ARRANGEMENTS WITH ONE BOUNDED COMPONENT AND p -ADIC INTEGRALS

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In this article we study arrangements $\mathcal{A}$, such that $\mathbb{R}^n \setminus \mathcal{A}$ has exactly one bounded component. We obtain a result about their structure which gives us a method to construct all combinatorially different such arrangements in a given dimension. (A complete list for dimensions 1,2,3 and 4 is included).

Furthermore we associate a p -adic integral to each such arrangement and proof that this integral can be written as a product of p -adic beta functions. This is analogous to results of Varchenko and Loeser for integrals over $\mathbb{R}$ and character sums over finite fields respectively.

Introduction

We study arrangements $\mathcal{A}$ of hyperplanes in $\mathbb{C}^n$ with real coefficients such that $\chi(\mathbb{C}^n \setminus \mathcal{A}) = \pm 1$, where χ denotes the Euler-Poincare characteristic. If the normal vectors to the hyperplanes of $\mathcal{A}$ span $\mathbb{C}^n$, the assumption $\chi(\mathbb{C}^n \setminus \mathcal{A}) = \pm 1$ is known to be equivalent with the number of bounded components of $\mathbb{R}^n \setminus \mathcal{A}$ being one. If the latter condition is satisfied, $\mathcal{A}$ will be called a real 1-arrangement.

To each real 1-arrangement, we associate a p -adic integral. Theorem A states that such integrals can be written as a product of beta functions. Analogous results have been proved in [V] for real integrals and in [L] for charactersums over finite fields. To prove theorem A, we need to know the structure of real 1-arrangements. The main result concerning this structure is stated in theorem B. We will see how this theorem implies that the number of hyperplanes $\#\mathcal{H}(\mathcal{A})$ of $\mathcal{A}$ is bounded by $n + 1 \leq \#\mathcal{H}(\mathcal{A}) \leq 2n$. Moreover, it gives a method to construct all real 1-arrangements of a given dimension. For lower dimensions, the construction is explicitly done. We also give a few examples of theorem A. Concerning the structure of the article, we have separated the main results, their proofs and the examples into 3 paragraphs.

Finally we'd like to thank Prof. J. Denef and Prof. F. Loeser for suggesting the problem and J. Denef for teaching us the necessary background material. We remark that the integrals we study in this article are related to Igusa's local zeta function [I] [D].

1. Announcement of the main results

1.1. Notations and definitions

We define a n -dimensional arrangement $\mathcal{A}$ to be a finite union of hyperplanes in $\mathbb{C}^n$. $\mathcal{A}$ will be called real if all the coefficients of the equations of the hyperplanes can be taken real and generating if the normal vectors to the hyperplanes span $\mathbb{C}^n$.

Let $\mathcal{A}$ be a n -dimensional real arrangement. By $BC(\mathbb{R}^n \setminus \mathcal{A})$ we will indicate the set of all bounded components of $\mathbb{R}^n \setminus \mathcal{A}$ and we will call $\mathcal{A}$ a real k -arrangement if $\#BC(\mathbb{R}^n \setminus \mathcal{A}) = k$. For an arbitrary arrangement $\mathcal{A}$, $\mathcal{H}(\mathcal{A})$ will denote the set of all hyperplanes of $\mathcal{A}$. To each hyperplane $H_r \in \mathcal{H}(\mathcal{A})$ we associate 2 arrangements, i.e. $\mathcal{A}'_{H_r} = \cup_{H \in \mathcal{H}(\mathcal{A}) \setminus \{H_r\}} H$ the deleted arrangement of $\mathcal{A}$ induced by H_r and $\mathcal{A}''_{H_r} = \cup_{H \in \mathcal{H}(\mathcal{A}) \setminus \{H_r\}} (H \cap H_r)$ the restricted arrangement of $\mathcal{A}$ to H_r . Let $\mathcal{A}$ be a real arrangement and C a component of $\mathbb{R}^n \setminus \mathcal{A}$ defined by the inequalities $f_1 \square_1 0, \dots, f_m \square_m 0$ where for each i , $\square_i \in \{<, >\}$ and $f_i = 0$ is an equation of a hyperplane H_i of $\mathcal{A}$. H_i will be a boundary of C if the component C changes when we omit the inequality $f_i \square_i 0$ from the inequalities of C . By a boundary of a n -dimensional real 1-arrangement $\mathcal{A}$, we mean a boundary of the unique bounded component of $\mathbb{R}^n \setminus \mathcal{A}$. The set of all boundaries of $\mathcal{A}$, will be denoted by $\beta(\mathcal{A})$. Finally we define two arrangements $\mathcal{A}$ and $\mathcal{B}$ to be combinatorially equivalent if there exists an inclusion preserving bijection between $L(\mathcal{A})$ and $L(\mathcal{B})$ where $L(\mathcal{A})$ denotes the set of all non-empty intersections of hyperplanes in $\mathcal{A}$.

1.2. The main results

Theorem A. *Let $\mathcal{A} = \cup_{i=1}^m H_i$ be a n -dimensional real arrangement such that*

- (i) *each of the H_i has rational coefficients*
- (ii) *$\chi(\mathbb{C}^n \setminus \mathcal{A}) = \pm 1$*
- (iii) *$\mathcal{A}$ is generating.*

Let H_i be given by the equation $a_i(x_1, \dots, x_n) = 0$. Then

$$\int_{Q_p^n} \prod_{i=1}^m |a_i(x_1 \dots x_n)|_p^{s_i-1} \prod_{i=1}^n |dx_i|_p$$

converges on a non-empty open set D of $\mathbb{R}^n$ and can be written on D as

$$p^{L(s)} \prod_{i=1}^n B(L_{i1}(s), L_{i2}(s))$$

where B is the p -adic beta function defined as in [S-T] namely

$B(r, t) = \int_{Q_p} |x|_p^r |1 - x|_p^t |dx|_p$; s denotes $(s_1, \dots, s_m)$; $L(s), L_{i1}(s), L_{i2}(s) \in \mathbb{Z}[s]$ and $\deg L(s) \leq 1$, $\deg L_{i1}(s) = \deg L_{i2}(s) = 1$.

The proof of theorem A depends on the following result about the structure of real 1-arrangements.

Theorem B. *Let $\mathcal{A}$ be a n -dimensional real 1-arrangement. Possibly after an affine change of coordinates, we have*

- (i) $x_n = 0$ is the equation of some boundary B of $\mathcal{A}$.
- (ii) the variable x_n appears in the equation of exactly one other hyperplane of $\mathcal{A}$.
- (iii) the equations of the hyperplanes in $\mathcal{H}(\mathcal{A})$ are all of the type

$$\sum_{i=1}^n \epsilon_i x_i - \epsilon = 0 \quad \text{where} \quad \epsilon_i, \epsilon \in \{0, 1\}.$$

- (iv) If $n \geq 2$, then properties (i), (ii), (iii) and (iv) remain true for the $n - 1$ dimensional arrangement $\mathcal{A}''_B$.

We will see how this theorem implies that $\#\mathcal{H}(\mathcal{A}) \leq 2n$ and enables us to construct all possible n -dimensional real 1-arrangements.

2. Proofs of theorem A and B

2.1. Real arrangements

In section 2.1, we will consider $\mathcal{A}$ to be a n -dimensional real arrangement. We define $\chi'(\mathbb{C}^n \setminus \mathcal{A})$ to be $(-1)^n \chi(\mathbb{C}^n \setminus \mathcal{A})$. By [O-T, p. 23], we have

$$\chi'(\mathbb{C}^n \setminus \mathcal{A}) = \chi'(\mathbb{C}^n \setminus \mathcal{A}'_H) + \chi'(\mathbb{C}^{n-1} \setminus \mathcal{A}''_H)$$

The following well known lemma states that, under certain conditions, the same formula is true for the number of bounded components of $\mathbb{R}^n \setminus \mathcal{A}$ [B-B-R, p. 179].

Lemma 2.1.1. *If $H \in \mathcal{H}(\mathcal{A})$ is such that $\mathcal{A}'_H$ is generating in $\mathbb{C}^n$, then $\#BC(\mathbb{R}^n \setminus \mathcal{A}) = \#BC(\mathbb{R}^n \setminus \mathcal{A}'_H) + \#BC(\mathbb{R}^{n-1} \setminus \mathcal{A}''_H)$.*

By induction on n this lemma yields the well known

Proposition 2.1.2. *Let $\mathcal{A}$ be an generating real arrangement, then $\chi'(\mathbb{C}^n \setminus \mathcal{A}) = \#BC(\mathbb{R}^n \setminus \mathcal{A})$.*

2.2. Real 1-arrangements

In this section, $\mathcal{A}$ will always be a n -dimensional real 1-arrangement.

Remark : for such arrangement $\mathcal{A}$, $\mathcal{A}'_H$ is generating for each H of $\mathcal{H}(\mathcal{A})$.

Lemma 2.2.1.

For each hyperplane H of $\mathcal{A}$, $\#BC(\mathbb{R}^n \setminus \mathcal{A}) = \#BC(\mathbb{R}^n \setminus \mathcal{A}'_H) + \#BC(\mathbb{R}^{n-1} \setminus \mathcal{A}''_H)$. Furthermore $\chi'(\mathbb{C}^n \setminus \mathcal{A}) = \#BC(\mathbb{R}^n \setminus \mathcal{A})$.

This follows easily from (2.1.1), (2.1.2) and the remark.

To recognize whether H is a boundary of $\mathcal{A}$ we can use, besides the definition, the following criterium.

Proposition 2.2.2. *Let H be a hyperplane of $\mathcal{A}$. The following properties are equivalent*

- (i) H is a boundary of $\mathcal{A}$.
- (ii) $\#BC(\mathbb{R}^n \setminus \mathcal{A}'_H) = 0$.
- (iii) $\#BC(\mathbb{R}^{n-1} \setminus \mathcal{A}''_H) = 1$.

Remark : From (2.2.1) and (2.2.2) it follows that $H \in \mathcal{H}(\mathcal{A}) \setminus \beta(\mathcal{A})$ iff $\#BC(\mathbb{R}^n \setminus \mathcal{A}'_H) = 1$ iff $\#BC(\mathbb{R}^{n-1} \setminus \mathcal{A}''_H) = 0$.

Proof of 2.2.2.

By 2.2.1 it is sufficient to prove that (i) is equivalent with (ii). The fact that (ii) implies (i) follows immediately from the definition. Suppose that H is a boundary of $\mathcal{A}$ such that $\#BC(\mathbb{R}^n \setminus \mathcal{A}'_H) = 1$. Since $H \in \beta(\mathcal{A})$ we may conclude that $\{A_1\} = BC(\mathbb{R}^n \setminus \mathcal{A}) \neq BC(\mathbb{R}^n \setminus \mathcal{A}'_H) = \{A_2\}$, and thus that $A_1 \subsetneq A_2$ which is impossible.

Furthermore, we obtain from 2.2.1.

Lemma 2.2.3. *Let $H \in \mathcal{H}(\mathcal{A}) \setminus \beta(\mathcal{A})$ and $G \in \mathcal{H}(\mathcal{A})$ then $G \in \beta(\mathcal{A}'_H)$ if and only if $G \in \beta(\mathcal{A})$.*

Proof : From the remark after 2.2.2 we obtain that $\#BC(\mathbb{R}^n \setminus \mathcal{A}'_H) = 1$. Suppose $G \in \beta(\mathcal{A}'_H)$. From 2.2.2 it follows that $\#BC(\mathbb{R}^{n-1} \setminus (\mathcal{A}'_H)_G'') = 1$. Hence also $\#BC(\mathbb{R}^{n-1} \setminus \mathcal{A}''_G) = 1$ which implies $G \in \beta(\mathcal{A})$. Now suppose $G \in \beta(\mathcal{A})$. Then by 2.2.2 we have $\#BC(\mathbb{R}^n \setminus \mathcal{A}'_G) = 0$. Hence $\#BC(\mathbb{R}^n \setminus (\mathcal{A}'_H)_G') = \#BC(\mathbb{R}^n \setminus (\mathcal{A}'_G)_H') = 0$. Using again the remark after 2.2.2 applied to the hyperplane G of the real 1-arrangement $\mathcal{A}'_H$ this implies that $G \in \beta(\mathcal{A}'_H)$.

The following propositions will tell us something about the relative position of the hyperplanes of a real 1-arrangement.

Proposition 2.2.4. *The intersection of 3 different boundaries of $\mathcal{A}$ has dimension at most $n - 3$.*

Proof. Let B_1, B_2 and B_3 be 3 different boundaries of $\mathcal{A}$. Suppose $\dim(B_1 \cap B_2 \cap B_3) = n - 2$. We change affine coordinates such that the equations of B_1, B_2 and B_3 become respectively $x_1 = 0, x_2 = 0$ and $x_1 + x_2 = 0$. The unique bounded component A of $\mathbb{R}^n \setminus \mathcal{A}$ is given by $x_1 \square_1 0, x_2 \square_2 0, x_1 + x_2 \square_3 0, a_i(x_1 \dots x_n) \square_i 0$ with $4 \leq i \leq m$ and $\square_i \in \{<, >\}$. Checking all possibilities for $\square_1, \square_2$ and $\square_3$, one proves that there is always one of the 3 first inequalities that can be omitted without changing A . This contradicts the fact that B_1, B_2 and B_3 are boundaries.

Proposition 2.2.5. *There exist no 3 parallel hyperplanes of $\mathcal{A}$.*

Proof (By induction on n).

The case $n = 1$ is trivial. Suppose $n \geq 2$ and suppose that H_1, H_2 and H_3 are 3 different parallel hyperplanes of $\mathcal{A}$. It is sufficient to prove that there exists at least

one boundary B of $\mathcal{A}$ such that $B \cap H_i \neq \emptyset$ for $i = 1, 2, 3$ (since the contradiction then follows from an induction argument on $\mathcal{A}''_B$).

Let $\tilde{\mathcal{A}}$ be the union of all boundaries of $\mathcal{A}$. From 2.2.3, and the remark after 2.2.2, it follows that $\tilde{\mathcal{A}}$ is a real 1-arrangement. So it is impossible that all $H \in \mathcal{H}(\tilde{\mathcal{A}})$ are parallel. Take $B \in \mathcal{H}(\tilde{\mathcal{A}}) \setminus \{H_1\}$ such that $B \cap H_1 \neq \emptyset$. Since H_1, H_2 and H_3 are parallel it follows that also $B \cap H_2 \neq \emptyset$ and $B \cap H_3 \neq \emptyset$.

2.3. Some special real 0-arrangement

The following lemma, concerning the structure of some special real 0-arrangement is due to Varchenko [V].

Lemma 2.3.1. *Let $\mathcal{A}$ be a n -dimensional real 0-arrangement and $H \in \mathcal{H}(\mathcal{A})$ such that $\#BC(\mathbb{R}^{n-1} \setminus \mathcal{A}''_H) \geq 1$. Then there exists an affine change of coordinates such that H is given by the equation $x_n = 0$ and such that the variable x_n doesn't appear in the equation of any hyperplane H' of $\mathcal{A}$ other than H .*

Proof. We eventually make an affine change of coordinates such that H is given by the equation $x_n = 0$ and such that $x_i = 0$, $1 \leq i \leq n-1$, are the equations of some other hyperplanes $H_i \in \mathcal{H}(\mathcal{A})$. (This is possible since $\#BC(\mathbb{R}^{n-1} \setminus \mathcal{A}''_H) \geq 1$).

We show that x_n doesn't appear in the equation of any $H' \in \mathcal{H}(\mathcal{A}'_H)$. Suppose it does. Then some $H' \in \mathcal{H}(\mathcal{A}'_H)$ is defined by $a(x_1, \dots, x_{n-1}) + x_n = 0$.

Then the normal vectors to $H_1 \dots H_{n-1}, H'$ are linearly independent. So by 2.1.1 we obtain

$$\#BC(\mathbb{R}^n \setminus \mathcal{A}) = \#BC(\mathbb{R}^n \setminus \mathcal{A}'_H) + \#BC(\mathbb{R}^{n-1} \setminus \mathcal{A}''_H) \geq 1$$

which contradicts the fact that $\mathcal{A}$ is a real 0-arrangement.

2.4. Proof of theorem B

Proof. By induction on n . The case $n = 1$ is trivial. So assume $n \geq 2$. We consider two cases.

Case 1. There exist 2 boundaries B, B' of $\mathcal{A}$ and a non-boundary H of $\mathcal{A}$ such that $H \cap B = B' \cap B$. From 2.2.2 and 2.2.3 we obtain that $\#BC(\mathbb{R}^n \setminus \mathcal{A}'_{B'}) = 0$ and $\#BC(\mathbb{R}^{n-1} \setminus (\mathcal{A}'_{B'})''_B) = 1$. So by 2.3.1, we can make a change of coordinates such that B is given by the equation $x_n = 0$ and the variable x_n doesn't appear in any equation of an hyperplane $H' \in \mathcal{H}(\mathcal{A}) \setminus \{B, B'\}$. Since $\#BC(\mathbb{R}^n \setminus \mathcal{A}) = 1$, it is obvious that every variable has to appear in the equation of at least 2 hyperplanes of $\mathcal{A}$. So B' is given by

$$\lambda x_n + a(x_1, \dots, x_{n-1}) = 0 \quad \text{with } \lambda \neq 0.$$

Since $\#BC(\mathbb{R}^{n-1} \setminus \mathcal{A}''_B) = 1$, we can use an induction argument to get an affine change of coordinates which leaves x_n invariant and makes the equations of $\mathcal{H}(\mathcal{A}''_B)$ satisfy (i) - (iv).

Rescaling the x_n -axis, we may finally assume that $\lambda = 1$ which completes the proof in this case.

Case 2. There exist no B, B' and H as in case 1. Let B be an arbitrary boundary of $\mathcal{A}$. We make an affine change of coordinates such that B is given by $x_1 = 0$. An induction argument enables us to make another affine change of coordinates, which leaves x_1 invariant, such that $x_n = 0$ is the equation of a boundary of $\mathcal{A}''_B$ and the variable x_n appears in the equation of exactly one other hyperplane of $\mathcal{H}(\mathcal{A}''_B)$. Because of 2.2.4 and the hypothesis of case 2, this means that x_n appears in the equation of exactly 2 hyperplanes of $\mathcal{H}(\mathcal{A})$ and one of them is given by $\mu x_1 + x_n = 0$. If $\mu \neq 0$ we make another affine change of coordinates such that $x_1, \dots, x_{n-1}$ remain invariant and the equation of $\mu x_1 + x_n = 0$ becomes $x_n = 0$. Let B be the boundary of $\mathcal{A}$ with equation $x_n = 0$. An induction argument on $\mathcal{A}''_B$ completes the proof.

Corollary 2.4.1. *If $\mathcal{A}$ is a n -dimensional real 1-arrangement, then $n + 1 \leq \#\mathcal{H}(\mathcal{A}) \leq 2n$.*

Proof. The first inequality is obvious. We prove the second by induction on n . The case $n = 1$ is trivial. So assume $n \geq 2$ and $\#\mathcal{H}(\mathcal{A}) \geq 2n + 1$. We make an affine change of coordinates such that the equations of the hyperplanes of $\mathcal{A}$ satisfy (i), (ii), (iii) and (iv) of theorem B. Cutting $\mathcal{A}$ with the B from theorem B, we obtain $\#BC(\mathbb{R}^{n-1} \setminus \mathcal{A}''_B) = 1$ and $\#\mathcal{H}(\mathcal{A}''_B) \geq 2n - 1$ which contradicts the induction argument.

Remark. Note that both bounds are reached respectively by the simplex and the hypercube. Furthermore, theorem B implies the following construction method for a complete list of combinatorially different real 1-arrangements of a given dimension.

Corollary 2.4.2. *Let $\mathcal{A}$ be a n -dimensional real 1-arrangement. Then there exists an affine change of coordinates such that $\mathcal{A}$ is defined by either*

- (i) $a_1(x_1, \dots, x_{n-1}) = 0; \dots; a_m(x_1, \dots, x_{n-1}) = 0; x_n = 0; x_n - 1 = 0$; or
 - (ii) $a_1(x_1, \dots, x_{n-1}) = 0; \dots; a_m(x_1, \dots, x_{n-1}) = 0; x_n = 0; a_m(x_1, \dots, x_{n-1}) + x_n = 0$
or
 - (iii) $a_1(x_1, \dots, x_{n-1}) = 0; \dots; a_{m-1}(x_1, \dots, x_{n-1}) = 0; x_n = 0;$
 $a_m(x_1, \dots, x_{n-1}) + x_n = 0$
- where $a_1(x_1, \dots, x_{n-1}) = 0; \dots; a_m(x_1, \dots, x_{n-1}) = 0$ are the equations of a $(n - 1)$ -dimensional real 1-arrangement.

Conversely, every arrangement, constructed from a $(n - 1)$ -dimensional real 1-arrangement with equations $a_1(x_1, \dots, x_{n-1}) = 0; \dots; a_m(x_1, \dots, x_{n-1}) = 0$ by applying (i), (ii) or (iii) is a n -dimensional real 1-arrangement.

Proof. Theorem B and proposition 2.2.2 imply the first statement, while the latter follows from 2.1.1.

2.5. Proof of theorem A

In the proof of theorem A, we will use the next result which follows easily from a change of coordinates.

Lemma 2.5.1. *Let $a_1(\bar{x}), \dots, a_m(\bar{x}), b(\bar{x})$ be polynomials in $Q_p[x_1, \dots, x_{n-1}]$. Then*

$$\begin{aligned} & \int_{Q_p^n} |a_1(\bar{x})|_p^{s_1-1} \dots |a_m(\bar{x})|_p^{s_m-1} |b(\bar{x}) + x_n|_p^{s_{m+1}-1} |x_n|_p^{s_{m+2}-1} |dx_1|_p \dots |dx_n|_p \\ &= \int_{Q_p^{n-1}} |a_1(\bar{x})|_p^{s_1-1} \dots |a_m(\bar{x})|_p^{s_m-1} |b(\bar{x})|_p^{s_{m+1}+s_{m+2}-1} |dx_1|_p \dots |dx_{n-1}|_p \\ & \cdot \int_{Q_p} |x_n - 1|_p^{s_{m+1}-1} |x_n|_p^{s_{m+2}-1} |dx_n|_p \end{aligned}$$

on the domain in $\mathbb{C}^{m+2}$ where the last 2 integrals exist.

Definition 2.5.2. *Let $\mathcal{A}$ be a n -dimensional real 1-arrangement, $\mathcal{A} = \cup_{i=1}^m H_i$ where H_i is given by $a_i(x) = 0$, $x = (x_1, \dots, x_n)$, and let $s_1, \dots, s_m \in \mathbb{R}$. We define the p -adic integral associated to $\mathcal{A}$*

$$I_{\mathcal{A}} = I(a_1, s_1; a_2, s_2; \dots; a_m, s_m)$$

to be

$$\int_{Q_p^n} \prod_{i=1}^m |a_i(x)|_p^{s_i-1} \prod_{i=1}^n |dx_i|_p.$$

Proposition 2.5.3. *Let $\mathcal{A}$ and $I_{\mathcal{A}}$ be as in the previous definition. Suppose furthermore that the equations of $\mathcal{A}$ satisfy (i), (ii), (iii) and (iv) from theorem B. Then there exists an open set $D \subset \mathbb{R}^m$ such that*

- (i) $D \cap H(\mathbb{R}^m) \neq \emptyset$ where $H(\mathbb{R}^m) = \{(s_1, \dots, s_m) \in \mathbb{R}^m \mid 0 < s_i < 1, \forall i\}$,
- (ii) $I_{\mathcal{A}}$ converges on D ,
- (iii) $I_{\mathcal{A}}$ can be written as a product of p -adic beta-functions on D .

Proof. By induction on n .

It is well-known that $D = \{(s_1, s_2) \in \mathbb{R}^2 \mid 0 < s_1, s_2, s_1 + s_2 < 1\}$ satisfies the conditions for $n = 1$ (cfr. [S]).

Suppose $n \geq 2$. By theorem B we may assume that $a_m(x_1, \dots, x_n) = x_n$, $a_{m-1}(x_1, \dots, x_n) = b(x_1, \dots, x_{n-1}, 0) + x_n$, and $a_\ell(x_1, \dots, x_n) = a_\ell(x_1, \dots, x_{n-1}, 0)$, for all $\ell \leq m-2$ and all a_i are as in theorem B.

There are 3 cases to consider :

Case 1. $b(x_1, \dots, x_n) = -1$. We then formally get

$$I_{\mathcal{A}} = I(a_1, s_1; \dots, a_{m-2}, s_{m-2}) B(s_{m-1}, s_m).$$

We take $D = D_1 \cap D_2$ where D_1 is obtained by an induction argument and

$$D_2 = \{(s_1, \dots, s_m) \mid 0 < s_{m-1}, s_m, s_{m-1} + s_m < 1\}.$$

Case 2. $b(x_1, \dots, x_n) \neq -1$ and $b(x_1, \dots, x_n) \neq a_i(x_1, \dots, x_n)$ for all $i \leq m-2$. By 2.5.1 we formally get

$$I_{\mathcal{A}} = I_{\mathcal{A}''_{H_m}}(a_1, s_1; \dots; a_{m-2}, s_{m-2}; b, s_{m-1} + s_m). B(s_{m-1}, s_m).$$

By an induction argument on $I_{\mathcal{A}''_{H_m}}$ we obtain $D_1 \subset \mathbb{R}^{m-1}$ such that D_1 satisfies (i), (ii) and (iii). Let D_2 be $\{(s_1, s_2) \in \mathbb{R}^2 \mid 0 < s_1, s_2, s_1 + s_2 < 1\}$. We define the continuous maps

$$\begin{aligned} f : \mathbb{R}^m &\rightarrow \mathbb{R}^{m-1} : (s_1, \dots, s_m) \mapsto (s_1, \dots, s_{m-2}, s_{m-1} + s_m). \\ p : \mathbb{R}^m &\rightarrow \mathbb{R}^2 : (s_1, \dots, s_m) \mapsto (s_{m-1}, s_m). \end{aligned}$$

It is easy to see that $D = f^{-1}(D_1) \cap p^{-1}(D_2)$ satisfies the conditions.

Case 3. If $b(x_1, \dots, x_n) = a_i(x_1, \dots, x_n)$ for one of the $a_i(x_1, \dots, x_n)$; the lemma follows from a similar argument as in case 2.

Proof of theorem A.

Conditions (i), (ii), (iii) and proposition 2.1.2 imply that $\#BC(\mathbb{R}^n \setminus \mathcal{A}) = 1$. Theorem B enables us to make an affine change of coordinates such that $\mathcal{A}$ satisfies the conditions of 2.5.3. Note that by the change of coordinates, a rational constant and an exponential function may appear.

3. Examples

3.1. Construction of a 4-dimensional real 1-arrangement

Applying consecutively (ii), (i) and (iii) of Corollary 2.4.2 to the real 1-arrangement $\mathcal{A}_1 : x_1, x_1 - 1$, gives respectively the following 2,3 and 4-dimensional real 1-arrangements:

$\mathcal{A}_2 : x_1, -1 + x_1, x_2, -1 + x_1 + x_2$; $\mathcal{A}_3 : x_1, -1 + x_1, x_2, -1 + x_1 + x_2, x_3, -1 + x_3$; $\mathcal{A}_4 : x_1, -1 + x_1, x_2, -1 + x_1 + x_2, x_3, x_4, -1 + x_3 + x_4$. Applying consecutively (i), (ii) and (iii) of corollary 2.4.2 to the same real 1-arrangement gives respectively the following 2,3 and 4-dimensional real 1-arrangements :

$\mathcal{A}'_2 : x_1, -1 + x_1, x_2, -1 + x_2$; $\mathcal{A}'_3 : x_1, -1 + x_1, x_2, -1 + x_2, x_3, -1 + x_2 + x_3$; $\mathcal{A}'_4 : x_1, x_2, -1 + x_2, x_3, -1 + x_2 + x_3, x_4, -1 + x_1 + x_4$. This shows that we can get combinatorially equivalent real 4-dimensional 1-arrangements starting from the same real 1-dimensional 1-arrangement but applying another order of construction.

3.2. Complete list of all combinatorially different real 1-arrangements in low dimensions

In each dimension, we will classify the arrangements by their number of hyperplanes and their Poincaré-polynomial. (The Poincaré polynomial $P(\mathcal{A})$ of an arrangement $\mathcal{A}$ is defined as in [O-T] p. 54).

In dimensions 1, 2 and 3, each line contains the following data associated to an arrangement $\mathcal{A}$: the number of hyperplanes of $\mathcal{A}$: the Poincaré polynomial of $\mathcal{A}$, the coefficients of the equations of the hyperplanes of $\mathcal{A}$. We will omit the equations $x_i = 0$ since they always occur. The equation $a + \sum_{i=1}^n a_i x_i = 0$ will be given as $(a, a_1, \dots, a_n)$.

3.2.1. Dimension 1

$$2; 1 + 2t; (-1, 1)$$

3.2.2. Dimension 2

$$\begin{aligned} &3; 1 + 3t + 3t^2; (-1, 1, 1) \\ &4; 1 + 4t + 4t^2; (-1, 1, 0), (-1, 0, 1) \\ &4; 1 + 4t + 4t^2; (-1, 1, 0), (-1, 1, 1) \end{aligned}$$

3.2.3. Dimension 3

$$\begin{aligned} &4; 1 + 4t + 6t^2 + 4t^3; (-1, 1, 1, 1) \\ &5; 1 + 5t + 9t^2 + 6t^3; (-1, 1, 1, 0), (-1, 0, 0, 1) \\ &5; 1 + 5t + 9t^2 + 6t^3; (-1, 1, 1, 1), (-1, 1, 1, 0) \\ &5; 1 + 5t + 9t^2 + 6t^3; (-1, 1, 0, 0), (-1, 1, 1, 1) \\ &5; 1 + 5t + 10t^2 + 7t^3; (-1, 1, 1, 0), (-1, 1, 0, 1) \\ &6; 1 + 6t + 12t^2 + 8t^3; (-1, 1, 0, 0), (-1, 0, 1, 0), (-1, 0, 0, 1) \\ &6; 1 + 6t + 12t^2 + 8t^3; (-1, 1, 1, 0), (-1, 1, 0, 0), (-1, 0, 0, 1) \\ &6; 1 + 6t + 12t^2 + 8t^3; (-1, 1, 1, 0), (-1, 1, 0, 1), (-1, 1, 0, 0) \\ &6; 1 + 6t + 12t^2 + 8t^3; (-1, 1, 1, 1), (-1, 1, 0, 0), (-1, 1, 1, 0) \\ &6; 1 + 6t + 12t^2 + 8t^3; (-1, 1, 1, 0), (-1, 1, 0, 0), (0, 1, 0, 1) \end{aligned}$$

3.2.4. Dimension 4

5 Hyperplanes

$$\begin{aligned} \text{Poincare-polynomial : } &1 + 5t + 10t^2 + 10t^3 + 5t^4 \\ &(-1, 1, 1, 1, 1) \end{aligned}$$

6 Hyperplanes

Poincare-polynomial

Poincare-polynomial

$$1 + 6t + 14t^2 + 16t^3 + 8t^4$$

$$1 + 6t + 15t^2 + 18t^3 + 9t^4$$

$$\begin{aligned} &(-1, 1, 1, 1, 0); (-1, 0, 0, 0, 1) \\ &(-1, 1, 1, 1, 1); (-1, 1, 0, 0, 0) \\ &(-1, 1, 1, 1, 1); (-1, 1, 1, 1, 0) \end{aligned}$$

$$\begin{aligned} &(-1, 1, 1, 0, 0), (-1, 0, 0, 1, 1) \\ &(-1, 1, 1, 1, 1), (-1, 1, 1, 0, 0) \end{aligned}$$

$$\text{Poincare-polynomial : } 1 + 6t + 15t^2 + 19t^3 + 10t^4$$

$$\begin{aligned} &(-1, 1, 1, 1, 1); (-1, 1, 1, 0, 1) \\ &(-1, 1, 1, 0, 0); (-1, 1, 0, 1, 1) \end{aligned}$$

7 Hyperplanes

Poincare-polynomial : $1 + 7t + 19t^2 + 24t^3 + 12t^4$

$(-1, 1, 1, 0, 0), (-1, 0, 0, 1, 0), (-1, 0, 0, 0, 1)$
 $(-1, 1, 1, 1, 0), (-1, 1, 1, 0, 0), (-1, 0, 0, 0, 1)$
 $(-1, 1, 1, 1, 0), (-1, 1, 0, 0, 0), (-1, 0, 0, 0, 1)$
 $(-1, 1, 1, 1, 1), (-1, 1, 0, 0, 0), (-1, 1, 1, 1, 0)$
 $(0, 1, 0, 0, 1), (-1, 1, 1, 1, 0), (-1, 1, 0, 0, 0)$
 $(-1, 1, 1, 1, 1), (-1, 1, 0, 0, 0), (-1, 1, 1, 0, 0)$
 $(-1, 1, 1, 0, 0), (-1, 0, 0, 1, 0), (-1, 0, 0, 1, 1)$
 $(-1, 1, 1, 0, 0), (-1, 1, 0, 1, 1), (-1, 1, 0, 0, 0)$
 $(-1, 1, 1, 1, 0), (-1, 1, 1, 0, 0), (-1, 1, 1, 0, 1)$
 $(-1, 1, 1, 1, 0), (-1, 1, 1, 1, 1), (-1, 1, 1, 0, 0)$
 $(0, 1, 0, 0, 1), (-1, 1, 1, 1, 0), (-1, 1, 1, 0, 0)$

Poincare-polynomial : $1 + 7t + 20t^2 + 27t^3 + 14t^4$

$(-1, 1, 1, 0, 0), (-1, 1, 0, 1, 0), (-1, 0, 0, 0, 1)$
 $(-1, 1, 1, 1, 0), (-1, 1, 0, 0, 0), (-1, 1, 1, 0, 1)$
 $(-1, 1, 1, 0, 0), (-1, 1, 0, 1, 1), (-1, 1, 0, 1, 0)$
 $(0, 1, 0, 0, 1), (-1, 1, 1, 0, 0), (-1, 1, 0, 1, 0)$

Poincare-polynomial : $1 + 7t + 21t^2 + 29t^3 + 14t^4$

$(-1, 1, 1, 0, 0), (-1, 1, 0, 1, 0), (-1, 1, 0, 0, 1)$

8 Hyperplanes

Poincare-polynomial : $1 + 8t + 24t^2 + 32t^3 + 16t^4$

$(-1, 1, 1, 0, 0), (-1, 1, 0, 1, 0), (-1, 1, 0, 0, 1), (-1, 1, 0, 0, 0)$
 $(-1, 1, 1, 0, 0), (-1, 1, 0, 1, 1), (-1, 1, 0, 1, 0), (-1, 1, 0, 0, 0)$
 $(0, 1, 0, 0, 1), (-1, 1, 1, 0, 0), (-1, 1, 0, 1, 0), (-1, 1, 0, 0, 0)$
 $(-1, 1, 1, 1, 0), (-1, 1, 0, 0, 0), (-1, 1, 1, 0, 1), (-1, 1, 1, 0, 0)$
 $(-1, 1, 1, 1, 0), (-1, 1, 0, 0, 0), (-1, 1, 1, 0, 0), (-1, 1, 1, 0, 1)$
 $(0, 0, 1, 0, 1), (-1, 1, 1, 1, 0), (-1, 1, 0, 0, 0), (-1, 1, 1, 0, 0)$
 $(0, 1, 0, 0, 1), (-1, 1, 1, 1, 0), (-1, 1, 0, 0, 0), (-1, 1, 1, 0, 0)$
 $(-1, 1, 1, 0, 0), (-1, 0, 0, 1, 1), (-1, 1, 0, 0, 0), (-1, 0, 0, 1, 0)$
 $(-1, 1, 1, 0, 0), (-1, 1, 0, 1, 0), (-1, 1, 0, 0, 0), (-1, 0, 0, 0, 1)$
 $(-1, 1, 1, 1, 0), (-1, 1, 0, 0, 0), (-1, 1, 1, 0, 0), (-1, 0, 0, 0, 1)$
 $(-1, 1, 1, 0, 0), (-1, 1, 0, 0, 0), (0, 1, 0, 1, 0), (-1, 0, 0, 0, 1)$
 $(-1, 1, 0, 0, 0), (-1, 0, 1, 0, 0), (-1, 0, 0, 1, 0), (-1, 0, 0, 1, 1)$
 $(-1, 1, 0, 0, 0), (-1, 0, 1, 0, 0), (-1, 0, 0, 1, 0), (-1, 0, 0, 0, 1)$

3.2.5. Remarks

We see there are respectively 1, 3, 10 and 37 combinatorially different types of real 1-arrangements in dimension 1, 2, 3 and 4. It is still unknown how this number increases with the dimension. Many combinatorially different types turn out to

on a non-empty open set.

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