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Linear stability of ideal MHD configurations. II. Results for stationary equilibrium configurations

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In this paper, we continue exploring the consequences of the general equation of motion (EOM) governing all Lagrangian perturbations ξ about a time-dependent, ideal magnetohydrodynamic (MHD) configuration, which includes self-gravity, external gravity, pressure gradients, compressibility, inertial effects, and anisotropic Lorentz force. We here address the specific case of MHD stability for 3D stationary equilibria, where the perturbed EOM features a symmetric operator $\mathcal{F}$ and an antisymmetric Doppler-Coriolis operator $\mathbf{v} \cdot \nabla$. For this case, we state and prove the general properties for the solutions ξ of the governing dynamical system. For axisymmetric perturbations about axisymmetric equilibria with purely toroidal, or purely poloidal magnetic fields, specific stability theorems can be formulated. We derive a useful integral expression for the quadratic quantity given by the inner product $\langle \xi, \mathcal{F}[\xi] \rangle$. For deriving stability statements on MHD states where self-gravity is involved as well, we provide an upper bound on the perturbed self-gravitational energy associated with the displacement ξ . The resulting expression elucidates the role of potentially stabilizing versus destabilizing contributions and shows the role of gravity, entropy gradients, velocity shear, currents, Lorentz forces, inertia, and pressure gradients in offering many routes to unstable behavior in flowing gases and plasmas. These have historically mostly been studied for static $\mathbf{v} = 0$ configurations, looking at stability of exactly force-balanced states, or by assuming stationarity similar to our approach here (i.e., $\partial_t \equiv 0$ for the state we perturb), but typically in combination with some reduced dimensionality on the configuration of interest (translational or axisymmetry). We show that in these limits, we find and generalize expressions well-known from, e.g., the study of ideal MHD stability of tokamak plasmas or from Schwarzschild's criteria controlling convection in hydrodynamic, (external) gravitating systems. When applied to stationary, axisymmetric configurations in motion, we can use our upper bound to derive a sufficient stability criterion for perturbations of arbitrary azimuthal mode number m used in $\xi(\mathbf{r}) = \eta(r, z) \exp(im\phi)$.
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I. INTRODUCTION

In a previous paper,¹ we pointed out the generality of magnetohydrodynamic (MHD) spectroscopy, i.e., the effort to categorize all normal mode solutions for the Lagrangian displacement field $\xi(\mathbf{r}, t) \propto \exp(-i\omega t)$, when given a completely general ideal magnetohydrodynamic (MHD) configuration as described by its instantaneous density, pressure (or entropy), velocity, and magnetic field vector in the MHD state $(\rho(\mathbf{r}, t), p(\mathbf{r}, t), \mathbf{v}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t))$. This MHD state represents a given solution of the full set of nonlinear, time-dependent MHD equations allowing for both external (potential Φ_{ext}) and self-gravity (potential Φ_{int}), extracted at an arbitrary time t from a real evolution happening on a specific domain with appropriate boundary conditions, or on the entire 3-space. For the time-reversible, ideal MHD system, the spatio-temporal variation of the components of this MHD state vector encodes that they jointly obey entropy advection (written using the variable $S = p\rho^{-\gamma}$ where γ is the ratio of specific heats), conservation of mass, Faraday's law, Newton's and Poisson's law (with gravitational constant G), and the monopole constraint, written as

$$\begin{aligned} \frac{d}{dt}S &= 0, \quad \frac{d}{dt}\rho = -\rho\nabla \cdot \mathbf{v}, \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \\ \rho \frac{d\mathbf{v}}{dt} &= -\nabla p + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B} - \nabla \left(\frac{B^2}{2\mu_0} \right) - \rho \nabla (\Phi_{ext} + \Phi_{int}), \\ \nabla^2 \Phi_{int} &= 4\pi G \rho, \quad \nabla \cdot \mathbf{B} = 0. \end{aligned} \quad (1)$$

These equations feature the Lagrangian derivative $\frac{d}{dt} = \partial_t + \mathbf{v} \cdot \nabla$. Our previous paper¹ attempted to bring together insights gathered in seemingly separately progressing research fields where MHD stability is of particular interest: the study of laboratory fusion (mostly tokamak) plasmas on one hand, along with the study of (self-)gravitating, differentially rotating plasma states encountered in astrophysics on the other hand. We pointed out that both research fields, since the 1960s, employ a formalism for linear stability that in essence looks at Lagrangian displacement fields ξ which obey a linearized equation of motion (EOM) expressed as

$$\frac{d^2 \xi}{dt^2} = \mathcal{G}[\xi]. \quad (2)$$

The most striking property of this EOM is that the operator $\mathcal{G}$ can be written purely based on the instantaneous density, pressure, and magnetic field configuration, $(\rho(\mathbf{r}, t), p(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t))$,

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$\mathbf{B}(\mathbf{r}, t)$), and is for all arbitrary times t , a self-adjoint operator. This operator relates to the Frieman-Rosenbluth formalism,² which governs the stability of stationary MHD configurations, and unifies early work done in fusion as well as astrophysical contexts.^{3–5}

Obviously, most MHD spectroscopic efforts to date—which encompasses all results relating to unmagnetized, gas dynamic evolutions governed by the Euler equations, extended with (self-)gravity—have diagnosed stability for at most stationary equilibria (i.e., $\partial_t = 0$ for the equilibrium quantities, but still obeying the full nonlinear MHD equations). In this paper, we will therefore also adopt this assumption, and we aim to formulate results on MHD stability that connect and possibly generalize several of the many earlier findings. This paper is then organized as follows: in Section II, we briefly recall especially those findings relevant for stationary equilibrium states, and we formulate a theorem that directly applies to the dynamical system that governs the spatio-temporal evolution of ξ . In Section III, we prove a general finding on the perturbational energy associated with ξ . We proceed in Section IV to relate the theory to selected, previously known variational principles for stationary, axisymmetric configurations. Section V will concentrate on rewriting the quadratic expression $\langle \xi, \mathcal{F}[\xi] \rangle$ in a form useful for deriving stability criteria. The remainder of the paper will then (1) make contact with selected results throughout the literature, by studying its consequences in various limiting cases; and (2) formulate a general stability result for axisymmetric, stationary configurations.

II. LAGRANGIAN DISPLACEMENTS: GENERAL THEORY

A. General definitions

We recall from our previous paper¹ (and references therein) that the Lagrangian displacement field ξ actually quantifies the instantaneous difference in the positions $\mathbf{r}$ and $\mathbf{r}'$ attained by two different MHD states $(\rho(\mathbf{r}, t), p(\mathbf{r}, t), \mathbf{v}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t))$ and $(\rho'(\mathbf{r}, t), p'(\mathbf{r}, t), \mathbf{v}'(\mathbf{r}, t), \mathbf{B}'(\mathbf{r}, t))$ that started off at an earlier reference time with exactly the same density, pressure, and magnetic field topology. Indeed, the positions $\mathbf{r}$ and $\mathbf{r}'$ correspond to the intersections of (a) the worldlines (in 3+1 spacetime) of an arbitrary fluid parcel selected from the shared initial condition, with (b) the fixed time coordinate hyperplane, and this for the two differently evolving MHD states. We found that all Lagrangian displacement fields that allow for this interpretation necessarily obey the linear EOM given by Eq. (2). In general, the EOM results from considering the consequence of a principle of least action, i.e., a stationary point for the action

$$\mathcal{S} = \int \int \mathcal{L} d^3\mathbf{r} dt, \quad (3)$$

where the integrand over space-time evaluates the Lagrangian density

$$\mathcal{L} = \rho \left(\frac{1}{2} v^2 - \frac{1}{2} \Phi_{int} - \Phi_{ext} - e \right). \quad (4)$$

In this Lagrangian density, we recognize the kinetic energy contribution, the potential energy related to self-gravity ($\frac{d\mathbf{v}}{dt} = -\nabla \Phi_{int}$ in Newton's law) and external gravity ($\frac{d\mathbf{v}}{dt} = -\nabla \Phi_{ext}$ in Newton's law), along with the specific energy

$$e = e^*(S, \rho^{-1}) + \frac{B^2}{2\mu_0\rho} = e^*(S, \rho^{-1}) + v_A^2/2, \quad (5)$$

with the usual Alfvén speed $v_A = B/\sqrt{\mu_0\rho}$, appearing in a term added to the specific energy $e^* = \frac{N k_B T}{2 \mu m_p} = \frac{N}{2} \frac{\mathcal{R}}{\mu} T = \frac{N}{2} \frac{p}{\rho}$. We thus adopt an ideal gas law with N degrees of freedom per particle such that $\gamma = 1 + 2/N$, along with the usual meanings for constants like vacuum permeability μ_0 , Boltzmann constant k_B , proton mass m_p , gas constant $\mathcal{R}$, and mean molecular weight μ .

B. Perturbed EOM about stationary MHD states

The linear EOM follows from evaluating the action for the perturbed flow as in

$$\begin{aligned} \mathcal{S}' &= \int \int \left(\frac{1}{2} v'^2 - \frac{1}{2} \Phi'_{int} - \Phi'_{ext} - e' \right) (\mathbf{r}', t) \rho'(\mathbf{r}', t) d^3\mathbf{r}' dt \\ &= \int \int \mathcal{L}' \left(\xi^i, \frac{\partial \xi^i}{\partial r^j}, \frac{\partial \xi^i}{\partial t} \right) d^3\mathbf{r} dt, \end{aligned} \quad (6)$$

where $\mathcal{L}'$ is the original Lagrangian density but now evaluated for the displaced flow, and the second line has already changed variables used for the spatial integration. The variation will be done over ξ and its derivatives, and the algebra proceeds by expanding $\mathcal{L}'$ to second order in ξ and its spatial derivatives (i.e., truncating to $\mathcal{L}'[\xi^i, \partial_j \xi^i, \partial_t \xi^i]$), and using the Euler-Lagrange equations to find the stationary point for the action $\mathcal{S}'$. The zeroth order terms disappear since they express the equation of motion (Newton's law) for the state vector quantities themselves, while the first order part is found to be (for a detailed derivation see our previous paper¹ and Refs. 3 and 4)

$$\partial_t^2 \xi + 2(\mathbf{v} \cdot \nabla) \partial_t \xi = \mathcal{F}[\xi] = \mathcal{G}[\xi] - (\mathbf{v} \cdot \nabla)^2 \xi. \quad (7)$$

Note that in Eq. (7), we specified Eq. (2) to stationary MHD states, i.e., time-independent (i.e., $\partial_t \equiv 0$) solutions of the full set of equations (1), and can be applied to 1D (e.g., cylindrical), 2D (e.g., with axisymmetric or translational symmetry), or true 3D states. The instantaneous state density, pressure, and magnetic field configuration define the self-adjoint operator $\mathcal{G}$ through

$$\begin{aligned} \rho \mathcal{G}(\xi) &= \nabla \cdot \left[\left((\gamma - 1)p + \frac{B^2}{2\mu_0} \right) \nabla \cdot \xi \right] \\ &+ \nabla \xi \cdot \nabla \left(p + \frac{B^2}{2\mu_0} \right) + \left[p + \frac{B^2}{2\mu_0} \right] \nabla (\nabla \cdot \xi) \\ &+ \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) [(\mathbf{B} \cdot \nabla) \xi - (\nabla \cdot \xi) \mathbf{B}] \\ &- \frac{1}{\mu_0} \nabla [\mathbf{B} \cdot ((\mathbf{B} \cdot \nabla) \xi)] \\ &- \rho (\xi \cdot \nabla) \nabla (\Phi_{ext} + \Phi_{int}) - \rho \nabla \delta \Phi_{int, \xi}. \end{aligned} \quad (8)$$

The term $\Phi_{int,\xi}(\mathbf{r}) = \int d^3\mathbf{x} G \nabla \cdot (\rho \xi) / |\mathbf{r} - \mathbf{x}|$, and this expression is to be complemented by

$$\nabla^2 \delta \Phi_{int,\xi} = -4\pi G \nabla \cdot (\rho \xi). \quad (9)$$

The self-adjointness of $\mathcal{G}$ relates to the usual definition for the inner product between two displacement fields ξ and η written as

$$\langle \eta, \xi \rangle = \int \rho (\eta^* \cdot \xi) dV, \quad (10)$$

which is an integral over the total plasma volume (which can be the entire 3-space). This introduces a norm through $\|\xi\|^2 = \langle \xi, \xi \rangle$. Note that the density ρ appears as weighting function, and that we wrote the complex conjugate as η^* , although actual Lagrangian displacement fields are real by definition since they quantify position differences $\mathbf{r}' - \mathbf{r}$. For this inner product and the particular situation of a 3D, stationary MHD state, the Doppler-Coriolis operator $\mathbf{v} \cdot \nabla$ is an antisymmetric operator, making also $\mathcal{F}$ self-adjoint.

It is also common to exploit expressions for flow quantities that compare the two states $(\rho(\mathbf{r}, t), p(\mathbf{r}, t), \mathbf{v}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t))$ and $(\rho'(\mathbf{r}, t), p'(\mathbf{r}, t), \mathbf{v}'(\mathbf{r}, t), \mathbf{B}'(\mathbf{r}, t))$ that are separated by the Lagrangian displacement field ξ , but do this comparison not between the positions $\mathbf{r}'$ and $\mathbf{r}$ that are separated by ξ , but rather compare quantities in the two states on fixed locations. In practice, this introduces local flow quantities like $\delta\rho$, measuring the difference between $\rho'(\mathbf{r}, t)$ and $\rho(\mathbf{r}, t)$, which each connects in spacetime views to typically different initial fluid parcels (as opposed to the ξ itself, which quantifies the spatial separation achieved between two worldlines that start on the same fluid parcel at the reference time). We recall that they can be written for density, entropy (with $s \propto \ln S$), velocity, and magnetic field as

$$\delta\rho = -\nabla \cdot (\rho \xi), \quad (11)$$

$$\delta s = -\xi \cdot \nabla s, \quad (12)$$

$$\delta\mathbf{v} = \partial_t \xi + \mathbf{v} \cdot \nabla \xi - (\xi \cdot \nabla) \mathbf{v}, \quad (13)$$

$$\delta\mathbf{B} = \nabla \times (\xi \times \mathbf{B}). \quad (14)$$

We can also quantify this for pressure and total pressure, through

$$\delta p = -\gamma p (\nabla \cdot \xi) - \xi \cdot \nabla p, \quad (15)$$

$$\delta P = \delta p + \frac{1}{\mu_0} \mathbf{B} \cdot \delta\mathbf{B}. \quad (16)$$

These quantities obey the linearized versions of our governing ideal MHD equations. When we use these local differences between states connected by a Lagrangian displacement field, we find that the governing EOM is also identifiable with

$$\begin{aligned} & \partial_t^2 \xi + 2(\mathbf{v} \cdot \nabla) \partial_t \xi \\ &= \delta \left(\frac{d\mathbf{v}}{dt} \right) - (\mathbf{v} \cdot \nabla)^2 \xi + (\xi \cdot \nabla) ([\mathbf{v} \cdot \nabla] \mathbf{v}) \\ &= -\delta \left(\frac{\nabla p}{\rho} + \nabla \Phi - \frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}) \times \mathbf{B} \right) \\ & \quad - (\mathbf{v} \cdot \nabla)^2 \xi + (\xi \cdot \nabla) ([\mathbf{v} \cdot \nabla] \mathbf{v}). \end{aligned} \quad (17)$$

The gravitational potential in this latter expression is $\Phi = \Phi_{int} + \Phi_{ext}$, and the Lorentz force is $\mathbf{F}_m = (\nabla \times \mathbf{B}) \times \mathbf{B} / \mu_0$. This Eq. (17) is directly connected to earlier analysis by Lynden-Bell and Ostriker,⁵ when we recast it (using the stationarity such that $[\mathbf{v} \cdot \nabla] \mathbf{v} = -\frac{\nabla p}{\rho} - \nabla \Phi + \frac{\mathbf{F}_m}{\rho}$) as

$$\frac{d^2 \xi}{dt^2} = (\delta + \xi \cdot \nabla) \left[-\frac{\nabla p}{\rho} - \nabla \Phi + \frac{\mathbf{F}_m}{\rho} \right], \quad (18)$$

and recognize the left hand side term as the acceleration. Turning to a rotating reference frame, we then get their Equation (17), where the acceleration rewrites to account for the uniform rotation with angular velocity vector Ω as

$$\begin{aligned} & \frac{d^2 \xi}{dt^2} + 2\Omega \times \frac{d\xi}{dt} + \Omega \times (\Omega \times \xi) \\ &= (\delta + \xi \cdot \nabla) \left[-\frac{\nabla p}{\rho} - \nabla \Phi + \frac{\mathbf{F}_m}{\rho} \right]. \end{aligned} \quad (19)$$

In our first paper,¹ we also mentioned a generalization to allow for local perturbations as follows

$$\delta s = -\xi \cdot \nabla s + \Delta s, \quad (20)$$

$$\delta\mathbf{B} = \nabla \times (\xi \times \mathbf{B}) + \Delta\mathbf{B}. \quad (21)$$

The additional magnetic displacement $\Delta\mathbf{B}$ represents an arbitrary function, which evolves according to the induction equation with the unperturbed flow $\mathbf{v}$ as in

$$\partial_t \Delta\mathbf{B} = \nabla \times (\mathbf{v} \times \Delta\mathbf{B}). \quad (22)$$

Similarly, the entropy displacement $\Delta s(\mathbf{r}, t)$ must obey $\frac{d}{dt} \Delta s = 0$. In such a generalized case, we can write for the case of a stationary equilibrium, the extended EOM as

$$\begin{aligned} & \rho (\partial_t^2 \xi_i + 2\mathbf{v} \cdot \nabla \partial_t \xi_i) \\ &= \rho (\mathcal{F}[\xi])_i - \partial_i \left(\left(\frac{\partial p}{\partial s} \right)_\rho \Delta s + \frac{1}{2\mu_0} \mathbf{B} \cdot \Delta\mathbf{B} \right) \\ & \quad + \frac{1}{\mu_0} (\Delta B^j \partial_j B_i + B^j \partial_j \Delta B_i), \end{aligned} \quad (23)$$

which is a dynamical system of the form $\partial_t^2 \xi(t) + 2\tilde{A} \partial_t \xi(t) = \tilde{S} \xi(t) + \mathbf{f}(t)$ with the effect of the magnetic and entropy displacements contained in a kind of driving term $\mathbf{f}(t)$ and where $\tilde{S}$ (i.e., our operator $\mathcal{F}$) is symmetric and $\tilde{A}$ (i.e., $\mathbf{v} \cdot \nabla$) is antisymmetric. This generalization is relevant when wanting to consider magnetic field perturbations about an unmagnetized state vector.

C. Solution properties of the governing dynamical system

We can formulate the following theorem in connection with general dynamical systems of the form $x''(t) + 2\tilde{A} x'(t) = \tilde{S} x(t) + \mathbf{f}(t)$, collecting properties of the solutions of the (modified) EOM, along with statements on its spectrum.⁶

Theorem 1. Let H be a complex inner product space, $\tilde{S} : H \rightarrow H$ a symmetric operator and $\tilde{A} : H \rightarrow H$ an anti-symmetric operator. Let $f(t) \in H$ be a smooth function. Consider the dynamical system

$$x''(t) + 2\tilde{A}x'(t) = \tilde{S}x(t) + f(t). \quad (24)$$

Then, we have the following:

- (1) The energy $E_x = \langle x'(t), x'(t) \rangle - \langle x(t), \tilde{S}x(t) \rangle - 2\operatorname{Re} \sum_{n=0}^{\infty} (-1)^n \langle f^{(n)}(t), x^{(-n)}(t) \rangle$ is a conserved quantity. Here, for $n \geq 0$ the notation $f^{(n)}(t)$ is used for the n -th time derivative of $f(t)$ and the notation $x^{(-n)}(t)$ is used for the function which satisfies $x^{(-n)}(0) = 0$ and $\frac{d}{dt}x^{(-n)}(t) = x^{(-n+1)}(t)$.

In the case where the driver $f(t) \equiv f$ is time-independent, this energy simplifies to

$$E_x = \langle x'(t), x'(t) \rangle - \langle x(t), \tilde{S}x(t) \rangle - 2\operatorname{Re} \langle f, x(t) \rangle.$$

- (2) If $\tilde{S}$ is negative semidefinite, i.e., $\forall x \in H: \langle x, \tilde{S}x \rangle \leq 0$, and there is no driver $f \equiv 0$, then for any solution $y(t)$ of Eq. (24) with energy E_y we have $\langle y'(t), y'(t) \rangle \leq E_y$.
- (3) If $\tilde{S}$ is uniformly negative definite, i.e., $\exists \lambda > 0: \forall x \in H: \langle x, \tilde{S}x \rangle \leq -\lambda^2 \langle x, x \rangle$, and $f(t) \equiv f$ is time-independent, then for any solution $y(t)$ of Eq. (24) with energy E_y we have $\langle y'(t), y'(t) \rangle \leq E_y + \frac{\|f\|^2}{\lambda^2}$ and $\|y(t)\| \leq \frac{(\|f\|^2 + \lambda^2 E_y)^{\frac{1}{2}} + \|f\|}{\lambda^2}$.
- (4) If $\tilde{A} = 0 \equiv f$ and if $\tilde{S}$ is not negative semidefinite, i.e., $\exists x \in H: \langle x, \tilde{S}x \rangle = \gamma^2 \langle x, x \rangle$ for some $\gamma > 0$, then there exists a solution $y(t)$ of Eq. (24) which satisfies

$$\langle y(t), y(t) \rangle \geq \langle y(0), y(0) \rangle \exp(2\gamma t), \quad (25)$$

and

$$\langle y'(t), y'(t) \rangle \geq \gamma^2 \langle y(0), y(0) \rangle \exp(2\gamma t). \quad (26)$$

So this solution either diverges in finite time or grows unbounded.

- (5) If $\tilde{A} = 0$, then the spectrum σ of the homogeneous version ($f \equiv 0$) of the dynamical system Eq. (24), i.e.,

$$\sigma = \{\lambda \in \mathbb{C} | \lambda^2 I - \tilde{S} \text{ has dense range} + \text{bounded inverse}\}^c, \quad (27)$$

is contained in the union of the real and imaginary axes.

- (6) If $\tilde{A} = 0$ and $f(t) \equiv f = a + b$ where $a \in \ker \tilde{S}$ and $b \in (\ker \tilde{S})^\perp$, then the general solution of (24) is given by

$$x(t) = r + st + \frac{at^2}{2} + z(t), \quad (28)$$

where $r, s \in \ker \tilde{S}$ and $z(t) \perp \ker \tilde{S}$ is a solution of $z''(t) = \tilde{S}z(t) + b$. So if either a or s is nonzero, this solution grows unbounded in the sense that

$$\|x(t)\|^2 = \left\| r + st + \frac{at^2}{2} \right\|^2 + \|z(t)\|^2 \geq \left\| r + st + \frac{at^2}{2} \right\|^2. \quad (29)$$

$$\|x'(t)\|^2 = \|s + at\|^2 + \|z'(t)\|^2 \geq \|s + at\|^2. \quad (30)$$

The proof of this theorem is provided in the Appendix. This theorem tells us in various ways that dynamical systems of the form (24) are stable if the driving force $\tilde{S}$ is negative-

semidefinite or preferably even uniformly negative definite. Under stricter circumstances (when $\tilde{A}$ is zero, the operator which for our purposes will always correspond to the Doppler-Coriolis term $\tilde{A}x' \equiv (\mathbf{v} \cdot \nabla) \partial_t \xi$), there is also a converse saying that $\tilde{S}$ being not negative-semidefinite implies that the dynamical system is exponentially unstable. Note that the first three statements of this theorem are relevant for the full system under study here, relating to the generalized EOM for Lagrangian displacement fields about a stationary MHD state as given by Eq. (23). The latter three statements relate to the special case when we happen to have a static equilibrium configuration where $\mathbf{v} = 0$ and thus have no Doppler-Coriolis operator to deal with. That case is extensively studied for Grad-Shafranov (balancing pressure gradient and Lorentz forces) equilibria for decades. E.g., statement (5) expresses the known fact that the spectrum of a static MHD state when writing $\xi \propto \exp(-i\omega t)$ is completely found on the real and imaginary axes of the complex ω plane, due to the self-adjointness of the standard force operator.

It is perhaps useful to especially restate the first and third aspect of the theorem for the case without the extra entropic or magnetic displacements, i.e., for the general case of a stationary MHD state and a purely Lagrangian displacement field, governed by its EOM as in Eq. (7) or its equivalent expression in Eq. (17). Indeed, the first item of this theorem gives us a powerful statement, namely, we find that for any solution $\xi(\mathbf{r}, t)$, we have a conserved energy quantity being

$$E_\xi = \langle \partial_t \xi, \partial_t \xi \rangle - \langle \xi, \mathcal{F}[\xi] \rangle. \quad (31)$$

This is a well-known result and expresses the fact that the Hamiltonian of the perturbations is conserved.⁷ Furthermore, the third item of this theorem states that if we can find a real number λ such that all our solutions obey

$$\langle \xi, \mathcal{F}[\xi] \rangle \leq -\lambda^2 \langle \xi, \xi \rangle, \quad (32)$$

then we can be sure of the following bounds

$$K[\xi] \equiv \langle \partial_t \xi, \partial_t \xi \rangle \leq E_\xi$$

$$(I[\xi])^{\frac{1}{2}} \equiv \langle \xi, \xi \rangle^{\frac{1}{2}} \leq \sqrt{\frac{E_\xi}{\lambda^2}}. \quad (33)$$

Hence, then all our solutions would have a finite, or bounded kinetic energy content $K[\xi]$, and remain square integrable (or have a finite squared norm $I[\xi]$) at all times. Note that for actual normal mode solutions, having a bounded norm or having a bounded kinetic energy are very similar statements, since then $K = |\omega|^2 I$. We further recall⁷ that in a normal mode analysis, one can use the governing equation (7) to formally get a quadratic equation for the (possibly complex) eigenvalues ω and their eigenfunctions ξ . This is done by dotting Eq. (7) with ξ^* and integrating according to the weighted inner product rule from Eq. (10). This gives an expression of the form $\omega^2 I[\xi] + 2\omega V[\xi] + W[\xi] = 0$, where $V = \int \rho \xi^* \cdot [i(\mathbf{v} \cdot \nabla) \xi] dV$ and $W = \langle \xi, \mathcal{F}[\xi] \rangle$. Since $\mathbf{v} \cdot \nabla$ is

antisymmetric and our displacement fields are real, $[i(\mathbf{v} \cdot \nabla)]$ is a symmetric operator, just like $\mathcal{F}$, making all coefficients I , V , and W real-valued.⁸ The solutions to this equation show that normal modes divide in stable waves where ω is purely real, and instabilities of the form $\omega = \sigma + i\nu$ with $\nu \neq 0$. The exponentially growing instabilities are such that the increase in kinetic energy balances the decrease in potential energy to keep the constant $E_\xi = 0 = K - W$. It is also known that the sign of the potential energy term given by $\langle \xi, \mathcal{F}[\xi] \rangle$ alone does not determine necessary, nor sufficient conditions for (in)stability. These insights on how normal mode eigenvalue-eigenfunction pairs (at least those discrete modes with a bounded norm $I[\xi]$) must obey these energy considerations have led to the introduction of the spectral web,^{9–11} as a convenient means to locate the eigenvalues of stationary MHD states in the complex plane. This can be done for any stationary MHD state, although it has thus far been implemented and showcased for essentially

one-dimensional equilibrium states (e.g., shear flow and sheared magnetic field in planar stratified slab configurations, or first extensions to cylindrical configurations) only.

III. PERTURBATIONAL ENERGY CONSERVATION

In this section and throughout the remainder of this paper, we will consider the case without driving terms due to extra entropic or magnetic displacement terms introduced in Eqs. (20)–(21). When the MHD state is stationary, all coefficients e , Φ , ... that appear in the actual expression for the Lagrangian density $\mathcal{L}'$ in Equation (6) are time-independent. This then also applies to the related $\tilde{\mathcal{L}}[\xi^i, \partial_j \xi^i, \partial_t \xi^i]$, which follows by only considering terms which are at most second order in ξ and $\partial \xi$, and which led to the perturbed EOM. For such “time-invariant” field theories, there is a conserved scalar which can be thought of as energy. This *Lagrangian perturbational energy scalar* is given by $\partial_t \xi^i \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_t \xi^i)} - \tilde{\mathcal{L}}$ and we can show its conservation as

$$\partial_t \left(\partial_t \xi^i \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_t \xi^i)} - \tilde{\mathcal{L}} \right) = \partial_t^2 \xi^i \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_t \xi^i)} + \partial_t \xi^i \partial_t \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_t \xi^i)} - \partial_t^2 \xi^i \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_t \xi^i)} - \partial_t \partial_j \xi^i \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_j \xi^i)} - \partial_t \xi^i \frac{\partial \tilde{\mathcal{L}}}{\partial \xi^i} - \underbrace{\frac{\partial \tilde{\mathcal{L}}}{\partial t}}_{=0 \text{ time-invariant}}, \quad (34)$$

$$= \partial_t \xi^i \left(\partial_t \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_t \xi^i)} + \partial_j \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_j \xi^i)} - \frac{\partial \tilde{\mathcal{L}}}{\partial \xi^i} \right) - \partial_j \left(\partial_t \xi^i \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_j \xi^i)} \right), \quad (35)$$

=perturbed EOM, hence 0

$$= -\partial_j \left(\partial_t \xi^i \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_j \xi^i)} \right). \quad (36)$$

This Lagrangian perturbational energy scalar $\partial_t \xi^i \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_t \xi^i)} - \tilde{\mathcal{L}}$ can easily be computed from the expressions given in our previous paper,¹ Section 3.3. We simply state the end result here as

$$\begin{aligned} \partial_t \xi^i \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_t \xi^i)} - \tilde{\mathcal{L}} &= \rho \partial_t \xi \cdot \left(\mathbf{v} + \frac{d\xi}{dt} \right) - \frac{1}{2} \rho \left(\mathbf{v} + \frac{d\xi}{dt} \right)^2 + T_i^j \partial_j \xi^i + \xi^i \rho \partial_i \Phi + \frac{\rho}{2} \Phi_{int} + \rho \Phi_{ext} + \rho e \\ &+ \frac{1}{2} \left(V_{ik}^{jl} \partial_j \xi^i \partial_l \xi^k + \rho \xi^i \xi^j \partial_i \partial_j \Phi + \rho \xi^i \partial_i \delta \Phi_{int, \xi} \right), \end{aligned} \quad (37)$$

where we must recall the notation T_i^j for the usual ideal MHD stress tensor, i.e.

$$\bar{\mathbf{T}} = \frac{\mathbf{B}\mathbf{B}}{\mu_0} - \left(p + \frac{B^2}{2\mu_0} \right) \hat{\mathbf{I}}, \quad (38)$$

along with the operator V_{ik}^{jl} that enters the $\mathcal{G}$ expression written as $\rho \mathcal{G}[\xi] = \partial_j [V_{ik}^{jl} \partial_l \xi^k] - \rho (\xi \cdot \nabla) \nabla \Phi - \rho \nabla \delta \Phi_{int, \xi}$, namely

$$\begin{aligned} V_{ik}^{jl} &= \left(\rho \left(\frac{\partial p}{\partial \rho} \right)_s - p + \frac{B^2}{2\mu_0} \right) \delta_i^j \delta_k^l + \left(p + \frac{B^2}{2\mu_0} \right) \delta_i^l \delta_k^j \\ &+ \frac{1}{\mu_0} \left(B^j B^l \delta_{ik} - \delta_i^j B_k B^l - \delta_k^l B_i B^j \right). \end{aligned} \quad (39)$$

We thus see that the Lagrangian perturbational energy scalar (37) has three contributions

- A term E_0 that is zeroth order in ξ : this is the integral of the Lagrangian density $\rho v^2/2 - \rho \Phi_{int}/2 - \rho \Phi_{ext} - \rho e$ of the unperturbed flow (i.e., the stationary equilibrium state), which is separately conserved because this Lagrangian density is time-independent.
- A term that is first order in ξ , which vanishes, as we will show. Indeed, the first order contribution is

$$E_1 = \int \left(\rho \mathbf{v} \cdot \left[\partial_t \xi - \frac{d\xi}{dt} \right] + T_i^j \partial_j \xi^i + \xi^i \rho \partial_i \Phi \right) dV \quad (40)$$

$$= \int (-\rho \mathbf{v} \cdot [(\mathbf{v} \cdot \nabla) \xi] + T_i^j \partial_j \xi^i + \xi^i \rho \partial_i \Phi) dV \quad (41)$$

$$= \int (-\rho v^i v^j \partial_j \xi_i + T_i^j \partial_j \xi^i + \xi^i \rho \partial_i \Phi) dV \quad (42)$$

$$= \int \xi_i \left[\underbrace{\partial_j (\rho v^j)}_{=0 \text{ mass conservation}} v^i + \rho v^j \partial_j v^i - \partial_j T_i^j + \rho \partial_i \Phi \right] dV \quad (43)$$

$$= \int \xi^i \underbrace{(\rho v^j \partial_j v^i - \partial_j T_i^j + \rho \partial_i \Phi)}_{=\text{equilibrium EOM, hence 0}} dV. \quad (44)$$

Note that we performed an integration by parts and assumed the boundary terms to yield no contributions.

- A second order energy contribution given by:

$$E_2 = \int \left[\rho \left(\partial_t \xi \cdot \frac{d\xi}{dt} - \frac{1}{2} \left(\frac{d\xi}{dt} \right)^2 \right) + \frac{1}{2} V_{ik}^{jl} \partial_j \xi^i \partial_l \xi^k + \rho \frac{1}{2} \xi^i \xi^j \partial_i \partial_j \Phi + \frac{1}{2} \rho \xi^i \partial_i \delta \Phi_{int, \xi} \right] dV \\ = \frac{1}{2} \int \left[\rho \left((\partial_t \xi)^2 - [(\mathbf{v} \cdot \nabla) \xi]^2 \right) + V_{ik}^{jl} \partial_j \xi^i \partial_l \xi^k + \rho \xi^i \xi^j \partial_i \partial_j \Phi + \rho \xi^i \partial_i \delta \Phi_{int, \xi} \right] dV. \quad (45)$$

So we have proven

The first order Lagrangian perturbational energy E_1 from Eq. (40) of an ideal stationary flow vanishes for solutions of the perturbed equation of motion (7). This also means that the second-order energy perturbation E_2 from Eq. (45) is separately conserved.

IV. VARIATIONAL PRINCIPLES FOR AXIALLY SYMMETRIC, ROTATING, IDEAL MHD EQUILIBRIA

In this section, we will now use the expressions presented so far and apply them to derive and generalize variational principles that apply specifically to axisymmetric, rotating stationary equilibria. Hence, we here consider perturbations about time-independent ($\partial_t = 0$), axisymmetric ($\partial_\phi = 0$) solutions to the full set of Eq. (1) with only a toroidal flow $\mathbf{v} = r\Omega(r, z)\hat{\phi}$. In this section, we will additionally assume that the perturbations are also axisymmetric. Thus, we look at equilibria where $(\rho(r, z), p(r, z), \mathbf{v} = r\Omega(r, z)\hat{\phi}, \mathbf{B}(r, z))$, with perturbations that are axially symmetric $\xi(r, z, t) = \xi_r(r, z, t)\hat{r} + \xi_\phi(r, z, t)\hat{\phi} + \xi_z(r, z, t)\hat{z}$. Indeed, we then can write $[\mathbf{v} \cdot \nabla]\mathbf{v} = -r\Omega^2\hat{r}$, and the inertial term as expressed in Eq. (17) can be written more specifically (recalling a general result from our previous paper in the first equality) as

$$\delta \left(\frac{d\mathbf{v}}{dt} \right) = \frac{d^2 \xi}{dt^2} - (\xi \cdot \nabla) \frac{d\mathbf{v}}{dt} \\ = \partial_t^2 \xi - \underbrace{2\Omega(\partial_t \xi_\phi \hat{r} - \partial_t \xi_r \hat{\phi})}_{2(\mathbf{v} \cdot \nabla) \partial_t \xi} + \hat{r} r (\xi \cdot \nabla) \Omega^2. \quad (46)$$

We can also use the ϕ -invariance for ξ and $\mathbf{B}$ to calculate the prevailing expression for the magnetic field perturbation, writing $\mathbf{B} = \mathbf{B}_p + B_\phi \hat{\phi}$ in its poloidal and toroidal part, to get

$$\delta \mathbf{B} = \nabla \times (\xi \times \mathbf{B}) = \frac{1}{r} \nabla (r(\xi_z B_r - \xi_r B_z)) \times \hat{\phi} \\ + r \nabla \cdot \left(\frac{\xi_\phi \mathbf{B}_p - \xi_p \mathbf{B}_\phi}{r} \right) \hat{\phi}. \quad (47)$$

Another useful expression is what remains of the azimuthal component of the Frieman-Rosenbluth system for such

equilibria and axisymmetric perturbations. Two cases can be distinguished, depending on the equilibrium magnetic field topology.

Case $\mathbf{B}_p = 0$, or purely toroidal field.

When there is no poloidal field $\mathbf{B}_p = 0$, we can observe from Eq. (47) that we may write $\delta B_\phi = -r \nabla \cdot \left(\frac{\xi_p \mathbf{B}_\phi}{r} \right)$. We further obtain the azimuthal component of the perturbed EOM from perturbing the total equation for azimuthal motion in axisymmetric systems. This total equation writes generally as

$$\rho \left(\frac{d\mathbf{v}}{dt} \right)_\phi = \frac{1}{r\mu_0} (\mathbf{B}_p \cdot \nabla) (r B_\phi). \quad (48)$$

Under axisymmetric perturbations, the flow remains axisymmetric and we can perturb the individual terms and factors in this equation. When $\mathbf{B}_p = 0$, the linear contribution on the RHS becomes

$$\delta \frac{1}{r\mu_0} (\mathbf{B}_p \cdot \nabla) (r B_\phi) = \frac{1}{r\mu_0} (\delta \mathbf{B}_p \cdot \nabla) (r B_\phi). \quad (49)$$

However, from Eq. (47), we note that in this case $\delta \mathbf{B}_p = 0$. All that remains is the perturbation of the LHS, which we find from Eq. (46) to reduce to

$$0 = \left[\delta \left(\rho \frac{d\mathbf{v}}{dt} \right) \right]_\phi = \delta \rho ([\mathbf{v} \cdot \nabla] \mathbf{v})_\phi + \rho \left(\delta \frac{d\mathbf{v}}{dt} \right)_\phi \\ = \rho \left(\delta \frac{d\mathbf{v}}{dt} \right)_\phi = \rho \left(\partial_t^2 \xi_\phi + 2\Omega \partial_t \xi_r \right). \quad (50)$$

Case $B_\phi = 0$, or purely poloidal field.

For the case where the toroidal field vanishes $B_\phi = 0$, the perturbation of the RHS of Eq. (48) writes to

$$\delta \frac{1}{r\mu_0} (\mathbf{B}_p \cdot \nabla) (r B_\phi) = \frac{1}{r\mu_0} (\mathbf{B}_p \cdot \nabla) (r \delta B_\phi). \quad (51)$$

Now

$$\delta B_\phi = [(\mathbf{B} \cdot \nabla) \xi]_\phi - \underbrace{[(\nabla \cdot \xi) \mathbf{B}]_\phi}_{=0} - [(\xi \cdot \nabla) \mathbf{B}]_\phi \quad (52)$$

$$= (\mathbf{B}_p \cdot \nabla) \xi_\phi - \frac{1}{r} \xi_\phi B_r = r(\mathbf{B}_p \cdot \nabla) \left(\frac{\xi_\phi}{r} \right). \quad (53)$$

Inserting this expression into the linearized Lorentz force from Eq. (51), together with the inertial term, allows us to write the azimuthal component of the perturbed EOM as

$$\rho \left[\partial_t^2 \xi_\phi + 2\Omega \partial_t \xi_r \right] = \frac{1}{\mu_0 r} (\mathbf{B}_p \cdot \nabla) \left[r^2 (\mathbf{B}_p \cdot \nabla) \left(\frac{\xi_\phi}{r} \right) \right]. \quad (54)$$

These two cases are of relevance for the following two variational principles. The first one can be found in the book by Tassoul¹² (pp. 137–141), where it was discussed for axisymmetric oscillations of non-magnetized stars, as investigated by a large number of researchers.^{5,13,14} Here, it is generalized to include azimuthal magnetic fields.

Theorem 2. Let $(\rho(r, z), p(r, z), \Omega(r, z), B_\phi(r, z))$ be an axisymmetric rotating system with only a toroidal magnetic field and let H consist of the smooth, square integrable, two-component functions $\xi_p(r, z) = (\xi_r(r, z), \xi_z(r, z))$ of (an open subset) of $\mathbb{R}^+ \times \mathbb{R}$ with inner product

$$\begin{aligned} \langle \eta_p, \xi_p \rangle &= \int \eta_p^*(r, z) \cdot \xi_p(r, z) \rho(r, z) dV \\ &= 2\pi \int_0^\infty r dr \int_{-\infty}^\infty dz \rho(r, z) \eta_p^*(r, z) \cdot \xi_p(r, z), \end{aligned} \quad (55)$$

and let L be the following symmetric operator

$$\begin{aligned} L[\xi] &= (\mathcal{F}[\xi]_r - 4\Omega^2 \xi_r) \hat{\mathbf{r}} + \mathcal{F}[\xi]_z \hat{\mathbf{z}} \\ &= -\delta \left(\frac{\nabla P}{\rho} + \nabla \Phi + \frac{B_\phi^2}{r\mu_0 \rho} \hat{\mathbf{r}} \right) - \hat{\mathbf{r}} \frac{1}{r^3} (\xi_p \cdot \nabla) (r^4 \Omega^2), \end{aligned} \quad (56)$$

where $P = p + \frac{B_\phi^2}{2\mu_0}$ is the total pressure, for $\delta\rho$, δp , $\delta\Phi$ the usual expressions apply and $\delta B_\phi = -r\nabla \cdot \left(\frac{\xi_p B_\phi}{r} \right)$. Then, the poloidal motion of axisymmetric perturbations is described by the dynamical system

$$\partial_t^2 \xi_p = L[\xi_p] + f(r, z) \hat{\mathbf{r}},$$

where $f(r, z)$ is any smooth, square integrable function. The results of our general theorem stated in Section II C apply for this dynamical system under the identification $\tilde{S} = L$, $\tilde{A} = 0$ and $a + b = (f(r, z), 0)$.

Proof. We first note it is a matter of algebra to transform the expression

$$\begin{aligned} &-(\mathbf{v} \cdot \nabla)^2 \xi + (\xi \cdot \nabla)([\mathbf{v} \cdot \nabla] \mathbf{v}) - 4\Omega^2 \xi_r \hat{\mathbf{r}} \\ &= -\hat{\mathbf{r}} \left[(\xi_p \cdot \nabla) (r\Omega^2) + 3\Omega^2 \xi_r \right] = -\hat{\mathbf{r}} \frac{1}{r^3} (\xi_p \cdot \nabla) (r^4 \Omega^2). \end{aligned} \quad (57)$$

Similarly, one can write

$$-(\mathbf{v} \cdot \nabla)^2 \xi + (\xi \cdot \nabla)([\mathbf{v} \cdot \nabla] \mathbf{v}) = -\hat{\mathbf{r}} r (\xi \cdot \nabla) \Omega^2. \quad (58)$$

Clearly, we have to elaborate again on

$$\delta \left(\frac{d\mathbf{v}}{dt} \right) = -\delta \left(\frac{\nabla p}{\rho} + \nabla \Phi - \frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}) \times \mathbf{B} \right). \quad (59)$$

The Lorentz force is in this special case

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{2} \nabla (B_\phi^2) - \frac{B_\phi^2}{r} \hat{\mathbf{r}}. \quad (60)$$

Noting again that the magnetic field after axisymmetric perturbation remains azimuthal, we now write out all three components of the perturbed EOM as (see Eq. (50))

$$\begin{cases} \partial_t^2 \xi_r - 2\Omega \partial_t \xi_\phi = -\delta \left(\frac{\partial_r P}{\rho} + \partial_r \Phi + \frac{B_\phi^2}{r\mu_0 \rho} \right) - r(\xi \cdot \nabla) \Omega^2 \\ \partial_t^2 \xi_\phi + 2\Omega \partial_t \xi_r = 0 = (\mathcal{F}[\xi])_\phi \\ \partial_t^2 \xi_z = -\delta \left(\frac{\partial_z P}{\rho} + \partial_z \Phi \right) = (\mathcal{F}[\xi])_z. \end{cases} \quad (61)$$

We can integrate the azimuthal equation with respect to t to obtain

$$\partial_t \xi_\phi + 2\Omega \xi_r = \frac{f(z, r)}{2\Omega}, \quad (62)$$

for some arbitrary smooth, square integrable function f . Plugging into the poloidal equations we finally get

$$\begin{aligned} \partial_t^2 \xi_r &= (\mathcal{F}[\xi])_r + 2\Omega \partial_t \xi_\phi \\ &= (\mathcal{F}[\xi])_r - 4\Omega^2 \xi_r + f = (L[\xi])_r + f, \end{aligned} \quad (63)$$

$$\partial_t^2 \xi_z = (\mathcal{F}[\xi])_z = (L[\xi])_z. \quad (64)$$

We still have to prove that L is symmetric. For this, it is enough to show that $\langle \xi, L[\xi] \rangle$ is real for any ξ , but

$$\begin{aligned} \langle \xi, L[\xi] \rangle &= \int dV \rho (\xi_r^* (\mathcal{F}[\xi])_r + \xi_z^* (\mathcal{F}[\xi])_z - 4\Omega^2 |\xi_r|^2) \\ &= \int dV \rho (\xi_r^* (\mathcal{F}[\xi])_r + \xi_\phi^* (\mathcal{F}[\xi])_\phi \\ &\quad + \xi_z^* (\mathcal{F}[\xi])_z - 4\Omega^2 |\xi_r|^2), \end{aligned} \quad (65)$$

for any square integrable and smooth $\xi_\phi(r, z)$ (and obviously $(\mathcal{F}[\xi])_\phi = 0$). The last expression is real by the symmetry of the force operator $\mathcal{F}$. $\square$

Note that the above variational principle can be used to investigate the Rayleigh criterion (first noted for inviscid incompressible fluids), generalized to the Solberg criterion for a rotating unmagnetized homentropic (i.e., constant entropy) star.¹² The hydrodynamic case was first discussed by Lebovitz¹⁵ and can be used to derive the Solberg-Høiland criteria. The above principle for axisymmetric perturbations of an axisymmetric, differentially rotating and (self-)gravitating body here includes toroidal magnetic field in the equilibrium, and there is no restrictive assumption made on the actual strength of the equilibrium magnetic field. This means that we can connect stability results where we vary the plasma beta values (ratio of gas to magnetic pressure) in the

equilibrium configuration, or more generally the relative strength of rotational, gravitational, pressure-related, or magnetic effects. Early work¹⁶ showed how weak axisymmetric toroidal fields can be sufficient to render stars unstable to axisymmetric modes.

A second theorem covers the case of rotational equilibria with only a poloidal magnetic field. It is related to an analysis made for non-self gravitating, axisymmetric, compressible, differentially rotating fluids with a poloidal magnetic field.¹⁷ Here, we also allow self-gravity and formulate its (in)stability property differently.

Theorem 3. Assume $(\rho, p, \mathbf{\Omega}, \mathbf{B}_p)$ is an axially symmetric rotating ideal MHD equilibrium that lacks a toroidal magnetic field. Suppose that we can find an axisymmetric perturbation $\boldsymbol{\eta}(\mathbf{r})$ for which

$$\langle \boldsymbol{\eta}, \mathcal{F}[\boldsymbol{\eta}] \rangle = E > 0. \quad (66)$$

Then, the perturbation $\boldsymbol{\xi}(\mathbf{r}, t)$ with initial conditions $\boldsymbol{\xi}(\mathbf{r}, 0) = \boldsymbol{\eta}(\mathbf{r})$ and $\frac{\partial \boldsymbol{\xi}}{\partial t}(\mathbf{r}, 0) = 0$ is unstable in the sense that

$$\begin{aligned} & Re[\langle \boldsymbol{\xi}, \partial_t \boldsymbol{\xi} \rangle - 2\langle \boldsymbol{\xi}_\varphi, \partial_t \boldsymbol{\xi}_\varphi \rangle - 2\langle \boldsymbol{\xi}_r, \boldsymbol{\Omega} \boldsymbol{\xi}_r \rangle] \\ & \geq -2Re\langle \boldsymbol{\xi}_\varphi, \boldsymbol{\Omega} \boldsymbol{\xi}_r \rangle_{t=0} + Et. \end{aligned} \quad (67)$$

Proof. An important aspect in the subsequent calculation is the conservation of energy for this perturbation, related to statement (I) in the general theorem presented in Section II C. Indeed, this proved the constancy of

$$-E_\xi = E = -\left\langle \frac{\partial \boldsymbol{\xi}}{\partial t}, \frac{\partial \boldsymbol{\xi}}{\partial t} \right\rangle + \langle \boldsymbol{\xi}, \mathcal{F}[\boldsymbol{\xi}] \rangle. \quad (68)$$

To prove Eq. (67), we embark on a term by term calculation, multiplying them all by 2 for convenience and adding them later. The time derivative of the first term gives

$$\begin{aligned} \frac{d}{dt} 2Re\langle \boldsymbol{\xi}, \partial_t \boldsymbol{\xi} \rangle &= \frac{d}{dt} [\langle \boldsymbol{\xi}, \partial_t \boldsymbol{\xi} \rangle + \langle \partial_t \boldsymbol{\xi}, \boldsymbol{\xi} \rangle] \\ &= 2\langle \partial_t \boldsymbol{\xi}, \partial_t \boldsymbol{\xi} \rangle + 2\langle \boldsymbol{\xi}, \mathcal{F}[\boldsymbol{\xi}] \rangle \\ &\quad - 4Re\langle \boldsymbol{\xi}, (\mathbf{v} \cdot \nabla) \partial_t \boldsymbol{\xi} \rangle \\ &= 4\langle \partial_t \boldsymbol{\xi}, \partial_t \boldsymbol{\xi} \rangle + 2E + 4Re\langle \boldsymbol{\xi}_r, \boldsymbol{\Omega} \partial_t \boldsymbol{\xi}_\varphi \rangle \\ &\quad - 4Re\langle \boldsymbol{\xi}_\varphi, \boldsymbol{\Omega} \partial_t \boldsymbol{\xi}_r \rangle, \end{aligned} \quad (69)$$

where we used that $\mathcal{F}[\boldsymbol{\xi}] = \partial_t^2 \boldsymbol{\xi} + 2\mathbf{v} \cdot \nabla \partial_t \boldsymbol{\xi}$, the symmetry properties of $\mathcal{F}$ and antisymmetry of $\mathbf{v} \cdot \nabla$, along with conservation of energy and the fact that the Coriolis term can be obtained as

$$\begin{aligned} & \langle \boldsymbol{\xi}, (\mathbf{v} \cdot \nabla) \partial_t \boldsymbol{\xi} \rangle \\ &= \int dV \rho [\zeta_r^* \hat{\mathbf{r}} + \zeta_\varphi^* \hat{\boldsymbol{\phi}} + \zeta_z^* \hat{\mathbf{z}}] \cdot \boldsymbol{\Omega} [-\partial_t \zeta_\varphi \hat{\mathbf{r}} + \partial_t \zeta_r \hat{\boldsymbol{\phi}}] \\ &= \int dV \rho \boldsymbol{\Omega} [-\zeta_r^* \partial_t \zeta_\varphi + \zeta_\varphi^* \partial_t \zeta_r] \\ &= \langle \boldsymbol{\xi}_\varphi, \boldsymbol{\Omega} \partial_t \boldsymbol{\xi}_r \rangle - \langle \boldsymbol{\xi}_r, \boldsymbol{\Omega} \partial_t \boldsymbol{\xi}_\varphi \rangle. \end{aligned} \quad (70)$$

The time derivative for the third term of Eq. (67) can be computed to become

$$\begin{aligned} -\frac{d}{dt} 4Re\langle \boldsymbol{\xi}_\varphi, \boldsymbol{\Omega} \boldsymbol{\xi}_r \rangle &= -\frac{d}{dt} 2[\langle \boldsymbol{\xi}_\varphi, \boldsymbol{\Omega} \boldsymbol{\xi}_r \rangle + \langle \boldsymbol{\Omega} \boldsymbol{\xi}_r, \boldsymbol{\xi}_\varphi \rangle] \\ &= -4Re\langle \boldsymbol{\xi}_\varphi, \boldsymbol{\Omega} \partial_t \boldsymbol{\xi}_r \rangle - 4Re\langle \boldsymbol{\xi}_r, \boldsymbol{\Omega} \partial_t \boldsymbol{\xi}_\varphi \rangle. \end{aligned} \quad (71)$$

Adding with the last two terms of (69), we see that a term $-8Re\langle \boldsymbol{\xi}_\varphi, \boldsymbol{\Omega} \partial_t \boldsymbol{\xi}_r \rangle$ remains. We now use (54) to get

$$-8Re\langle \boldsymbol{\xi}_\varphi, \boldsymbol{\Omega} \partial_t \boldsymbol{\xi}_r \rangle = -4Re\langle \boldsymbol{\xi}_\varphi, 2\boldsymbol{\Omega} \partial_t \boldsymbol{\xi}_r \rangle \quad (72)$$

$$\begin{aligned} &= 4Re \int dV \zeta_\varphi^* \left[\rho \partial_t^2 \zeta_\varphi - \frac{1}{\mu_0 r} (\mathbf{B}_p \cdot \nabla) \right. \\ &\quad \left. \times \left[r^2 (\mathbf{B}_p \cdot \nabla) \left(\frac{\zeta_\varphi}{r} \right) \right] \right] \end{aligned} \quad (73)$$

$$\begin{aligned} &= 4Re \int dV \left[\rho \zeta_\varphi^* \partial_t^2 \zeta_\varphi + \frac{r^2}{\mu_0} \left| \mathbf{B}_p \cdot \nabla \left(\frac{\zeta_\varphi}{r} \right) \right|^2 \right] \end{aligned} \quad (74)$$

$$= 4Re\langle \boldsymbol{\xi}_\varphi, \partial_t^2 \boldsymbol{\xi}_\varphi \rangle + 4 \int dV \frac{r^2}{\mu_0} \left| \mathbf{B}_p \cdot \nabla \left(\frac{\zeta_\varphi}{r} \right) \right|^2. \quad (75)$$

In this derivation, an integration by parts is performed, where boundary contributions proportional to $r \zeta_\varphi$ vanish. The last term to calculate is

$$\begin{aligned} -\frac{d}{dt} 4Re\langle \boldsymbol{\xi}_\varphi, \partial_t \boldsymbol{\xi}_\varphi \rangle &= -\frac{d}{dt} 2[\langle \boldsymbol{\xi}_\varphi, \partial_t \boldsymbol{\xi}_\varphi \rangle + \langle \partial_t \boldsymbol{\xi}_\varphi, \boldsymbol{\xi}_\varphi \rangle] \\ &= -4\langle \partial_t \boldsymbol{\xi}_\varphi, \partial_t \boldsymbol{\xi}_\varphi \rangle - 4Re\langle \boldsymbol{\xi}_\varphi, \partial_t^2 \boldsymbol{\xi}_\varphi \rangle. \end{aligned} \quad (76)$$

Putting everything together we see that the $\langle \boldsymbol{\xi}_\varphi, \partial_t^2 \boldsymbol{\xi}_\varphi \rangle$ -terms cancel to get

$$\frac{d}{dt} [2Re\langle \boldsymbol{\xi}, \partial_t \boldsymbol{\xi} \rangle - 4Re\langle \boldsymbol{\xi}_\varphi, \partial_t \boldsymbol{\xi}_\varphi \rangle - 4Re\langle \boldsymbol{\xi}_\varphi, \boldsymbol{\Omega} \boldsymbol{\xi}_r \rangle] \quad (77)$$

$$\begin{aligned} &= 2E + 4\langle \partial_t \boldsymbol{\xi}, \partial_t \boldsymbol{\xi} \rangle - 4\langle \partial_t \boldsymbol{\xi}_\varphi, \partial_t \boldsymbol{\xi}_\varphi \rangle \\ &\quad + 4 \int dV \frac{r^2}{\mu_0} \left| \mathbf{B}_p \cdot \nabla \left(\frac{\zeta_\varphi}{r} \right) \right|^2 \end{aligned} \quad (78)$$

$$\geq 2E. \quad (79)$$

Integrating with respect to t yields the desired result, where only the $t=0$ contribution from $\langle \boldsymbol{\xi}_\varphi, \boldsymbol{\Omega} \boldsymbol{\xi}_r \rangle$ remains due to the initial condition choice where $\partial_t \boldsymbol{\xi} = 0$. $\square$

The above principle is again general for axisymmetric perturbations about an axisymmetric, (differentially) rotating, gravitationally stratified configuration with a purely poloidal field, and these ingredients are known to suffice for the development of, e.g., magnetorotational instabilities (MRI).^{18–20} Previously,¹⁷ necessary and sufficient conditions for stability were derived for special cases (e.g., adopting a weak magnetic field, or for a potential, i.e., current-free, equilibrium magnetic field), along with statements on field strengths needed to stabilize the MRI. The theorems above also apply to axisymmetric perturbations in 1D, cylindrical accretion disk models, if one considers either purely toroidal (i.e., azimuthal) field (Theorem 2) or purely poloidal (i.e., axial) magnetic field (Theorem 3). Note that they do not

make a requirement on the strength of the magnetic field a priori, and previous work^{21,22} has already highlighted the role of up to equipartition magnetic fields, in a mixture of both poloidal and toroidal fields, and their effect on axisymmetric perturbations.

V. DIAGONALIZING THE PERTURBATIONAL ENERGY

In this section, we derive an expression for the “energy” related quantity given by $\langle \xi, \mathcal{F}[\xi] \rangle$, where we consider the general case of an arbitrary stationary MHD equilibrium and any type of perturbation obeying the perturbed EOM from Eq. (17). To start in general, we repeat this perturbed EOM here as

$$\begin{aligned} \delta \left(\frac{d\mathbf{v}}{dt} \right) &= \frac{d^2 \xi}{dt^2} - (\xi \cdot \nabla) \frac{d\mathbf{v}}{dt} \\ &= \partial_t^2 \xi + 2(\mathbf{v} \cdot \nabla) \partial_t \xi + (\mathbf{v} \cdot \nabla)^2 \xi - (\xi \cdot \nabla)([\mathbf{v} \cdot \nabla] \mathbf{v}) \end{aligned} \quad (80)$$

$$\begin{aligned} &= -\delta \frac{\nabla p}{\rho} - \nabla \delta \Phi + \delta \left(\frac{1}{\rho} \mathbf{F}_m \right) \\ &= -\underbrace{\frac{1}{\rho} \nabla \delta p}_I + \underbrace{\frac{\delta \rho}{\rho^2} \nabla p - \nabla \delta \Phi}_{III} - \underbrace{\frac{\delta \rho}{\rho^2} \mathbf{F}_m + \frac{1}{\rho} \delta \mathbf{F}_m}_{II} \end{aligned} \quad (81)$$

$$= \mathcal{F}[\xi] + (\mathbf{v} \cdot \nabla)^2 \xi - (\xi \cdot \nabla)([\mathbf{v} \cdot \nabla] \mathbf{v}), \quad (82)$$

where $\mathbf{F}_m$ is the Lorentz force. We can already remark here that although originally $\Phi = \Phi_{ext} + \Phi_{int}$, clearly $\delta \Phi$ only has

a contribution from internal gravity, which we denoted consistently with $\delta \Phi_{int, \xi}$ in earlier expressions. Since this is the only one that matters in the following algebra, we from here on understand $\delta \Phi \equiv \delta \Phi_{int, \xi}$. Now we take the dot product of this equation with $\rho \xi^*$, integrate over the entire plasma volume and finalize by taking the real part, noting that $\langle \xi, \mathcal{F}[\xi] \rangle$ is real anyway.

We will consider (minus) the term containing the perturbed pressure, i.e., term I

$$\begin{aligned} Re \int_V \xi^* \cdot \nabla \delta p dV \\ = -Re \int_V (\nabla \cdot \xi^*) \delta p dV = Re \int_V \frac{1}{\gamma p} (\delta p^* + \xi^* \cdot \nabla p) \delta p dV \end{aligned} \quad (83)$$

$$= Re \int_V \left[\frac{|\delta p|^2}{\gamma p} + \frac{1}{\gamma p} (\xi^* \cdot \nabla p) (-\xi \cdot \nabla p - \gamma p (\nabla \cdot \xi)) \right] dV \quad (84)$$

$$= \int_V \left[\frac{|\delta p|^2}{\gamma p} - Re \left\{ (\xi^* \cdot \nabla p) \left(\frac{\xi \cdot \nabla p}{\gamma p} - \frac{1}{\rho} (\delta \rho + \xi \cdot \nabla \rho) \right) \right\} \right] dV. \quad (85)$$

An integration by parts has been done in the first step, with no boundary contributions under normal circumstances. All other steps just manipulated with the expressions given in Eqs. (11)–(15). Next we turn to (minus) term II, where we get

$$Re \int_V \frac{\delta \rho}{\rho} \xi^* \cdot \mathbf{F}_m dV = -Re \int_V \left((\nabla \cdot \xi) + \frac{1}{\rho} \xi \cdot \nabla \rho \right) \xi^* \cdot \mathbf{F}_m dV \quad (86)$$

$$\begin{aligned} &= -Re \int_V \left(-\frac{1}{\gamma p} (\delta p + \xi \cdot \nabla p) + \frac{\xi \cdot \nabla \rho}{\rho} \right) \xi^* \cdot \mathbf{F}_m dV \\ &= \int_V \left[\frac{1}{\gamma p} Re[(\xi^* \cdot \mathbf{F}_m) \delta p] + Re \left\{ (\xi^* \cdot \mathbf{F}_m) \left(\frac{\xi \cdot \nabla p}{\gamma p} - \frac{\xi \cdot \nabla \rho}{\rho} \right) \right\} \right] dV. \end{aligned} \quad (87)$$

Adding all terms coming from I, II, and III, we thus get

$$\begin{aligned} &Re \int_V \xi^* \cdot \left(\nabla \delta p - \frac{\delta \rho}{\rho} \nabla p + \frac{\delta \rho}{\rho} \mathbf{F}_m \right) dV \\ &= \int_V \left[\frac{1}{\gamma p} \left(|\delta p|^2 + Re[(\xi^* \cdot \mathbf{F}_m) \delta p] \right) + Re \left\{ (\xi^* \cdot (\nabla p - \mathbf{F}_m)) \left(\frac{\xi \cdot \nabla \rho}{\rho} - \frac{\xi \cdot \nabla p}{\gamma p} \right) \right\} \right] dV \\ &= \int_V \left[\frac{1}{\gamma p} \left| \delta p + \frac{1}{2} (\xi \cdot \mathbf{F}_m) \right|^2 - \frac{1}{4\gamma p} |\xi \cdot \mathbf{F}_m|^2 + Re \left\{ (\xi^* \cdot (\nabla p - \mathbf{F}_m)) \left(\frac{\xi \cdot \nabla \rho}{\rho} - \frac{\xi \cdot \nabla p}{\gamma p} \right) \right\} \right] dV. \end{aligned} \quad (89)$$

We still have a term related to the perturbed Lorentz force, which we rework to

$$Re \int_V [\xi^* \cdot \delta \mathbf{F}_m] dV = Re \int_V \left[\xi^* \cdot \left(\frac{1}{\mu_0} (\nabla \times \delta \mathbf{B}) \times \mathbf{B} + \mathbf{J} \times \delta \mathbf{B} \right) \right] dV \quad (90)$$

$$= Re \int_V \left[-\frac{1}{\mu_0} (\xi^* \times \mathbf{B}) \cdot (\nabla \times \delta \mathbf{B}) + (\xi^* \times \mathbf{J}) \cdot \delta \mathbf{B} \right] dV \quad (91)$$

$$= Re \int_V \left[-\frac{1}{\mu_0} \nabla \times (\xi^* \times \mathbf{B}) \cdot \delta \mathbf{B} + (\xi^* \times \mathbf{J}) \cdot \delta \mathbf{B} \right] dV \quad (92)$$

$$= \int_V \left[-\frac{1}{\mu_0} |\delta \mathbf{B}|^2 + Re \{ (\boldsymbol{\xi}^* \times \mathbf{J}) \cdot \delta \mathbf{B} \} \right] dV \quad (93)$$

$$= \int_V \left[-\frac{1}{\mu_0} \left| \delta \mathbf{B} - \frac{\mu_0}{2} \boldsymbol{\xi} \times \mathbf{J} \right|^2 + \frac{\mu_0}{4} |\boldsymbol{\xi} \times \mathbf{J}|^2 \right] dV. \quad (94)$$

Finally, we manipulate the self-gravitational term to reveal that it is destabilising in all circumstances

$$\int_V \rho \boldsymbol{\xi}^* \cdot \nabla \delta \Phi dV = - \int_V \nabla \cdot (\rho \boldsymbol{\xi}^*) \delta \Phi dV = \int_V \delta \rho^* \delta \Phi dV = \frac{1}{4\pi G} \int_V \nabla^2 \delta \Phi^* \delta \Phi dV \quad (95)$$

$$= -\frac{1}{4\pi G} \left(\int_V |\nabla \delta \Phi|^2 dV - \int_{\partial V} \delta \Phi \nabla \delta \Phi^* d\mathbf{S} \right) \quad (96)$$

$$= -\frac{1}{4\pi G} \left(\int_V |\nabla \delta \Phi|^2 dV + \int_{\partial V^c} \delta \Phi \nabla \delta \Phi^* d\mathbf{S} \right),$$

where V^c is the surrounding vacuum,

$$= -\frac{1}{4\pi G} \left(\int_V |\nabla \delta \Phi|^2 dV + \int_{V^c} \nabla \cdot (\delta \Phi \nabla \delta \Phi^*) dV \right) \quad (97)$$

$$= -\frac{1}{4\pi G} \left(\int_V |\nabla \delta \Phi|^2 dV + \int_{V^c} \left(|\nabla \delta \Phi|^2 + \delta \Phi \underbrace{\nabla^2 \delta \Phi^*}_{\propto \delta \rho^* = 0 \text{ in vacuum}} \right) dV \right) \quad (98)$$

$$= -\frac{1}{4\pi G} \int_{\mathbb{R}^3} |\nabla \delta \Phi|^2 dV. \quad (99)$$

Now we collect all this to derive the expression for the energy associated with the force operator, from

$$\begin{aligned} \int_V dV \rho (\boldsymbol{\xi}^* \cdot \mathcal{F}[\boldsymbol{\xi}]) &= Re \int_V dV \rho (\boldsymbol{\xi}^* \cdot \mathcal{F}[\boldsymbol{\xi}]) = \int_V dV \left[-Re \left\{ \rho \boldsymbol{\xi}^* \cdot (\mathbf{v} \cdot \nabla)^2 \boldsymbol{\xi} \right\} - \frac{\left| \delta p + \frac{1}{2} \boldsymbol{\xi} \cdot \mathbf{F}_m \right|^2}{\gamma p} - \frac{1}{\mu_0} \left| \delta \mathbf{B} - \frac{\mu_0}{2} \boldsymbol{\xi} \times \mathbf{J} \right|^2 \right] \\ &+ \frac{1}{4\pi G} \int_{\mathbb{R}^3} |\nabla \delta \Phi|^2 + \int_V dV \left[\frac{1}{4\gamma p} |\boldsymbol{\xi} \cdot \mathbf{F}_m|^2 + \frac{\mu_0}{4} |\boldsymbol{\xi} \times \mathbf{J}|^2 \right] \\ &+ Re \left\{ \rho \boldsymbol{\xi}^* \cdot (\boldsymbol{\xi} \cdot \nabla) ([\mathbf{v} \cdot \nabla] \mathbf{v}) + (\boldsymbol{\xi}^* \cdot (\nabla p - \mathbf{F}_m)) \left(\frac{1}{\gamma p} (\boldsymbol{\xi} \cdot \nabla p) - \frac{1}{\rho} (\boldsymbol{\xi} \cdot \nabla \rho) \right) \right\}. \end{aligned} \quad (100)$$

This leads to the general and useful expression for the energy, namely

$$\begin{aligned} \langle \boldsymbol{\xi}, \mathcal{F}[\boldsymbol{\xi}] \rangle &= \int_V dV \left[\rho |(\mathbf{v} \cdot \nabla) \boldsymbol{\xi}|^2 - \frac{\left| \delta p + \frac{1}{2} \boldsymbol{\xi} \cdot \mathbf{F}_m \right|^2}{\gamma p} - \frac{1}{\mu_0} \left| \delta \mathbf{B} - \frac{\mu_0}{2} \boldsymbol{\xi} \times \mathbf{J} \right|^2 \right] + \frac{1}{4\pi G} \int_{\mathbb{R}^3} |\nabla \delta \Phi|^2 \\ &+ \int_V dV \left[\frac{1}{4\gamma p} |\boldsymbol{\xi} \cdot \mathbf{F}_m|^2 + \frac{\mu_0}{4} |\boldsymbol{\xi} \times \mathbf{J}|^2 \right] \\ &+ \int_V Re \left\{ \rho \boldsymbol{\xi}^* \cdot (\boldsymbol{\xi} \cdot \nabla) ([\mathbf{v} \cdot \nabla] \mathbf{v}) + \underbrace{\left(\boldsymbol{\xi}^* \cdot (\nabla p - \mathbf{F}_m) \right)}_I \underbrace{\left(\frac{1}{\gamma p} (\boldsymbol{\xi} \cdot \nabla p) - \frac{1}{\rho} (\boldsymbol{\xi} \cdot \nabla \rho) \right)}_{II} \right\}. \end{aligned} \quad (101)$$

This expression is organized such that the terms that involve $\partial_i \xi_j$ all appear on the first line and are in a form that tells us whether they are always negative or positive. The other terms, those which do not depend on the spatial derivatives of $\boldsymbol{\xi}$, come in the second and third line. Moreover, the last term on the third line has a factor I, which is just $\partial_i T_j^i$, or the divergence of the ideal MHD stress tensor of the equilibrium flow. We can understand factor II better by considering its thermodynamical meaning. If we define the Schwarzschild entropy from

$$s^\dagger(\mathbf{r}, \mathbf{r}') = \log \left(\frac{\rho(p(\mathbf{r}), s(\mathbf{r}'))}{\rho(\mathbf{r})} \right). \quad (102)$$

where $\rho(\mathbf{r}) = \rho(p(\mathbf{r}), s(\mathbf{r}))$, then

$$(\nabla_1 s^\dagger)(\mathbf{r}, \mathbf{r}) = \left(\frac{1}{\rho} \left(\frac{\partial \rho}{\partial p} \right)_s \nabla p - \frac{1}{\rho} \nabla \rho \right)(\mathbf{r}), \quad (103)$$

where ∇_1 indicates differentiation with respect to the first argument of $s^\dagger$. This appears in term II in the form $\xi \cdot \nabla s^\dagger$ (dropping the subscript 1), in fact generalized from a polytropic equation of state $p(\rho) \propto \rho^\gamma$ by replacing γp by $\rho \left(\frac{\partial p}{\partial \rho} \right)_s$ everywhere.

To use the general expression (101) for actual stability criteria, it is useful to provide a bound on the energy from self-gravitation. Since $\nabla \cdot (\rho \xi) = -\delta \rho = -\frac{1}{4\pi G} \nabla^2 \delta \Phi$, we have

$$\rho \xi = -\frac{1}{4\pi G} \nabla \delta \Phi + \mathbf{V}, \quad (104)$$

where $\mathbf{V} = \nabla \times \mathbf{W}$ is a divergenceless vector field (not really uniquely defined). This allows us to calculate

$$\begin{aligned} 4\pi G \int_V |\rho \xi|^2 dV &= 4\pi G \int_V \left(\left| \frac{1}{4\pi G} \nabla \delta \Phi \right|^2 + |\mathbf{V}|^2 \right) dV - 2Re \int_V \nabla \delta \Phi \cdot \mathbf{V} dV \\ &= 4\pi G \int_V \left(\left| \frac{1}{4\pi G} \nabla \delta \Phi \right|^2 + |\mathbf{V}|^2 \right) dV + 2Re \int_V \underbrace{\delta \Phi (\nabla \cdot \mathbf{V})}_{=0} dV - 2 \int_{\partial V} \delta \Phi \mathbf{V} \cdot d\mathbf{S} \\ &= 4\pi G \int_V \left(\left| \frac{1}{4\pi G} \nabla \delta \Phi \right|^2 + |\mathbf{V}|^2 \right) dV - 2 \int_{\partial V} \delta \Phi \mathbf{V} \cdot d\mathbf{S} \\ &\geq \frac{1}{4\pi G} \int_V |\nabla \delta \Phi|^2 dV - 2 \int_{\partial V} \delta \Phi \mathbf{V} \cdot d\mathbf{S}. \end{aligned} \quad (105)$$

However, the boundary condition $\mathbf{n} \cdot \xi = 0$ at the plasma boundary ∂V says that at the boundary ∂V we have $\mathbf{V} = \frac{1}{4\pi G} \nabla \delta \Phi + \mathbf{Z}$, where $\mathbf{n} \cdot \mathbf{Z} = 0$. Therefore, the last term now rewrites to

$$-2 \int_{\partial V} \delta \Phi \mathbf{V} \cdot d\mathbf{S} = -\frac{2}{4\pi G} \int_{\partial V} \delta \Phi \nabla \delta \Phi \cdot d\mathbf{S} \quad (106)$$

$$= 2 \frac{1}{4\pi G} \int_{\partial V^c} \delta \Phi \nabla \delta \Phi \cdot d\mathbf{S} \quad (107)$$

$$= 2 \frac{1}{4\pi G} \int_{V^c} \nabla \cdot (\delta \Phi \nabla \delta \Phi) dV \quad (108)$$

$$= 2 \frac{1}{4\pi G} \int_{V^c} \left(|\nabla \delta \Phi|^2 + \underbrace{\delta \Phi \nabla^2 \delta \Phi}_{=0 \text{ in vacuum}} \right) dV \quad (109)$$

$$= 2 \frac{1}{4\pi G} \int_{V^c} |\nabla \delta \Phi|^2 dV. \quad (110)$$

So the gravitational energy is bounded by

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^3} \frac{1}{4\pi G} |\nabla \delta \Phi|^2 dV \\ &\leq \frac{1}{4\pi G} \left(\int_V |\nabla \delta \Phi|^2 dV + 2 \int_{V^c} |\nabla \delta \Phi|^2 dV \right) \\ &\leq 4\pi G \int_V |\rho \xi|^2 dV. \end{aligned} \quad (111)$$

In the special case of axisymmetric equilibria and axisymmetric perturbations, we have $\nabla \cdot (\rho \xi) = \nabla \cdot (\rho \xi_p)$ leading similarly to

$$\rho \xi_p = -\frac{1}{4\pi G} \nabla \delta \Phi + \mathbf{V}', \quad (112)$$

for some different vector-field $\mathbf{V}'$. The whole line of reasoning can then be repeated for $\rho \xi_p$ to conclude finally that

$$0 \leq \int_{\mathbb{R}^3} \frac{1}{4\pi G} |\nabla \delta \Phi|^2 dV \leq 4\pi G \int_V dV |\rho \xi_p|^2. \quad (113)$$

Using this bound for the gravitational energy, we can estimate the total energy (101) as

$$\begin{aligned} \langle \xi, \mathcal{F}[\xi] \rangle &\leq \int_V dV \left[\rho |(\mathbf{v} \cdot \nabla) \xi|^2 - \frac{|\delta p'|^2}{\gamma p} - \frac{1}{\mu_0} |\delta \mathbf{B}'|^2 + 4\pi G |\rho \xi|^2 \right] \\ &\quad + \int_V dV \left[\frac{1}{4\gamma p} |\xi \cdot \mathbf{F}_m|^2 + \frac{\mu_0}{4} |\xi \times \mathbf{J}|^2 + Re \left\{ \rho \xi^* \cdot (\xi \cdot \nabla) ([\mathbf{v} \cdot \nabla] \mathbf{v}) + (\xi^* \cdot (\nabla p - \mathbf{F}_m)) \xi \cdot \nabla s^\dagger \right\} \right], \end{aligned} \quad (114)$$

where the term $4\pi G |\rho \xi|^2$ can be replaced by $4\pi G |\rho \xi_p|^2$ if ξ is axisymmetric. In this expression, we introduced shorthand notations $\delta p' = \delta p + \frac{1}{2} \xi \cdot \mathbf{F}_m$ and $\delta \mathbf{B}' = \delta \mathbf{B} - \frac{\mu_0}{2} \xi \times \mathbf{J}$.

VI. STABILITY THEORY

In this section, we will use the general expressions for the energy-related $\langle \xi, \mathcal{F}[\xi] \rangle$ given in Eq. (101), and its bound as stated in Eq. (114), for revisiting and generalizing several known stability results for static equilibria.

A. Static hydrodynamic equilibria

In its simplest limit, we adopt a static hydro equilibrium where (self-)gravity is balanced by pressure gradients. As $\mathbf{B} = 0 = \mathbf{v}$, we find that $\delta \mathbf{B}' = 0$ automatically, so we only have to mitigate the stabilizing effect of the $\delta p'$ -term. The general expression reduces in this unmagnetized, static limit to

$$\langle \xi, \mathcal{F}[\xi] \rangle = - \int_V dV \left[\frac{|\delta p|^2}{\gamma p} - Re(\xi^* \cdot \nabla p)(\xi \cdot \nabla s^\dagger) \right] + \frac{1}{4\pi G} \int_{\mathbb{R}^3} |\nabla \delta \Phi|^2. \quad (115)$$

We can introduce generally for some unit vector $\hat{\mathbf{z}}$, the (squared) Brunt-Väisälä frequency in that direction from

$$N_z^2 = -\frac{1}{\rho} (\hat{\mathbf{z}} \cdot \nabla p)(\hat{\mathbf{z}} \cdot \nabla s^\dagger). \quad (116)$$

In the case of purely external gravity, we have $\delta \Phi = 0$, and the only destabilizing effect is entirely related to this term. We can conclude generally that in a static, unmagnetized equilibrium that is stable, the Schwarzschild entropy $s^\dagger(\mathbf{r}, \mathbf{r}')$ always decreases as a function of $\mathbf{r}$ in directions where the pressure increases. Instability results from regions and directions with negative N_z^2 , occurring when ∇p and $\nabla s^\dagger$ are not precisely antiparallel. Note that this obviously connects to well-known convective-gravitational instability criteria, such as the Schwarzschild criterion for convective instability. It is to be noted that these are usually presented for specific 1D static equilibria (assuming spherical symmetry for stellar interiors, or considering a planar gravitationally stratified slab or atmosphere).

B. Static MHD equilibria

In static MHD, we can consult the energy given by (114) to get the following. Let $(\rho, p, \mathbf{B})$ be a static ideal MHD solution. It is stable in the sense that $\langle \xi, \mathcal{F}[\xi] \rangle \leq 0$ for all Lagrangian perturbation ξ if²³ $\forall \mathbf{r}$ and $\forall \mathbf{e} \in \mathbb{R}^3$

$$4\pi G(\rho \mathbf{e})^2 + \frac{1}{4\gamma p} (\mathbf{e} \cdot \mathbf{F}_m)^2 + \frac{\mu_0}{4} (\mathbf{e} \times \mathbf{J})^2 + (\mathbf{e} \cdot (\nabla p - \mathbf{F}_m))(\mathbf{e} \cdot \nabla s^\dagger) \leq 0. \quad (117)$$

This “would-be” stability theorem is actually only applicable whenever a current-free ($\mathbf{J} = 0$) equilibrium is around and when we take $G = 0$ (i.e., we neglect self-gravitation). In this case, the stability criterion reduces to the same thing as in static hydro, namely

$$(\mathbf{e} \cdot \nabla p)(\mathbf{e} \cdot \nabla s^\dagger) \leq 0. \quad (118)$$

Note that this is now a general result for non-self-gravitating, static MHD equilibria where the equilibrium magnetic field is potential (i.e., where $\mathbf{J} = 0$, note that we do have $\delta \mathbf{B} \neq 0$). This means essentially that ∇p and $\nabla s^\dagger$ should be antiparallel everywhere. In particular, it is clear that homogeneous plasmas (uniform $\mathbf{B}$, pressure and density) are always stable (no longer true when self-gravity is included,¹ which then gives the Jeans instability).

The following theorem may be useful as well.

Theorem 4. Let $(\rho, s, \mathbf{B})$ be a static MHD equilibrium and suppose that we can find a function $g > 0$ such that $\forall \xi \in K := (\ker \mathcal{F})^\perp$ we have

$$\int \left(\frac{1}{\gamma p} |\delta p'|^2 + \frac{1}{\mu_0} |\delta \mathbf{B}'|^2 \right) dV \geq \int \rho g |\xi|^2 dV. \quad (119)$$

Then, we conclude that for some $\lambda > 0$ and $\forall \xi \in K$ we have $\langle \xi, \mathcal{F}[\xi] \rangle + \lambda^2 \langle \xi, \xi \rangle \leq 0$ if the following condition is satisfied $\forall \mathbf{r}$ and $\forall \mathbf{e} \in \mathbb{R}^3$

$$-\rho g e^2 + 4\pi G(\rho \mathbf{e})^2 + \frac{1}{4\gamma p} (\mathbf{e} \cdot \mathbf{F}_m)^2 + \frac{\mu_0}{4} (\mathbf{e} \times \mathbf{J})^2 + (\mathbf{e} \cdot (\nabla p - \mathbf{F}_m))(\mathbf{e} \cdot \nabla s^\dagger) + \lambda^2 \rho e^2 \leq 0. \quad (120)$$

The energy expression as given above has consequences for magnetoconvection and magnetic buoyancy, where it is often stated that bubbles rise in their environment while adjusting the total pressure $P = p + \frac{B^2}{2\mu_0}$ to that of the environment. If by doing so the bubble density drops faster than the one of the environment, then it will continue to rise by buoyancy. This however neglects magnetic tension and takes only the isotropic part of the magnetic stress tensor into account. The above expression suggest that if an instability may arise it should rather be

$$\delta p' = \delta p + \frac{1}{2} \xi \cdot \mathbf{F}_m \neq \delta P, \quad (121)$$

which remains zero relative to the environment (i.e., the unperturbed state).

If we consider again the general case of a static, ideal MHD equilibrium where $\mathbf{v} = 0$ and the equilibrium obeys $\nabla p - \mathbf{F}_m = -\rho \nabla \Phi$, several well-known facts can be read off from the remaining expression

$$\begin{aligned} \langle \xi, \mathcal{F}[\xi] \rangle = & \int_V dV \left[-\frac{|\delta p|^2}{\gamma p} - \frac{1}{\mu_0} |\delta \mathbf{B}'|^2 \right] \\ & + \frac{1}{4\pi G} \int_{\mathbb{R}^3} |\nabla \delta \Phi|^2 + \int_V dV \left[\frac{1}{4\gamma p} |\xi \cdot \mathbf{F}_m|^2 \right. \\ & \left. + \frac{\mu_0}{4} |\xi \times \mathbf{J}|^2 + Re\{(-\rho \xi^* \cdot \nabla \Phi) \xi \cdot \nabla s^\dagger\} \right]. \end{aligned} \quad (122)$$

The expression shows that gravity, pressure gradients, and current density can act destabilizing, a well-known result. When gravity is neglected (both self-gravity and external gravity), we have $\nabla p = \mathbf{F}_m$ in the equilibrium and only two

terms can cause instabilities. Many results have originally been investigated for 1D cylindrical equilibria of this kind, or for axially symmetric equilibria where the equilibrium balance between pressure gradient and Lorentz force gives

the Grad-Shafranov equation used to describe static tokamak equilibria. In fact, textbooks²⁴ emphasize a different means of writing the expression (122), which is manipulated into the general form

$$\langle \xi, \mathcal{F}[\xi] \rangle = \int_V dV \left\{ -\gamma p |\nabla \cdot \xi|^2 - \frac{1}{\mu_0} |\delta \mathbf{B}|^2 - (\xi \cdot \nabla p) \nabla \cdot \xi^* + (\xi^* \cdot \nabla \Phi) \nabla \cdot (\rho \xi) - \mathbf{J} \cdot (\xi^* \times \delta \mathbf{B}) + \nabla \cdot [\xi^* (\xi \cdot \nabla) p + (\xi^* \times \mathbf{B}) \times \delta \mathbf{B} + \gamma p (\nabla \cdot \xi) \xi^*] \right\},$$

where we allowed for external gravity only, i.e., equilibria $\nabla p - \mathbf{F}_m = -\rho \nabla \Phi$. This way of writing emphasizes the stabilizing role played by energy needed for compression (acoustic term) and in magnetic energy on the first line, and has all potential destabilizing effects due to pressure gradients, external gravity, and currents on the second line. The third line is reorganized to a divergence term, which will give no contribution after application of the Gauss theorem, when the plasma obeys the standard boundary conditions $\mathbf{n} \cdot \mathbf{B} = 0$, $\mathbf{n} \cdot \xi = 0$, making $\xi^* \times \mathbf{B} \parallel \mathbf{n}$ such that $\mathbf{n} \cdot [(\xi^* \times \mathbf{B}) \times \delta \mathbf{B}] = 0$. We note that the equivalent expression (122) has a nice interpretation for pure Grad-Shafranov equilibria (i.e., axially symmetric balances of $\nabla p = \mathbf{F}_m$). Indeed, then equilibrium fieldlines with tangent vectors $\mathbf{B}$ and $\mathbf{J}$ live on nested flux surfaces (tori for tokamaks) that are the isosurfaces of pressure, thus defining three directions of interest: the normal to the flux surfaces (direction of $\nabla p = \mathbf{F}_m$), as well as $\mathbf{B}$ and $\mathbf{J}$ that span the tangent plane to the toroidal flux surfaces. Perturbations that are purely in this plane make $\xi \cdot \mathbf{F}_m$ vanish, but can still be destabilizing from misalignment with the local current $\mathbf{J}$ direction. Perturbations ξ purely aligned with the current $\mathbf{J}$ are still orthogonal to $\mathbf{F}_m$ and suffer no destabilizing effect at all. In fact, the pure force-free limit (where there is no pressure but only $\mathbf{F}_m = 0$ defines the equilibrium) has only destabilization from misaligned ξ and $\mathbf{J}$.

VII. SUFFICIENT CRITERIA FOR STABILITY OF AXISYMMETRIC, STATIONARY MHD CONFIGURATIONS

Here, we will consider the special case of axisymmetric, stationary MHD configurations. It is well known that for an

axially symmetric, flowing MHD stationary state, the poloidal parts of $\mathbf{v}$ and $\mathbf{B}$ are perfectly aligned

$$\rho \mathbf{v}_p = k \mathbf{B}_p, \quad (123)$$

where the function $k(r, z)$ is called the *mass-loading*. We may then define a “non-aligned rotation frequency” $\Omega'(r, z)$ (note that the prime does not mean a derivative with respect to time, but is just a label to distinguish it from Ω related to $v_\phi = r\Omega$) as follows:

$$\rho \mathbf{v} = k \mathbf{B} + \rho r \Omega' \hat{\phi}. \quad (124)$$

A. Perturbations with arbitrary m

By linearity, it is enough to investigate the stability of perturbations with definite angular wavenumbers m

$$\begin{aligned} \xi(r, \varphi, z) &= (\eta_r(r, z) \hat{\mathbf{r}} + \eta_\phi(r, z) \hat{\phi} + \eta_z(r, z) \hat{\mathbf{z}}) \exp(im\varphi) \\ &=: \boldsymbol{\eta}(r, z) \exp(im\varphi), \end{aligned} \quad (125)$$

and where we suppose that $\boldsymbol{\eta}$ is a real function (again no restriction).

Then, the upper bound on the energy (114) becomes (using the finding of Eq. (113))

$$\langle \xi, \mathcal{F}[\xi] \rangle \quad (126)$$

$$\begin{aligned} &\leq \int_V dV \left[\rho |(\mathbf{v} \cdot \nabla) \xi|^2 - \frac{|\delta p'_\xi|^2}{\gamma p} - \frac{1}{\mu_0} |\delta \mathbf{B}'_\xi|^2 + 4\pi G \rho^2 (|\xi|^2 - \delta_0^m |\xi_\phi|^2) \right] \\ &+ \int_V dV \left[\frac{1}{4\gamma p} |\boldsymbol{\eta} \cdot \mathbf{F}_m|^2 + \frac{\mu_0}{4} |\boldsymbol{\eta} \times \mathbf{J}|^2 + \rho \boldsymbol{\eta} \cdot [(\boldsymbol{\eta} \cdot \nabla)(\mathbf{v} \cdot \nabla) \mathbf{v}] + (\boldsymbol{\eta} \cdot (\nabla p - \mathbf{F}_m)) (\boldsymbol{\eta} \cdot \nabla s^\dagger) \right]. \end{aligned} \quad (127)$$

The first part of our program will now be to separate contributions which grow as $O(m)$ or $O(m^2)$ from those which do not depend on m . To this end, we have

$$(\mathbf{v} \cdot \nabla) \xi = \left[\underbrace{(\mathbf{v} \cdot \nabla) \boldsymbol{\eta}}_{\in \mathbb{R}} + \underbrace{i \Omega m \boldsymbol{\eta}}_{\in i\mathbb{R}} \right] \exp(im\varphi). \quad (128)$$

Note that we wrote Ω here (and not Ω'). Hence

$$\rho |(\mathbf{v} \cdot \nabla) \boldsymbol{\xi}|^2 = \rho (|(\mathbf{v} \cdot \nabla) \boldsymbol{\eta}|^2 + m^2 \Omega^2 |\boldsymbol{\eta}|^2). \quad (129)$$

Next in line, we have

$$\delta p'_\xi = -(\gamma p (\nabla \cdot \boldsymbol{\xi}) + \boldsymbol{\xi} \cdot \nabla p) + \frac{1}{2} \boldsymbol{\xi} \cdot \mathbf{F}_m \quad (130)$$

$$= \left[-\left(\gamma p \left(\frac{im}{r} \eta_\phi + (\nabla \cdot \boldsymbol{\eta}) \right) + \boldsymbol{\eta} \cdot \nabla p \right) + \frac{1}{2} \boldsymbol{\eta} \cdot \mathbf{F}_m \right] \exp(im\varphi) \quad (131)$$

$$= \left[\underbrace{\frac{-im\gamma p}{r} \eta_\phi}_{\in i\mathbb{R}} + \underbrace{\delta p'_\eta}_{\in \mathbb{R}} \right] \exp(im\varphi). \quad (132)$$

So we have $|\delta p'_\xi|^2 = \left(\frac{m\gamma p}{r} \eta_\phi \right)^2 + |\delta p'_\eta|^2$. Also we have

$$\begin{aligned} \delta \mathbf{B}'_\xi &= (\mathbf{B} \cdot \nabla) \boldsymbol{\xi} - (\nabla \cdot \boldsymbol{\xi}) \mathbf{B} - (\boldsymbol{\xi} \cdot \nabla) \mathbf{B} - \frac{\mu_0}{2} \boldsymbol{\xi} \times \mathbf{J} \\ &= \left[\frac{imB_\phi}{r} \boldsymbol{\eta} + (\mathbf{B} \cdot \nabla) \boldsymbol{\eta} - \frac{im}{r} \eta_\phi \mathbf{B} - (\nabla \cdot \boldsymbol{\eta}) \mathbf{B} - (\boldsymbol{\eta} \cdot \nabla) \mathbf{B} - \frac{\mu_0}{2} \boldsymbol{\eta} \times \mathbf{J} \right] \exp(im\varphi) \\ &= \left[\underbrace{\frac{im}{r} (B_\phi \boldsymbol{\eta} - \eta_\phi \mathbf{B})}_{\in i\mathbb{R}} + \underbrace{\delta \mathbf{B}'_\eta}_{\in \mathbb{R}} \right] \exp(im\varphi), \end{aligned} \quad (133)$$

hence $|\delta \mathbf{B}'_\xi|^2 = \left(\frac{m}{r} (B_\phi \boldsymbol{\eta} - \eta_\phi \mathbf{B}) \right)^2 + |\delta \mathbf{B}'_\eta|^2$. So, regrouping the terms of the energy a bit and noting that $|\boldsymbol{\xi}| = |\boldsymbol{\eta}|$ and $|\xi_\phi| = |\eta_\phi|$, we get

$$\begin{aligned} \langle \boldsymbol{\xi}, \mathcal{F}[\boldsymbol{\xi}] \rangle &\leq \int_V dV \left[\rho ((\mathbf{v} \cdot \nabla) \boldsymbol{\eta})^2 - \frac{(\delta p'_\eta)^2}{\gamma p} - \frac{1}{\mu_0} (\delta \mathbf{B}'_\eta)^2 + 4\pi G \rho^2 (\boldsymbol{\eta}^2 - \delta_0^m \eta_\phi^2) \right] \\ &+ \int_V dV \left[\rho \boldsymbol{\eta} \cdot [(\boldsymbol{\eta} \cdot \nabla) (\mathbf{v} \cdot \nabla) \mathbf{v}] + \frac{1}{4\gamma p} |\boldsymbol{\eta} \cdot \mathbf{F}_m|^2 + (\boldsymbol{\eta} \cdot (\nabla p - \mathbf{F}_m)) \left(\frac{1}{\gamma p} (\boldsymbol{\eta} \cdot \nabla p) - \frac{1}{\rho} (\boldsymbol{\eta} \cdot \nabla \rho) \right) + \frac{\mu_0}{4} |\boldsymbol{\eta} \times \mathbf{J}|^2 \right] \\ &+ m^2 \int_V dV \left[\rho \Omega^2 \boldsymbol{\eta}^2 - \gamma p \left(\frac{\eta_\phi}{r} \right)^2 - \frac{1}{r^2} (B_\phi \boldsymbol{\eta} - \eta_\phi \mathbf{B})^2 \right]. \end{aligned} \quad (134)$$

As discussed before, special attention should be devoted to the destabilising term with $|(\mathbf{v} \cdot \nabla) \boldsymbol{\eta}|^2$. We have

$$\rho (\mathbf{v} \cdot \nabla) \boldsymbol{\eta} = k (\mathbf{B} \cdot \nabla) \boldsymbol{\eta} + \rho \Omega' \hat{\mathbf{z}} \times \boldsymbol{\eta} \quad (135)$$

$$= k (\delta \mathbf{B}_\eta + (\nabla \cdot \boldsymbol{\eta}) \mathbf{B} + \boldsymbol{\eta} \cdot \nabla \mathbf{B}) + \rho \Omega' \hat{\mathbf{z}} \times \boldsymbol{\eta} \quad (136)$$

$$= k \left(\delta \mathbf{B}_\eta - \frac{1}{\gamma p} (\delta p_\eta + \boldsymbol{\eta} \cdot \nabla p) \mathbf{B} + \boldsymbol{\eta} \cdot \nabla \mathbf{B} \right) + \rho \Omega' \hat{\mathbf{z}} \times \boldsymbol{\eta}, \quad (137)$$

where we used (14) and (15), respectively. Manipulating still a bit further to change $\delta \mathbf{B}_\eta$ to $\delta \mathbf{B}'_\eta$ and δp_η to $\delta p'_\eta$ as they are featured in (101) or (114), we get

$$\rho (\mathbf{v} \cdot \nabla) \boldsymbol{\eta} = k \left(\delta \mathbf{B}'_\eta - \frac{1}{\gamma p} \left(\delta p'_\eta - \frac{1}{2} \boldsymbol{\eta} \cdot \mathbf{F}_m + \boldsymbol{\eta} \cdot \nabla p \right) \mathbf{B} + \frac{\mu_0}{2} \boldsymbol{\eta} \times \mathbf{J} + \boldsymbol{\eta} \cdot \nabla \mathbf{B} \right) + \rho \Omega' \hat{\mathbf{z}} \times \boldsymbol{\eta}, \quad (138)$$

where it is for the moment just important to remember that it is of the form

$$\rho (\mathbf{v} \cdot \nabla) \boldsymbol{\eta} = k \left(\delta \mathbf{B}' - \frac{1}{\gamma p} \delta p' \mathbf{B} \right) + \sqrt{\rho} \mathbf{q}(\boldsymbol{\eta}), \quad (139)$$

where q depends only on $\boldsymbol{\eta}$ and not on its spatial derivatives. This means that, when we introduce

$$\mathbf{C} = \frac{1}{\sqrt{\mu_0}} \delta \mathbf{B}'_\eta, \quad \mathbf{D} = \frac{\mathbf{B}}{\sqrt{\gamma p B}} \delta p'_\eta, \quad \alpha = k \sqrt{\frac{\mu_0}{\rho}}, \quad \beta = k \frac{B}{\sqrt{\gamma p \rho}}, \quad (140)$$

we can manipulate, with some algebra, the following expression

$$\begin{aligned}
 -\frac{1}{\rho} |(\rho \mathbf{v} \cdot \nabla) \boldsymbol{\eta}|^2 + \frac{1}{\gamma p} |\delta p'|^2 + \frac{1}{\mu_0} |\delta \mathbf{B}'|^2 &= C^2 + D^2 - (\alpha C + \beta D + \mathbf{q})^2 = \underbrace{\frac{1}{1-\alpha^2} [(1-\alpha^2)C - \alpha\beta D - \alpha\mathbf{q}]^2}_I \\
 &+ \underbrace{\frac{1}{(1-\alpha^2)(1-\alpha^2-\beta^2)} [(1-\alpha^2-\beta^2)D - \beta\hat{\mathbf{B}} \cdot \mathbf{q}]^2}_{II} \\
 &- \underbrace{\frac{q^2}{1-\alpha^2}}_{III} - \underbrace{\frac{\beta^2}{(1-\alpha^2)(1-\alpha^2-\beta^2)} (\hat{\mathbf{B}} \cdot \mathbf{q})^2}_{IV}, \quad (141)
 \end{aligned}$$

where $\hat{\mathbf{B}}$ is the unit vector along $\mathbf{B}$, and it is important in this calculation to note that it is only valid as long as $1 - \alpha^2 - \beta^2 > 0$ which amounts to

$$\frac{\rho\gamma p - k^2(\mu_0\gamma p + B^2)}{\rho\gamma p} > 0, \quad (142)$$

requiring a low mass loading (k) as expected. Under this condition, the squares I and II still contain all $\partial_j \eta^i$ -terms of the force operator and are still stabilising. So a sufficient condition for stability is for all the other terms in (114) apart from

the squares I and II to add up to a negative number. This means we have the following general theorem:

Theorem 5. Let $(\rho, p, \mathbf{v}, \mathbf{B})$ be an axisymmetric stationary ideal MHD equilibrium and let k be the mass loading which is sufficiently small in the sense that

$$\rho\gamma p - k^2(\mu_0\gamma p + B^2) \geq 0, \quad (143)$$

and let Ω' be the non-aligned rotation frequency. The equilibrium is stable against Lagrangian perturbations with wave-number m (in the sense that $\langle \xi, \mathcal{F}[\xi] \rangle \leq 0$) if $\forall \mathbf{r}, \forall \mathbf{e} \in \mathbb{R}^3$

$$\begin{aligned}
 &\frac{\gamma p}{\rho\gamma p - k^2\mu_0\gamma p} \left(\underbrace{\left[k\mathbf{e} \cdot \left(\frac{1}{2\gamma p} \mathbf{F}_m - \frac{\nabla p}{\gamma p} + \nabla \right) \mathbf{B} + \mu_0 \frac{k}{2} \mathbf{e} \times \mathbf{J} + \rho\Omega' \hat{\mathbf{z}} \times \mathbf{e} \right]^2}_I \right. \\
 &+ \underbrace{\frac{k^2 B^2}{\rho\gamma p - k^2(\mu_0\gamma p + B^2)} \left[\hat{\mathbf{B}} \cdot \left(k\mathbf{e} \cdot \left(\frac{1}{2\gamma p} \mathbf{F}_m - \frac{\nabla p}{\gamma p} + \nabla \right) \mathbf{B} + \mu_0 \frac{k}{2} \mathbf{e} \times \mathbf{J} + \rho\Omega' \hat{\mathbf{z}} \times \mathbf{e} \right) \right]^2}_{II} \\
 &+ 4\pi G \rho^2 (\mathbf{e}^2 - \delta_0^m e_\varphi^2) + \underbrace{\frac{1}{4\gamma p} |\mathbf{e} \cdot \mathbf{F}_m|^2}_{III} + \underbrace{\frac{\mu_0}{4} |\mathbf{e} \times \mathbf{J}|^2}_{IV} + \underbrace{(\mathbf{e} \cdot (\nabla p - \mathbf{F}_m)) \mathbf{e} \cdot \nabla s^\dagger}_{IV} + \underbrace{\rho \mathbf{e} \cdot (\mathbf{e} \cdot \nabla) [\mathbf{v} \cdot \nabla] \mathbf{v}}_V \\
 &+ m^2 \left(\rho\Omega^2 \mathbf{e}^2 - \gamma p \left(\frac{e_\varphi}{r} \right)^2 - \frac{1}{r^2 \mu_0} (B_\varphi \mathbf{e} - e_\varphi \mathbf{B})^2 \right) \leq 0. \quad (144)
 \end{aligned}$$

This is only a sufficient condition. Note that when we have the opposite $\langle \xi, \mathcal{F}[\xi] \rangle > 0$, this tells us *a priori* nothing about the growth and time evolution of ξ . Translated to the words of Theorem 1, we have a nontrivial operator $\hat{A}$, the coriolis-term, which impedes us to make such claims. The above stability criterion simplifies a lot whenever the equilibrium is rotational ($k=0$ and $\Omega' = \Omega$, so $\mathbf{v} = r\Omega\hat{\boldsymbol{\phi}}$), when terms $\rho\Omega^2(\hat{\mathbf{z}} \times \mathbf{e})^2 + \rho \mathbf{e} \cdot (\mathbf{e} \cdot \nabla) [\mathbf{v} \cdot \nabla] \mathbf{v}$ combine to $-\rho r e_r \mathbf{e} \cdot \nabla \Omega^2$ to get

$$\begin{aligned}
 0 &\geq 4\pi G \rho^2 (\mathbf{e}^2 - \delta_0^m e_\varphi^2) - \rho r e_r (\mathbf{e} \cdot \nabla) \Omega^2 + \underbrace{\frac{1}{4\gamma p} |\mathbf{e} \cdot \mathbf{F}_m|^2}_{III} + \underbrace{\frac{\mu_0}{4} |\mathbf{e} \times \mathbf{J}|^2}_{IV} + (\mathbf{e} \cdot (\nabla p - \mathbf{F}_m)) \mathbf{e} \cdot \nabla s^\dagger \\
 &+ m^2 \left(\rho\Omega^2 \mathbf{e}^2 - \gamma p \left(\frac{e_\varphi}{r} \right)^2 - \frac{1}{r^2 \mu_0} (B_\varphi \mathbf{e} - e_\varphi \mathbf{B})^2 \right). \quad (145)
 \end{aligned}$$

A quick observation can be made by comparing this expression with the more general (144): if we consider axisymmetric toroidal perturbations ($m=0$, $\mathbf{e} = e_\varphi \hat{\boldsymbol{\phi}}$) we end up with the (positive) square underbraced by I contributing for a rotational equilibrium while in the more general criterion terms I, II, III, and V may all contribute. Term V may give a negative contribution. For a rotational equilibrium however we conclude that $\frac{\mu_0}{4} |\mathbf{e} \times \mathbf{J}|^2 \geq 0$ can only be identically zero if $\mathbf{J} = J_\varphi \hat{\boldsymbol{\phi}}$, which is equivalent to $\mathbf{B} = \mathbf{B}_p$. The opposite case of $\mathbf{B} = B_\varphi \hat{\boldsymbol{\phi}}$ and $\mathbf{J} = \mathbf{J}_p$ can be given a special treatment by Theorem 2. All cases in between where $B_\varphi \neq 0$ and $\mathbf{B}_p \neq 0$ however require some polar flow $\mathbf{v}_p \neq 0$ if we are to conclude stability from the above criteria.

B. Discussion

Using the above result for general perturbations about axisymmetric, stationary MHD configurations (as well as the specific results for axisymmetric perturbations of axisymmetric, purely rotating states with either poloidal or toroidal magnetic field from Section IV) require the knowledge of the equilibrium quantities $(\rho, p, \mathbf{v}, \mathbf{B})$. It is nontrivial to obtain axisymmetric, stationary equilibrium solutions to the full set of ideal MHD equations although some aspects are worth mentioning here. A general analysis,^{7,25} allowing for an external gravitational field of a central massive object of mass M_* , showed how 5 arbitrary 1D functions (plus the constant GM_*) of the flux function $\psi(r, z)$ (determining the poloidal magnetic field through $\mathbf{B}_p = \frac{1}{r} \hat{\phi} \times \nabla \psi$) enter: the derivative of the poloidal velocity stream function, the Bernoulli function, the entropy $S(\psi)$, the poloidal vorticity-current density stream function, and minus the derivative of the electric potential. The relation (123) can be rewritten as $\mathbf{v}_p = M \mathbf{B}_p / \sqrt{\rho}$, with M the poloidal Alfvén Mach number ($M^2 = \rho v_p^2 / B_p^2$), and a rescaling allows to formulate the problem as a nonlinear PDE for $\psi(r, z)$, with an algebraic Bernoulli equation for $M^2(r, z)$. The PDE generalizes the Grad-Shafranov equation (i.e., an elliptic PDE for ideal axisymmetric MHD equilibria without flow) to cases with flow, but unlike that static case, it can have elliptic as well as hyperbolic flow regimes. Only in certain simplified cases, one gets an elliptic PDE, such as when restricting attention to incompressible flows,²⁶ where the density is necessarily then also a flux function $\rho(\psi)$. This special case has also been shown²⁶ to not allow for incompressible equilibria with purely poloidal flows. The more general, compressible case is amenable to a numerical approach,²⁷ when forcing the solution to be in one of the three possible elliptic flow regimes. A most general analysis (albeit without external or self-gravity) of axisymmetric MHD equilibria and their stability used a Lyapunov analysis.²⁸ There, with isentropic poloidal flux surfaces where entropy $S(\psi)$, it is shown how stationary equilibria relate to the critical points of a nonlinearly conserved Lyapunov functional. That functional adds up the total energy (or Hamiltonian H which integrates kinetic, magnetic, and internal energy over the plasma volume) with 4 additional invariants related to mass, cross-helicity, toroidal flux, and angular momentum. Moreover, one can derive sufficient conditions for linear Lyapunov stability of such an axisymmetric, flowing MHD state by considering the second variation of the functional. This second variation of the nonlinear Lyapunov functional is conserved by the linearized dynamics, and was written as an algebraic quadratic form involving an 8×8 matrix collecting perturbations in density, velocity, magnetic field, and flux function ψ . From considering the definiteness of the quadratic form, conditions for linearized Lyapunov stability could be stated more directly,²⁸ such as equilibrium flow to be subsonic on every corotating flux surface frame, only have the flow in the first elliptic flow regime, etc. Note that we found conservation of a second-order energy perturbation in Section III, being a quadratic expression involving ξ , and both results may well relate closely. Our result from Section III is for any

general time-independent MHD state (i.e., possibly 3D, not assuming axisymmetry).

VIII. SUMMARY AND OUTLOOK

We continued our study of linear ideal MHD stability, involving the Frieman-Rosenbluth operator $\mathcal{G}$, and derived several results of interest for stationary equilibrium configurations. We formulated a general theorem on the governing dynamical system. We made links to known results for static and selected stationary cases (axisymmetric, rotating configurations). One may now use our general results to revisit stability limits obtained in simplified geometries, or how they connect with known results of MHD stability of, e.g., astrophysical jet or accretion disk configurations.

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APPENDIX: GENERAL THEOREM FOR PERTURBATIONS ABOUT A STATIONARY MHD EQUILIBRIUM

We here provide the proof of the theorem stated in Section II C.

Proof. To establish the conservation of energy expressed in (1), calculate

$$0 = \langle x', x'' + 2\tilde{A}x' - \tilde{S}x - f \rangle + \langle x'' + 2\tilde{A}x' - \tilde{S}x - f, x' \rangle \quad (\text{A1})$$

$$= (\langle x', x'' \rangle + \langle x'', x' \rangle) - (\langle x', \tilde{S}x \rangle + \langle x, \tilde{S}x' \rangle) - 2\text{Re}\langle f, x' \rangle, \quad (\text{A2})$$

where we used the antisymmetry of $\tilde{A}$, and the symmetry of $\tilde{S}$. This then rewrites to

$$0 = \frac{d}{dt} (\langle x', x' \rangle - \langle x, \tilde{S}x \rangle - 2\text{Re}\langle f, x \rangle) + 2\text{Re}\langle f', x \rangle. \quad (\text{A3})$$

One can check furthermore that (formally)

$$\begin{aligned} \frac{d}{dt} & \left[-2\text{Re} \sum_{n=1}^{\infty} (-1)^n \langle f^{(n)}, x^{(-n)} \rangle \right] \\ &= -2\text{Re} \sum_{n=1}^{\infty} (-1)^n \left(\langle f^{(n+1)}, x^{(-n)} \rangle + \langle f^{(n)}, x^{(-n+1)} \rangle \right) \\ &= 2\text{Re}\langle f', x \rangle. \end{aligned} \quad (\text{A4})$$

Combining all these results yields the expression for the conserved energy.

This energy conservation (1) immediately yields (2). To prove (3), note that the energy is for such $\tilde{S}$ bounded from below as follows:

$$E_y = \|y'\|^2 - \langle y, \tilde{S}y \rangle - 2\operatorname{Re}\langle f, y \rangle \\ \geq \|y'\|^2 + \lambda^2 \|y\|^2 - 2\|f\| \|y\|, \quad (\text{A5})$$

where we also used the Cauchy-Schwarz inequality. Combining this with $(\lambda^2 \|y\| - \|f\|)^2 \geq 0$ leading to

$$\lambda^2 \|y\|^2 - 2\|f\| \|y\| \geq -\frac{\|f\|^2}{\lambda^2}, \quad (\text{A6})$$

we can deduce the first bound stated in (3), namely, $\|y'\|^2 \leq E_y + \frac{\|f\|^2}{\lambda^2}$. The second statement is found when using $\|y'\|^2 \geq 0$ to write $E_y \geq \lambda^2 \|y\|^2 - 2\|f\| \|y\|$ and rewriting this quadratic in $\|y\|$ to

$$0 \geq \lambda^2 \left(\|y\| - \frac{\|f\| + \sqrt{\|f\|^2 + \lambda^2 E_y}}{\lambda^2} \right) \\ \times \left(\|y\| - \frac{\|f\| - \sqrt{\|f\|^2 + \lambda^2 E_y}}{\lambda^2} \right), \quad (\text{A7})$$

to deduce $\|y\| \leq \frac{\|f\| + \sqrt{\|f\|^2 + \lambda^2 E_y}}{\lambda^2}$.

To show (4), take $y(t)$ to be the solution of (24) with $y(0) = x$ and $y'(0) = \gamma x$ and define $I(t) := \log \left(\frac{\langle y(t), y(t) \rangle}{\langle y(0), y(0) \rangle} \right)$. One should first note that this solution has zero energy, i.e., $\forall t$

$$E_y = \langle y'(t), y'(t) \rangle - \langle y(t), \tilde{S}y(t) \rangle \\ = \langle y'(0), y'(0) \rangle - \langle y(0), \tilde{S}y(0) \rangle = 0. \quad (\text{A8})$$

Also

$$I(0) = 0. \quad (\text{A9})$$

Next, we have

$$I'(t) = \frac{\langle y(t), y'(t) \rangle + \langle y'(t), y(t) \rangle}{\langle y(t), y(t) \rangle}. \quad (\text{A10})$$

So in particular, $I'(0) = 2\gamma$. Next, we have

$$I''(t) = \frac{(\langle y''(t), y(t) \rangle + 2\langle y'(t), y'(t) \rangle + \langle y(t), y''(t) \rangle) \langle y(t), y(t) \rangle - (\langle y(t), y'(t) \rangle + \langle y'(t), y(t) \rangle)^2}{\langle y(t), y(t) \rangle^2} \\ = \frac{(2\langle y(t), \tilde{S}y(t) \rangle + 2\langle y'(t), y'(t) \rangle) \langle y(t), y(t) \rangle - (\langle y(t), y'(t) \rangle + \langle y'(t), y(t) \rangle)^2}{\langle y(t), y(t) \rangle^2} \\ = \frac{4\langle y'(t), y'(t) \rangle \langle y(t), y(t) \rangle - (\langle y(t), y'(t) \rangle + \langle y'(t), y(t) \rangle)^2}{\langle y(t), y(t) \rangle^2} \geq 0, \quad (\text{A11})$$

where the last step is a consequence of the Cauchy-Schwarz inequality. Hence

$$I'(t) \geq I'(0) = 2\gamma, \quad (\text{A12})$$

$$I(t) \geq I(0) + 2\gamma t = 2\gamma t. \quad (\text{A13})$$

So $\frac{\langle y(t), y(t) \rangle}{\langle y(0), y(0) \rangle} \geq \exp(2\gamma t)$. Also, by (A12), we have

$$4\gamma^2 \langle y(t), y(t) \rangle^2 \leq (\langle y(t), y'(t) \rangle + \langle y'(t), y(t) \rangle)^2 \quad (\text{A14})$$

$$\leq 4\langle y(t), y(t) \rangle \langle y'(t), y'(t) \rangle. \quad (\text{A15})$$

Hence

$$\langle y'(t), y'(t) \rangle \geq \gamma^2 \langle y(t), y(t) \rangle \quad (\text{A16})$$

$$\geq \gamma^2 \langle y(0), y(0) \rangle \exp(2\gamma t). \quad (\text{A17})$$

Now we prove (5). In this case, only a symmetric operator $\tilde{S}: H \rightarrow H$ is involved, and we first prove generally that for any symmetric $\tilde{S}$, its kernel is orthogonal to its range, i.e., $\ker \tilde{S} = (\operatorname{Ran} \tilde{S})^\perp$. Indeed, take $x \in \ker \tilde{S}$ (i.e., $\tilde{S}x = 0$) and any $y \in \operatorname{Ran} \tilde{S}$, for which exists z such that $y = \tilde{S}z$. Then, $\langle x, y \rangle = \langle x, \tilde{S}z \rangle = \langle \tilde{S}x, z \rangle = 0$, hence $x \in (\operatorname{Ran} \tilde{S})^\perp$. Conversely, take $x \in (\operatorname{Ran} \tilde{S})^\perp$ and take any

$y \in \operatorname{Ran} \tilde{S}, y = \tilde{S}z$. Then, $0 = \langle x, y \rangle = \langle x, \tilde{S}z \rangle = \langle \tilde{S}x, z \rangle$ where z is arbitrary. Hence, $\tilde{S}x = 0$ and $x \in \ker \tilde{S}$. This finalizes the proof of $\ker \tilde{S} = (\operatorname{Ran} \tilde{S})^\perp$.

Now we prove that the spectrum σ of the dynamical system $x''(t) = \tilde{S}x(t)$ is found purely on the axes of the complex plane. Suppose $\lambda \in \sigma$, writing $\mu := \lambda^2$, we need to prove that $\mu \in \mathbb{R}$. Suppose $\operatorname{Im} \mu \neq 0$, then we will come to a contradiction when we show that $\mu I - \tilde{S}$ has dense range and a bounded inverse (i.e., λ belongs to the complement of σ). To show injectivity of $\mu I - \tilde{S}$ and boundedness of its inverse, we can write for any unit vector x

$$0 < (\operatorname{Im} \mu)^2 \leq \|\operatorname{Im} \mu x\|^2 + \|\operatorname{Re} \mu I - \tilde{S}\| x\|^2 \\ = \|(\mu I - \tilde{S})x\|^2, \quad (\text{A18})$$

which shows that the kernel of $(\mu I - \tilde{S})$ is trivial, as all unit vectors have an image with a strictly positive norm. This implies that the mapping through $(\mu I - \tilde{S})$ is injective: two unequal vectors y, z with the same image would give a unit vector $(y - z)/\|y - z\|$ in $\ker(\mu I - \tilde{S})$. The bounded inverse theorem then ensures existence and boundedness of the inverse. Using the finding that $\ker \tilde{S} = (\operatorname{Ran} \tilde{S})^\perp$ for any symmetric operator (here used on $\mu I - \tilde{S}$) and the standard result (valid in any inner product space) that for subspaces $K \subset H$ we have $(K^\perp)^\perp = \overline{K}$, we write

$$\begin{aligned}\overline{\text{Ran}(\mu I - \tilde{S})} &= ((\text{Ran}(\mu I - \tilde{S}))^\perp)^\perp = (\ker(\mu I - \tilde{S}))^\perp \\ &= \{0\}^\perp = H,\end{aligned}\quad (\text{A19})$$

showing that the range of $\mu I - \tilde{S}$ is indeed dense.

To show (6), we silently assume uniqueness of solutions associated with particular initial conditions. We can generally consider the solution $x(t)$ to have the initial condition $x(0) = r + v$, $x'(0) = s + w$ where $r, s \in \ker \tilde{S}$ and $v, w \perp \ker \tilde{S}$. Let $z(t)$ solve $z''(t) = \tilde{S}z(t) + b$ with the initial conditions $z(0) = v$ and $z'(0) = w$. We show that $\forall t: z(t) \perp \ker \tilde{S}$. Choose arbitrary $c \in \ker \tilde{S}$ and define $P(t) = \langle c, z(t) \rangle$, then by the initial conditions $P(0) = P'(0) = 0$. Moreover

$$\frac{d^2}{dt^2} P(t) = \langle c, \tilde{S}z(t) + b \rangle = \underbrace{\langle \tilde{S}c, z(t) \rangle}_{=0} + \underbrace{\langle c, b \rangle}_{=0} = 0. \quad (\text{A20})$$

Hence, $P \equiv 0$ meaning that $z(t) \perp \ker \tilde{S}$ as c was arbitrary. One can easily check that $x(t) = r + st + \frac{at^2}{2} + z(t)$ is then indeed a full solution to (24), as we note that $\tilde{S}x(t) = \tilde{S}[r + st + at^2/2 + z(t)] = \tilde{S}z(t)$ since $r, s, a \in \ker \tilde{S}$. Finally, $\|x(t)\|^2 = \langle x(t), x(t) \rangle = \langle (r + st + \frac{at^2}{2}) + z(t), (r + st + \frac{at^2}{2}) + z(t) \rangle = \|r + st + \frac{at^2}{2}\|^2 + \|z(t)\|^2$ since $z(t) \perp \ker \tilde{S}$. $\square$

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nontrivial kernel; (2) the continuous spectrum, for which $\lambda I - \mathcal{O}$ has a trivial kernel, yet its inverse $(\lambda I - \mathcal{O})^{-1}$ fails to be bounded; and (3) the residual spectrum for which $\lambda I - \mathcal{O}$ has a trivial kernel and a bounded inverse, but where $\lambda I - \mathcal{O}$ has no dense range.

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