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Properties of interpolatory quadrature with equidistant nodes on the unit circle*

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Abstract. In this paper we study the computation of the moments associated to rational weight functions given as a power spectrum with known or unknown poles of any order in the interior of the unit disc. A recursive algebraic procedure is derived that computes the moments in a finite number of steps. We also study the associated interpolatory quadrature formulas with equidistant nodes on the unit circle. Explicit expressions are given for the positive quadrature weights in the case of a polynomial weight function. For rational weight functions with simple poles, mostly real or uniformly distributed on a circle in the open unit disc, we also obtain expressions for the quadrature weights and sufficient conditions that guarantee that they are positive. The Poisson kernel is a simple example of a rational weight function, and in a last section we derive an asymptotic expansion of the quadrature error.

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1 Introduction

Let us define $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disc and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle.

*Dedicated to the memory of Pablo González-Vera

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Our general framework considers the following integrals on the unit circle

$$I(f) = \int_{-\pi}^{\pi} f(e^{it})w(t)dt, \tag{1}$$

where $w(t)$ is a given weight function. They are approximated by quadrature formulas of the form

$$Q_n(f) = \sum_{j=1}^{n} c_{j,n} f(z_{j,n}). \tag{2}$$

The coefficients $c_{j,n}$ are called quadrature weights and the $z_{j,n}$ are the quadrature nodes. The most popular way to construct quadrature formulas (2) is to replace the integrand function f in (1) by some interpolating function and then take the integral of this interpolating function as the approximate value of the integral. In this way we get interpolatory quadrature formulas. One has to choose a set of basis functions for the interpolation. The algebraic polynomials are not dense with the sup-norm in the space of continuous functions on $\mathbb{T}$ so that they are not appropiate. Laurent polynomials are dense and this motivates the use of such functions. They are functions of the form $\sum_{-p}^{q} a_j z^j$, where p and q are non-negative integers and $a_j \in \mathbb{C}$. We describe next the interpolatory quadrature formulas based on Laurent polynomials. See e.g. [6] for details.

For a given natural number $n = 1, 2, \ldots$, we consider two arbitrary non-negative integers p_n and q_n such that $p_n + q_n = n - 1$. We denote by Λ_{-p_n,q_n} the subset of Laurent polynomials

$$\Lambda_{-p_n,q_n} = \left\{ \sum_{j=-p_n}^{q_n} a_j z^j, \; a_j \in \mathbb{C} \right\}.$$

We fix a set of n different quadrature nodes $\{z_{j,n}\}_{j=1}^{n}$ on $\mathbb{T}$ and define the interpolating Laurent polynomial

$$L_n(z) = \sum_{j=1}^{n} L_{j,n} f(z_{j,n}) \in \Lambda_{-p_n,q_n},$$

where

$$L_{j,n}(z) = \frac{z_{j,n}^{p_n} N_n(z)}{z^{p_n}(z - z_{j,n}) N_n'(z_{j,n})} \in \Lambda_{-p_n,q_n}, \quad 1 \le j \le n, \tag{3}$$

are the fundamental Lagrange polynomials and $N_n(z) = (z - z_{1,n}) \cdots (z - z_{n,n})$ is the nodal polynomial. Clearly, $L_n(z_{j,n}) = f(z_{j,n}),\ j = 1, 2, \ldots, n$. We then approximate $I(f)$ by the integral of the Lagrange interpolating polynomial, i.e.,

$$I(f) \approx I(L_n) = Q_n(f) = \sum_{j=1}^{n} c_{j,n} f(z_{j,n})$$

where the quadrature weights $c_{j,n}$ are given by the integrals of the fundamental Lagrange polynomials:

$$c_{j,n} = \int_{-\pi}^{\pi} L_{j,n}(e^{it})w(t)dt, \quad j = 1, 2, \ldots, n. \tag{4}$$

The quadrature nodes $\{z_{j,n}\}_{j=1}^{n}$ are often chosen to be uniformly distributed on $\mathbb{T}$. I.e., they form the set of roots of $z^n - k_n = 0,\ k_n \in \mathbb{T}$. For the quadrature formulas $Q_n(f)$ of interpolatory type in $\Lambda_{-p_n,q_n},\ p_n + q_n = n-1$, and with equidistant nodes on the unit circle, it is known (see [18]) that the quadrature weights are given by

$$c_{j,n} = \frac{1}{n} \sum_{k=-p_n}^{q_n} m_k \frac{\omega^{(1-j)k}}{z_{1,n}^k}, \tag{5}$$

or equivalently

$$c_{j,n} = \frac{1}{n}\left(m_0 + \sum_{k=1}^{q_n} m_k \frac{\omega^{(1-j)k}}{z_{1,n}^k} + \sum_{k=q_n+1}^{n-1} m_{k-n} \frac{\omega^{(1-j)k}}{z_{1,n}^k} \right), \quad 1 \le j \le n, \tag{6}$$

where $\omega = e^{2\pi i/n}$, $z_{j,n} = \omega^{j-1} z_{1,n}, \quad 1 \le j \le n$, and m_k are the so-called moments

$$m_k = \int_{-\pi}^{\pi} e^{ikt} w(t)dt,\ k = 0, \pm1, \pm2, \ldots,$$

for the weight function $w(t)$.

Let us consider the expression for $c_{j,n}$ in the most common case of $k_n = 1,\ p_n = q_n$ and $m_k = m_{-k} \in \mathbb{R},\ k = 0, 1, 2, \ldots$. If $k_n = 1$ one naturally takes $z_{1,n} = 1,\ n \ge 1$. Furthermore, if $p_n = q_n$, we can write

$$c_{j,n} = \frac{1}{n}\left(m_0 + \sum_{k=1}^{p_n} m_k \omega^{(1-j)k} + \sum_{k=1}^{p_n} m_{-k} \omega^{-(1-j)k} \right), \quad 1 \le j \le n.$$

Now, if $m_k = m_{-k}$, we get the simplified expression

$$c_{j,n} = \frac{1}{n}\left(m_0 + \sum_{k=1}^{p_n} m_k \left(\omega^{(1-j)k} + \frac{1}{\omega^{(1-j)k}} \right) \right), \quad 1 \le j \le n. \tag{7}$$

Observe from (4) and (6) that when constructing the quadrature weights in this way, it is fundamental to compute the moments as a first step.

Once these moments are known, our main focus is on quadrature formulas

$$Q_n(f) = \sum_{j=1}^{n} c_{j,n} f(z_{j,n}),$$

of interpolatory type in Λ_{-p_n,q_n}, $p_n+q_n=n-1$, and with equidistant quadrature nodes on $\mathbb{T}$. We are interested in efficients expressions to compute the quadrature weights $c_{j,n}$, and in particular positive weights are desirable since that will imply smaller roundoff errors in numerical computation. See e.g. [16, page 72].

So, the outline of the paper is as follows. In Section 2 we are concerned with the computation of the moments m_k associated to rational weight functions $w(t)=1/|h(e^{it})|^2$ where $h(z)$ is an algebraic polynomial with all its roots in $\mathbb{D}$. Clearly, when the roots of $h(z)$ are known, the moments can be calculated by the Residue Theorem. When the roots of $h(z)$ are not known then problems arise. In any of these two cases we give a recursive algebraic procedure to compute the moments in a finite number of steps.

The quadrature weights $c_{j,n}$ for polynomial and rational weight functions are considered in Section 3 where we also derive conditions that guarantee their positivity. For polynomial weight functions, $w(t)=|h(e^{it})|^2$ where $h(z)$ is an algebraic polynomial with real coefficients and all its roots in $\mathbb{D}$ we give an explicit expression for the quadrature weights which implies their positive character if we just take p_n and q_n not smaller than the degree of $h(z)$. For the case $p_n=q_n$ this is generalized to other non-polynomial weight functions $w(t)$ having real moments satisfying $m_k=m_{-k}$ and associated Laurent polynomial $P_n(z)=m_0+\sum_{k=1}^{p_n} m_k(z^k+1/z^k)$ positive on the unit circle. For rational weight functions $w(t)=1/|h(e^{it})|^2$ where $h(z)=(z-\alpha_1)\cdots(z-\alpha_N)$, is an algebraic polynomial with simple roots α_ℓ in $\mathbb{D}\setminus\{0\}$ we give computable expressions of the quadrature weights and sufficient conditions on the poles α_ℓ to assure positive quadrature weights.

Finally in Section 4 we consider the Poisson kernel as a particular case of a rational weight. An asymptotic expansion for the quadrature error is obtained, suitable for a modified Romberg type quadrature technique. The integrand f is assumed to satisfy a regularity condition to ensure the pointwise convergence of its Fourier expansion.

2 The computation of moments for rational weights

2.1 The Åström algorithm

We consider the computation of the moments

$$(8)\qquad m_k=\int_{-\pi}^{\pi}\frac{e^{ikt}}{|h(e^{it})|^2}dt,\quad k=0,\pm1,\pm2,\ldots,$$

of the rational weight function $w(t)=1/|h(e^{it})|^2$ where $h(z)=h_0z^N+h_1z^{N-1}+\cdots+h_N$ is a polynomial with real coefficients of degree N, $h_0\neq0$. The roots of $h(z)$ may be known or unknown and of any order but it is assumed that they are all in $\mathbb{D}$. To analyze if an algebraic polynomial has all its roots inside the unit disk one can use the Schur-Cohn criterion, see e.g. [20] or the Jury test [14] or the Bistritz test [3, 4, 5]. To compute the moments we propose a recursive procedure based on an existing algorithm due to Åström. His algorithm computes integrals of the form

$$(9)\qquad I=\frac{1}{2\pi i}\int_T\frac{B(z)B(1/z)}{A(z)A(1/z)}\frac{dz}{z}$$

where $A(z)$ and $B(z)$ are polynomials of degree N with real coefficients

$$A(z) = a_0 z^N + a_1 z^{N-1} + \cdots + a_N,$$
$$B(z) = b_0 z^N + b_1 z^{N-1} + \cdots + b_N.$$

These integrals occur in control problems, see e.g. [2, 23]. The Åström algorithm computes these integrals by a recursion as follows. Let $p^*(z) = z^n p(1/z)$ for any polynomial of degree n and define $A_N(z) = A(z)$ and $B_N(z) = B(z)$. Then sequences of polynomials

$$A_k(z) = a_0^{(k)} z^k + a_1^{(k)} z^{k-1} + \cdots + a_k^{(k)},$$
$$B_k(z) = b_0^{(k)} z^k + b_1^{(k)} z^{k-1} + \cdots + b_k^{(k)},$$

of decreasing degree k are defined recursively by

$$A_{k-1}(z) = (1/z)\left(A_k(z) - \alpha_k A_k^*(z)\right), \quad \alpha_k = a_k^{(k)}/a_0^{(k)},$$
$$B_{k-1}(z) = (1/z)\left(B_k(z) - \beta_k A_k^*(z)\right), \quad \beta_k = b_k^{(k)}/a_0^{(k)}.$$

The following Theorem is essentially the Schur-Cohn test interpretation of the recursion.

Theorem 1 *(See [2]) The polynomial $A(z)$ has all its zeros inside the unit circle if and only if $a_0^{(k)} > 0$ for all k.*

However, Åström noted that the recursion can also be used to get a recursion for the following integrals

$$I_k = \frac{1}{2\pi i} \int_T \frac{B_k(z) B_k(1/z)}{A_k(z) A_k(1/z)} \frac{dz}{z},$$

since indeed starting with $I = I_N$ we have

$$(1 - \alpha_k^2) I_{k-1} = I_k - \beta_k^2, \quad k = N, N-1, \ldots, 1.$$

This recursively relates I_0 and the yet unknown value I_N. But direct computation gives $I_0 = \beta_0^2$, and this is obtained from the descending recursion for the A_k and B_k since $\beta_0 = b_0^{(0)}/a_0^{(0)}$. Hence, knowing I_0, we can run the recursion for the I_k in ascending order and eventually get $I = I_N$. For more details see [2].

Going back to the problem of computing the moments m_k given by (8) we observe that the imaginary part is

$$\mathrm{Im}(m_{-k}) = -\mathrm{Im}(m_k) = -\int_{-\pi}^{\pi} \frac{\sin kt}{|h(e^{it})|^2} dt, \quad k = 0, 1, 2, \ldots.$$

The denominator $|h(e^{it})|^2$ is a cosine trigonometric polynomial since $h(z)$ has real coefficients. Thus $\mathrm{Im}(m_k)$ is the integral of an odd function $\sin kt/|h(e^{it})|^2$ on the interval $(-\pi, \pi)$ and therefore $\mathrm{Im}(m_k) = 0,\ k = 0, 1, 2, \ldots$. We conclude that $m_{-k} = m_k\ (\in \mathbb{R})$, and that it is sufficient to compute m_k for $k = 0, 1, 2, \ldots$. Based on Åström's algorithm we propose the following Procedure to compute the moments m_k for $k = 0, 1, 2, \ldots$.

Procedure 1

1. To compute m_0 take $A(z) = h(z)$ and $B(z) = z^N$ in (9). For this selection observe that $I = m_0/(2\pi)$. We proceed then by computing the corresponding integral I by the Åström algorithm. Once I is computed we get m_0 from $m_0 = 2\pi I$.

2. To compute m_k for $k = 1, 2, \ldots, N$, take $A(z) = h(z)$ and $B(z) = z^N + z^{N-k}$ in (9). Observe that $I = (m_0 + m_k)/\pi$. Compute the corresponding integral I by the Åström algorithm. Then $m_k = \pi I - m_0$.

3. To compute m_k for $k = N+1, N+2, \ldots$, take $A(z) = z^{k-N}h(z)$ and $B(z) = z^k + 1$ in (9). For this selection one obtains $I = (m_0 + m_k)/\pi$. Compute the corresponding integral I by the Åström algorithm. Then $m_k = \pi I - m_0$.

Example 1 Let us consider the computation of the moments

$$m_k = \int_{-\pi}^{\pi} \frac{e^{ikt}}{|h(e^{it})|^2} dt, \ k = 0, 1, 2, 3, 4,$$

where $h(z) = z^2 - 3z/4 + 1/8$.

In order to compute m_0 we consider $A(z) = h(z)$ and $B(z) = z^2$. Applying the Åström algorithm for these polynomials gives the following computations.

$$\begin{aligned}
&A_2(z) = a_0^{(2)}z^2 + a_1^{(2)}z + a_2^{(2)} = z^2 - \tfrac{3}{4}z + \tfrac{1}{8} \Rightarrow \alpha_2 = \tfrac{a_2^{(2)}}{a_0^{(2)}} = \tfrac{1/8}{1} = \tfrac{1}{8} \\
&B_2(z) = b_0^{(2)}z^2 + b_1^{(2)}z + b_2^{(2)} = z^2 + 0z + 0 \Rightarrow \beta_2 = \tfrac{b_2^{(2)}}{a_0^{(2)}} = \tfrac{0}{1} = 0 \\
&A_1(z) = \tfrac{1}{z}[A_2(z) - \alpha_2 A_2^*(z)] = a_0^{(1)}z + a_1^{(1)} = \tfrac{63}{64}z - \tfrac{21}{32} \Rightarrow \alpha_1 = \tfrac{a_1^{(1)}}{a_0^{(1)}} = -\tfrac{2}{3} \\
&B_1(z) = \tfrac{1}{z}[B_2(z) - \beta_2 A_2^*(z)] = z + 0 \Rightarrow \beta_1 = \tfrac{b_1^{(1)}}{a_0^{(1)}} = \tfrac{0}{63/64} = 0 \\
&A_0(z) = \tfrac{1}{z}[A_1(z) - \alpha_1 A_1^*(z)] = a_0^{(0)} = \tfrac{35}{64} \Rightarrow \alpha_0 = \tfrac{a_0^{(0)}}{a_0^{(0)}} = 1 \\
&B_0(z) = \tfrac{1}{z}[A_1(z) - \beta_1 A_1^*(z)] = b_0^{(0)} = 1 \Rightarrow \beta_0 = \tfrac{b_0^{(0)}}{a_0^{(0)}} = \tfrac{1}{35/64} = \tfrac{64}{35} \\
&I_0 = \beta_0^2 = (\tfrac{64}{35})^2 \Rightarrow I_1 = (1 - (\tfrac{2}{3})^2)I_0 \Rightarrow I_2 = (1 - (\tfrac{1}{8})^2)I_1 = \tfrac{64}{35}
\end{aligned}$$

and this eventually gives $m_0 = 2\pi I_0 = \frac{128\pi}{35}$.

In a similar way we can compute the other moments. For m_1 start with $A(z) = h(z)$ and $B(z) = z^2 + z$ giving $I_2 = \frac{128}{21}$ and thus $m_1 = \pi I_2 - m_0 = \frac{256\pi}{105}$.

For m_2 use $A(z) = h(z)$ and $B(z) = z^2 + 1$ to get $I_2 = \frac{176}{35}$ and $m_2 = \pi I_2 - m_0 = \frac{48\pi}{35}$.

For m_3, we need $A(z) = zh(z)$ and $B(z) = z^3 + 1$ to find $I_3 = \frac{92}{21}$ and $m_3 = \pi I_3 - m_0 = \frac{76\pi}{105}$.

Finally for m_4, $A(z) = z^2h(z)$, $B(z) = z^4 + 1$, $I_4 = \frac{141}{35}$ and $m_4 = \pi I_4 - m_0 = \frac{13\pi}{35}$.

The obtained values of m_0, m_1, m_2, m_3 and m_4 are the ones that one can obtain by the Residue Theorem in this elementary Example

$$m_k = \int_{-\pi}^{\pi} \frac{e^{ikt}}{|h(e^{it})|^2} dt = \frac{1}{i} \int_{\mathbb{T}} \frac{z^{k-1}}{|h(z)|^2} dz, \ k = 0, 1, 2 \ldots,$$

and using $h(z) = (z - 1/4)(z - 1/2)$.

The advantage of the Åström precedure is that we never have to know or compute the zeros of $h(z)$.

2.2 Application to the calculation of integrals on the unit circle

We can also apply Procedure 1 to compute the integral $\int_{-\pi}^{\pi} q(e^{it})/p(e^{it})dt$ over the unit circle of any rational function $q(z)/p(z)$ provided that the polynomial $p(z)$ has real coefficients and all its roots in $\mathbb{D}$. Indeed, let us rewrite the integral in the form

$$\int_{-\pi}^{\pi} \frac{q(e^{it})}{p(e^{it})}dt = \int_{-\pi}^{\pi} \frac{q(e^{it})p(e^{-it})}{|p(e^{it})|^2}dt.$$

We can decompose it into a linear combination of moments determined by the product $q(z)p(1/z)$. After that, apply Procedure 1 to compute the necessary moments $\int_{-\pi}^{\pi} e^{ikt}/|p(e^{it})|^2dt$. Finally calculate the sum determined by the linear combination.

If the polynomial $p(z) = p_0z^N + p_1z^{N-1} + \cdots + p_N$ has all its roots in $\mathbb{D}$ but it has complex coefficients one can still compute the integral $\int_{-\pi}^{\pi} q(e^{it})/p(e^{it})dt$ by means of Procedure 1. Indeed, let us consider the polynomial $\overline{p}(z) = \overline{p_0}z^N + \overline{p_1}z^{N-1} + \cdots + \overline{p_N}$. The product $p(z)\overline{p}(z)$ is given by

$$p(z)\overline{p}(z) = \sum_{k=0}^{2N} c_k z^{2N-k} \quad \text{with} \quad c_k = \sum_{\substack{0\le \ell,j\le N,\\ \ell+j=k}} p_\ell \overline{p_j}, \quad \text{which are real values.}$$

If $p(z)$ has all its roots in $\mathbb{D}$ then $p(z)\overline{p}(z)$ has also all its roots in $\mathbb{D}$. So, let us rewrite the integral to be calculated in the form

$$\int_{-\pi}^{\pi} \frac{q(e^{it})}{p(e^{it})}dt = \int_{-\pi}^{\pi} \frac{q(e^{it})\overline{p}(e^{it})\overline{p}(e^{-it})p(e^{-it})}{|p(e^{it})\overline{p}(e^{it})|^2}dt.$$

Now one can proceed similarly as in the previous case of real coefficients.

3 Positive quadrature weights

3.1 Polynomial weight function

Let us consider weight functions $w(t) = P(t)$ where

$$P(t) = a_0 + a_1 \cos t + \cdots + a_N \cos Nt \ge 0, \quad a_\ell \in \mathbb{R}, \quad \ell = 0, 1, \ldots, N, \tag{10}$$

is a non-negative cosine trigonometric polynomial of degree N. By a corollary of Theorem 1.2.2 in [21], this is equivalent with another formulation which is perhaps more frequently encountered in the literature where one considers weight functions $P(t) = |h(e^{it})|^2$ where $h(z)$ is an algebraic polynomial of degree N with real coefficients. Assuming $h(z)$ has all its roots in $\mathbb{D}$ and $h(0) > 0$ makes $h(z)$ unique.

Theorem 2 *Consider integrals* $I(f) = \int_{-\pi}^{\pi} f(e^{it})w(t)dt$, *where* $w(t) = P(t)$ *is given by (10) and the quadrature formulas* $Q_n(f) = \sum_{j=1}^{n} c_{j,n} f(z_{j,n})$, *of interpolatory type in* Λ_{-p_n,q_n}, $p_n + q_n = n-1$, *with quadrature nodes* $z_{j,n} = e^{it_{j,n}}$, *the roots of* $z^n - k_n = 0$, $k_n \in \mathbb{T}$. *If both parameters* $p_n, q_n \geq N$, *then the quadrature weights are non-negative and they are given by*

$$c_{j,n} = \frac{2\pi}{n} P(t_{j,n}) \geq 0, \quad j = 1, 2, \ldots, n, \quad n = 1, 2, \ldots.$$

Proof. The moments $m_k = \int_{-\pi}^{\pi} e^{ikt} P(t)dt$, are given by $m_0 = 2\pi a_0$, $m_k = \pi a_{|k|}$, $1 \leq |k| \leq N$, and $m_k = 0$ for $|k| \geq N+1$. Denote the roots $z_{j,n}$ of $z^n - k_n = 0$, $|k_n| = 1$, as $z_{j,n} = e^{it_{j,n}} = \omega^{j-1} z_{1,n}$, where $\omega = e^{2\pi i/n}$. Plug these values into (5), then for $p_n, q_n \geq N$, we can write

$$c_{j,n} = \frac{1}{n} \left(2\pi a_0 + \pi \sum_{k=1}^{N} a_k \left(z_{j,n}^k + \frac{1}{z_{j,n}^k} \right) \right) = \frac{2\pi}{n} P(t_{j,n}).$$

The proof follows. □

Remark 1 Note that this is also a consequence of Szegő quadrature. The nodes $z_{j,n}$, equidistributed on $\mathbb{T}$ and the weights $\frac{2\pi}{n}$ are just the nodes and weights of the Szegő quadrature formula for the Lebesgue measure, and hence it is exact in the space of all Laurent polynomials $\Lambda_{-(n-1),n-1} = \Lambda_{-(p_n+q_n),p_n+q_n}$. Thus, if $p_n, q_n \geq N$, it is at least exact in $\Lambda_{-2N,2N}$ and therefore will integrate exactly all the weights as given by equation (4)

$$c_{j,n} = \int_{-\pi}^{\pi} L_{j,n}(e^{it})w(t)dt = \frac{2\pi}{n} \sum_{k=1}^{n} L_{j,n}(e^{it_{k,n}}) P(t_{k,n}) = \frac{2\pi}{n} \delta_{j,k} P(t_{k,n}) = \frac{2\pi}{n} P(t_{j,n}).$$

In fact

$$\sum_{j=1}^{n} c_{j,n} f(z_{j,n}) = \frac{2\pi}{n} \sum_{j=1}^{n} P(t_{j,n}) f(e^{it_{j,n}})$$

is just the Szegő quadrature formula to approximate the integral $\int_{-\pi}^{\pi} g(t)dt$ where $g(t) = P(t)f(e^{it})$.

Even though the Szegő quadrature for $f(e^{it})P(t)$ with weight 1 gives the same value as the interpolatory quadrature for $f(e^{it})$ and weight $P(t)$, the analysis needed for the error estimation is different. Such analysis requires usually considering the behaviour in the complex plane of the integrand. As we have said just before Theorem 2, $P(t)$ is frequently given as the restriction for $z \in \mathbb{T}$ of the complex function $|h_N(z)|^2$ where $h_N(z)$ is an algebraic polynomial of degree N with real coefficients and all its roots in $\mathbb{D}$, that is, $P(t) = |h_N(e^{it})|^2$. Thus in the case of a Szegő formula for weight 1 and integrand $g(t) = P(t)f(e^{it}) = |h_N(e^{it})|^2 f(e^{it})$ we have to deal in the error analysis with an integrand $|h_N(z))|^2 f(z)$ where $|h_N(z))|^2$ is not analytic everywhere. The advantage of our interpolatory quadrature approach is that the non-analytic part $|h_N(z)|^2$ is moved to the weight function $w(t)$ while retaining as integrand only $f(z)$. This simplifies the estimation of the quadrature error for smooth functions $f(z)$.

Example 2 Let us take from [11] the weight functions $w(t) = P_i(t),\ i = 1,2,3$, where $P_1(t) = (1+\cos t)/(2\pi),\ P_2(t) = (1-\cos t)/(2\pi)$ and $P_3(t) = (1-\cos 2t)/(4\pi)$. Consider the construction of quadrature formulas of interpolatory type in $\Lambda_{-p_n,q_n},\ p_n+q_n = n-1$, and with quadrature nodes $z_{j,n} = e^{it_{j,n}} = \omega^{j-1}z_{1,n} = e^{2\pi i(j-1)/n}e^{i\phi/n}$ the roots of $z^n - k_n = 0,\ k_n = e^{i\phi}$, where then $t_{j,n} = (2\pi(j-1)+\phi)/n$. The corresponding quadrature weights $c_{j,n}$ were obtained in [11] from formula (4). They obtained for all $j = 1,2,\ldots,n$, that

$$\text{for } P_1(t),\ c_{j,n} = \begin{cases} (2z_{j,n}+1)/(2nz_{j,n}), \text{ if } p_n = 0, \\ (1+\cos(t_{j,n}))/n = (2\pi/n)P_1(t_{j,n}), \text{ if } 1 \le p_n \le n-2, \\ (2+z_{j,n})/(2n), \text{ if } p_n = n-1, \end{cases}$$

$$\text{for } P_2(t),\ c_{j,n} = \begin{cases} (2z_{j,n}-1)/(2nz_{j,n}), \text{ if } p_n = 0, \\ (1-\cos(t_{j,n}))/n = (2\pi/n)P_2(t_{j,n}), \text{ if } 1 \le p_n \le n-2, \\ (2-z_{j,n})/(2n), \text{ if } p_n = n-1, \end{cases}$$

$$\text{and for } P_3(t),\ c_{j,n} = \begin{cases} (2z_{j,n}^2-1)/(4nz_{j,n}^2), \text{ if } p_n = 0,1, \\ (1-\cos(2t_{j,n}))/(2n) = (2\pi/n)P_3(t_{j,n}), \text{ if } 2 \le p_n \le n-3, \\ (2-z_{j,n}^2)/(4n), \text{ if } p_n = n-2, n-1. \end{cases}$$

Clearly, the cosine trigonometric polynomials $P_i,\ i = 1,2,3$ are non-negative. The polynomials $P_1(t)$ and $P_2(t)$ are of degree $N = 1$ and $P_3(t)$ is of degree $N = 2$. We note that for the three weight functions and for both parameters $p_n, q_n \ge N$, Theorem 2 reproduces the value of the quadrature weights obtained in [11].

Unless stated otherwise, we shall in the sequel restrict the quadrature formulas to the case where $p_n = q_n$, hence $2p_n = n-1$ and take as nodes the roots of $z^n - 1 = 0$, thus $z_{j,n} = \omega^{j-1},\ \omega = e^{2\pi i/n},\ j = 1,\ldots,n$.
Theorem 2 can be generalized to other non-polynomial weight functions.

Theorem 3 *Consider quadrature formulas $Q_n(f) = \sum_{j=1}^n c_{j,n}f(z_{j,n})$ of interpolatory type in Λ_{-p_n,p_n}, with $2p_n = n-1$ and quadrature nodes $z_{j,n} = \omega^{j-1},\ \omega = e^{2\pi i/n}$. These approximate integrals $I(f) = \int_{-\pi}^{\pi} f(e^{it})w(t)dt$ where $w(t)$ is a weight function having real moments satisfying $m_k = m_{-k},\ k = 0,1,2,\ldots$. Define $P_n(z) = m_0 + \sum_{k=1}^{p_n} m_k(z^k + 1/z^k)$. Then the quadrature weights are given by $c_{j,n} = \frac{1}{n}P_n(\omega^{j-1}),\ j = 1,2,\ldots,n$ and if $P_n(z) > 0,\ z \in \mathbb{T}$, then these quadrature weights are positive.*

Proof. From (7) we can write for each j, $1 \le j \le n$

$$c_{j,n} = \frac{1}{n}\left(m_0 + \sum_{k=1}^{p_n} m_k\left(\omega^{(1-j)k} + \frac{1}{\omega^{(1-j)k}}\right)\right) = \frac{1}{n}P_n(\omega^{1-j}) = \frac{1}{n}P_n(\omega^{j-1}).$$

The proof follows. □

Observe that Theorem 3 holds for any weight (polynomial or not) that satisfies the conditions of the theorem. However, if $w(t) = P(t)$ is a weight function as considered in Theorem 2 with $N = n$, then by the relation between the coefficients a_k and the moments

m_k, given in its proof, it follows that $P(t) = \frac{1}{2\pi} P_n(e^{it})$ with $P_n(z)$ the polynomial introduced in Theorem 3. Thus Theorem 2 is reproduced as a particular case of Theorem 3.

Example 3 Consider the Rogers-Szegő weight function $w(t,q)$ defined by

$$w(t,q) = \frac{1}{\sqrt{2\pi \log(1/q)}} \sum_{j\in\mathbb{Z}} \exp\left(-\frac{(t-2\pi j)^2}{2\log(1/q)}\right), \ 0 < q < 1.$$

In [10] the authors studied the estimation of integrals $\int_{-\pi}^{\pi} f(e^{it}) w(t,q) dt$ by means of Szegő's quadrature and quadrature of interpolatory type. This weight appeared as a so-called wrapped Gaussian measure after applying a transformation for the integral $\int_{-\infty}^{\infty} f(x)\sigma_\zeta(x)dx$, where $\sigma_\zeta(x) = \sqrt{2/\pi} e^{-\zeta x^2}$, $\zeta > 0$.

Our interest in the Rogers-Szegő weight function is to show how Theorem 3 can be applied to construct quadrature formulas with positive weights. The moments associated with the Rogers-Szegő weight function are given by, see [9],

$$m_k = \int_{-\pi}^{\pi} e^{ik\theta} w(t,q) dt = q^{k^2/2}, \ k \in \mathbb{Z}.$$

Then the polynomial $P_n(z)$ introduced in Theorem 3 and associated with $w(t,q)$ is given by

$$(11) \qquad P_n(z) = 1 + \sum_{k=1}^{p_n} q^{k^2/2}(z^k + 1/z^k).$$

Given $q \in (0,1)$, the problem is to find out how large p_n can be chosen and still have $P_n(z) > 0$ for $z \in \mathbb{T}$. Clearly the simple bound

$$P_n(z) \geq 1 - 2\sum_{k=1}^{p_n} q^{k^2/2}, \ z \in \mathbb{T}$$

is not very useful. A much better and satisfactory condition on p_n to assure the positive character of $P_n(z)$, $z \in \mathbb{T}$, can be obtained by means of the Jacobi triple product identity, given by (see e.g. [13, 1])

$$\sum_{k=-\infty}^{\infty} x^{k^2} z^{2k} = \prod_{k=1}^{\infty} (1 - x^{2k})(1 + x^{2k-1}z^2)(1 + \frac{x^{2k-1}}{z^2})$$

where x and z are complex numbers, $|x| < 1$ and $z \neq 0$. The series treated next are absolutely convergent provided that additionally $|x| < |z| < |x|^{-1}$. For the particular values $x = q^{1/2}$ and $z = e^{it/2}$, it results

$$\sum_{k=-\infty}^{\infty} q^{k^2/2} e^{ikt} = \prod_{k=1}^{\infty} (1 - q^k)(1 + 2q^{(2k-1)/2}\cos t + q^{2k-1})$$

from where

$$(12)\qquad 1+\sum_{k=1}^{p_n} q^{k^2/2}(e^{ikt}+\frac{1}{e^{ikt}})+\sum_{k,|k|\geq p_n+1} q^{k^2/2}e^{ikt}\geq \prod_{k=1}^{\infty}(1-q^k)(1-q^{(2k-1)/2})^2.$$

For $p_n>0$ it holds that

$$(13)\qquad \sum_{k,|k|\geq p_n+1} q^{k^2/2}e^{ikt}=2\sum_{k\geq p_n+1} q^{k^2/2}\cos(kt)\leq 2\sum_{k\geq p_n+1} q^{k^2/2}\leq 2\sum_{k\geq p_n+1} q^{k}=\frac{2q^{p_n+1}}{1-q}.$$

Then from (11)-(13)

$$(14)\qquad P_n(e^{it})=1+\sum_{k=1}^{p_n} q^{k^2/2}(e^{ikt}+\frac{1}{e^{ikt}})\geq \prod_{k=1}^{\infty}(1-q^k)(1-q^{(2k-1)/2})^2-\frac{2q^{p_n+1}}{1-q}.$$

By the Jacobi triple product identity for $x=q^{1/2}$ and $z=i$ we can write

$$(15)\qquad \Pi_\infty:=\prod_{k=1}^{\infty}(1-q^k)(1-q^{(2k-1)/2})^2=\sum_{k=-\infty}^{\infty}(-1)^k q^{k^2/2}=1+2\sum_{k=1}^{\infty}(-1)^k q^{k^2/2}=:S_\infty.$$

It is known, see e.g. [15, p. 220] that if for every k, $a_k\geq 0$ then the product $\prod(1-a_k)$ is convergent if and only if $\sum a_k$ converges. Since the series $\sum_{k=1}^{\infty} 1-(1-q^k)(1-q^{(2k-1)/2})^2$ clearly converges, also the infinite product Π_∞ and thus the infinite sum S_∞ will converge. Since none of the factors of the product vanishes, the result will be positive as long as q is bounded away from 1. Replacing (15) into (14)

$$P_n(e^{it})\geq 1+2\sum_{k=1}^{\infty}(-1)^k q^{k^2/2}-\frac{2q^{p_n+1}}{1-q}.$$

We are now in a key position to say that for given q we can find then N such that the odd partial sum $S_{2N+1}=1+2\sum_{k=1}^{2N+1}(-1)^k q^{k^2/2}$ is positive. Indeed, if all odd partial sums S_{2N+1} were negative, them $S_\infty=\lim_{N\to\infty}S_{2N+1}\leq 0$. But since $S_\infty=\Pi_\infty$, a product of positive factors, this is a contradiction. Finally, observe that when one takes odd partial sums $S_{2N+1}>0$ then the remainder $\sum_{k=2(N+1)}^{\infty}(-1)^k q^{k^2/2}$ is also positive. Consider then N such that $S_{2N+1}>0$. Thus

$$P_n(e^{it})\geq S_{2N+1}-\frac{2q^{p_n+1}}{1-q}+\sum_{k=2(N+1)}^{\infty}(-1)^k q^{k^2/2}.$$

Now, if one takes a natural number $p_n>0$ such that $S_{2N+1}-2q^{p_n+1}/(1-q)>0$, that is, $p_n>-1+(\ln((1-q)S_{2N+1}/2))/\ln q$ then the polynomial $P_n(z)$ will be positive on the unit circle.

We can then construct quadrature formulas $Q_n(f)=\sum_{j=1}^{n} c_{j,n}f(z_{j,n})$ as in Theorem 3 that are of interpolatory type in Λ_{-p_n,p_n}, $2p_n=n-1$ with quadrature nodes $z_{j,n}=\omega^{j-1}$, $\omega=$

$e^{2\pi i/n}$, to approximate integrals on the unit circle $I(f) = \int_{-\pi}^{\pi} f(e^{it})w(t,q)dt$ with the Rogers-Szegő weight function and that will have positive quadrature weights $c_{j,n}$. Once the minimum N is computed such that $S_{2N+1} > 0$, one has to consider the minimum natural value $p_n > 0$ such that $S_{2N+1} - 2q^{p_n+1}/(1-q) > 0$. From Theorem 3 if follows that then the quadrature weights are positive and that they are given by $c_{j,n} = \frac{1}{n}P_n(\omega^{j-1}) > 0,\ j = 1, 2, \ldots, n$ for all $n \geq 2p_n + 1$.

In Table 1 we show for some example values of the parameter q, the minimum value of N for which $S_{2N+1} > 0$, the value of this sum S_{2N+1} and the minimum value p_n necessary to achieve $S_{2N+1} - 2q^{p_n+1}/(1-q) > 0$.

q	N	S_{2N+1}	p_n
0.1	0	0.3675	1
0.2	0	0.1056	1
0.3	1	$0.7568 \cdot 10^{-1}$	3
0.4	1	$0.2271 \cdot 10^{-1}$	5
0.5	2	$0.4865 \cdot 10^{-2}$	9
0.6	2	$0.2513 \cdot 10^{-3}$	19
0.7	4	$0.8192 \cdot 10^{-5}$	38
0.8	6	$0.2027 \cdot 10^{-8}$	100
0.9	14	$0.6547 \cdot 10^{-19}$	447

Table 1: Some examples of the proposed method to calculate p_n in order to construct quadrature formulas as in Theorem 3 with positive quadrature weights for the Rogers-Szegő weight function. For a number of nodes $n \geq 2p_n + 1$ the quadrature weights are all positive.

Note that for q close to 1, the infinite product, and thus also the infinite sum will be close to zero. Therefore the first value of N for which S_{2N+1} is positive may be very large. Since for $q \approx 1$ many terms in this sum are oscillating between approximately 1 and -1, and thus causing an accumulation of rounding errors, finding this N may well be out of reach of the machine precision of the computer. In that case, computing S_∞ as the infinite product $\Pi_\infty = \prod_{k=1}^{\infty}(1-q^k)(1-q^{(2k-1)/2})^2$ will be numerically much more reliable. There is obviously also a numerical limitation to the number of factors that can be multiplied, and a finite product will not give a mathematically proven lower bound for p_n. Nevertheless for q close to 1, computing p_n such that $p_n > -1 + (\ln((1-q)\Pi_M/2))/\ln q$ where Π_M multiplies only the first M factors of Π_∞ will give a good estimate if M is large enough.

Remark 2 Observe from (6) that the set of quadrature weights $\{c_{1,n}, c_{2,n}, \ldots, c_{n,n}\}$ of interpolatory quadrature formulas with uniformly distributed nodes on the unit circle are the finite Fourier transform of the data $\{m_0, m_1\omega/z_{1,n}, \ldots, m_q(\omega/z_{1,n})^q, m_{-p}(\omega/z_{1,n})^{q+1}, \ldots, m_{-1}(\omega/z_{1,n})^{n-1}\}$, see [18]. Thus, one can apply the Fast Fourier Transform (FFT) algorithm to compute them successfully. We also observe that the expression $c_j = \frac{1}{n}P_n(w^{1-j})$ found in Theorem 3 does not provide us with a new or better way to compute the moments than the FFT but we wanted to point out the relation between the positivity of the quadrature weights and the positivity on the unit circle of the polynomial $P_n(z)$ defined in this Theorem.

Nevertheless, for some weight functions such as the rational weight functions treated in the next Subsection 3.2, it is possible to get a closed form expression for the sum (6)-(7) that defines the quadrature weights. In these cases direct evaluation of this expression is simpler than computing the quadrature weights by the FFT. A second advantage of this calculated expression for the quadrature weights is the possibility of study its positive character.

3.2 Rational weight function

We next consider weight functions $w(t) = 1/|h(e^{it})|^2$ where

$$h(z) = (z-\alpha_1)\cdots(z-\alpha_N),\ \alpha_\ell \in \mathbb{D}\setminus\{0\},\ \alpha_\ell \neq \alpha_j \text{ if } \ell \neq j,\ \ell, j = 1, 2, \ldots, N,$$

is an algebraic polynomial with simple roots in $\mathbb{D}\setminus\{0\}$. Clearly, it is not of interest to consider a root at $z = 0$. To compute the associated moments

$$m_k = \int_{-\pi}^{\pi} \frac{e^{ikt}}{|h(e^{it})|^2} dt = \frac{1}{i}\int_{\mathbb{T}} \frac{z^{k-1}}{|h(z)|^2} dz = \frac{1}{i}\int_{\mathbb{T}} \frac{z^{N+k-1}}{h(z)h^*(z)} dz,\ k = 0, \pm 1, \pm 2, \ldots$$

where $h^*(z) = z^N \overline{h}(1/z) = \prod_{\ell=1}^{N}(1-\overline{\alpha_\ell} z),\ z \in \mathbb{C}$, we make use of the Residue Theorem.

For $k \geq 0$, the integrand has only simple poles in $\mathbb{D}$ given by the α_ℓ, $\ell = 1, \ldots, N$ because $h^*(z)$ has all its zeros outside the closed unit disk and $N + k - 1 \geq 0$. Thus

$$m_k = 2\pi \sum_{\ell=1}^{N} \operatorname{Res}\left[\frac{z^{N+k-1}}{h(z)h^*(z)}\right]_{\alpha_\ell} = 2\pi \sum_{\ell=1}^{N} \frac{\alpha_\ell^{N+k-1}}{h'(\alpha_\ell)h^*(\alpha_\ell)}, \quad k = 0, 1, 2, \ldots.$$

Finally, for negative values of k observe that $w(t) = |h(e^{it})|^{-2}$ is real, and thus

$$m_{-k} = \int_{-\pi}^{\pi} e^{-ikt} w(t) dt = \overline{\int_{-\pi}^{\pi} e^{ikt} w(t) dt} = \overline{m_k}.$$

Note that $m_0 = \int_{-\pi}^{\pi} w(t)dt$ is real (and positive).

We summarize our formulas for the moments in the following Theorem.

Theorem 4 *Let $h(z) = (z-\alpha_1)\cdots(z-\alpha_N)$, be a polynomial with simple roots in $\mathbb{D}\setminus\{0\}$. Then the moments*

$$m_k = \int_{-\pi}^{\pi} \frac{e^{ikt}}{|h(e^{it})|^2} dt,\ k = 0, \pm 1, \pm 2, \ldots,$$

are given by

$$m_k = 2\pi \sum_{\ell=1}^{N} \frac{\alpha_\ell^{N+k-1}}{h'(\alpha_\ell)h^*(\alpha_\ell)},\ m_{-k} = \overline{m_k},\ k = 0, 1, 2, \ldots.$$

The following Theorem about the positivity of the moments associated to a rational modification of a weight function will be of interest later.

Theorem 5 *Let $w(t)$ be a weight function with positive moments $\int_{-\pi}^{\pi} e^{ikt}w(t)dt \geq 0$, $k = 0, \pm 1, \pm 2, \ldots$. Then, if $\alpha \in (0,1)$, also the moments for the weight function $w(t)/|e^{it}-\alpha|^2$ will be positive, i.e.,*

$$m_k = \int_{-\pi}^{\pi} e^{ikt} \frac{w(t)}{|e^{it}-\alpha|^2} dt \geq 0, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Proof.

$$\begin{aligned} m_k &= \int_{-\pi}^{\pi} e^{ikt} \frac{w(t)}{|e^{it}-\alpha|^2} dt = \int_{-\pi}^{\pi} e^{ikt} w(t) \frac{1}{(1-\alpha/e^{it})(1-\alpha e^{it})} dt \\ &= \int_{-\pi}^{\pi} e^{ikt} w(t) \sum_{\ell=0}^{\infty} \left(\frac{\alpha}{e^{it}}\right)^{\ell} \sum_{j=0}^{\infty} \left(\alpha e^{it}\right)^j dt = \sum_{\ell=0}^{\infty}\sum_{j=0}^{\infty} \alpha^{\ell+j} \int_{-\pi}^{\pi} e^{i(k+j-\ell)t} w(t) dt. \end{aligned}$$

The proof follows. □

Corollary 1 *The weight function $w(t) = 1/|h(e^{it})|^2$ where $h(z) = (z-\alpha_1)\cdots(z-\alpha_N)$, with simple positive roots $\alpha_\ell \in (0,1)$, $\ell = 1, 2, \ldots, N$, has non-negative moments $m_k = \int_{-\pi}^{\pi} e^{ikt}/|h(e^{it})|^2 dt \geq 0$, $k = 0, \pm 1, \pm 2, \ldots$.*

Proof. The weight function $w(t) = 1$ has non-negative moments, $m_0 = 2\pi$, and $m_k = 0$ for $k = \pm 1, \pm 2, \ldots$ so that the result follows from the previous theorem. □

3.2.1 Case $N = 1$ of one pole

In this subsection we consider the case $N = 1$ of one real or complex pole. We are dealing then (apart from a multiplicative constant) with the Poisson weight function defined by

$$w(t) = K_\alpha(t) = \frac{1-|\alpha|^2}{|e^{it}-\alpha|^2}, \quad \alpha \in \mathbb{C}, \quad 0 \leq |\alpha| < 1, \quad -\pi \leq t < \pi. \tag{16}$$

Following Theorem 4, the associated moments $m_k = \int_{-\pi}^{\pi} e^{ikt} K_\alpha(t) dt$, are given by $m_0 = 2\pi$, and $m_k = 2\pi\alpha^k$, $m_{-k} = 2\pi\overline{\alpha}^k$ for $k = 1, 2, \ldots$.

Theorem 6 *([18]) Consider integrals of the form $I(f) = \int_{-\pi}^{\pi} f(e^{it}) K_\alpha(t) dt$, where $K_\alpha(t)$ is the Poisson weight function with $0 \leq \alpha < 1$. Denote by $Q_n(f) = \sum_{j=1}^{n} c_{j,n} f(z_{j,n})$ the corresponding quadrature formulas of interpolatory type in Λ_{-p_n,p_n} with $2p_n = n-1$ whose nodes are $z_{j,n} = \omega^{j-1}$, $\omega = e^{2\pi i/n}$. Then for $n \geq 1$*

$$c_{j,n} = \frac{2\pi}{n} \frac{(1-\alpha)\left(1+\alpha-(-1)^{j-1} 2\alpha^{p_n+1}\cos(\pi(j-1)/n)\right)}{1-2\alpha\cos(2\pi(j-1)/n)+\alpha^2} > 0, \quad 1 \leq j \leq n. \tag{17}$$

Quadrature formulas constructed as in Theorem 6 but with α not located in the interval $0 \leq \alpha < 1$, may have negative quadrature weights.

Example 4 Consider the weight function $K_\alpha(t) = (1 - |\alpha|^2)/|e^{it} - \alpha|^2$ for $\alpha = -2/3$. The $n = 3$ points quadrature formula

$$Q_3(f) = c_{1,3}f(z_{1,3}) + c_{2,3}f(z_{2,3}) + c_{3,3}f(z_{3,3}) \approx I(f) = \int_{-\pi}^{\pi} f(e^{it})\frac{5/9}{|e^{it}+2/3|^2}dt,$$

of interpolatory type in Λ_{-p_3,p_3}, $p_3 = 1$, $2p_3 = n - 1 = 2$, and with quadrature nodes $z_{j,3}$, $j = 1, 2, 3$, the roots of $z^3 - 1 = 0$. Thus $z_{j,3} = \omega^{j-1}$, $j = 1, 2, 3$, where $\omega = e^{2\pi i/3}$. We calculate the quadrature weights from (3) and (4) giving

$$c_{j,3} = z_{j,3}\int_{-\pi}^{\pi} \frac{N_3(z)}{z(z - z_{j,3})N_3'(z_{j,3})}\frac{5/9}{|e^{it}+2/3|^2}dt, \quad j = 1, 2, 3,$$

where the nodal polynomial is $N_3(z) = z^3 - 1$. After some calculations we obtain $c_{1,3} = -2\pi/9 < 0$, and, $c_{2,3} = c_{3,3} = 10\pi/9 > 0$. Hence not all quadrature weights are positive.

We give next a sufficient condition for the positivity of the quadrature weights.

Theorem 7 *Consider quadrature formulas $Q_n(f) = \sum_{j=1}^n c_{j,n}f(z_{j,n})$ of interpolatory type in Λ_{-p_n,p_n}, $2p_n = n - 1$ with quadrature nodes $z_{j,n} = \omega^{j-1}$, $\omega = e^{2\pi i/n}$, to approximate the integrals $I(f) = \int_{-\pi}^{\pi} f(e^{it})K_\alpha(t)dt$, where $K_\alpha(t)$ is the Poisson weight function with, $\alpha = |\alpha|e^{i\theta_\alpha} \in \mathbb{D}$. Then the quadrature weights are given by*

$$(18) \qquad c_{j,n} = \frac{2\pi}{n}\frac{1 - |\alpha|^2 + 2|\alpha|^{p_n+1}\mathrm{Re}\left((e^{i\theta_\alpha}\omega^{1-j})^{p_n}(-e^{i\theta_\alpha}\omega^{1-j} + |\alpha|)\right)}{|1 - \alpha\omega^{1-j}|^2}, \ 1 \le j \le n, \ n \ge 1.$$

As a consequence, if $1 - |\alpha| - 2|\alpha|^{p_n+1} > 0$ then $c_{j,n} > 0$, $1 \le j \le n$.

Proof. From (6) with $k_n = 1$ and $z_{1,n} = 1$,

$$c_{j,n} = \frac{2\pi}{n}\left(1 + \sum_{k=1}^{p_n} \alpha^k\omega^{(1-j)k} + \sum_{k=p_n+1}^{n-1} \overline{\alpha}^{n-k}\omega^{(1-j)k}\right), \quad 1 \le j \le n.$$

Thus, since $\omega^j = \overline{\omega^{n-j}}$,

$$c_{j,n} = \frac{2\pi}{n}\left(1 + \sum_{k=1}^{p_n} \alpha^k\omega^{(1-j)k} + \overline{\sum_{k=1}^{p_n} \alpha^k\omega^{(1-j)k}}\right).$$

The summation is a geometric sum and hence

$$c_{j,n} = \frac{2\pi}{n}\left(1 + \frac{\alpha\omega^{1-j} - \alpha^{p_n+1}\omega^{(1-j)(p_n+1)}}{1 - \alpha\omega^{1-j}} + \frac{\overline{\alpha}\omega^{-1+j} - \overline{\alpha}^{p_n+1}\omega^{(-1+j)(p_n+1)}}{1 - \overline{\alpha}\omega^{-1+j}}\right).$$

After some calculations one obtains the expression

$$c_{j,n} = \frac{2\pi}{n}\frac{1 - |\alpha|^2 + (\alpha\omega^{1-j})^{p_n}(-\alpha\omega^{1-j} + |\alpha|^2) + (\overline{\alpha}\omega^{-1+j})^{p_n}(-\overline{\alpha}\omega^{-1+j} + |\alpha|^2)}{|1 - \alpha\omega^{1-j}|^2}.$$

Taking into account that $\alpha = |\alpha|e^{i\theta_\alpha}$,

$$c_{j,n} = \frac{2\pi}{n}\frac{1-|\alpha|^2+|\alpha|^{p_n+1}A_{j,n}}{|1-\alpha\omega^{1-j}|^2},$$

where the auxiliary parameter $A_{j,n}$ is given by

$$A_{j,n} = (e^{i\theta_\alpha}\omega^{1-j})^{p_n}(-e^{i\theta_\alpha}\omega^{1-j}+|\alpha|) + (e^{-i\theta_\alpha}\omega^{-1+j})^{p_n}(-e^{-i\theta_\alpha}\omega^{-1+j}+|\alpha|).$$

Thus

$$c_{j,n} = \frac{2\pi}{n}\frac{1-|\alpha|^2+2|\alpha|^{p_n+1}\mathrm{Re}\left((e^{i\theta_\alpha}\omega^{1-j})^{p_n}(-e^{i\theta_\alpha}\omega^{1-j}+|\alpha|)\right)}{|1-\alpha\omega^{1-j}|^2}.$$

Observe that

$$\mathrm{Re}\left((e^{i\theta_\alpha}\omega^{1-j})^{p_n}(-e^{i\theta_\alpha}\omega^{1-j}+|\alpha|)\right) \geq -(1+|\alpha|).$$

Then

$$c_{j,n} \geq \frac{2\pi}{n}\frac{1-|\alpha|^2-2|\alpha|^{p_n+1}(1+|\alpha|)}{|1-\alpha\omega^{1-j}|^2} = \frac{2\pi}{n}\frac{(1+|\alpha|)(1-|\alpha|-2|\alpha|^{p_n+1})}{|1-\alpha\omega^{1-j}|^2}.$$

The proof follows. □

Formula (18) for the values $\alpha = -2/3$ and $p_n = 1$ considered in Example 4 reproduces the quadrature weights obtained there. Furthermore note that the sufficient condition $1 - |\alpha| - 2|\alpha|^{p_n+1} > 0$ for positive quadrature weights does *not* hold for the values considered in Example 4. On the other hand, if $0 \leq \alpha < 1$ then formula (18) reduces to formula (17), and the quadrature weights are positive under this hypothesis.

3.2.2 Case of $N \geq 2$ real poles

We now consider the case of $N \geq 2$ real and simple poles on the open interval $(-1,1)$ excluding the origin: $\{\alpha_\ell\}_{\ell=1}^N \subset (-1,1)\setminus\{0\}$. Under this assumption the polynomial $h(z)$ has real coefficients and then from Theorem 4 one deduces $m_k = m_{-k} \in \mathbb{R},\ k = 0,1,2,\ldots$.

Theorem 8 *Consider quadrature formulas $Q_n(f) = \sum_{j=1}^n c_{j,n}f(z_{j,n})$ of interpolatory type in Λ_{-p_n,p_n}, $2p_n = n-1$, with quadrature nodes $z_{j,n} = \omega^{j-1}$, $\omega = e^{2\pi i/n}$, to approximate integrals $\int_{-\pi}^{\pi} f(e^{it})/|h(e^{it})|^2dt$, where $h(z) = (z-\alpha_1)\cdots(z-\alpha_N)$ is an algebraic polynomial with simple real roots $\alpha_1,\alpha_2,\ldots,\alpha_N$ in $(-1,1)\setminus\{0\}$. Then the quadrature weights $c_{j,n}$ satisfy*

$$(19)\qquad c_{j,n} = \sum_{\ell=1}^{N}\frac{\alpha_\ell^{N-1}}{h'(\alpha_\ell)h^*(\alpha_\ell)}c_{j,n}(\alpha_\ell),\quad j=1,2,\ldots,n,\quad n=1,2,\ldots,$$

where $c_{j,n}(\alpha_\ell)$ denotes the quadrature weights given by (18) for $\alpha = \alpha_\ell$.

Proof. We can use (7) to compute the weights and replace the moments by their expressions from Theorem 4 to get

$$c_{j,n} = \frac{2\pi}{n} \sum_{\ell=1}^{N} \frac{\alpha_\ell^{N-1}}{h'(\alpha_\ell)h^*(\alpha_\ell)} \left(1 + \sum_{k=1}^{p_n} \alpha_\ell^k (\omega^{(1-j)k} + \overline{\omega^{(1-j)k}})\right).$$

According to the proof of Theorem 7, the expression between the parenthesis is $\frac{n}{2\pi} c_{j,n}(\alpha_\ell)$. □

Corollary 2 *With the hypothesis of Theorem 8, if* $N = 2$, $-1 < \alpha_1 < 0 < \alpha_2 < 1$, *and* $1 - |\alpha_1| - 2|\alpha_1|^{p_n+1} > 0$, *then the quadrature weights* $c_{j,n} > 0$, *are positive.*

Proof. For $N = 2$ and $-1 < \alpha_1 < 0 < \alpha_2 < 1$, it holds $\alpha_\ell^{N-1}/(h'(\alpha_\ell)h^*(\alpha_\ell)) > 0$, $\ell = 1, 2$. The proof follows from Theorems 6 and 7. □

Theorem 9 *Consider quadrature formulas* $Q_n(f) = \sum_{j=1}^{n} c_{j,n} f(z_{j,n})$ *of interpolatory type in* Λ_{-p_n,p_n}, $2p_n = n-1$, *with quadrature nodes* $z_{j,n} = \omega^{j-1}$, $\omega = e^{2\pi i/n}$, *to approximate integrals* $I(f) = \int_{-\pi}^{\pi} f(e^{it})/|h(e^{it})|^2 dt$, *where* $h(z) = (z - \alpha_1)\cdots(z - \alpha_N)$ *is an algebraic polynomial with simple real roots* $\alpha_1, \ldots, \alpha_N$ *in* $(-1, 1) \setminus \{0\}$. *If the moments* m_k *for the weight function* $1/|h(e^{it})|^2$ *are non-negative for* $k = 1, 2, \ldots, p_n$, *and if*

$$\sum_{\ell=1}^{N} \frac{\alpha_\ell^{N-1}(1 - 3\alpha_\ell + 2\alpha_\ell^{p_n+1})}{h'(\alpha_\ell)h^*(\alpha_\ell)(1 - \alpha_\ell)} > 0,$$

then all the quadrature weights are positive and given by $c_{j,n} = \frac{1}{n} P_n(\omega^{j-1}) > 0$, $j = 1, 2, \ldots, n$, *where* $P_n(z) = m_0 + \sum_{k=1}^{p_n} m_k(z^k + 1/z^k)$.

Proof. For non-negative moments the Laurent polynomial $P_n(z)$ satisfies on the unit circle $P_n(z) \geq m_0 - 2\sum_{k=1}^{p_n} m_k$. Taking into account Theorem 4 we write

$$P_n(z) \geq 2\pi \sum_{\ell=1}^{N} \frac{\alpha_\ell^{N-1}}{h'(\alpha_\ell)h^*(\alpha_\ell)} \left(1 - 2\sum_{k=1}^{p_n} \alpha_\ell^k\right) = 2\pi \sum_{\ell=1}^{N} \frac{\alpha_\ell^{N-1}(1 - 3\alpha_\ell + 2\alpha_\ell^{p_n+1})}{h'(\alpha_\ell)h^*(\alpha_\ell)(1 - \alpha_\ell)} > 0.$$

The proof follows from Theorem 3. □

Example 5 Consider $h(z) = (z + 0.3)(z + 0.1)(z - 0.5)$, i.e., $N = 3$, $\alpha_1 = -0.3$, $\alpha_2 = -0.1$, $\alpha_3 = 0.5$. We take for example $p_n = 10$ so that $n = 21$. The moments for the weight function $1/|h(e^{it})|^2$ are

$$m_k = \sum_{\ell=1}^{N} \frac{\alpha_\ell^{N+k-1}}{h'(\alpha_\ell)h^*(\alpha_\ell)} > 0 \quad \text{for } k = 1, 2, \ldots, 10.$$

Furthermore also

$$\sum_{\ell=1}^{N} \frac{\alpha_\ell^{N-1}(1 - 3\alpha_\ell + 2\alpha_\ell^{p_n+1})}{h'(\alpha_\ell)h^*(\alpha_\ell)(1 - \alpha_\ell)} > 0.$$

Therefore all the quadrature weights are positive and given by $c_{j,21} = \frac{1}{21} P_{21}(\omega^{j-1})$, $j = 1, 2, \ldots, 21$, where $P_{21}(z) = m_0 + \sum_{k=1}^{10} m_k(z^k + 1/z^k)$.

Corollary 3 *With the hypothesis of Theorem 9 for* $h(z) = (z - \alpha_1) \cdots (z - \alpha_N)$ *with simple and positive roots* $\alpha_1, \dots, \alpha_N$ *in* $(0, 1)$ *and*

$$\sum_{\ell=1}^{N} \frac{\alpha_\ell^{N-1}(1 - 3\alpha_\ell + 2\alpha_\ell^{p_n+1})}{h'(\alpha_\ell)h^*(\alpha_\ell)(1 - \alpha_\ell)} > 0,$$

then the quadrature weights are positive and given by $c_{j,n} = \frac{1}{n}P_n(\omega^{j-1}) > 0$, $j = 1, 2, \dots, n$ *where* $P_n(z) = m_0 + \sum_{k=1}^{p_n} m_k(z^k + 1/z^k)$.

Proof. For positive $\alpha_1, \dots, \alpha_N$ the moments are non-negative by virtue of Corollary 1. The proof follows. □

Theorem 10 *Consider quadrature formulas* $Q_n(f) = \sum_{j=1}^{n} c_{j,n} f(z_{j,n})$ *of interpolatory type in* Λ_{-p_n,p_n}, $2p_n = n - 1$ *with quadrature nodes* $z_{j,n} = \omega^{j-1}$, $\omega = e^{2\pi i/n}$ *for the integrals* $\int_{-\pi}^{\pi} f(e^{it})/|h(e^{it})|^2 dt$, *where* $h(z) = (z - \alpha_1) \cdots (z - \alpha_N)$ *is an algebraic polynomial with simple positive roots* $\alpha_1, \dots, \alpha_N$ *in* $(0, 1)$. *If*

$$\sum_{\ell=1}^{N} \frac{\alpha_\ell^{N-1} d_{\ell,n}}{h'(\alpha_\ell)h^*(\alpha_\ell)} > 0,$$

where

$$d_{\ell,n} = \begin{cases} \dfrac{1 + \alpha_\ell + 2\alpha_\ell^{p_n+1}}{1 - \alpha_\ell}, & \text{if } \ell N \text{ odd, i.e., } N \text{ even and } \ell \text{ odd or } N \text{ odd and } \ell \text{ even,} \\[2ex] \dfrac{(1 - \alpha_\ell)(1 + \alpha_\ell - 2\alpha_\ell^{p_n+1})}{(1 + \alpha_\ell)^2}, & \text{if } \ell N \text{ even, i.e., } N \text{ and } \ell \text{ both even or both odd,} \end{cases}$$

then the quadrature weights are positive.

Proof. From (19) we have

$$c_{j,n} = \sum_{\ell=1}^{N} \frac{\alpha_\ell^{N-1}}{h'(\alpha_\ell)h^*(\alpha_\ell)} c_{j,n}(\alpha_\ell), \quad j = 1, 2, \dots, n, \quad n = 1, 2, \dots,$$

where $c_{j,n}(\alpha_\ell)$ denotes the quadrature weights given by (17) for $\alpha = \alpha_\ell$. Note that

$$\frac{2\pi}{n} \frac{(1 - \alpha_\ell)(1 + \alpha_\ell - 2\alpha_\ell^{p_n+1})}{(1 + \alpha_\ell)^2} \le c_{j,n}(\alpha_\ell) \le \frac{2\pi}{n} \frac{1 + \alpha_\ell + 2\alpha_\ell^{p_n+1}}{1 - \alpha_\ell}.$$

If $0 < \alpha_\ell < 1$ then $h^*(\alpha_\ell) > 0$. The sign of $\alpha_\ell^{N-1}/(h'(\alpha_\ell)h^*(\alpha_\ell))$ is then the same as the sign of $h'(\alpha_\ell)$. Thus we can say that if

$$\sum_{\ell=1}^{N} \frac{\alpha_\ell^{N-1} d_{\ell,n}}{h'(\alpha_\ell)h^*(\alpha_\ell)} > 0$$

where

$$d_{\ell,n} = \begin{cases} \dfrac{1+\alpha_\ell+2\alpha_\ell^{p_n+1}}{1-\alpha_\ell}, & \text{if } h'(\alpha_\ell)<0, \\ \\ \dfrac{(1-\alpha_\ell)(1+\alpha_\ell-2\alpha_\ell^{p_n+1})}{(1+\alpha_\ell)^2}, & \text{if } h'(\alpha_\ell)>0, \end{cases}$$

then the quadrature weights are positive. The sign of $h'(\alpha_\ell)$ alternates with $\ell = 1, 2, \ldots, N$. Furthermore the sign of $h'(\alpha_1)$ depends on the parity of N. The proof easily follows. □

Example 6 Let us consider quadrature formulas as described in Corollary 3 and Theorem 10 for the case of simple positive roots $\alpha_1, \ldots, \alpha_N$ of $h(z)$. We are interested in the sign of the numbers

$$(20) \qquad A_n = \sum_{\ell=1}^{N} \frac{\alpha_\ell^{N-1}(1-3\alpha_\ell+2\alpha_\ell^{p_n+1})}{h'(\alpha_\ell)h^*(\alpha_\ell)(1-\alpha_\ell)} \quad \text{and} \quad B_n = \sum_{\ell=1}^{N} \frac{\alpha_\ell^{N-1} d_{\ell,n}}{h'(\alpha_\ell)h^*(\alpha_\ell)}$$

given in Corollary 3 and Theorem 10 respectively for the positive character of the quadrature weights. Let us consider as example $N = 2$ and $p_n = 20$. The results are summarized in Table 2.

α_1	α_2	A_n	B_n	Conclusion
0.1	0.7	<0	>0	B_n guarantees positive quadrature weights
0.5	0.7	<0	<0	Neither A_n nor B_n guarantee positive quadrature weights
0.1	0.12	>0	<0	A_n guarantees positive quadrature weights
0.05	0.2	>0	>0	Both A_n and B_n guarantee positive quadrature weights

Table 2: Some examples to show that sometimes A_n and sometimes B_n from (20) can guarantee the positivity of the quadrature weights, depending on the location of the poles α_1 and α_2.

3.2.3 Case of complex poles uniformly distributed on a circle

Let us now consider the case $h(z) = z^N - r$, where, $0 < r < 1$, $N = 1, 2, \ldots$. The roots of $h(z)$ are then the complex numbers $\alpha_\ell = \omega^{\ell-1} r^{1/N}$, $\ell = 1, 2, \ldots, N$, $\omega = e^{2\pi i/N}$, being uniformly distributed on the circle $|z| = r$. We deal then with the rational weight function

$$(21) \qquad w(t) = K_{N,r}(t) = \frac{1-r^2}{|e^{iNt}-r|^2}, \quad N = 1, 2, \ldots, \quad 0 < r < 1.$$

For real parameter $0 < r < 1$, this weight function represents a generalization of the Poisson weight function ($N = 1$) studied in subsection 3.2.1.

We can determine its associated moments $m_k = \int_{-\pi}^{\pi} e^{ikt} K_{N,r}(t)dt$ from Theorem 4. Indeed, let us first consider $k = 0, 1, 2, \ldots$. One gets

$$m_k = (1-r^2)2\pi \sum_{\ell=1}^{N} \frac{\alpha_\ell^{N+k-1}}{h'(\alpha_\ell)h^*(\alpha_\ell)} = \frac{2\pi}{N}\sum_{\ell=1}^{N} \alpha_\ell^k = \frac{2\pi r^{k/N}}{N}\sum_{\ell=1}^{N} (\omega^k)^{\ell-1}.$$

By the geometric sum formula

$$m_k = \begin{cases} 2\pi r^s, \text{ if } k = sN, \ s = 0,1,2,\ldots, \\ 0, \text{ if } k \neq sN, \end{cases} \qquad k = 0,1,2\ldots. \tag{22}$$

From Theorem 4 and (22) we observe $m_{-k} = \overline{m_k} = m_k, \ k = 1, 2, \ldots$, having then determined all the moments.

Theorem 11 *Consider quadrature formulas $Q_n(f) = \sum_{j=1}^{n} c_{j,n} f(z_{j,n})$ of interpolatory type in Λ_{-p_n,p_n}, $2p_n = n-1$, with quadrature nodes $z_{j,n} = \omega^{j-1}$, $\omega = e^{2\pi i/n}$ for the integrals $I(f) = \int_{-\pi}^{\pi} f(e^{it})K_{N,r}(t)dt$. Then the quadrature weights are given by*

$$c_{j,n} = \frac{2\pi}{n} \frac{1 - r^{\lfloor p_n/N \rfloor+1}\left(\overline{d_{j,n}} + d_{j,n}\right) + r^{\lfloor p_n/N \rfloor+2}\left(\overline{e_{j,n}} + e_{j,n}\right) - r^2}{\left|1 - r\omega^{(j-1)N}\right|^2} \tag{23}$$

where

$$d_{j,n} = \omega^{(j-1)N(\lfloor p_n/N \rfloor+1)} \quad \text{and} \quad e_{j,n} = \omega^{(j-1)N\lfloor p_n/N \rfloor}, \quad j = 1, 2, \ldots, n.$$

Proof. From (7) we can write

$$c_{j,n} = \frac{2\pi}{n}\left(1 + \sum_{s=1}^{\lfloor p_n/N \rfloor}\left(r\overline{\omega}^{(j-1)N}\right)^s + \sum_{s=1}^{\lfloor p_n/N \rfloor}\left(r\omega^{(j-1)N}\right)^s\right).$$

Each summation is a geometric sum and thus

$$c_{j,n} = \frac{2\pi}{n}\left(1 + \frac{r\overline{\omega}^{(j-1)N} - r^{\lfloor p_n/N \rfloor+1}\overline{d_{j,n}}}{1 - r\overline{\omega}^{(j-1)N}} + \frac{r\omega^{(j-1)N} - r^{\lfloor p_n/N \rfloor+1}d_{j,n}}{1 - r\omega^{(j-1)N}}\right).$$

The proof follows after some calculations. □

Theorem 12 *With the hypothesis of Theorem 11 and if $1 - r - 2r^{\lfloor p_n/N \rfloor+1} > 0$, then the quadrature weights are positive: $c_{j,n} > 0, \ j = 1, 2, \ldots, n$.*

Proof. From (23) we can write

$$c_{j,n} \geq \frac{2\pi}{n}\frac{1 - 2r^{\lfloor p_n/N \rfloor+1} - 2r^{\lfloor p_n/N \rfloor+2} - r^2}{\left|1 - r\omega^{(j-1)N}\right|^2} = \frac{2\pi}{n}\frac{(1+r)\left(1 - r - 2r^{\lfloor p_n/N \rfloor+1}\right)}{\left|1 - r\omega^{(j-1)N}\right|^2}.$$

The proof follows. □

Theorem 13 *With the hypothesis of Theorem 11 and if* $N = 1, 3, 5, \ldots,\ n = (2\ell+1)N,\ \ell = 0, 1, 2, \ldots,$ *then the quadrature weights are positive:* $c_{j,n} > 0,\ j = 1, 2, \ldots, n$.

Proof. Since $2p_n = n-1$, $p_n = (n-1)/2$. From $n = (2\ell+1)N$, we get $p_n = \ell N + (N-1)/2$. So $\lfloor p_n/N \rfloor = \ell$. This observation and taking into account $\omega^n = 1$ this permits to write

$$\overline{d_{j,n}} = \overline{\omega}^{(j-1)N(\ell+1)} = \overline{\omega}^{(j-1)(n-N\ell)} = \overline{\omega}^{(j-1)(-N\ell)} = \omega^{(j-1)N\lfloor p_n/N \rfloor} = e_{j,n}.$$

Substituting $\overline{d_{j,n}} = e_{j,n}$ into (23) we get

$$c_{j,n} = \frac{2\pi}{n} \frac{(1-r)\left(1 + r - r^{\lfloor p/N \rfloor + 1}(e_{j,n} + \overline{e_{j,n}})\right)}{|1 - r\omega^{(j-1)N}|^2}.$$

Thus

$$c_{j,n} \geq \frac{2\pi}{n} \frac{(1-r)\left(1 + r - 2r^{\lfloor p_n/N \rfloor + 1}\right)}{|1 - r\omega^{(j-1)N}|^2} > 0.$$

The proof ends. □

Observe that Theorem 13 reproduces the result of Theorem 6 about the positive character of the quadrature weights for the Poisson weight function ($N = 1$).

Remark 3 For the weight function $K_{N,r}(t) = (1-r^2)/|e^{iNt} - r|^2$ studied in this subsection it is possible to give an alternative expression to (23) for the quadrature weights.

Indeed, although $h(z) = z^N - r$ has roots α_ℓ that are not real, it is easily verified that there is enough symmetry so that the expression (19) of Theorem 8 still holds in this case. Note that $h'(z) = Nz^{N-1}$ so that $h'(\alpha_\ell) = N\alpha_\ell^{N-1}$. On the other hand $h^*(\alpha_\ell) = (1 - r\alpha_\ell^N) = 1 - r^2$. Thus

$$c_{j,n} = \sum_{\ell=1}^{N} \frac{\alpha_\ell^{N-1}}{h'(\alpha_\ell)h^*(\alpha_\ell)} c_{j,n}(\alpha_\ell) = \frac{1}{N(1-r^2)} \sum_{\ell=1}^{N} c_{j,n}(\alpha_\ell), \quad j = 1, 2, \ldots, n,$$

where $c_{j,n}(\alpha_\ell)$ denotes the quadrature weights given by (18) for $\alpha = \alpha_\ell$.

Concerning the positivity of the quadrature weights we can take Theorem 7 into account saying that if $1 - r^{1/N} - 2r^{(p_n+1)/N} > 0$ then $c_{j,n} > 0,\quad 1 \leq j \leq n$. However, the sufficient condition $1 - r - 2r^{\lfloor p_n/N \rfloor + 1} > 0$ given in Theorem 12 is better because $1 - r - 2r^{\lfloor p_n/N \rfloor + 1} \geq 1 - r^{1/N} - 2r^{(p_n+1)/N}$. This inequality may suggest that $1 - r - 2r^{\lfloor p_n/N \rfloor + 1} > 0$, is a sharp condition. Indeed, let us consider for example $N = 4$ and $p_n = 15$ ($n = 31$). One can obtain that $1 - r - 2r^{\lfloor p_n/N \rfloor + 1} = 1 - r - 2r^4 > 0$ for $0 < r < 0.6477\ldots$. For a nearby but larger value $r = 0.66$ outside this region where positive quadrature weights are guaranteed, we have found a negative quadrature weight $c_{5,31} = -0.001719\ldots < 0$.

Remark 4 For periodic weight functions like (21) treated in this subsection we can reduce the number of quadrature weights to be computed when one takes a suitable selection of uniformly distributed quadrature nodes. Indeed, from (3)-(4) we can write for our usual case of quadrature formulas $Q_n(f) = \sum_{j=1}^{n} c_{j,n} f(z_{j,n})$ of interpolatory type in Λ_{-p_n,p_n}, $2p_n = n-1$ with quadrature nodes $z_{j,n} = \omega^{j-1}$, $\omega = e^{2\pi i/n}$, that

$$c_{j,n} = \int_{-\pi}^{\pi} \frac{z_{j,n}^{p_n} N_n(z)}{z^{p_n}(z - z_{j,n}) N_n'(z_{j,n})} w(t)dt = \frac{\omega^{(j-1)p_n}}{n\omega^{(j-1)(n-1)}} \int_{-\pi}^{\pi} \frac{z^n - 1}{z^{p_n}(z - \omega^{j-1})} w(t)dt.$$

Make the change of variable $\theta = t - \frac{2\pi}{n}(j-1)$. Then after some calculations one can obtain,

$$c_{j,n} = \frac{1}{n} \int_{-\pi-\frac{2\pi}{n}(j-1)}^{\pi-\frac{2\pi}{n}(j-1)} \frac{z^n - 1}{z^{p_n}(z-1)} w\left(\theta + \frac{2\pi}{n}(j-1)\right) d\theta, \ z = e^{i\theta}, \ j = 1, 2, \ldots, n.$$

For periodic weight functions of period $2\pi/N$, $N = 1, 2, \ldots$, and a number $n = mN$ of quadrature nodes that is a multiple of N, the product $(z^n - 1)/(z^{p_n}(z-1))w\left(\theta + \frac{2\pi}{n}(j-1)\right)$ is a periodic function of period 2π an then we can write

$$c_{j,n} = \frac{1}{n} \int_{-\pi}^{\pi} \frac{z^n - 1}{z^{p_n}(z-1)} w\left(\theta + \frac{2\pi}{n}(j-1)\right) d\theta, \quad z = e^{i\theta}, \quad j = 1, 2, \ldots, n.$$

This expression tells us that the quadrature weights have period m, that is, $c_{k,n} = c_{k+\ell m,n}$, $k = 1, \ldots, m$, $\ell = 1, \ldots, N-1$.

This is a computational advantageous since one only needs to compute m quadrature weights, say $c_{1,n}, c_{2,n}, \ldots, c_{m,n}$. The quadrature formula can then be written in the simplified form

$$Q_n(f) = \sum_{\ell=1}^{m} c_{\ell,n} \sum_{k=0}^{N-1} f(z_{km+\ell,n}).$$

4 Asymptotic expansion

In this section we deal with the estimation of integrals

$$I(f) = \int_{-\pi}^{\pi} f(e^{it}) K_\alpha(t) dt$$

where $K_\alpha(t)$ is the Poisson weight function given in (16). We consider in this section $0 \leq \alpha < 1$. The integrand can be expressed in terms of its real and imaginary part: $f(e^{it}) = f_1(t) + i f_2(t)$ with $f_1(t) = \operatorname{Re} f(e^{it})$ and $f_2(t) = \operatorname{Im} f(e^{it})$ real functions. On the other hand each real function can be decomposed in terms of an even plus an odd function. Then $f_j(t) = f_{j,e}(t) + f_{j,o}(t)$, where $f_{j,e}(t) = (f_j(t) + f_j(-t))/2$ is the even part and $f_{j,o}(t) = (f_j(t) - f_j(-t))/2$ the odd part, $j = 1, 2$. Since $K_\alpha(t)$ is an even function the integral of the odd parts vanishes and then

$$\begin{aligned} I(f) &= \int_{-\pi}^{\pi} f(e^{it}) K_\alpha(t) dt = \int_{-\pi}^{\pi} f_{1,e}(t) K_\alpha(t) dt + i \int_{-\pi}^{\pi} f_{2,e}(t) K_\alpha(t) dt \\ &= \int_{-\pi}^{\pi} f_e(t) K_\alpha(t) dt, \quad f_e(t) = f_{1,e}(t) + i f_{2,e}(t). \end{aligned}$$

To estimate $I(f)$ we use an n-point quadrature formula $I_n(f)$ with nodes the roots of unity $z_{j,n}^n = 1$, $(z_{1,n} = 1)$ and with domain of validity Λ_{-p_n,p_n}, $n = 2p_n + 1$. We will obtain an asymptotic expansion for the quadrature error suitable for a modified Romberg type quadrature technique. For convenience in order to take advantage on symmetry we write the quadrature formula by virtue of [18, Corollary 2.2] in the form

$$I_n(f) = c_{1,n} f(1) + \sum_{j=2}^{p_n+1} c_{j,n} \left(f(z_{j,n}) + f(\overline{z_{j,n}}) \right), \quad z_{j,n} = e^{2\pi i (j-1)/n},$$

and the quadrature coefficients $c_{j,n}$ are real numbers, see [18, Theorem 2.1].

Thus to approximate integrals $Q(h) = \int_{-\pi}^{\pi} h(t) K_\alpha(t) dt$ for a real function $h(t)$, we consider the quadrature rule

$$Q_n(h) = c_{1,n} h(t_{1,n}) + \sum_{j=2}^{p_n+1} c_{j,n} (h(t_{j,n}) + h(-t_{j,n})), \quad t_{j,n} = 2\pi \frac{j-1}{n}, \ n = 2p_n + 1.$$

For $\alpha = 0$ is $c_{j,n} = 2\pi/n$, see (17), and then the quadrature rule $Q_n(h)$ is just the classical midpoint rule (for n odd), see e.g. [12, Section 2.1].

There is an obvious natural relation between the rules I_n and Q_n. Note that if $h(t)$ is an odd function then $Q_n(h) = 0$. Taking this into account we can write

$$I_n(f) = Q_n(f_e) = Q_n(f_{1,e} + i f_{2,e}) = Q_n(f_{1,e}) + i Q_n(f_{2,e}). \tag{24}$$

Furthermore

$$\begin{aligned} \operatorname{Re} I_n(f) &= I_n(\operatorname{Re} f) = Q_n(f_1) = Q_n(f_{1,e}) = Q_n(\operatorname{Re} f_e) \\ \operatorname{Im} I_n(f) &= I_n(\operatorname{Im} f) = Q_n(f_2) = Q_n(f_{2,e}) = Q_n(\operatorname{Im} f_e), \end{aligned} \tag{25}$$

and similarly $\operatorname{Re} I(f) = Q(f_1) = Q(f_{1,e})$ and $\operatorname{Im} I(f) = Q(f_2) = Q(f_{2,e})$. Thus to study the asymptotic error, we essentially need to analyse the asymptotic error for even real functions $f_{1,e}$ and $f_{2,e}$.

We consider functions $f(e^{it})$ such that $f_{j,e}(t) \in C^\nu[-\pi, \pi]$, with $\nu \geq 2$, for $j = 1, 2$. This regularity consideration is sufficient to assure the pointwise convergence of the Fourier series to $f_{j,e}(t)$ (more precisely, to its 2π-periodic extension). Since the discussion of $f_{1,e}(t)$ and for $f_{2,e}(t)$ is the same, let us consider an even periodic function $h(t) \in C^\nu[-\pi, \pi]$, $\nu \geq 2$, which can be either of the $f_{j,e}(t)$ $j = 1, 2$. So, let us consider the Fourier series expansion of the even function $h(t)$ given by

$$h(t) \sim \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) \tag{26}$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} h(t) \cos(kt) dt, \ k = 0, 1, \ldots$$

and the notation $\sim$ in (26) refers to pointwise convergence.

Consider the inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(e^{it})\overline{g(e^{it})}w(t)dt$, where $f, g \in L_2^w(\mathbb{T})$ and $w(t)$ is a given weight function. A sequence $\{P_k(z)\}_{k=0}^{\infty}$ is said to be a sequence of Szegő polynomials with respect to the weight function $w(t)$ if $P_k(z)$ is a polynomial of degree k, $\langle P_k, P_k\rangle \neq 0$, and for $k \geq 1$ is $\langle P_k, z^j\rangle = 0$ with $j = 0, 1, \dots, k-1$. See, e.g. [19, 21]. It is immediately checked that the monic orthogonal polynomials associated with the Poisson weight $w(t) = K_\alpha(t)$ are given by $P_0(z) = 1,\ P_k(z) = z^k - \alpha z^{k-1},\ k = 1, 2, \dots$ (see [22]). Some calculations show that the powers $z^k,\ k = 0, 1, 2, \dots$, can be expressed in this orthogonal system by

$$z^k = P_k(z) + \alpha P_{k-1}(z) + \alpha^2 P_{k-2} + \cdots + \alpha^{k-1}P_1(z) + \alpha^k P_0(z). \tag{27}$$

Taking into account that $\cos(kt) = (e^{ikt} + e^{-ikt})/2$ and (27) we can rewrite the Fourier series of $h(t)$ in the form

$$h(t) \sim g(e^{it}) + \overline{g(e^{it})} \tag{28}$$

where

$$2g(e^{it}) = P_0(e^{it})\left(\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k\alpha^k\right) + \sum_{j=1}^{\infty} P_j(e^{it})\sum_{k=j}^{\infty} a_k\alpha^{k-j}.$$

Let

$$d_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(t)K_\alpha(t)dt.$$

Taking into account (26) and $\int_{-\pi}^{\pi} e^{ikt}K_\alpha(t)dt = 2\pi\alpha^{|k|},\ k = 0, \pm 1, \dots$, we get

$$d_0 = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k\alpha^k. \tag{29}$$

Furthermore let us define

$$d_k = \frac{1}{\pi}\int_{-\pi}^{\pi} h(t)P_k(e^{it})K_\alpha(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} h(t)(e^{ikt} - \alpha e^{i(k-1)t})K_\alpha(t)dt. \tag{30}$$

From (26) we obtain

$$\frac{1}{\pi}\int_{-\pi}^{\pi} h(t)e^{ikt}K_\alpha(t)dt = a_0\alpha^k + \sum_{j=1}^{\infty} a_j(\alpha^{j+k} + \alpha^{|j-k|}),\ k = 1, 2, \dots,$$

and that permits us to obtain after some calculations

$$d_k = (1-\alpha^2)\sum_{j=k}^{\infty} a_j\alpha^{j-k},\ k = 1, 2, \dots. \tag{31}$$

Taking into account (29) and (31) we write the function $g(e^{it})$ as

$$(32) \qquad 2g(e^{it}) = d_0 P_0(e^{it}) + \frac{1}{1-\alpha^2} \sum_{k=1}^{\infty} d_k P_k(e^{it})$$

and then from (28) and (32) one gets the complex form of the Fourier series of $h(t)$ with respect to the weight $K_\alpha(t)$:

$$(33) \qquad h(t) \sim g(e^{it}) + \overline{g(e^{it})} = d_0 P_0(e^{it}) + \frac{1}{2(1-\alpha^2)} \sum_{k=1}^{\infty} d_k \left(P_k(e^{it}) + \overline{P_k(e^{it})} \right).$$

For $\alpha = 0$ this form reduces to the well known complex form of the Fourier series of a real and even function. This form is an interesting reorganization of the absolutely convergent Fourier series expansion of $h(t)$. It represents the basis for our study of the quadrature error. By writing (33) as

$$h(t) \sim d_0 \gamma_0 + \sum_{k=1}^{\infty} \frac{d_k}{1-\alpha^2} \gamma_k$$

it is clear that it is the expansion of the even function $h(t)$ in the system $\gamma_0 \cup \{\gamma_k, \beta_k\}_{k=1}^{\infty}$ where $\gamma_0 = 1$ and for $k \geq 1$, $\gamma_k = \cos kt - \alpha \cos(k-1)t$, $\beta_k = \sin kt - \alpha \sin(k-1)t$. This system is known to be bi-orthonormal for the Poisson weight function $K_\alpha(t)$. See e.g. [8].

From the pointwise convergence of the Fourier series and (25) and (33) we write

$$(34) \qquad \begin{aligned} Q_n(h) = & \ d_0 Q_n(1) + \frac{1}{2(1-\alpha^2)} \sum_{k=1}^{\infty} d_k \, Q_n(R_k), \quad R_k(t) = 2\mathrm{Re}\, P_k(e^{it}) \\ = & \ d_0 I_n(1) + \frac{1}{2(1-\alpha^2)} \sum_{k=1}^{\infty} d_k \left(I_n(P_k) + I_n(\overline{P_k}) \right). \end{aligned}$$

From the definition of d_0 and since the quadrature formula I_n integrates constants exactly because its domain of validity is Λ_{-p_n,p_n} with $p_n \geq 1$, we get

$$Q_n(h) - Q(h) = \frac{1}{2(1-\alpha^2)} \sum_{k=1}^{\infty} d_k \left(I_n(P_k) + I_n(\overline{P_k}) \right).$$

Taking into account $P_k(z) = z^k - \alpha z^{k-1}$ and $z_{1,n} = 1$,

$$\begin{aligned} I_n(\overline{P_k}) &= c_{1,n} \overline{P_k(z_{1,n})} + \sum_{j=2}^{p_n+1} c_{j,n} \left(\overline{P_k(z_{j,n})} + \overline{P_k(\overline{z_{j,n}})} \right) \\ &= c_{1,n} P_k(z_{1,n}) + \sum_{j=2}^{p_n+1} c_{j,n} \left(P_k(\overline{z_{j,n}}) + P_k(z_{j,n}) \right) = I_n(P_k). \end{aligned}$$

Thus

$$Q_n(h) - Q(h) = \frac{1}{1-\alpha^2} \sum_{k=1}^{\infty} d_k I_n(P_k).$$

Some terms in this sum must vanish due to the aliasing property $z_{j,n}^{n\sigma} = 1$ of the roots of unity. Indeed, let us decompose $k = n\sigma + m$, where $m = -p_n, \dots, p_n$, and $\sigma = 0, 1, \dots$. Since Λ_{-p_n,p_n} is the domain of validity of $I_n(f)$ we find

$$\begin{aligned}
I_n(P_k) &= c_{1,n}(1-\alpha) + \sum_{j=2}^{p_n+1} c_{j,n} \left(z_{j,n}^{n\sigma+m} - \alpha z_{j,n}^{n\sigma+m-1} + z_{j,n}^{-(n\sigma+m)} - \alpha z_{j,n}^{-(n\sigma+m-1)} \right) \\
&= c_{1,n}(1-\alpha) + \sum_{j=2}^{p_n+1} c_{j,n} \left(z_{j,n}^{m} - \alpha z_{j,n}^{m-1} + z_{j,n}^{-m} - \alpha z_{j,n}^{-(m-1)} \right) \\
&= I_n(z^m) - \alpha I_n(z^{m-1}) = I(z^m) - \alpha I(z^{m-1}).
\end{aligned}$$

From where $I_n(P_k) = 0$ for the corresponding positive values of $m = 1, 2, \dots, p_n$, and $I_n(P_k) = 2\pi(1-\alpha^2)\alpha^{-m}$ for $m = 0, -1, \dots, -p_n$. (For $\alpha = 0$ and $m = 0$ it is understood that $\alpha^m = 1$.) The quadrature error becomes

$$Q_n(h) - Q(h) = 2\pi \sum_{k=1}^{\infty} \sum_{m=0}^{p_n} \alpha^m d_{nk-m}.$$

The interior sum can be simplified. Indeed, taking into account the integral representation (30) of d_{nk-m} and the formula for a geometric sum

$$\begin{aligned}
Q_n(h) - Q(h) &= 2 \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} h(t) K_\alpha(t) \left(\sum_{m=0}^{p_n} \alpha^m P_{nk-m}(e^{it}) \right) dt \\
&= 2 \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} h(t) K_\alpha(t) \left(e^{itnk} - \alpha^{p_n+1} e^{it(nk-p_n-1)} \right) dt \\
&= 2 \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} h(t) K_\alpha(t) \left(e^{itnk} - \alpha^{p_n+1} e^{-it/2} e^{itn(k-1/2)} \right) dt.
\end{aligned} \tag{35}$$

For functions $\phi \in C^\nu[a,b]$ the following formula known as the Fourier coefficient asymptotic expansion holds, see e.g. [17],

$$\begin{aligned}
\int_a^b \phi(t) e^{it\ell} dt = &- e^{ib\ell} \left(\frac{i}{\ell}\phi(b) + (\frac{i}{\ell})^2 \phi^{(1)}(b) + \cdots + (\frac{i}{\ell})^\nu \phi^{(\nu-1)}(b) \right) \\
&+ e^{ia\ell} \left(\frac{i}{\ell}\phi(a) + (\frac{i}{\ell})^2 \phi^{(1)}(a) + \cdots + (\frac{i}{\ell})^\nu \phi^{(\nu-1)}(a) \right) \\
&+ (\frac{i}{\ell})^\nu \int_a^b \phi^{(\nu)}(t) e^{it\ell} dt.
\end{aligned} \tag{36}$$

We next apply this formula to $\phi_1(t) = h(t)K_\alpha(t)$ and $\ell = nk$. The function $\phi_1(t) \in C^\nu[-\pi, \pi]$. Taking into account that $n = 2p_n + 1$ is odd, so that $e^{\pm i\pi n} = -1$, hence $e^{\pm ink} = (-1)^k$, and since $\phi_1(t)$ is an even function we write

$$\begin{aligned}\int_{-\pi}^{\pi} \phi_1(t)e^{itnk}dt &= \sum_{j=1}^{\nu}(\frac{i}{n})^j \frac{(-1)^k}{k^j}\Delta^-_{-\pi,\pi}\phi_1^{(j-1)}(t) + (\frac{i}{nk})^\nu \int_{-\pi}^{\pi} \phi_1^{(\nu)}(t)e^{itnk}dt \\ &= \sum_{j=1}^{\lfloor\nu/2\rfloor} \frac{(-1)^j}{n^{2j}}\frac{(-1)^k}{k^{2j}}\Delta^-_{-\pi,\pi}\phi_1^{(2j-1)}(t) + (\frac{i}{nk})^\nu \int_{-\pi}^{\pi} \phi_1^{(\nu)}(t)e^{itnk}dt\end{aligned}$$

where $\Delta^\pm_{a,b}g(t)$ denotes $\Delta^\pm_{a,b}g(t) = g(a) \pm g(b)$, for any function $g(t)$.

We apply again (36) but now to $\phi_2(t) = \phi_1(t)e^{-it/2}$ and $\ell = n(k-1/2)$. Taking into account that $e^{i\pi n(k-1/2)} = i(-1)^{k+p_n+1}$, (and hence $e^{-i\pi n(k-1/2)} = i(-1)^{k+p_n}$) and $\Delta^+_{-\pi,\pi}\phi_2(t) = 0$, one gets

$$\begin{aligned}\int_{-\pi}^{\pi} \phi_2(t)e^{itn(k-1/2)}dt =& i(-1)^{p_n}\sum_{j=2}^{\nu}(\frac{i}{n})^j \frac{(-1)^k}{(k-1/2)^j}\Delta^+_{-\pi,\pi}\phi_2^{(j-1)}(t) \\ &+ (\frac{i}{n(k-1/2)})^\nu \int_{-\pi}^{\pi} \phi_2^{(\nu)}(t)e^{itn(k-1/2)}dt.\end{aligned}$$

The quadrature error (35) takes the following expression after interchanging the order of summation

$$\begin{aligned}\frac{1}{2}(Q_n(h) - Q(h)) = &\sum_{j=1}^{\lfloor\nu/2\rfloor} \frac{(-1)^j}{n^{2j}}E_{2j}\Delta^-_{-\pi,\pi}\phi_1^{(2j-1)}(t) \\ &- i(-1)^{p_n}\alpha^{p_n+1}\sum_{j=2}^{\nu}(\frac{i}{n})^j C_j \Delta^+_{-\pi,\pi}\phi_2^{(j-1)}(t) + D_{n,\nu}\end{aligned}$$

where the values

$$E_j = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^j}, \quad C_j = \sum_{k=1}^{\infty}\frac{(-1)^k}{(k-1/2)^j}, \quad j \geq 2 \tag{37}$$

are the sum of absolutely convergent series and

$$D_{n,\nu} = \sum_{k=1}^{\infty}\left(\frac{i}{nk}\right)^\nu \int_{-\pi}^{\pi}\phi_1^{(\nu)}(t)e^{itnk}dt - \alpha^{p_n+1}\sum_{k=1}^{\infty}\left(\frac{i}{n(k-1/2)}\right)^\nu \int_{-\pi}^{\pi}\phi_2^{(\nu)}(t)e^{itn(k-1/2)}dt. \tag{38}$$

Thus

$$\frac{1}{2}(Q_n(h)-Q(h))=\sum_{j=1}^{\lfloor\nu/2\rfloor}\frac{(-1)^j}{n^{2j}}E_{2j}\Delta^-_{-\pi,\pi}\phi_1^{(2j-1)}(t)$$
$$-i(-1)^{p_n}\alpha^{p_n+1}\left\{\sum_{j=1}^{\lfloor\nu/2\rfloor}\frac{(-1)^j}{n^{2j}}C_{2j}\Delta^+_{-\pi,\pi}\phi_2^{(2j-1)}(t)\right.$$
$$\left.+i\sum_{j=1}^{\kappa}\frac{(-1)^j}{n^{2j+1}}C_{2j+1}\Delta^+_{-\pi,\pi}\phi_2^{(2j)}(t)\right\}+D_{n,\nu}$$

where $\kappa=\lfloor(\nu-1)/2\rfloor$. Then, using that $\phi_2(t)=\phi_1(t)e^{-it/2}=\phi_1(t)(\cos(t/2)-i\sin(t/2))$, we get

$$\frac{1}{2}(Q_n(h)-Q(h))=\sum_{j=1}^{\lfloor\nu/2\rfloor}\frac{(-1)^j}{n^{2j}}E_{2j}\Delta^-_{-\pi,\pi}(h(t)K_\alpha(t))^{(2j-1)} \tag{39}$$
$$+(-1)^{p_n}\alpha^{p_n+1}\left\{\sum_{j=1}^{\lfloor\nu/2\rfloor}\frac{(-1)^j}{n^{2j}}C_{2j}\Delta^+_{-\pi,\pi}(h(t)K_\alpha(t)\sin(t/2))^{(2j-1)}\right.$$
$$\left.+\sum_{j=1}^{\kappa}\frac{(-1)^j}{n^{2j+1}}C_{2j+1}\Delta^+_{-\pi,\pi}(h(t)K_\alpha(t)\cos(t/2))^{(2j)}\right\}+D_{n,\nu}.$$

Observe that $D_{n,\nu}$ given by (38) is a real value depending on $h(t)$.

Remark 5 The particular case of this formula when $\alpha=0$ is well known. Indeed, for $\alpha=0$ is $K_\alpha(t)\equiv 1$. Consider a function $\psi\in C^\nu[-\pi,\pi]$, $\nu\geq 2$, with even part ψ_e and odd part ψ_o. Since $Q_n(\psi)-Q(\psi)=Q_n(\psi_e)-Q(\psi_e)$, we can use the previous relation with $\alpha=0$, and ψ_e in the role of h to get

$$Q_n(\psi)-Q(\psi)=2\sum_{j=1}^{\lfloor\nu/2\rfloor}\frac{(-1)^j}{n^{2j}}E_{2j}\Delta^-_{-\pi,\pi}\psi_e^{(2j-1)}(t)+2\sum_{k=1}^{\infty}\left(\frac{i}{nk}\right)^\nu\int_{-\pi}^{\pi}\psi_e^{(\nu)}(t)e^{itnk}dt \tag{40}$$
$$=2\sum_{j=1}^{\lfloor\nu/2\rfloor}\frac{(-1)^{j+1}}{n^{2j}}E_{2j}\Big(\psi^{(2j-1)}(\pi)-\psi^{(2j-1)}(-\pi)\Big) \tag{41}$$
$$+2\sum_{k=1}^{\infty}\left(\frac{i}{nk}\right)^\nu\int_{-\pi}^{\pi}\psi_e^{(\nu)}(t)e^{itnk}dt \tag{42}$$

where we have used in (41) that $\Delta^-_{-\pi,\pi}\psi_o^{(2j-1)}(t)=0$. The coefficient $2(-1)^{j+1}E_{2j}$ of $1/n^{2j}$

in Eq. (41) admits the alternative expression

$$(43)\qquad \begin{aligned} 2(-1)^{j+1}E_{2j} =& 2(-1)^{j+1}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{2j}} \\ =& 2(-1)^{j+1}\Big(-\sum_{k=1}^{\infty}\frac{1}{k^{2j}}+2\sum_{k=1}^{\infty}\frac{1}{(2k)^{2j}}\Big) \\ =& 2(-1)^{j+1}(-1+2^{1-2j})\sum_{k=1}^{\infty}\frac{1}{k^{2j}}. \end{aligned}$$

We recall the property

$$(44)\qquad \zeta(2j)=\sum_{k=1}^{\infty}\frac{1}{k^{2j}}=\frac{(-1)^{j+1}(2\pi)^{2j}B_{2j}}{2(2j)!}$$

of the Riemann zeta function $\zeta(z)$ for positive integers j, and where B_{2j} are the even Bernoulli numbers. See e.g. [7] and the references in there. From (43) and (44) we get

$$2(-1)^{j+1}E_{2j}=-\frac{(1-2^{1-2j})(2\pi)^{2j}}{(2j)!}B_{2j}.$$

So we get an alternative expression for Eq. (41)

$$2\sum_{j=1}^{\lfloor\nu/2\rfloor}\frac{(-1)^{j+1}E_{2j}}{n^{2j}}\Big(\psi^{(2j-1)}(\pi)-\psi^{(2j-1)}(-\pi)\Big)=\sum_{j=1}^{\lfloor\nu/2\rfloor}\frac{c_{2j}\delta^{2j}}{(2j)!}\Big(\psi^{(2j-1)}(\pi)-\psi^{(2j-1)}(-\pi)\Big)$$

where $\delta=2\pi/n,\ c_{2j}=-(1-2^{1-2j})B_{2j}$.

Observe since $\psi_e\in C^{\nu}[-\pi,\pi]$ and $\nu\ge 2$ the term in Eq. (42), $2\sum_{k=1}^{\infty}\left(\frac{i}{nk}\right)^{\nu}\int_{-\pi}^{\pi}\psi_e^{(\nu)}(t)e^{itnk}dt$, is of order $O(1/n^{\nu})$.

For functions $\psi\in C^{\infty}[-\pi,\pi]$ our quadrature error formula (40) for the midpoint rule reproduces then a well known asymptotic formula

$$Q_n(\psi)-Q(\psi)\approx\sum_{j=1}^{\infty}\frac{c_{2j}}{(2j)!}\delta^{2j}\Big(\psi^{(2j-1)}(\pi)-\psi^{(2j-1)}(-\pi)\Big)$$

for the quadrature error in the midpoint rule, see e.g. [12, page 139].

Concerning quadrature on the unit circle we have obtained the following Theorem.

Theorem 14 *Let* $I_n(f)=c_{1,n}f(z_{1,n})+\sum_{j=2}^{p_n+1}c_{j,n}\left(f(z_{j,n})+f(\overline{z_{j,n}})\right)$, *be the* n*-point quadrature formula with nodes the roots of unity* $z_{j,n}^n=1$, *(where* $z_{1,n}=1$*), and with domain of validity* Λ_{-p_n,p_n}, $n=2p_n+1$, *to approximate integrals*

$$I(f)=\int_{-\pi}^{\pi}f(e^{it})K_\alpha(t)dt$$

where $K_\alpha(t)$ is the Poisson weight function (16) with $0 \le \alpha < 1$. For functions $f(e^{it})$ of class C^ν in $t \in [-\pi, \pi]$ the quadrature error $I_n(f) - I(f)$ admits the expression

$$I_n(f) - I(f) = 2\left\{U_{n,\nu} + (-1)^{p_n}\alpha^{p_n+1}(V_{n,\nu} + W_{n,\nu}) + D_{n,\nu}\right\} \tag{45}$$

where

$$U_{n,\nu} = \sum_{j=1}^{\lfloor \nu/2 \rfloor} \frac{(-1)^j}{n^{2j}} E_{2j} \Delta^-_{-\pi,\pi} \left(f_e(t)K_\alpha(t)\right)^{(2j-1)} \tag{46}$$

$$V_{n,\nu} = \sum_{j=1}^{\lfloor \nu/2 \rfloor} \frac{(-1)^j}{n^{2j}} C_{2j} \Delta^+_{-\pi,\pi} \left(f_e(t)K_\alpha(t)\sin(t/2)\right)^{(2j-1)} \tag{47}$$

$$W_{n,\nu} = \sum_{j=1}^{\kappa} \frac{(-1)^j}{n^{2j+1}} C_{2j+1} \Delta^+_{-\pi,\pi} \left(f_e(t)K_\alpha(t)\cos(t/2)\right)^{(2j)} \tag{48}$$

and $f_e(t) = f_{1,e}(t) + i f_{2,e}(t)$, $\kappa = \lfloor(\nu-1)/2\rfloor$. The E_j and C_j are given by (37) and

$$D_{n,\nu} = \sum_{k=1}^{\infty} \left(\frac{i}{nk}\right)^\nu \int_{-\pi}^{\pi} h_1^{(\nu)}(t) e^{itnk} dt - \alpha^{p_n+1} \sum_{k=1}^{\infty} \left(\frac{i}{n(k-1/2)}\right)^\nu \int_{-\pi}^{\pi} h_2^{(\nu)}(t) e^{itn(k-1/2)} dt \tag{49}$$

where $h_1(t) = f_e(t)K_\alpha(t)$ and $h_2(t) = h_1(t)e^{-it/2}$.

Proof. As it was commented at the beginning of this Section $f(e^{it}) = f_{1,e}(t) + f_{1,o}(t) + i(f_{2,e}(t) + f_{2,o}(t))$. Apply (39) to $f_{1,e}(t)$ and $f_{2,e}(t)$ the even real and imaginary parts and take into account (24). □

For analytic functions $f(z)$ on the open disc $|z| < R$ with $R > 1$ or in the annulus $r < |z| < R$ where $0 < r < 1$ and $R > 1$, the even functions $f_{j,e}(t)$, $j = 1, 2$, are 2π-periodic functions of class $C^\infty(\mathbb{R})$. Since $K_\alpha(t)$ has the same property, also $g(t) = f_e(t)K_\alpha(t)$ will be even and 2π-periodic and all its derivatives exist and are 2π-periodic.

Because an odd order derivative of an even function is odd, and by the mentioned properties of $g(t)$, it follows that $g^{(2\ell-1)}(\pi) = g^{(2\ell-1)}(-\pi) = 0$. As a result $U_{n,\nu} = 0$.

On the other hand, all odd derivatives of $\sin(t/2)$ vanish at $t = -\pi$ and $t = \pi$. Then, by Leibniz rule all terms in the derivative $(g(t)\sin(t/2))^{(2j-1)}$ contain at least the odd derivative of one of the two functions and hence

$$\Big(g(t)\sin(t/2)\Big)^{(2j-1)}\bigg|_{t=\pi} = \Big(g(t)\sin(t/2)\Big)^{(2j-1)}\bigg|_{t=-\pi} = 0.$$

Thus $V_{n,\nu} = 0$. By a similar reasoning $W_{n,\nu} = 0$. For analytic functions our asymptotic formula vanishes but suggests a convergence rate faster than any power of $1/n$.

Next, we want to show that the quadrature error expression obtained in (45) can be appropiately expressed in a form suitable to apply a Romberg type quadrature technique.

Indeed, although the coefficients of $1/n^j$ in the three sums $U_{n,\nu}, V_{n,\nu}$ and $W_{n,\nu}$ do not depend on n, a fundamental need in Romberg quadrature, we note however that the common multiplicative factor α^{p_n+1} in front of the sums $V_{n,\nu}$ and $W_{n,\nu}$ depends on p_n (hence on n). We can use this as follows. Consider the positive function $y = \alpha^{1/x}$, $x \in (0,2)$. We are interested in showing that it has a convergent Taylor series around $x = 1$ for $x \in (0,2)$. It holds

$$\log y = \frac{1}{x}\log\alpha = \frac{1}{1-(1-x)}\log\alpha = \log\alpha\sum_{\ell=0}^{\infty}(1-x)^{\ell}$$

and the geometric series converges for $|1-x| < 1$ that is $x \in (0,2)$. The exponential function has a convergent Taylor series around $x = 0$ for $x \in \mathbb{R}$ then

$$y = \prod_{\ell=0}^{\infty} e^{(1-x)^{\ell}\log\alpha} = \alpha\prod_{\ell=1}^{\infty}\sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{k!}(1-x)^{\ell k}.$$

Expanding terms we can write the function $y = \sum_{\ell=0}^{\infty} a_\ell (x-1)^\ell$ as a power series around $x = 1$ that is convergent in $x \in (0,2)$. Since $y \in C^\infty(0,2)$ this power series is the Taylor series around $x = 1$ with $a_\ell = y^{(\ell)}(1)/\ell!$.

For $x = 2/n \in (0,2)$, $n = 2p_n + 1$, we deduce

$$\alpha^{p_n+1} = \alpha^{1/2}\alpha^{n/2} = \alpha^{1/2}\sum_{\ell=0}^{\infty} a_\ell\left(\frac{2}{n}-1\right)^\ell = \sum_{\ell=0}^{\infty} b_\ell\frac{1}{n^\ell} \tag{50}$$

where

$$b_\ell = 2^\ell\alpha^{1/2}\sum_{j=\ell}^{\infty}(-1)^{j-\ell}\binom{j}{j-\ell}a_j, \quad \ell \ge 0.$$

We have then a convergent expansion of α^{p_n+1} in powers of $1/n$. Next we rewrite the quadrature error (45) in powers of $1/n$. Indeed, since $h_1(t)$ and $h_2(t)$ given in Theorem 14 are functions of class $C^\nu[-\pi,\pi]$, and $\nu \ge 2$, the terms

$$\sum_{k=1}^{\infty}\left(\frac{i}{nk}\right)^{\nu}\int_{-\pi}^{\pi} h_1^{(\nu)}(t)e^{itnk}dt$$

and

$$\alpha^{p_n+1}\sum_{k=1}^{\infty}\left(\frac{i}{n(k-1/2)}\right)^{\nu}\int_{-\pi}^{\pi} h_2^{(\nu)}(t)e^{itn(k-1/2)}dt$$

of $D_{n,\nu}$ in (49) are of order $O(1/n^\nu)$. So $D_{n,\nu}$ is of order $O(1/n^\nu)$. We can write then (45) in the form

$$I_n(f) - I(f) = 2\left\{U_{n,\nu} + (-1)^{p_n}\alpha^{p_n+1}(V_{n,\nu} + W_{n,\nu})\right\} + O(1/n^\nu). \tag{51}$$

From (47), (48) and (50) we can write

$$\alpha^{p_n+1}(V_{n,\nu}+W_{n,\nu}) = \sum_{\ell=0}^{\infty} b_\ell \frac{1}{n^\ell}(V_{n,\nu}+W_{n,\nu}) = \sum_{j=2}^{\nu-1}\frac{d_j}{n^j} + \sum_{j=3}^{\nu-1}\frac{e_j}{n^j} + O(\frac{1}{n^\nu})$$

where

$$d_j = \sum_{\ell=1}^{\lfloor j/2 \rfloor} v_{2\ell} b_{j-2\ell}, \quad e_j = \sum_{\ell=1}^{\kappa} w_{2\ell+1} b_{j-(2\ell+1)},$$

$\kappa = \lfloor (j-1)/2 \rfloor$, and

$$\begin{aligned} v_{2\ell} =&(-1)^\ell C_{2\ell}\Delta^+_{-\pi,\pi}\left(f_e(t)K_\alpha(t)\sin(t/2)\right)^{(2\ell-1)}, \\ w_{2\ell+1} =&(-1)^\ell C_{2\ell+1}\Delta^+_{-\pi,\pi}\left(f_e(t)K_\alpha(t)\cos(t/2)\right)^{(2\ell)}. \end{aligned}$$

We denote $u_{2j} = (-1)^j E_{2j}\Delta^-_{-\pi,\pi}\left(f_e(t)K_\alpha(t)\right)^{(2j-1)}$.

These calculations allows us to write (51) in the form

$$\begin{aligned} I_n(f) - I(f) =& 2\sum_{j=1}^{\lfloor \nu/2 \rfloor -1}\frac{u_{2j}}{n^{2j}} + 2(-1)^{p_n}\left(\sum_{j=2}^{\nu-1}\frac{d_j}{n^j} + \sum_{j=3}^{\nu-1}\frac{e_j}{n^j}\right) + O(\frac{1}{n^\nu}) \\ =& \sum_{j=2}^{\nu-1}\frac{A_j}{n^j} + O(\frac{1}{n^\nu}) \end{aligned}$$

where $A_2 = 2(u_2 + (-1)^{p_n} d_2)$, $A_3 = 2(-1)^{p_n}(d_3+e_3)$ and for $j \geq 2$ is $A_{2j} = 2(u_{2j} + (-1)^{p_n}(d_{2j}+e_{2j}))$ and $A_{2j+1} = 2(-1)^{p_n}(d_{2j+1}+e_{2j+1})$. Taking, as example, $p_n = 2\tau$ as an even value we remove the dependence of the $A_j's$ on n. In that case $n = 2p_n + 1 = 4\tau + 1$.

For functions $f(e^{it})$ of class $C^\infty[-\pi,\pi]$ we have obtained then an asymptotic expansion for the quadrature error given by

$$I_n(f) - I(f) = \frac{A_2}{n^2} + \frac{A_3}{n^3} + \frac{A_4}{n^4} + \cdots, \ n = 4\tau+1, \ \tau = 1,2,3,\ldots$$

where the coefficients A_j do not depend on n.

We end noting that this asymptotic expansion has the typical form required for the Richardson extrapolation technique leading to a modified Romberg type quadrature method. As is well known this technique will permit us to obtain a sequence of integral estimation with increasing rate of convergence by removing successively powers of $1/n$.

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