L_1 + \alpha_2 L_e$ | 2484.67 | 2602.89 | −4.76 |
| 49 | 0.2 | $L_1$ | 4591.06 | 4556.96 | 0.74 |
| | | $L_2$ | 5797.55 | 5714.42 | 1.43 |
| | | $L_e$ | 4600.67 | 4883.11 | −6.14 |
| | | $\alpha_1 L_1 + \alpha_2 L_2$ | 5556.25 | 5482.93 | 1.32 |
| | | $\alpha_1 L_1 + \alpha_2 L_e$ | 4598.75 | 4817.88 | −4.77 |
| | 0.5 | $L_1$ | 4191.26 | 4260.46 | −1.65 |
| | | $L_2$ | 6338.43 | 5978.23 | 5.68 |
| | | $L_e$ | 4656.45 | 4931.01 | −5.90 |
| | | $\alpha_1 L_1 + \alpha_2 L_2$ | 5264.84 | 5119.34 | 2.76 |
| | | $\alpha_1 L_1 + \alpha_2 L_e$ | 4423.86 | 4595.73 | −3.89 |
| | 0.8 | $L_1$ | 4002.67 | 4021.03 | −0.46 |
| | | $L_2$ | 8254.94 | 7952.12 | 3.67 |
| | | $L_e$ | 5717.78 | 5948.01 | −4.03 |
| | | $\alpha_1 L_1 + \alpha_2 L_2$ | 4853.12 | 4807.25 | 0.95 |
| | | $\alpha_1 L_1 + \alpha_2 L_e$ | 4345.69 | 4406.43 | −1.40 |

Table 3 contains the maximum cost between a client site and the assigned facility in the first stage ($L_1$), in the worst-case scenario of the second stage ($L_2$), and the expected value for all scenarios in the second stage ($L_e$). We also report the weighted sums $\alpha_1 L_1 + \alpha_2 L_2$ and $\alpha_1 L_1 + \alpha_2 L_e$. The column "Improvement" gives the relative change in the values when changing from 2SSO to 2SRO. All results are the average of 10 experiments, divided by $10^4$.

Table 3 clearly illustrates the different goals of these two models. Not surprisingly, solutions to 2SRO are better in their worst-case scenarios ($L_2$), while 2SSO has better expected values ($L_e$) in the second stage. More importantly, however, we see that in all cases, the differences in the different objectives for the different stage are quite small. We conclude that our model 2SRO is not overly conservative.

### 6.5. *Comparison with a two-stage robust model for the reliable p-median problem*

As a final comparison, we look into the differences between the solutions produced by 2SRO and the two-stage robust model for the $p$-median problem in 2SRO-PMEDIAN. The parameter values are again $|K| = 4$, $p = 8$, and $\alpha_1 = 0.2$, 0.5 and 0.8. In Table 4, we include the total cost ($cost_1$) and the maximum cost ($L_1$) in the first stage, and also the worst-case total cost ($cost_2$) and maximum cost ($L_2$) in the second stage. The column "Improvement" is the relative change from robust $p$-median to robust $p$-center. The results are divided by $10^5$.

The two models turn out to both have their benefits. In order to minimize the maximum cost $L_1$, the reliable $p$-center model sacrifices on total cost. The results indicate that, instead of focusing on cost robustness in 2SRO-PMEDIAN, 2SRO provides an alternative reliable strategy for practical usage when reliability for every single client has priority. We also notice that the objective value for one stage can be relatively bad, so as to achieve a better score on the total objective value in a two-stage recourse setting, and this especially when the weight for the corresponding stage ($\alpha_1$ or $\alpha_2$) is small. In the first line of

Table 4: Comparison between 2-stage robust $p$-center model and 2-stage robust $p$-median model

| $\lvert I\rvert$ | $\alpha_1$ | Value | Robust $p$-median | Robust $p$-center | Improvement (%) |
|---|---|---|---|---|---|
| 25 | 0.2 | $cost_1$ | 1717.76 | 1578.81 | 8.09 |
| | | $cost_2$ | 2151.43 | 2770.08 | −28.76 |
| | | $L_1$ | 470.41 | 237.74 | 49.46 |
| | | $L_2$ | 828.12 | 782.39 | 5.52 |
| | 0.5 | $cost_1$ | 1502.94 | 1586.36 | −5.55 |
| | | $cost_2$ | 2213.99 | 2739.51 | −23.74 |
| | | $L_1$ | 284.39 | 225.27 | 20.79 |
| | | $L_2$ | 856.71 | 711.57 | 16.94 |
| | 0.8 | $cost_1$ | 1394.55 | 1462.09 | −4.84 |
| | | $cost_2$ | 2443.45 | 2777.21 | −13.66 |
| | | $L_1$ | 208.15 | 192.98 | 7.29 |
| | | $L_2$ | 924.96 | 838.79 | 9.32 |
| 49 | 0.2 | $cost_1$ | 6003.23 | 6347.74 | −5.74 |
| | | $cost_2$ | 5971.06 | 6842.62 | −14.60 |
| | | $L_1$ | 517.03 | 443.61 | 14.20 |
| | | $L_2$ | 800.14 | 680.12 | 15.00 |
| | 0.5 | $cost_1$ | 5824.19 | 6123.87 | −5.15 |
| | | $cost_2$ | 5874.53 | 6444.91 | −9.71 |
| | | $L_1$ | 505.42 | 407.29 | 19.42 |
| | | $L_2$ | 760.79 | 625.85 | 17.74 |
| | 0.8 | $cost_1$ | 5693.73 | 6280.79 | −10.31 |
| | | $cost_2$ | 6386.72 | 7114.43 | −11.39 |
| | | $L_1$ | 451.18 | 405.81 | 10.06 |
| | | $L_2$ | 1096.21 | 664.71 | 39.36 |

Table 4, for instance, $cost_1$ is lower for robust $p$-center than for robust $p$-median, but the objective value $\alpha_1 cost_1 + \alpha_2 cost_2$ is lower for robust $p$-median than for robust $p$-center due to the effect of $cost_2$. A similar phenomenon occurred in Table 3 for $|I| = 25$, $\alpha_1 = 0.8$, where $L_e$ is lower for 2SRO than for 2SSO.

## 7. Conclusion

In this paper, we have proposed a two-stage robust model for a reliable $p$-center facility location problem. The model is suitable for practical environments without detailed probabilistic information, and when reliability and fairness should be considered simultaneously. Three solution methods have been developed and implemented to solve the model, namely a standard linear reformulation, a Benders dual cutting plane method and a column-and-constraint generation method. The computational results show that the performance of the Benders dual method is the best for most of the instances, while the column-and-constraint generation algorithm is competitive, and is faster for the largest instances, which is probably due to the slower convergence of the Benders dual method. We have also compared our model with the classic uncapacitated vertex $p$-center model, with a two-stage stochastic optimization model and with a two-stage robust model for the $p$-median problem. Our experiments indicate that the solutions produced by our model are not overly conservative, which is known to be a disadvantage for some robust optimization applications.

One logical direction for future research is to study the capacitated variant of the $p$-center problem within a two-stage robust optimization framework. Another avenue for further work is to investigate how different uncertainty sets can be applied in our problem. Finally, a variety of solution methods for two-stage robust optimization can be considered for solving larger instances, where scenario relaxation (Talla Nobibon and Leus, 2014) as a heuristic framework might be one option to achieve better performance than the current exact methods.

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**FACULTY OF ECONOMICS AND BUSINESS**
Naamsestraat 69 bus 3500
3000 LEUVEN, BELGIË
tel. + 32 16 32 66 12
fax + 32 16 32 67 91
info@econ.kuleuven.be
www.econ.kuleuven.be