The effect of generalized force correlations on the response statistics of a harmonically driven random system

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Abstract

If the physical properties of a structural component are sufficiently random then the statistical distribution of the natural frequencies and mode shapes tends to a universal distribution associated with the Gaussian Orthogonal Ensemble (GOE) of random matrices. Previous work has exploited this result to yield expressions for the relative variance of the energy of the response of a random system to harmonic excitation. The derivation of these expressions employed random point process theory, and in the theoretical development it was assumed that the modal generalised forces were uncorrelated. Although this assumption is often valid, there are cases in which correlations between the generalised forces can significantly affect the response variance, and in the present work the existing theory is extended to include correlations of this type. The extended theory is applicable to both single frequency responses and to band average responses, and the developed closed form expressions are validated by comparison with direct simulations for a random plate structure.

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1. Introduction

The vibrational response of an engineering system can be very sensitive to imperfections, and there is much practical interest in being able to predict not just the response of a nominally perfect structure but also the statistical spread of the response arising from uncertainties. This can be an extremely difficult task, since a computational model of a car or an aeroplane may contain millions of degrees of freedom, and the statistics of the uncertainties in the properties of the system may not be known. In many cases however the system can be sufficiently random for certain universal statistical laws to apply to the distribution of the natural frequencies and the mode shapes and this means that, almost counterintuitively, the statistics of the response can be predicted without any detailed knowledge of the uncertainties in the system. Early work in this area was performed by Lyon [1], who derived formulae for the mean and variance of the power input to a system subjected to single point harmonic forcing on the assumption that the natural frequencies constitute a Poisson random point process. The effect of repulsion between natural frequencies (not included in the Poisson model) was also considered in reference [1], and this work was extended by Davy [2] to the case of multiple excitation and response points. It was later recognised that in many practical cases the natural frequencies of a random system (without symmetries) do not have the distribution assumed in references [1,2]; more commonly, the natural frequencies have the same distribution as the eigenvalues of a random matrix ensemble known as the Gaussian Orthogonal Ensemble (GOE) [3], as demonstrated experimentally for an aluminium block by Weaver [4]. The applicability of this distribution is reviewed in reference [5], and the occurrence of the distribution has been exploited to predict of the response statistics of single structural components (for example [6-8]) and acoustic volumes (for example [9-10]). Work of this type has also been extended to the prediction of the response of built-up systems within the context of the Statistical Energy Analysis (SEA) approach to structural dynamics [11]. As described below, the present work provides an enhancement of previous work on the application of the GOE to vibrating systems by relaxing one of the assumptions employed in the analysis, thereby allowing the approach to be applied to a wider range of problems. Of course, the literature regarding the response of random dynamic systems is vast and there are many approaches that can be employed to predict the statistics of the response of such a system. Because the present work is narrowly focussed on extending the range of an existing method, the previous work referenced here is exclusively focussed on material directly

related to the earlier results. Wider views on the treatment of uncertainties in computational models can be found in texts such as [12].

The usual approach to the analysis of the dynamic response of a linear structural component is to represent the response as a sum of modal contributions; each modal contribution involves the frequency response function of the mode, the generalised force in the mode, and the mode shape. If the system natural frequencies are taken to be random, then the modal sum is exactly the type of quantity that is amenable to analysis by random point process theory [8-10]. In this theory the response is viewed as a sum of frequency response functions that have random amplitudes, and given the distribution of the natural frequencies (arising from the GOE for example), the theory yields expressions for the statistical moments of the response. This approach was used in references [7,8] to yield closed form expressions for the mean and the variance of the energy of the response, and the results were compared favourably with both detailed simulations and experimental measurements. Reference [7] was concerned with the response at a single forcing frequency, whereas reference [8] extended this result to the statistics of a band-averaged response. A key feature of random point process theory, as presented in references [13-15] and exploited in references [7,8], is that the random amplitudes in the modal sum are taken to be statistically independent and identically distributed. Although this is a perfectly reasonable assumption in many cases, it has been found in the course of the present work that significant errors can arise if the system is subjected to random loading and the modal overlap is relatively high. It has also been found that the key failing in the existing theory is the neglect of correlations between the random amplitudes, arising from correlations between the generalised forces, and hence the objective of the present work is to extend the current theory for single frequency and band-averaged responses by including the effect of such correlations. The theory is extended in a general way, so that the correlation between the random amplitudes may have any cause, although particular attention is paid to the example of correlations arising from random forces. To add a practical context to the work, it can be noted that harmonic loading with a random or uncertain amplitude can arise (for example) from rotating or reciprocating machinery. The frequency of operation may be well defined, but the loading generated by imbalance or internal mechanisms may vary randomly across different operating conditions and across different machines from the same production line.

In Section 2 below, the existing theory for predicting the response variance of a structural component is summarised, and an example is given for which the theory provides

poor results. Random point process theory is then extended in Section 3 to include the effect of correlations between terms in the modal sum, and a new closed form expression is obtained for the relative variance under harmonic excitation. The analysis is extended in Section 4 to the relative variance of the band-averaged response, and in Section 5 both new results are compared with numerical simulations of a random plate. The findings of the work are summarised in Section 6.

2. Preliminary considerations

The response of a proportionally damped linear system to harmonic forcing at frequency ω can be written as a modal sum in the form

$$u(\omega, \mathbf{x}) = \sum_n \frac{i\omega g_n \phi_n(\mathbf{x})}{\omega_n^2 - \omega^2 + i\eta\omega_n^2}, \quad g_n = \int_R P(\mathbf{x}) \phi_n(\mathbf{x}) d\mathbf{x}, \quad (1,2)$$

where $u(\omega, \mathbf{x})$ is the complex amplitude of the velocity at spatial point $\mathbf{x}$, ω_n and $\phi_n(\mathbf{x})$ are respectively the natural frequency and the mode shape of the n th mode of vibration, η is the loss factor, $P(\mathbf{x})$ is a distributed load, and g_n is the resulting generalised force in the n th mode. For simplicity both the velocity and the mode shape have been taken to be scalar quantities in Eq. (1), but both could readily be generalised to vector quantities. The time averaged kinetic energy of the system is given by

$$T(\omega) = (1/4) \int_R \rho(\mathbf{x}) |u(\omega, \mathbf{x})|^2 d\mathbf{x}, \quad (3)$$

where $\rho(\mathbf{x})$ is the mass density and R is the spatial region occupied by the system. Equations (1-3) allow the kinetic energy to be written in the convenient form

$$T(\omega) = \sum_n a_n H(\omega, \omega_n), \quad (4)$$

where

$$a_n = \left| \int_R P(\mathbf{x}) \phi_n(\mathbf{x}) d\mathbf{x} \right|^2, \quad H(\omega, \omega_n) = \frac{(\omega^2/4)}{(\omega_n^2 - \omega^2)^2 + (\eta\omega_n^2)^2}. \quad (5a,b)$$

If the system under consideration has uncertain properties then the natural frequencies and the mode shapes will be random quantities. As discussed in [5], if the system is sufficiently random then the principle of universality implies that there is a strong tendency for the statistics of the natural frequencies to have a distribution that conforms to the Gaussian Orthogonal Ensemble (GOE). In this case the joint probability density function of the natural frequencies is a function of a single parameter, the modal density $\nu(\omega)$, which is defined as

the average number of natural frequencies which fall within a unit frequency band centred on ω . In this case it has been shown using random point process theory [7] that the ensemble mean and relative variance of the kinetic energy are given by

$$E[T] = \frac{E[a_n] \pi \nu(\omega)}{8M\eta\omega}, \quad (6)$$

$$\text{RelVar}[T] = \frac{[\alpha + q(m)]}{\pi m}, \quad (7)$$

where

$$q(m) = -1 + \frac{1}{2\pi m} [1 - \exp(-2\pi m)] + E_1(\pi m) \left[\cosh(\pi m) - \frac{1}{\pi m} \sinh(\pi m) \right], \quad (8)$$

$$\alpha = \frac{E[a_n^2]}{E[a_n]^2}, \quad (9)$$

and the parameter $m = \omega\eta\nu$ is termed the modal overlap of the system. If the modal overlap factor is greater than around 0.2, then a very good approximation to equation (7) is [7]

$$\text{RelVar}[T] \approx \frac{(\alpha - 1)}{\pi m} + \frac{1}{(\pi m)^2}. \quad (10)$$

Although Eqs. (6) and (7) have been validated for many examples, it should be noted that the equations are based on an assumption regarding the nature of the applied loading: it is assumed that the generalised forces are such that the coefficients a_n in Eq. (4) are uncorrelated, and the validation cases generally meet this condition. The limitations of this assumption can be explored by considering a single point force with complex amplitude F applied at the location $\mathbf{x}_0$, in which case Eq. (5) yields

$$a_n = |F|^2 \phi_n^2(\mathbf{x}_0). \quad (11)$$

Assuming that any two mode shapes ϕ_n and ϕ_m are uncorrelated (which is a consequence of GOE statistics), it is readily shown that

$$\gamma \equiv \frac{\text{Cov}[a_n, a_m]}{E[a_n]E[a_m]} = \frac{E[|F|^4]}{E[|F|^2]^2} - 1 \quad (n \neq m), \quad (12)$$

so that a_n and a_m are uncorrelated ($\gamma = 0$) for a deterministic force (or for a force with a deterministic amplitude and a random phase), but more generally they are correlated. If F is Gaussian and real then $\gamma = 2$, while if F is circularly-symmetric complex Gaussian then $\gamma = 1$. The effect of γ on the variance of the response can be explored by considering the example of a simply supported plate that is randomised by the addition of small point masses.

Full details of this example are given in Section 5, but in brief the plate is made of Aluminium (Young's modulus $E = 71 \times 10^9 \text{ Nm}^{-2}$, density $\rho = 2700 \text{ kg m}^{-3}$, Poisson's ratio $\bar{\nu} = 0.33$) and has dimensions $1.4\text{m} \times 1.3\text{m}$, thickness 1.25 mm , and loss factor $\eta = 1\%$. A random ensemble of 200 plates has been generated by adding 20 point masses in random positions, with each mass having 1% of the mass of the bare plate. The plate is subjected to a harmonic point load with either a fixed amplitude ($\gamma = 0$) or a complex Gaussian amplitude ($\gamma = 1$); when the point load is random, a random amplitude is selected for each plate in the ensemble, and this is held constant over all frequencies. Also, the location of the point load is selected randomly and the same location is used for all plates at all frequencies. Results for the relative variance of the response, computed using the Lagrange-Rayleigh-Ritz method to model the plate (see section 5) are compared to Eq. (7) in Figure 1. The deterministic load has a unit amplitude, while the real and imaginary parts of the complex Gaussian load are uncorrelated, with each having zero mean and unit variance.

Fig. 1: Mean (left) and relative variance (right) of the energy of a plate excited by one complex Gaussian force or by one deterministic force as a function of modal overlap. The curves are: benchmark model for the complex Gaussian load (fluctuating thin black line), benchmark model for the deterministic load (fluctuating thin grey line), standard SEA for the complex Gaussian load (thick black line), standard SEA for the deterministic load (thick grey line).

It can be seen that Eq. (7) yields a very good prediction of the relative variance for the uncorrelated case, but also that the correlation of the amplitudes a_n arising for the case of the complex Gaussian force has a very significant effect on the response statistics. The aim of the following section is to extend the theory behind Eq. (7) to deal with correlation effects of this type.

3. Response variance theory including the effect of correlations

The statistics of the natural frequencies $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_N)$ of an N -degree-of-freedom random system are fully determined if the joint probability density function (jpdf) $p(\boldsymbol{\omega})$ is known. It is usual to consider the natural frequencies $\boldsymbol{\omega}$ to be an ordered sequence, with $\omega_{j+1} \geq \omega_j$, but in the present context it is more convenient to consider the natural frequencies to be *unordered*, so that any particular natural frequency, ω_k say, is equally likely to be any member of the ordered set. Stratonovich [14] has defined an extremely useful set of functions by the relation

$$f_k(\omega_1, \omega_2, \dots, \omega_k) = \frac{N!}{(N-k)!} \int_0^\infty \dots \int_0^\infty p(\boldsymbol{\omega}) d\omega_{k+1} \dots d\omega_N, \quad (13)$$

where f_k is known as the k th distribution function. It can be noted that the same set of functions is employed in random matrix theory [3], but in that case f_k is denoted by R_k and is referred to as the k -point correlation function. A key property of f_k is that the probability of finding a natural frequency in each of the k regions $\omega_j < \omega < \omega_j + d\omega_j$ (where $j = 1, 2, \dots, k$ and $d\omega_j$ is a small interval), without regard for the location of the other $N - k$ natural frequencies, is $f_k(\omega_1, \omega_2, \dots, \omega_k) d\omega_1 d\omega_2 \dots d\omega_k$. The ensemble average of the kinetic energy given by Eq. (4) can be expressed in terms of the first distribution function by noting that

$$E[T] = \sum_n E[a_n] \int_0^\infty \dots \int_0^\infty H(\omega, \omega_n) p(\boldsymbol{\omega}) d\boldsymbol{\omega} = E[a_n] \int_0^\infty H(\omega, \omega_n) f_1(\omega_n) d\omega_n, \quad (14)$$

where it has been assumed that the coefficients a_n are statistically independent of the natural frequencies and that each coefficient has the same ensemble average value. Equation (14) leads immediately to Eq. (6), noting that the first distribution function f_1 is by definition equal to the modal density ν . Likewise, the mean squared value of the kinetic energy can be expressed initially in the form

$$E[T^2] = \left\{ \sum_{n=m} + \sum_n \sum_{m \neq n} \right\} E[a_n a_m] \int_0^\infty \dots \int_0^\infty H(\omega, \omega_n) H(\omega, \omega_m) p(\boldsymbol{\omega}) d\boldsymbol{\omega}, \quad (15)$$

and the application of Eq. (13) then yields

$$E[T^2] = E[a_n^2] \int_0^\infty [H(\omega, \omega_n)]^2 f_1(\omega_n) d\omega_n + E[a_n a_m] \int_0^\infty \int_0^\infty H(\omega, \omega_n) H(\omega, \omega_m) f_2(\omega_n, \omega_m) d\omega_n d\omega_m. \quad (16)$$

Stratonovich [14] has introduced a further set of functions g_k (referred to as correlation functions by Stratonovich [14], cumulant functions by Lin [15], and with a small change in notation, cluster functions in random matrix theory [3]) which are related to the functions f_k . For the present purposes it is sufficient to note that the function g_2 is defined by the relation

$$f_2(\omega_n, \omega_m) = g_2(\omega_n, \omega_m) + f_1(\omega_n) f_1(\omega_m), \quad (17)$$

so that equations (14), (16) and (17) can be combined to yield the variance of the energy in the form

$$\begin{aligned} \text{Var}[T] = & E[a_n^2] \int_0^\infty [H(\omega, \omega_n)]^2 f_1(\omega_n) d\omega_n + \text{Cov}[a_n, a_m] \left\{ \int_0^\infty H(\omega, \omega_n) f_1(\omega_n) d\omega_n \right\}^2 \dots \\ & + \left\{ \text{Cov}[a_n, a_m] + E[a_n]^2 \right\} \int_0^\infty \int_0^\infty H(\omega, \omega_n) H(\omega, \omega_m) g_2(\omega_n, \omega_m) d\omega_n d\omega_m. \end{aligned} \quad (18)$$

As noted above, the function f_1 is equal to the modal density of the system; if it is assumed that the natural frequencies have GOE statistics, then the function g_2 is available from random matrix theory [3]. Each of the three integrals that appear in Eq. (18) have been evaluated by Langley and Brown [7], and so the details are not repeated here. Having performed these integrals, Eqs. (14) and (18) can be combined to yield the relative variance of the kinetic energy in the form

$$\text{RelVar}[T] = \frac{[\alpha + (\gamma + 1)q(m)]}{\pi m} + \gamma, \quad (19)$$

where m is the modal overlap factor, α and γ are given respectively by Eqs. (9) and (12), and the function $q(m)$ is given by Eq. (8).

Equation (19) represents an extension of the existing result, Eq. (7), to the case where correlations between the coefficients a_n cannot be neglected. The application of Eq. (19) to the simple example considered in Section 2 yields a curve in good agreement with the Gaussian force case shown in Fig. 1, although the presentation of such results is deferred until Section 5.

If the modal overlap factor is greater than around 0.2, then Eq. (19) can be approximated by

$$\text{RelVar}[T] \approx \frac{(\alpha - \gamma - 1)}{\pi m} + \frac{(\gamma + 1)}{(\pi m)^2} + \gamma, \quad (20)$$

which represents a generalisation of Eq. (10). It is clear from this result that the effect of correlations will be particularly important when the modal overlap factor is high: Eq. (10) has a limit of zero for very high modal overlap factor, whereas Eq. (20) tends to γ .

4. Extension to the band-averaged response

In vibro-acoustics the quantity of engineering interest is often the frequency band-averaged response of the system, rather than the response at a single frequency. The band-averaged kinetic energy can be expressed in the form

$$\bar{T}(\omega, \Delta) = (1/\Delta) \int_{\omega-\Delta/2}^{\omega+\Delta/2} T(\omega) d\omega, \quad (21)$$

where Δ is the averaging band-width, centred on the frequency ω . The relative variance of the band averaged energy can be derived by noting that Eqs. (14) and (18) remain valid for this case providing the frequency response function H is replaced by the band-average value $\bar{H}$, defined by

$$\bar{H}(\omega, \omega_n, \Delta) = (1/\Delta) \int_{\omega-\Delta/2}^{\omega+\Delta/2} H(\omega, \omega_n) d\omega. \quad (22)$$

The ensemble averaged energy is unchanged by band-averaging, but the three integrals which appear in Eq. (18) are significantly modified. Langley and Brown [8] have performed these integrals for the case where the modal overlap is not small (say $m > 0.2$), and employing the reported results leads to the expression

$$\text{RelVar}[\bar{T}] = \left(\frac{\alpha - \gamma - 1}{\pi m B^2} \right) \left\{ 2B \tan^{-1} B - \ln(1 + B^2) \right\} + \left[\frac{\gamma + 1}{(\pi m B)^2} \right] \ln(1 + B^2) + \gamma, \quad (23)$$

where

$$B = \Delta / (\omega \eta). \quad (24)$$

It can be noted that Eq. (23) does not reduce to Eq. (19) for $B=0$; rather, it reduces to the approximate result, Eq. (20), due to the fact that the derivation is based on the assumption that the modal overlap is not small. Equation (23) represents an extension of Eq. (26) of reference [8] to include the effect of correlations, and this result is compared with numerical simulations in the following section. It can be noted that in deriving Eq. (23) it has been assumed that the applied forces are at least approximately constant with frequency.

5. Numerical examples

In this section the results for the relative variance given by equations (19) and (23) are compared with numerical simulations for a random plate that is subjected to various forms of point loading. For the case of J point loads, Eq. (5) yields

$$P(\mathbf{x}) = \sum_{r=1}^J F_r \delta(\mathbf{x} - \mathbf{x}_r) \Rightarrow a_n = \sum_{r=1}^J \sum_{s=1}^J F_r F_s^* \phi_n(\mathbf{x}_r) \phi_n(\mathbf{x}_s), \quad (25)$$

where F_r and $\mathbf{x}_r$ are respectively the complex amplitude and the location of the r th point load. From a practical point of view, multiple point loads may arise from equipment attached to the system at J discrete mounts. If the point loads are statistically independent, then it can be shown that the coefficients α and γ that appear in Eqs. (19) and (23) are given by

$$\alpha = \frac{J-1}{J} \left(2 + \frac{|m_{c2}|^2}{m_2^2} \right) + \frac{\alpha_F \alpha_\phi}{J}, \quad \gamma = \frac{\alpha_F - 1}{J}, \quad (26)$$

where

$$m_{c2} = E[F_r^2], \quad m_2 = E[|F_r|^2], \quad \alpha_F = E[|F_r|^4] / m_2^2, \quad \alpha_\phi = E[\phi_n^4] / E[\phi_n^2]^2. \quad (27-30)$$

These equations will be used to determine the parameters of the relative variance equations (Eq. (19) and Eq. (23)) for the different loadings considered.

The geometrical and mechanical properties of the considered plate are the same as those discussed in Section 2. Only bending motion is considered and the frequency range of interest is 50-1000 Hz and there are 575 resonant modes over this range.

In order to provide a benchmark for the proposed approach, an ensemble of random plates has been generated. The Lagrange-Rayleigh-Ritz method [16] has been employed to formulate the equations of motion for a simply supported plate with point masses attached at random locations. As described in Section 2, twenty point masses were considered, each having 1% of the mass of the bare plate. Earlier work has considered the use of random masses to promote the occurrence of GOE natural frequency statistics in plate systems, and the number and size of the masses employed in the present work has previously been found to yield the required modal statistics over a broad frequency range [7,8,17,18]. For each loading case investigated, an ensemble of 200 or 2000 systems (chosen to yield convergent statistics depending on the case investigated) was generated by varying the positions of the point masses. The relative variance obtained for the ensemble of random systems has been compared to that obtained with standard SEA theory and with the proposed approach.

5.1 Gaussian point forces

In this case the plate is excited by a number of uncorrelated complex Gaussian forces applied at fixed locations. Each force has zero mean, and real and imaginary parts are uncorrelated with a variance of 1N^2 . Four different cases have been considered, comprising 1, 7, 100 and 1000 point forces.

The mean and the relative variance of the vibrational energy of the plate as predicted by the benchmark model and standard SEA, Eq. (7), are compared for different numbers of point forces in Fig. 2. It can be seen that the ensemble average value is predicted accurately, but the relative variance is in general underestimated. When the modal overlap is low, the relative variance is reasonably accurate, but very poor results are obtained at high modal overlap. The modal overlap gives a rough indication of the number of modes that contribute to the response at a particular forcing frequency, and it is not surprising that correlations between the modal generalised forces become increasingly important as the modal overlap increases. It can also be observed in Figure 2 that the relative variance prediction is less accurate for a lower number of input forces. Moreover, as the number of forces increases the relative variance prediction becomes highly accurate. This is consistent with Eq. (26), which predicts that the correlation parameter γ is small for a large number of point loads.

Fig. 2: Mean (left) and relative variance (right) of the energy of a plate versus modal overlap considering 1 (black), 7, 100 and 1000 (lightest grey) complex Gaussian input forces, predicted by the benchmark model (thin oscillating lines) and standard SEA (thick dashed lines).

Results obtained using the present theory, Eq. (19), are shown in Fig. 3. It can be observed that the revised equation leads to an accurate prediction of the relative variance, irrespective of the number of point forces considered. This demonstrates that: (i) the failure in the existing SEA theory is due to the neglect of correlations, (ii) these effects are captured accurately by the present theory.

Energy variables can often be assumed (at least approximately) to have a log-normal distribution [7,19]. This assumption is assessed for the present example in Fig. 4, where an empirical probability density function obtained from the benchmark study at each of the points highlighted in Fig. 3 is compared with a lognormal distribution having the same mean and variance. Each row of Fig. 4 shows the result for a fixed number of input forces ($J=1, 7, 100$ and 1000) and each column considers a fixed modal overlap factor ($0.5, 1$ and 3). In general, good agreement between the empirical distribution and the lognormal distribution is obtained.

Fig. 3: Mean (left) and relative variance (right) of the energy of a plate versus modal overlap considering 1 (black), 7, 100 and 1000 (lightest grey) complex Gaussian input forces, predicted by the benchmark model (thin oscillating lines) and the proposed approach (thick lines). The points highlighted in the mean results are those at which energy distributions are displayed in Fig. 4.

Fig. 4: Histograms of the vibrational energy of the plate yielded by the Monte Carlo studies compared to log-normal curves for $J=1, 7, 100$ and 1000 complex Gaussian input forces (rows) at three modal overlap factors $m=0.5, 1$ and 3 (columns).

Clearly, Fig. 4 provides strong support for the log-normal assumption, and on this basis the mean and variance yielded by the present approach have been used to predict response confidence intervals under this assumption. The results obtained for the 5% and 95% confidence levels are shown in Fig. 5 for the four loading cases. Very good agreement with the confidence intervals estimated from the benchmark model is observed in general. One exception is the 5% confidence level for the single point loading; this result is consistent with Fig. 4, where it can be seen that the log-normal distribution is not too accurate at low energy levels for this case. Nonetheless the 95% confidence level is well predicted even for this case.

Fig. 5: 5% and 95% confidence levels yielded by the conditional probability approach (thick dashed-dotted curves) and benchmark Monte Carlo results (thin curves) for the plate excited with $J=1, 7, 100$ and 1000 complex Gaussian input forces.

5.2 Other input force statistics

This section investigates the effect of different input forces statistics, considering forces that are: (i) complex Gaussian, (ii) zero mean real Gaussian, (iii) the square of a zero mean real Gaussian variable, and (iv) deterministic. The values of α_F corresponding to these cases are respectively 2, 3, $35/3$, and 1. Results yielded by the present theory for 1 and 7 input forces are shown in Fig. 6, where a comparison is made with the benchmark model results. In the benchmark model, 200 samples were used for the complex Gaussian and deterministic force cases, and 2000 samples were used for the real Gaussian and squared Gaussian cases. Convergent statistics and good agreement with the present theory is obtained for all cases except for the case of a single squared Gaussian input force, where more samples would be required to obtain an accurate result. As an alternative, when only 1 input force is considered, the result for the squared Gaussian case can be deduced from the result for the deterministic case: the energy in the former case is simply the energy in the latter case multiplied by the square of the random input force. The relative variance is then

$\alpha_F(1 + r_{\text{det}}^2) - 1$ with r_{det}^2 being the relative variance for the deterministic case. It can be seen in Fig. 6 that results obtained using this approach are in excellent agreement with the theoretical prediction.

Fig. 6: Relative variance of the energy of the plate for different type of input forces ($J=1$ black and $J=7$ grey): (i) complex Gaussian (top left); (ii) Real Gaussian (top right); (iii) Real Gaussian Squared (bottom left) and (iv) Deterministic (bottom right) predicted by the benchmark model (thin lines) and proposed SEA approach (thick lines). The thin black line with diamond markers in (iii) represents the results for 1 real squared Gaussian point force obtained from the benchmark case with 1 deterministic point force.

5.3 Band-average variance

Results for the relative variance of the response obtained for $J=1$ and $J=7$ complex Gaussian input forces using frequency-averaging bandwidths of $\Delta=10$ Hz and $\Delta=100$ Hz are shown in Figs 7 and 8, respectively. In each case the random forces are assumed to be constant with frequency, i.e. the same forces are applied at each frequency. It can be noted that the average number of modes in a band is given by the product of B and the modal overlap, being equal to the product of Δ and ν , with ν being the modal density of the plate. When $\Delta = 10$ Hz,

approximately 5 modes are included in the band, and when $\Delta = 100$ Hz about 50 modes are included. As expected from Eq. (23), as the modal overlap increases and/or for the averaging bandwidth, the relative variance tends to γ . The results obtained with the proposed approach are in very good agreement with those yielded by the benchmark model.

Fig. 7: Band average relative variance for $\Delta=10$ Hz using $J=1$ and $J=7$ complex Gaussian forces.

Fig. 8: Band average relative variance for $\Delta=100$ Hz using $J=1$ and $J=7$ complex Gaussian forces

6. Conclusions

The main results of this paper are contained in Eqs. (19) and (23): Eq. (19) gives the relative variance of the energy of the response of a random structural component that is subjected to harmonic excitation, while Eq. (23) gives the corresponding result for the band-averaged energy. These equations represent an extension of previous results [7,8] to allow for the effect of correlations between the modal generalised forces acting on the system, and the extension is particularly important for random loads when there is high modal overlap or a large averaging bandwidth is employed. Conversely, if the forces acting on the system are deterministic then the additional terms vanish and the previous results are recovered. It has also been demonstrated that the predicted relative variance can be used in conjunction with the mean response and the assumption of a log-normal distribution to yield confidence intervals on the response. Although Eqs. (19) and (23) relate to a single structural component, they can be employed to compute the variance of the response of a complex built-up structure by simply updating the relative variance expressions employed in the SEA analysis contained in reference [11]. The validity of Eqs. (19) and (23) rests on the applicability of GOE statistics to the modal properties of the component, and this requires a sufficient degree of randomness in the structural properties, as discussed in references [5-8,18].

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