On the equivalence of dynamic relaxation and the Newton-Raphson method

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SUMMARY

Dynamic relaxation is an iterative method to solve nonlinear systems of equations, which is frequently used for form finding and analysis of structures that undergo large displacements. It is based on the solution of a fictitious dynamic problem where the vibrations of the structure are traced by means of a time integration scheme until a static equilibrium is reached. Fictitious values are used for the mass and damping parameters. Heuristic rules exist to determine these values in such a way that the time integration procedure converges rapidly without becoming unstable. Central to these heuristic rules is the assumption that the highest convergence rate is achieved when the ratio of the highest and lowest eigenfrequency of the structure is minimal.

This short communication shows that all eigenfrequencies become identical when a fictitious mass matrix proportional to the stiffness matrix is used. If, in addition, specific values are used for the fictitious damping parameters and the time integration step, the dynamic relaxation method becomes completely equivalent to the Newton-Raphson method. The Newton-Raphson method can therefore be regarded as a specific form of dynamic relaxation. This insight may help to interpret and improve nonlinear solvers based on dynamic relaxation and/or the Newton-Raphson method. Copyright © 2017 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The first version of the Newton-Raphson method has been described by Newton in 1669 and generalized by Raphson in 1690 [1]. The method was originally conceived as a general iterative method to approximate the roots of real-valued functions. Its application in structural analysis came later around the 1960s, following the emergence of the finite-element method. The method is described in almost any textbook on nonlinear structural analysis, such as [2] or [3]. In every iteration, the method makes an estimate of the displacements of the structure under influence of the externally applied loads, by assuming the response of the structure to be linear. As the actual response of the structure is not linear, the resulting internal forces will not be in equilibrium with the externally applied loads, and a residual force remains. A subsequent estimate is made by updating the stiffness and loading the deformed structure again with the remaining residual force. This process is repeated until the residual force is below a chosen threshold and convergence is reached. The Newton-Raphson method is well-known for its fast convergence. However, the method requires

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the inverse of the tangent stiffness matrix at each iteration, which is computationally expensive. Many researchers have come up with various ways to improve the calculation speed of the Newton-Raphson method. Zienkiewicz et al. [4] proposed a modified Newton-Raphson method in which the stiffness matrix is calculated only during the first iteration, consequently reducing the computational cost of other iterations. An alternative to reduce the computational cost of an iteration in the Newton-Raphson scheme was suggested by Dennis and Moré [5]. They proposed to apply a quasi-Newton method. This method approximates the tangent stiffness matrix in each iteration, using information of previous iterations. Matthies and Strang [6] provided an important contribution for the application of quasi-Newton methods in the field of nonlinear structural analysis. A comparison of quasi-Newton methods is given by Papadrakakis and Pantazopoulos [7]. By using an approximation of the tangent stiffness matrix, the quadratic convergence, typical for the Newton-Raphson method, is generally lost. In an attempt to increase the convergence speed of these Newton-type methods, several researchers have proposed some kind of acceleration scheme. Aitken [8] proposed a diagonal acceleration matrix to be multiplied by the iterative vector. Similarly, Nayak and Zienkiewicz [9] introduced a diagonal acceleration matrix at every second iteration. Jennings [10] replaced Aitken's diagonal acceleration matrix by a scalar. Crisfield [11] proposed an acceleration method in which the deflection increment is a weighted average of the previous deflection increment and the usual unaccelerated increment. For challenging nonlinear problems, these acceleration methods can be used to increase the numerical stability of the discussed Newton-type methods, instead of increasing the convergence rate. Comparisons of various acceleration schemes are given by Crisfield [12] and Ma and May [13]. A general comparison of procedures for the solution of incremental finite element equations in nonlinear analysis is given by Bathe and Cimento [14].

Dynamic relaxation was first proposed by Day [15] in 1965 as an explicit solver for the static analysis of structures discretized in space by finite elements. The method originates from the approach adopted by Otter and Day [16] for the computation of tidal flows, which was combined with a finite difference discretization in space. Although originally developed as an analysis tool, dynamic relaxation became a popular method in the context of form finding because of its effectiveness in handling large displacements and highly nonlinear behavior. The method was first proposed for the form finding of hanging roofs by Day and Bunce [17]. It was later also applied to the form finding of other tension/compression structures such as cable nets, membranes and tensegrity structures, with important contributions by Barnes [18], Topping [19], Papadrakakis [20] and Wakefield [21]. The method solves a static problem by reformulating it as a dynamic problem, and solving the reformulated problem iteratively. In its initial position, the internal forces of the structure are not in equilibrium with the externally applied loads. In the dynamic relaxation solution scheme, the structure starts to oscillate because of the applied loads, until, under influence of damping, the structure finally comes to rest in the sought equilibrium position. The movement of the structure is traced step by step. The number of time steps needed to reach an equilibrium position depends on the convergence parameters, more precisely on the inertia/mass of the structure, the damping and the time step. These convergence parameters only affect the structure's dynamic behavior; they do not change the final, static equilibrium position. Therefore, the convergence parameters can be chosen freely to minimize the number of iterations. The only restriction is related to the numerical stability of the method, which depends on the highest eigenfrequency of the structure. Similar to, but independently from the Newton-Raphson method, numerous researchers have suggested ways to improve the convergence of the method. Most of these improvements are based on the assumption that the highest convergence rate is achieved when the ratio of the highest and lowest eigenfrequency is minimal. Papadrakakis [22] proposed a method for the automatic evaluation of the dynamic relaxation parameters through an approximation of the highest and the lowest eigenfrequency. Similarly, Underwood [23] described an adaptive dynamic relaxation approach. This approach was modified and improved by Zhang et al. [24, 25]. Kadkhodayan [26] proposed a way to update the time step by minimizing the residual force after each iteration. Rezaiee-Pajand et al. [27, 28] introduced new relations for the fictitious mass and damping matrix, allowing the time step to change each iteration. Alamatian [29] presented new fictitious masses for a varying time step in combination with kinetic damping.

Generally, dynamic relaxation and the Newton-Raphson method are considered as two distinct methods. Indeed, Newton-Raphson can be categorized as an implicit solver*, as it requires the inversion of a full tangential stiffness matrix, whereas dynamic relaxation is generally formulated as an explicit solver*. However, if the fictitious masses used in the dynamic relaxation method are not restricted to be lumped, the method effectively becomes implicit. In the literature, comparisons between dynamic relaxation and Newton-Raphson primarily focus on the performance of the two methods [31, 32, 33]. However, their performance is very problem-dependent, and no general conclusions can be drawn [2]. Interestingly, Felippa [33] already touched on the equivalence of dynamic relaxation and other iterative solvers in 1976. He stated that the biggest strength of dynamic relaxation is that it inherently encompasses all other methods. However, no more details are given. Generally, it is said that dynamic relaxation is better in handling strongly nonlinear behavior, as the inertia forces smoothen out the response [14], and the method is easier to implement, because no stiffness matrix needs to be calculated [2, 33]. Typically, in its explicit form, dynamic relaxation requires more iterations to converge, but each iteration is computationally cheaper.

Often in structural analysis, these nonlinear solvers are combined with an overlaying incremental strategy such as load control or arc length methods to trace the response of a given structure under varying loads. An overview of some important path-finding procedures is given by Rezaiee-Pajand et al. [34, 35]. If the analyst is only interested in the final equilibrium of a structure for a fixed load, an incremental strategy is unnecessary and the nonlinear solvers can be applied directly to find the correct equilibrium. However, for numerically challenging problems, nonlinear solvers can have difficulties to converge, and incremental strategies can be applied purely to stabilize the calculation. Alternatively, an acceleration scheme can be used to stabilize the solvers directly.

This short communication discusses the equivalence of dynamic relaxation and the Newton-Raphson method. A novel strategy is applied to choose the fictitious mass matrix for the dynamic relaxation method, based on a stability analysis of the time integration scheme used in dynamic relaxation. An analytical comparison between the resulting dynamic relaxation scheme and the Newton-Raphson method is performed to study the equivalence of the two methods.

The remainder of this short communication is organized as follows. Section 2 describes the solution procedure of the Newton-Raphson method. Section 3 describes the solution procedure of the dynamic relaxation method. Section 4 analyzes the stability of the dynamic relaxation method using modal analysis, leading to a novel approach for the choice of the fictitious mass matrix. Finally, section 5 compares the resulting dynamic relaxation approach to the Newton-Raphson method from an analytical point of view. Conclusions are drawn in section 6.

2. THE NEWTON-RAPHSON METHOD

The Newton-Raphson method is an iterative way to solve the following nonlinear set of equations, describing the static equilibrium of a structure under externally applied loads, for the unknown displacements $\mathbf{u}$:

$$\mathbf{f}(\mathbf{u}) = \mathbf{p} \quad (1)$$

where $\mathbf{f}$ is a vector containing the internal forces, as a nonlinear function of the displacements $\mathbf{u}$, and $\mathbf{p}$ is the vector containing all externally applied loads. Note that $\mathbf{p}$ can be the actually applied load, or a fraction of that load, corresponding to a certain load step. This short communication only considers solution procedures for a single load step; overlaying incremental strategies are beyond its scope. In the Newton-Raphson method, the displacement vector $\mathbf{u}$ is updated using the following equation:

$$\mathbf{u}^{t+\Delta t} = \mathbf{u}^t + \mathbf{K}^{t-1} \mathbf{r}^t \quad (2)$$

*For the definition of implicit and explicit algorithms, we follow Hughes et al. [30]: "... implicit algorithms, in which a matrix system is solved, one or more times per step, to advance the solution; and explicit algorithms, in which the solution may be advanced without storing a matrix, or solving a system of equations."

where $\mathbf{u}^t$ and $\mathbf{u}^{t+\Delta t}$ are the displacement vectors at iteration t and $t + \Delta t$ respectively, $\mathbf{K}^t$ is the structure's tangent stiffness matrix at iteration t , and $\mathbf{r}^t$ is the residual force vector at iteration t , defined as

$$\mathbf{r}^t = \mathbf{p} - \mathbf{f}^t \quad (3)$$

Equation (2) is evaluated repeatedly until equation (1) is satisfied, within certain accuracy limits.

As discussed in the introduction, the classical Newton-Raphson method given above can be sped up by approximating the actual tangent stiffness matrix in equation (2). Two approaches exist; in the modified Newton-Raphson scheme, the stiffness matrix is only updated after a certain number of iterations instead of at each iteration, effectively limiting the number of computationally expensive stiffness matrix inversions. Quasi-Newton methods on the other hand approximate the stiffness matrix using the deformations and internal forces known from previous iterations. To speed up or stabilize the convergence rate, an acceleration/damping scheme can be used, such as the strategy proposed by Jennings [10], where equation (2) is reformulated as:

$$\mathbf{u}^{t+\Delta t} = \mathbf{u}^t + a\mathbf{K}^{t-1}\mathbf{r}^t \quad (4)$$

where a is a scalar. If $a < 1$, convergence is slowed down, and the numerical stability is increased. If $a > 1$, convergence is accelerated, but the numerical stability decreases, possibly leading to divergence. More elaborate acceleration/damping schemes are mentioned in the introduction [8, 11, 12, 13].

3. DYNAMIC RELAXATION

In the dynamic relaxation approach, the static problem is reformulated as a dynamic problem. In the linear case, the dynamic behavior of a structure is governed by the following equation:

$$\mathbf{M}\mathbf{a} + \mathbf{C}\mathbf{v} + \mathbf{K}\mathbf{u} = \mathbf{p} \quad (5)$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{a}$ is the acceleration vector, $\mathbf{C}$ is the damping matrix, $\mathbf{v}$ is the velocity vector, $\mathbf{K}$ is the stiffness matrix, $\mathbf{u}$ is the displacement vector, and $\mathbf{p}$ is the external load vector. Note that when the velocity and the acceleration vectors are zero, equation (5) describes the linear static equilibrium. The full dynamic equilibrium, however, is a set of coupled differential equations, as the velocities $\mathbf{v}$ are the time derivatives of the displacements $\mathbf{u}$, and the accelerations $\mathbf{a}$ are in turn the time derivatives of the velocities $\mathbf{v}$. The system of equations is solved for the displacements $\mathbf{u}$ by discretization in time using a central difference scheme. At a given moment in time, the velocity vector $\mathbf{v}$ is written in terms of the displacement vector $\mathbf{u}$ as

$$\mathbf{v}^{t+\Delta t/2} = \frac{\mathbf{u}^{t+\Delta t} - \mathbf{u}^t}{\Delta t} \quad (6)$$

Where Δt is the time step, $\mathbf{v}^{t+\Delta t/2}$ is the velocity vector at time $t + \Delta t/2$, $\mathbf{u}^{t+\Delta t}$ is the displacement vector at a time $t + \Delta t$, and $\mathbf{u}^t$ is the displacement vector at time t . Note that the velocities are defined at intermediate time intervals ($\Delta t/2$) as compared to the displacements. The velocities at a given time step are then simply defined by linear interpolation:

$$\mathbf{v}^t = \frac{\mathbf{v}^{t+\Delta t/2} + \mathbf{v}^{t-\Delta t/2}}{2} \quad (7)$$

The acceleration vector can also be written in terms of the velocity vector:

$$\mathbf{a}^t = \frac{\mathbf{v}^{t+\Delta t/2} - \mathbf{v}^{t-\Delta t/2}}{\Delta t} \quad (8)$$

Equation (5) is rewritten as

$$\mathbf{M}\mathbf{a}^t + \mathbf{C}\mathbf{v}^t = \mathbf{r}^t \quad (9)$$

Where $\mathbf{r}^t$ is the residual force vector at time t , defined as

$$\mathbf{r}^t = \mathbf{p} - \mathbf{K}\mathbf{u}^t \quad (10)$$

For nonlinear problems, this linear relation does not hold, and equation (10) is replaced by:

$$\mathbf{r}^t = \mathbf{p} - \mathbf{f}^t \quad (11)$$

Note that this equation is identical to equation (3).

Given a time step Δt , damping matrix $\mathbf{C}$ and mass matrix $\mathbf{M}$, the displacements $\mathbf{u}$ are traced step by step until equilibrium is found under the influence of damping. Initially, displacements $\mathbf{u}$, velocities $\mathbf{v}$ and accelerations $\mathbf{a}$ are set to zero or some other arbitrary value. By combining equations (7), (8), and (9), the following updating expression is obtained for the velocities $\mathbf{v}$:

$$\mathbf{v}^{t+\Delta t/2} = \left(\frac{\mathbf{M}}{\Delta t} + \frac{\mathbf{C}}{2} \right)^{-1} \left(\frac{\mathbf{M}}{\Delta t} - \frac{\mathbf{C}}{2} \right) \mathbf{v}^{t-\Delta t/2} + \left(\frac{\mathbf{M}}{\Delta t} + \frac{\mathbf{C}}{2} \right)^{-1} \mathbf{r}^t \quad (12)$$

Equation (6) is used to determine the displacements $\mathbf{u}$. When the residual forces $\mathbf{r}$ become sufficiently small after a number of time steps, the system is said to have converged.

4. STABILITY ANALYSIS

The time step Δt , damping matrix $\mathbf{C}$, and mass matrix $\mathbf{M}$ are usually chosen in such a way as to minimize the number of iterations needed for the system to converge. However, an ill-considered choice may result in numerical instability and consequently divergence. This section presents an analytical stability study, based on modal analysis, of the time integration scheme used in the dynamic relaxation method. Based on this study, a novel choice for the fictitious masses is proposed.

For the stability analysis given below, we follow the approach described by Bathe [36], assuming a linear structural response. According to Bathe [36], a time integration scheme is numerically stable if the amplitude of any given oscillation in the structure, caused by some initial perturbation, remains the same or decreases over time when no external loads are applied. This requirement imposes a limit on the time step. Therefore, to derive the stability limit, the external load vector $\mathbf{p}$ in equation (5) is replaced by the zero vector:

$$\mathbf{M}\mathbf{a}^t + \mathbf{C}\mathbf{v}^t + \mathbf{K}\mathbf{u}^t = \mathbf{0} \quad (13)$$

Equation (13) is now modally decomposed into a set of decoupled differential equations. If the method is stable for all decoupled differential equations, it is also stable for the original equation (13). Therefore, the stability analysis is performed for one decoupled differential equation, corresponding to mode i :

$$\phi_i^T \mathbf{M} \phi_i a_i^{*t} + \phi_i^T \mathbf{C} \phi_i v_i^{*t} + \phi_i^T \mathbf{K} \phi_i u_i^{*t} = 0 \quad (14)$$

where ϕ_i is the eigenvector corresponding to mode i , u_i^* is the modal displacement, v_i^* is the modal velocity, and a_i^* is the modal acceleration. Assuming mass-normalized eigenvectors and classical damping, equation (14) becomes

$$a_i^{*t} + 2\xi_i \omega_i v_i^{*t} + \omega_i^2 u_i^{*t} = 0 \quad (15)$$

where ω_i is the i th eigenfrequency, and ξ_i is the modal damping ratio. Equations (6), (7), and (8) also apply to the modal displacement u_i^* , modal velocity v_i^* and modal acceleration a_i^* . By combining equations (6), (7), (8), and (15), the modal acceleration a_i^* is eliminated and the modal displacement u_i^* and velocity v_i^* at the next time step are formulated in terms of the modal displacement and velocity at the current time step:

$$u_i^{*t+\Delta t} = u_i^{*t} + \Delta t \frac{(1 - \Delta t \xi_i \omega_i) v_i^{*t-\Delta t/2} - \Delta t \omega_i^2 u_i^{*t}}{1 + \Delta t \xi_i \omega_i} \quad (16)$$

$$v_i^{*t+\Delta t/2} = \frac{(1 - \Delta t \xi_i \omega_i) v_i^{*t-\Delta t/2} - \Delta t \omega_i^2 u_i^{*t}}{1 + \Delta t \xi_i \omega_i} \quad (17)$$

Equations (16) and (17) are put in matrix form:

$$\begin{bmatrix} u_i^{*t+\Delta t} \\ v_i^{*t+\Delta t/2} \end{bmatrix} = \mathbf{A} \begin{bmatrix} u_i^{*t} \\ v_i^{*t-\Delta t/2} \end{bmatrix} \quad \text{or} \quad \hat{\mathbf{u}}_i^{*t+\Delta t} = \mathbf{A} \hat{\mathbf{u}}_i^{*t} \quad (18)$$

where $\mathbf{A}$ is called the amplification matrix:

$$\mathbf{A} = \begin{bmatrix} 1 - \frac{\Delta t^2 \omega_i^2}{1 + \Delta t \xi_i \omega_i} & \frac{\Delta t - \Delta t^2 \xi_i \omega_i}{1 + \Delta t \xi_i \omega_i} \\ -\frac{\Delta t \omega_i^2}{1 + \Delta t \xi_i \omega_i} & \frac{1 - \Delta t \xi_i \omega_i}{1 + \Delta t \xi_i \omega_i} \end{bmatrix} \quad (19)$$

At time step $t + n\Delta t$, the modal velocities v_i^* and displacements u_i^* can be expressed in terms of modal velocities and displacements at time t as follows:

$$\begin{bmatrix} u_i^{*t+n\Delta t} \\ v_i^{*t+n\Delta t-\Delta t/2} \end{bmatrix} = \mathbf{A}^n \begin{bmatrix} u_i^{*t} \\ v_i^{*t-\Delta t/2} \end{bmatrix} \quad \text{or} \quad \hat{\mathbf{u}}_i^{*t+n\Delta t} = \mathbf{A}^n \hat{\mathbf{u}}_i^{*t} \quad (20)$$

Spectral decomposition of $\mathbf{A}^n$ gives

$$\mathbf{A}^n = \Phi \Lambda^n \Phi^T \quad (21)$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (22)$$

with λ_1 and λ_2 eigenvalues of $\mathbf{A}$. The solution $\hat{\mathbf{u}}_i^{*t+n\Delta t}$ will remain bounded for n going to infinity if and only if the absolute values of the eigenvalues λ_1 and λ_2 of $\mathbf{A}$ are less than or equal to one:

$$\max |\lambda_{1,2}| \leq 1 \quad (23)$$

This imposes a limit on the time step Δt . The eigenvalues are found as:

$$\lambda_{1,2} = \frac{1 - \frac{\Delta t^2 \omega_i^2}{2} \pm \sqrt{(\xi_i^2 - 1) \Delta t^2 \omega_i^2 + \frac{\Delta t^4 \omega_i^4}{4}}}{1 + \Delta t \xi_i \omega_i} \quad (24)$$

Combining equations (23) and (24) results in the following limit for the time step:

$$\Delta t \leq \frac{2}{\omega_i} \quad (25)$$

Equation (25) indirectly also imposes a restriction to the fictitious masses assumed in dynamic relaxation as the mass influences the structure's eigenfrequencies ω_i .

As equation (25) should hold for all modes i , the highest eigenfrequency will determine the maximum allowable time step. Because a larger time step implies that less iterations are needed to reach equilibrium, the time step should be as high as possible, and consequently, from equation (25), the highest eigenfrequency should be minimized. On the other hand, since lower eigenmodes attenuate more slowly, the lowest eigenfrequencies should be maximized. Ideally, therefore, all eigenfrequencies should coincide. The eigenfrequencies can be manipulated by the choice of the fictitious mass, which, as pointed out before, has no impact on the final equilibrium solution.

The fictitious mass that makes all eigenfrequencies coincide is determined as follows. If the dynamic behavior of the structure is governed by equation (5), the modes and eigenfrequencies are found by analyzing the free vibration. This means that no load is applied and damping is ignored:

$$\mathbf{M}\mathbf{a} + \mathbf{K}\mathbf{u} = \mathbf{0} \quad (26)$$

Non-trivial solutions to equation (26) are found by solving the following generalized eigenvalue problem:

$$\mathbf{K}\phi - \omega^2 \mathbf{M}\phi = 0 \quad (27)$$

Instead of finding the eigenfrequencies for given mass and stiffness matrices, a fictitious mass matrix is sought that makes all eigenfrequencies coincide. It can be seen from equation (27) that this is the case if the mass matrix is proportional to the stiffness matrix. Indeed if

$$\mathbf{M} = \frac{1}{\omega^{*2}} \mathbf{K} \quad (28)$$

with ω^* as the target eigenfrequency for all modes, equation (27) becomes

$$\mathbf{K}\phi - \frac{\omega^2}{\omega^{*2}} \mathbf{K}\phi = 0 \quad (29)$$

for which non-trivial solutions simply require that $\omega = \omega^*$. If all eigenfrequencies coincide, the eigenfrequency ω_i in equation (25) can be replaced by the target eigenfrequency ω^* .

For the derivation of the fictitious mass, the structural response was assumed to be linear. In a nonlinear case the tangent stiffness matrix $\mathbf{K}^t$ should be used to calculate the fictitious mass matrix, and updated each iteration. Furthermore, the structure's eigenfrequencies will shift somewhat, and a slightly more conservative choice for the time step is required to maintain stability.

Because the fictitious mass matrix given by equation (28) includes non-diagonal terms, the dynamic relaxation approach becomes implicit, as this matrix has to be inverted in every iteration, cfr. equation (12). Therefore, the computation cost of a single iteration increases. However, it is expected that less iterations are required for convergence. The influence of this trade-off is very problem dependent, and should therefore be evaluated for each application separately. This is beyond the scope of this short communication, which is kept application independent to maintain generality.

Finally, appropriate values for the damping parameters must be determined. Viscous damping is usually avoided in dynamic relaxation because the choice of a suitable damping matrix requires a trial analysis to estimate the lowest eigenfrequency. Instead, an artificial damping technique, called kinetic damping, is often applied. This technique resets the velocities to zero whenever the structure encounters a kinetic energy peak. However, when the fictitious mass matrix given in equation (28) is used, all eigenfrequencies coincide with the target eigenfrequency, and, consequently, viscous damping can be applied without the need for a trial analysis. Classical damping is assumed:

$$\Phi^T \mathbf{C} \Phi = \text{diag}(2\xi_i \omega_i) \quad i = 1 \dots n \quad (30)$$

where Φ is a matrix containing all eigenvectors, ξ_i is the damping ratio corresponding to mode i , and n is the total number of degrees of freedom. If all eigenfrequencies ω_i coincide, and the modal damping ratio is equal to ξ^* for all modes, equation (30) becomes

$$\Phi^T \mathbf{C} \Phi = 2\xi^* \omega^* \mathbf{I} \quad (31)$$

where $\mathbf{I}$ is an identity matrix. For mass-normalized eigenvectors, equation (31) results in the following expression for the damping matrix $\mathbf{C}$:

$$\mathbf{C} = 2\xi^* \omega^* \mathbf{M} \quad (32)$$

5. EQUIVALENCE OF DYNAMIC RELAXATION AND THE NEWTON-RAPHSON METHOD

For the implicit dynamic relaxation approach proposed in the previous section, a similar updating expression as for the Newton-Raphson method (equation (2)) is found by combining equations (6), (12), (28), and (32):

$$\mathbf{u}^{t+\Delta t} = \mathbf{u}^t + \Delta t \left(\frac{1 - \Delta t \xi^* \omega^*}{1 + \Delta t \xi^* \omega^*} \mathbf{v}^{t-\Delta t/2} + \frac{\omega^{*2} \Delta t}{1 + \Delta t \xi^* \omega^*} \mathbf{K}^{-1} \mathbf{r}^t \right) \quad (33)$$

This expression relates the displacements of the next time step to displacements, velocities, and residual forces, known from the current time step.

Assume that the following values are chosen for the time step Δt and the damping ratio ξ^* :

$$\begin{aligned}\Delta t &= \frac{\sqrt{2}}{\omega^*} \\ \xi^* &= \frac{1}{\sqrt{2}}\end{aligned}\tag{34}$$

The time step Δt is within the stability limits for linear problems (equation (25)), and the damping ratio ξ^* is chosen slightly lower than the critical damping ratio of 1. In this case, the updating expression (equation (33)) reduces to

$$\mathbf{u}^{t+\Delta t} = \mathbf{u}^t + \mathbf{K}^{t-1} \mathbf{r}^t\tag{35}$$

This expression is identical to the updating expression for the Newton Raphson method, given by equation (2). Therefore, the Newton-Raphson method can be regarded as a specific form of dynamic relaxation.

Assume that a parameter a is introduced into equation (34):

$$\begin{aligned}\Delta t &= \frac{\sqrt{2a}}{\omega^*} \\ \xi^* &= \frac{1}{\sqrt{2a}}\end{aligned}\tag{36}$$

In this case, the updating expression (equation (33)) becomes

$$\mathbf{u}^{t+\Delta t} = \mathbf{u}^t + a\mathbf{K}^{t-1} \mathbf{r}^t\tag{37}$$

This expression is identical to the damping/acceleration method given by equation (4), which is used to speed up or stabilize the Newton-Raphson method. An increase of the time step, combined with a decrease of the damping, will speed up the convergence rate, and a decrease of the time step, combined with an increase of the damping, will make the method more stable. By interpreting the Newton-Raphson method as a dynamic method, the convergence parameter a in equation (37) thus acquires a physical meaning, as indicated by equation (36). Moreover, it becomes clear that the convergence speed can not be increased infinitely using this damping/acceleration method, as the stability described by equation (25) imposes an upper bound to the time step, and therefore also to the constant a . In addition, a too large decrease of the damping ratio causes the system to be underdamped and, consequently, vibrations will slow down the convergence.

If the choice for the time step and damping ratio is only constrained by the stability limit, these two convergence parameters can be chosen independently to fine-tune the convergence behavior of the nonlinear solver according to the problem that needs to be solved.

Finally, the proven equivalence between dynamic relaxation and the Newton-Raphson method shows that the tangential stiffness matrix used in the Newton-Raphson method can be regarded as a well chosen fictitious mass matrix. Therefore, it can be easily understood that this stiffness matrix does not need to be exact (as the convergence parameters in dynamic relaxation have no impact on the final solution), and approximations can be made, as is the case for any modified Newton-Raphson or quasi-Newton method. Generally, a better prediction or approximation of the stiffness of the structure will cause the eigenfrequencies to lie closer together, and, consequently, faster convergence is obtained. However, good approximations of the stiffness are often computationally expensive, which confirms that there is a trade-off between the number of iterations and the computational cost per iteration.

6. CONCLUSION

This short communication presents an analytical stability study, based on modal analysis, of the time integration scheme used in the dynamic relaxation method. Based on this study, a fictitious mass matrix proportional to the stiffness matrix is proposed, as this makes all eigenfrequencies coincide. This fictitious mass matrix is combined with classical viscous damping. It is shown that the resulting implicit dynamic relaxation approach becomes identical to the Newton-Raphson method if a specific time step and damping ratio are chosen. The Newton-Raphson method can therefore be regarded as a specific form of dynamic relaxation. Choosing another damping ratio and time step accelerates or damps the convergence of the dynamic relaxation method, in a similar way as the damping/acceleration techniques developed for the Newton-Raphson method.

The proposed approach offers new insight in both dynamic relaxation and the Newton-Raphson method, which can help to understand and interpret these nonlinear iterative solution techniques. Moreover, the physical interpretation of Newton-type methods and corresponding convergence parameters facilitates a well-considered choice of these parameters, making it easier to adequately manipulate these method's convergence behavior.

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