Time-stepping scheme for mechanical systems with unilateral constraints and time-delays

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Summary. Many mechanical systems operate in environments with unilateral position constraints that induce impulsive dynamics, while time-delays appear in the force terms. The motion of these systems is expressed in terms of delay measure differential inclusions. The velocity of solutions then becomes a function of special bounded variation in time, and solutions will continue after accumulating impacts. We prove local existence of solutions in the single-constraint case and provide a simulation algorithm to compute the solutions. Sufficient conditions are presented on the system dynamics that guarantee that the algorithm converges. Numerical results illustrate the performance of the algorithm.

Introduction

Many mechanical systems are modelled effectively as multibody systems with unilateral constraints. Controlled applications of these systems contain time-delays. In this paper, the dynamics of these systems are modelled in terms of delay measure differential inclusions. For this class of measure differential inclusions, we investigate existence of solutions and provide a simulation method that combines the Moreau time-stepping scheme [1, 2, 3] with interpolation of the delayed variables.

To simulate mechanical systems with impulses, time-stepping methods [1] compute the impulse transferred by the constraint during one time-step, such that a fixed time-step method is attained which can be used in cases where accumulating impacts occur. Various effective algorithms to compute solutions of smooth delay differential equations (of retarded type, as typically occur in controlled applications) rely on the method-of-steps: given a numerical approximation of the history, solutions of the delay differential equation are expressed, in the next time step, as a non-autonomous, but finite-dimensional, differential equation, by interpolating in the values of the history [4].

This study presents a solution concept and a time-stepping scheme to find solutions of mechanical systems with unilateral position constraints and delays. The results are illustrated by means of an example of a controlled systems with two degrees of freedoms.

System model

We consider mechanical systems whose position vector q satisfies $q \in K$, with $K := \{q \in \mathbb{R}^d \,|\, g(q) \geq 0\}$ and g a twice continuously differentiable function. Considering velocities u as functions of special bounded variation in time, this allows to describe the system dynamics with the delay measure differential inclusion:

$$M(q)du = f\left(t, q(t), u(t), q(t-\tau_1(t)), u(t-\tau_1(t)), \ldots, q(t-\tau_s(t)), u(t-\tau_s(t))\right)dt + dr, \tag{1}$$

and $dq = u(t)dt$, where dr represents the constraint interaction, the piecewise continuous time-varying delay signals $\tau_k : \mathbb{R}_{\geq 0},\ k = 1, \ldots s$, satisfy $\dot{\tau}_k \leq 1$ a.e., and $t - \tau_k(t) \in [-H, 0)$ with some finite $H > 0$. The position-dependent mass matrix $M(q)$ is positive definite for all $q \in K$. We assume that function f is continuous and locally Lipschitz in its arguments. Let $W(q) = \left(\frac{dg}{dq}\right)^T$ and let $N_K(q)$ denote the normal cone to K. The effect of the constraint is captured as $dr = W(q)\lambda dt + W(q)\Lambda d\eta$, with $\lambda, \Lambda \geq 0$;

$$-dr \in N_K(q) \quad \text{and} \quad 0 \leq \Lambda \perp W^T(u^+(t) + \epsilon u^-(t)) \geq 0. \tag{2a}$$

Scalar $\epsilon \in [0, 1]$ is the restitution coefficient and $u^{\pm}$ are the right- and left-sided limits of u.

Existence of solutions

Theorem 1. *Consider system* (1), (2). *If*

(i) function f is Lipschitz in all arguments with Lipschitz constant L;

(ii) mass-matrix $M(q)$ is positive definite and Lipschitz continuous for all $q \in K$ and, in addition, there exist scalars $\alpha, \beta > 0$ such that

$$\alpha\|u\|^2 \leq u^T M(q)u \leq \beta\|u\|^2,\ \forall u; \tag{3}$$

(iii) scalar function $g(q)$ is twice continuously differentiable and $\nabla g(q) \neq 0$ if $g(q) = 0$; and

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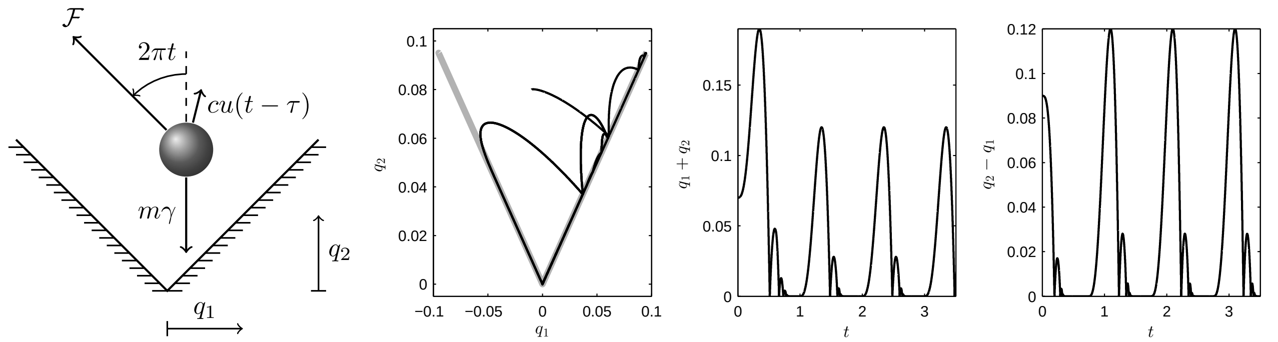

Figure 1: left: Two-dimensional mechanical system with gravitational acceleration γ, added damping $cu(t-\tau)$ and rotating force $\mathcal{F}$. right:Positions of the two-dimensional mechanical system with gravitational acceleration $\gamma = 9.81$, $c = 5$, $\tau = 0.1$ $\epsilon = 0.5$ and external force $\mathcal{F} = 10(\sin(2\pi t), \cos(2\pi t))$.

(iv) the time-varying delay signals $\tau_i : \mathbb{R}_{\geq 0},\ i = 1, \dots s$, are such that $t-\tau_i(t)$ is strictly increasing, almost everywhere continuous, and there exists $H > 0$ such that $t - \tau_i(t) \in [-H, 0]$ for all $t \in \mathbb{R}$,

then for any pair of initial functions Φ_q, Φ_u, with Φ_q absolutely continuous, $g(\Phi_q(t_0 + s)) \geq 0$ for all $s \in [-H, 0]$, and Φ_u having bounded variation, there exists $t^\star > t_0$ such that a solution exists on the time interval $[t_0, t^\star]$.

Sketch of numerical method

We discretize the solutions on the time interval $[t_0, t_f]$ with $n + 1$ equidistant time points, such that $t_i = t_0 + ih_n,\ i = 0, \dots n$, and discretize the initial function $(q, u) : [-H, 0] \to K \times \mathcal{T}_K(q)$, with $T_K(q)$ the polar cone to $N_K(q)$. For $k \in \{0, \dots, n\}$, we then compute the variables $\tilde{q}_1^{k+1} = \tilde{q}_1^k + h_n \tilde{u}_1^k,\quad \tilde{q}_j^{k+1} = \tilde{q}_{j-1}^k,\quad j = 2, 3, \dots \mathcal{N}$, that are used to approximate the mass term $M(q(t))$, cone $T_K(q(t))$ and force term with $M_{k+1} = M(\tilde{q}_1^{k+1})$, $W^{k+1} = W(\tilde{q}_1^{k+1})$, and

$$\begin{aligned} f_{k+1}^n = f\Big(&t_{k+1}, \tilde{q}_1^{k+1}, \tilde{u}_1^k, \bar{q}_1^{k+1}(t_{k+1} - \tau_1(t_{k+1})), \bar{u}^k(t_{k+1} - \tau_1(t_{k+1})), \dots, \\ &\tilde{q}_1^{k+1}(t_{k+1} - \tau_s(t_{k+1})), \bar{u}^k(t_{k+1} - \tau_s(t_{k+1}))\Big), \end{aligned} \tag{4a}$$

respectively. We then compute the next velocity as:

$$\tilde{u}_1^{k+1} = -\epsilon \tilde{u}_1^k + (1 + \epsilon)\, \mathrm{prox}_{T_K^{k+1}}^{M_{k+1}} \left(\tilde{u}_1^k + \tfrac{h_n}{1+\epsilon} M_{k+1}^{-1} f_{k+1}^n \right) \qquad \tilde{u}_j^{k+1} = \tilde{u}_{j-1}^k, \quad j = 2, 3, \dots \mathcal{N}. \tag{4b}$$

In this algorithm, the velocity term $\tilde{u}_1^k + \frac{h_n}{1+\epsilon} M_{k+1}^{-1} f_{k+1}^n$ is projected onto the cone of feasible velocities T_K^{k+1} of the next time step. Convergence of this algorithm is proven and implies Theorem 1.

Mechanical system with two constraints and two degrees of freedom

We apply algorithm (4) to a mechanical systems with two constraints and two degrees of freedom shown in Figure 1. This system is modelled with identity mass matrix, $f = (A \sin(2\pi t) - cu_1(t - \tau), A \cos(2\pi t) - cu_2(t - \tau) - \gamma)$, where we selected the parameters $A = 10, c = 0.5, \tau = 0.1, \gamma = 9.81$. The depicted solution shows accumulations of impacts.

Conclusions

Solutions are described of mechanical systems subject to an impulsive position constraint and forces that are functions of delayed position and velocity terms, leading to models in terms of delay measure differential inclusions. The velocity of the solutions will be functions of *special bounded variation* in time, such that they can feature accumulating discontinuities (or Zeno behaviour). Existence of solutions is shown and a numerical method is presented to simulate these mechanical systems with delays.

References

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