CLASSIFICATION OF A FAMILY OF NON ALMOST PERIODIC FREE ARAKI–WOODS FACTORS

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Abstract. We obtain a complete classification of a large class of non almost periodic free Araki–Woods factors $\Gamma(\mu, m)''$ up to isomorphism. We do this by showing that free Araki–Woods factors $\Gamma(\mu, m)''$ arising from finite symmetric Borel measures μ on $\mathbf{R}$ whose atomic part μ_a is nonzero and not concentrated on $\{0\}$ have the joint measure class $\mathcal{C}(\bigvee_{k\geq 1} \mu^{*k})$ as an invariant. Our key technical result is a deformation/rigidity criterion for the unitary conjugacy of two faithful normal states. We use this to also deduce rigidity and classification theorems for free product von Neumann algebras.

1. Introduction

Free Araki-Woods factors are a free probability analog of the type III hyperfinite factors, just like free group factors are free probability analogs of the hyperfinite II_1 factor. The classification of hyperfinite type III factors has a beautiful history originating in the work of Powers [Po67]. Through the works of Connes [Co72], Haagerup [Ha85] and Krieger [Kr75], the classification question was ultimately reduced to the classification of ergodic actions of the additive group of real numbers $\mathbf{R}$, i.e., to classification of virtual subgroups of $\mathbf{R}$ in the sense of Mackey.

Following [Sh96], to every orthogonal representation $(U_t)_{t\in\mathbf{R}}$ of $\mathbf{R}$ on a real Hilbert space $H_\mathbf{R}$ is associated the free Araki–Woods factor $\Gamma(U, H_\mathbf{R})''$. For almost periodic representations, i.e. when (U_t) is a direct sum of finite dimensional representations, the free Araki–Woods factors were completely classified in [Sh96] by Connes' Sd invariant [Co74], which is in this case equal to the subgroup of $(\mathbf{R}^*_+, \cdot)$ generated by the eigenvalues of (U_t). Beyond the almost periodic case, the classification of free Araki–Woods factors is a very intriguing open problem and there is not even a conjectural classification statement. So far, one could only distinguish between families of non almost periodic free Araki–Woods factors by computing their invariants, like Connes' τ-invariant (see [Sh97b, Sh02]), or by structural properties of their continuous core (see [Sh02, Ho08b, Ha15]). In this paper, we prove the first complete classification theorem for a large family of non almost periodic free Araki–Woods factors.

Orthogonal representations of $\mathbf{R}$ are classified by their spectral measure and multiplicity function. So, to any finite symmetric Borel measure μ on $\mathbf{R}$ and to any symmetric Borel multiplicity function $m : \mathbf{R} \to \mathbf{N} \cup \{+\infty\}$ (that we always assume to satisfy $m \geq 1$ μ-almost everywhere), we associate the free Araki–Woods factor $\Gamma(\mu, m)''$, which comes equipped with the free quasi-free state $\varphi_{\mu,m}$. The almost periodic case corresponds to μ being an atomic measure and then, by [Sh96], $\Gamma(\mu_1, m_1)''$ is isomorphic with $\Gamma(\mu_2, m_2)''$ if and only if the sets of atoms of μ_1 and μ_2 generate the same subgroup of $(\mathbf{R}, +)$.

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In this paper, we fully classify the free Araki–Woods factors in the case where the atomic part μ_a is nonzero and not concentrated on $\{0\}$ and where the continuous part μ_c satisfies $\mu_c * \mu_c \prec \mu_c$. We find in particular that in that case, the free Araki–Woods factor does not depend on the multiplicity function m. But we also show that in other cases, $\Gamma(\mu, m)''$ does depend on m.

In order to state our main results, we first introduce some terminology. For every σ-finite Borel measure μ on $\mathbf{R}$, we denote by $\mathcal{C}(\mu)$ the *measure class* of μ, defined as the set of all Borel sets $U \subset \mathbf{R}$ with $\mu(U) = 0$. Note that $\mathcal{C}(\mu) = \mathcal{C}(\nu)$ if and only if $\mu \sim \nu$, while $\mathcal{C}(\mu) \subset \mathcal{C}(\nu)$ if and only if $\nu \prec \mu$. For any sequence of measures $(\mu_k)_{k\in\mathbf{N}}$, we denote by $\bigvee_{k\in\mathbf{N}} \mu_k$ any measure with the property that $\mathcal{C}(\bigvee_{k\in\mathbf{N}} \mu_k) = \bigcap_{k\in\mathbf{N}} \mathcal{C}(\mu_k)$. We denote by $\mu = \mu_c + \mu_a$ the unique decomposition of a measure μ as the sum of a continuous and an atomic measure.

We show that free Araki–Woods factors $\Gamma(\mu, m)''$ arising from finite symmetric Borel measures μ on $\mathbf{R}$ whose atomic part μ_a is nonzero and not concentrated on $\{0\}$ have the joint measure class $\mathcal{C}(\bigvee_{k\geq 1} \mu^{*k})$ as an invariant. More precisely, we obtain the following result.

Theorem A. *Let μ, ν be finite symmetric Borel measures on $\mathbf{R}$ and $m, n : \mathbf{R} \to \mathbf{N} \cup \{+\infty\}$ symmetric Borel multiplicity functions. Assume that ν has at least one atom not equal to 0.*

If the free Araki–Woods factors $\Gamma(\mu, m)''$ and $\Gamma(\nu, n)''$ are isomorphic, then there exists an isomorphism that preserves the free quasi-free states. In particular, the joint measure classes $\mathcal{C}(\bigvee_{k\geq 1} \mu^{*k})$ and $\mathcal{C}(\bigvee_{k\geq 1} \nu^{*k})$ are equal.

Denote by $\mathcal{S}(\mathbf{R})$ the set of all finite symmetric Borel measures $\mu = \mu_c + \mu_a$ on $\mathbf{R}$ satisfying the following two properties:

(i) $\mu_c * \mu_c \prec \mu_c$ and
(ii) $\mu_a \neq 0$ and $\operatorname{supp}(\mu_a) \neq \{0\}$.

Denote by $\Lambda(\mu_a)$ the countable subgroup of $\mathbf{R}$ generated by the atoms of μ_a and by $\delta_{\Lambda(\mu_a)}$ a finite atomic measure on $\mathbf{R}$ whose set of atoms equals $\Lambda(\mu_a)$.

Combining our Theorem A with the isomorphism Theorem 4.1 below, we obtain a complete classification of the free Araki–Woods factors arising from measures in $\mathcal{S}(\mathbf{R})$. Here and elsewhere in this paper, we call *isomorphism* between von Neumann algebras M and N any bijective $*$-isomorphism. Even when M and N are equipped with distinguished faithful normal states, isomorphisms are not assumed to preserve these states.

Corollary B. *The set of free Araki–Woods factors*

$$\left\{\Gamma(\mu, m)'' : \mu \in \mathcal{S}(\mathbf{R}) \text{ and } m : \mathbf{R} \to \mathbf{N} \cup \{+\infty\} \text{ is a symmetric Borel multiplicity function}\right\}$$

is exactly classified, up to isomorphism, by the countable subgroup $\Lambda(\mu_a)$ and the measure class $\mathcal{C}(\mu_c * \delta_{\Lambda(\mu_a)})$.

Note that the measure class $\mathcal{C}(\mu_c * \delta_{\Lambda(\mu_a)})$ equals the set of Borel sets $U \subset \mathbf{R}$ satisfying $\mu_c(x + U) = 0$ for all $x \in \Lambda(\mu_a)$ and, in particular, does not depend on the choice of $\delta_{\Lambda(\mu_a)}$.

The family $\mathcal{S}(\mathbf{R})$ is large and provides many nonisomorphic free Araki–Woods factors having the same Connes' invariants and in particular the same τ-invariant, see Example 5.4. Note that previously, only two non almost periodic free Araki–Woods factors having the same τ-invariant could be distinguished, see [Sh02, Theorem 5.6].

Combining our Corollary B with [HI15, Theorem A], we also obtain a complete classification for tensor products of free Araki–Woods factors arising from measures in $\mathcal{S}(\mathbf{R})$.

We then show that free Araki–Woods factors $\Gamma(\mu, m)''$ arising from continuous finite symmetric Borel measures μ on $\mathbf{R}$ have all their centralizers amenable, i.e. the centralizer of any faithful normal state is amenable. More generally, we obtain the following result.

Corollary C. *Let μ be any finite symmetric Borel measure on $\mathbf{R}$ and $m : \mathbf{R} \to \mathbf{N} \cup \{+\infty\}$ any symmetric Borel multiplicity function. The free Araki–Woods factor $\Gamma(\mu, m)''$ has all its centralizers amenable if and only if the atomic part μ_a of μ is either zero or is concentrated on $\{0\}$ with $m(0) = 1$.*

By [Ho08a], all free Araki–Woods factors M satisfy Connes' bicentralizer conjecture (see [Co80]) and thus, by [Ha85, Theorem 3.1], admit faithful normal states φ such that $M^\varphi \subset M$ is an irreducible subfactor. So, having all centralizers amenable is the smallest centralizers can be in general.

In the setting of Corollary C and under the additional assumption that the Fourier transform of the continuous finite symmetric Borel measure μ_c vanishes at infinity, it was shown in [Ho08b, Theorem 1.2] that the continuous core of the corresponding free Araki–Woods factor $\Gamma(\mu, m)''$ is *solid* (see [Oz03]), meaning that the relative commutant of any diffuse subalgebra that is the range of a faithful normal conditional expectation is amenable. Any type III_1 factor whose continuous core is solid has all its centralizers amenable. Observe that there are many free Araki–Woods factors arising in Corollary C whose Connes' τ-invariant (see [Co74]) is not the usual topology on $\mathbf{R}$. In particular, these free Araki–Woods factors have a continuous core that is not *full* (see [Co74, Sh02]) and hence not solid (see [Oz03, Proposition 7] with $\mathcal{N}_0 = \mathcal{M}$). Therefore, Corollary C provides many new examples of type III_1 factors whose centralizers are all amenable.

The following is an immediate consequence of Corollary C.

Corollary D. *Let λ be the Lebesgue measure on $\mathbf{R}$. Then $\Gamma(\lambda + \delta_0, 1)'' \ncong \Gamma(\lambda + \delta_0, 2)''$. So in certain cases, the isomorphism class of $\Gamma(\mu, m)''$ depends on the multiplicity function m.*

Our main technical tool to prove the results mentioned so far is a deformation/rigidity criterion for the unitary conjugacy of two faithful normal states on a von Neumann algebra M. In Corollary 3.2 below, we prove that a corner of the state ψ is unitarily conjugate with a corner of the state φ if and only if in the continuous core $\mathrm{c}(M)$, there is a Popa intertwining bimodule (in the sense of [Po02, Po03]) between the canonical subalgebras $\mathrm{L}_\psi(\mathbf{R})$ and $\mathrm{L}_\varphi(\mathbf{R})$ of $\mathrm{c}(M)$ given by realizing $\mathrm{c}(M)$ as respectively $M \rtimes_{\sigma^\psi} \mathbf{R}$ and $M \rtimes_{\sigma^\varphi} \mathbf{R}$ (see Section 2 for details).

More generally, when $P \subset M$ is a von Neumann subalgebra that is the range of a faithful normal conditional expectation $\mathrm{E}_P : M \to P$, we provide in Theorem 3.1 below a deformation/rigidity criterion describing when a state ψ on M has a corner that is unitarily conjugate with a corner of a state of the form $\theta \circ \mathrm{E}_P$. Applying this criterion to a free product von Neumann algebra M, we obtain the following complete characterization of when M has all its centralizers amenable.

Theorem E. *For $i = 1, 2$, let (M_i, φ_i) be a von Neumann algebra with a faithful normal state. Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ their free product. Then M has all its centralizers amenable if and only if both M_1 and M_2 have all their centralizers amenable and M^φ is amenable.*

Finally, we use the same criterion, in combination with methods of [Sh97a], to prove the following classification result for a free product with a free Araki–Woods factor.

Theorem F. *Let μ be a continuous finite symmetric Borel measure on $\mathbf{R}$. Fix the free Araki–Woods factor $(M, \varphi) = (\Gamma(\mu, +\infty)'', \varphi_{\mu,+\infty})$, where $+\infty$ denotes the multiplicity function equal to $+\infty$ everywhere.*

(i) *If (A, τ) and (B, τ) are nonamenable II_1 factors with their tracial states, then the free products $(M, \varphi) * (A, \tau)$ and $(M, \varphi) * (B, \tau)$ are isomorphic (not necessarily in a state preserving way) if and only if there exists a $t > 0$ such that $A \cong B^t$.*

(ii) *If* (A_i, ψ_i), $i = 1, 2$, *are full type* III *factors with almost periodic states having a factorial centralizer* $A_i^{\psi_i}$, *then the free products* $(M, \varphi) * (A_1, \psi_1)$ *and* $(M, \varphi) * (A_2, \psi_2)$ *are isomorphic (again, not necessarily in a state preserving way) if and only if* $A_1 \cong A_2$.

By Theorem F, the free Araki–Woods factors $\Gamma((\lambda, +\infty) + (\delta_0, m))''$ are isomorphic for all $2 \leq m < +\infty$, but the question whether these are isomorphic with $\Gamma((\lambda, +\infty) + (\delta_0, +\infty))''$ is equivalent with the free group factor problem $\mathrm{L}(\mathbf{F}_m) \cong^? \mathrm{L}(\mathbf{F}_\infty)$.

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Disclaimer. Some of the results in this paper, in particular Corollary D, Theorem 4.1 and Proposition 7.5, were obtained in the preprint [Sh03] by the second named author. That preprint will remain unpublished and has been incorporated in this article.

Contents

2. Preliminaries

For any von Neumann algebra M, we denote by $\mathcal{U}(M)$ its group of unitaries. For any (possibly unbounded) positive selfadjoint closed operator A on a separable Hilbert space H, we denote by $\mathcal{C}(A)$ the measure class on $\mathbf{R}$ of the spectral measure of $\log(A)$, i.e. the set of all Borel sets $\mathcal{U} \subset \mathbf{R}$ such that the spectral projection $1_{\mathcal{U}}(\log(A))$ equals 0. Here $1_{\mathcal{U}}$ denotes the function that is equal to 1 on $\mathcal{U}$ and equal to 0 elsewhere. Note that we can always choose a measure μ on $\mathbf{R}$ such that $\mathcal{C}(A) = \mathcal{C}(\mu)$.

Free Araki–Woods factors. Following [Sh96], we associate to every orthogonal representation $(U_t)_{t \in \mathbf{R}}$ of $\mathbf{R}$ on the real Hilbert space $H_\mathbf{R}$ the free Araki–Woods factor $\Gamma(H_\mathbf{R}, U_t)''$, equipped with the free quasi-free state φ_U.

Denoting by $H = H_\mathbf{R} + \mathrm{i}H_\mathbf{R}$ the complexification of $H_\mathbf{R}$, define the positive nonsingular operator Δ on H such that $U_t = \Delta^{\mathrm{i}t}$ for all $t \in \mathbf{R}$. Also define the anti-unitary operator $J : H \to H : J(\xi + \mathrm{i}\eta) = \xi - \mathrm{i}\eta$ for all $\xi, \eta \in H_\mathbf{R}$. Then, $J\Delta J = \Delta^{-1}$. Therefore, the measure class $\mathcal{C}(\Delta)$ of the spectral measure of $\log(\Delta)$ and the multiplicity function $m : \mathbf{R} \to \mathbf{N} \cup \{+\infty\}$ of $\log(\Delta)$ are symmetric. This measure class and multiplicity function completely classify orthogonal representations of $\mathbf{R}$ on separable real Hilbert spaces. We therefore use the notation $(\Gamma(\mu, m)'', \varphi_{\mu,m})$ to denote the free Araki–Woods factor and its free quasi-free state associated with the unique orthogonal representation with spectral invariant (μ, m).

Background on σ-finite von Neumann algebras. Let M be any σ-finite von Neumann algebra with predual M_* and $\varphi \in M_*$ any faithful normal state. We denote by σ^φ the modular automorphism group of the state φ defined by the formula $\sigma_t^\varphi = \mathrm{Ad}(\Delta_\varphi^{it})$ for all $t \in \mathbf{R}$. The *centralizer* M^φ of the state φ is by definition the fixed point algebra of (M, σ^φ). The *continuous core* of M with respect to φ, denoted by $\mathrm{c}_\varphi(M)$, is the crossed product von Neumann algebra $M \rtimes_{\sigma^\varphi} \mathbf{R}$. The natural inclusion $\pi_\varphi : M \to \mathrm{c}_\varphi(M)$ and the unitary representation $\lambda_\varphi : \mathbf{R} \to \mathrm{c}_\varphi(M)$ satisfy the *covariance* relation

$$\lambda_\varphi(t)\pi_\varphi(x)\lambda_\varphi(t)^* = \pi_\varphi(\sigma_t^\varphi(x)) \quad \text{for all } x \in M \text{ and all } t \in \mathbf{R}.$$

Put $\mathrm{L}_\varphi(\mathbf{R}) := \lambda_\varphi(\mathbf{R})''$. There is a unique faithful normal conditional expectation $\mathrm{E}_{\mathrm{L}_\varphi(\mathbf{R})} : \mathrm{c}_\varphi(M) \to \mathrm{L}_\varphi(\mathbf{R})$ satisfying $\mathrm{E}_{\mathrm{L}_\varphi(\mathbf{R})}(\pi_\varphi(x)\lambda_\varphi(t)) = \varphi(x)\lambda_\varphi(t)$ for all $x \in M$ and all $t \in \mathbf{R}$. The faithful normal semifinite weight defined by $f \mapsto \int_{\mathbf{R}} \exp(-s) f(s)\, \mathrm{d}s$ on $\mathrm{L}^\infty(\mathbf{R})$ gives rise to a faithful normal semifinite weight Tr_φ on $\mathrm{L}_\varphi(\mathbf{R})$ via the Fourier transform. The formula $\mathrm{Tr}_\varphi = \mathrm{Tr}_\varphi \circ \mathrm{E}_{\mathrm{L}_\varphi(\mathbf{R})}$ extends it to a faithful normal semifinite trace on $\mathrm{c}_\varphi(M)$.

Because of Connes' Radon–Nikodym cocycle theorem [Co72, Théorème 1.2.1] (see also [Ta03, Theorem VIII.3.3]), the semifinite von Neumann algebra $\mathrm{c}_\varphi(M)$ together with its trace Tr_φ does not depend on the choice of φ in the following precise sense. If $\psi \in M_*$ is another faithful state, there is a canonical isomorphism $\Pi_{\varphi,\psi} : \mathrm{c}_\psi(M) \to \mathrm{c}_\varphi(M)$ of $\mathrm{c}_\psi(M$ onto $\mathrm{c}_\varphi(M)$ such that $\Pi_{\varphi,\psi} \circ \pi_\psi = \pi_\varphi$ and $\mathrm{Tr}_\varphi \circ \Pi_{\varphi,\psi} = \mathrm{Tr}_\psi$. Note however that $\Pi_{\varphi,\psi}$ does not map the subalgebra $\mathrm{L}_\psi(\mathbf{R}) \subset \mathrm{c}_\psi(M)$ onto the subalgebra $\mathrm{L}_\varphi(\mathbf{R}) \subset \mathrm{c}_\varphi(M)$ (and hence we use the symbol $\mathrm{L}_\varphi(\mathbf{R})$ instead of the usual $\mathrm{L}(\mathbf{R})$). We have $\Pi_{\varphi,\psi}(\lambda_\psi(t)) = \pi_\varphi(w_t)\lambda_\varphi(t)$ for every $t \in \mathbf{R}$, where $w_t = [\mathrm{D}\psi : \mathrm{D}\varphi]_t$ is Connes' Radon–Nikodym cocycle between ψ and φ.

Lemma 2.1. *Let M be any σ-finite von Neumann algebra and $\varphi \in M_*$ any faithful state such that M^φ is a II_1 factor. Let $M^\varphi \subset P \subset M$ be any intermediate von Neumann subalgebra that is globally invariant under the modular automorphism group σ^φ. Denote by $\mathrm{E}_P : M \to P$ the unique φ-preserving conditional expectation and write $M \ominus P := \ker(\mathrm{E}_P)$. Let $p \in M^\varphi$ be any nonzero projection and put $\varphi_p := \frac{\varphi(p \cdot p)}{\varphi(p)} \in (pMp)_*$.*

Then we have

$$\mathcal{C}(\Delta_\varphi) = \mathcal{C}(\Delta_{\varphi_p}) \quad \textit{and} \quad \mathcal{C}(\Delta_\varphi|_{\mathrm{L}^2(M)\ominus \mathrm{L}^2(P)}) = \mathcal{C}(\Delta_{\varphi_p}|_{\mathrm{L}^2(pMp)\ominus \mathrm{L}^2(pPp)}).$$

Proof. Fix a standard representation $M \subset \mathrm{B}(H)$ and denote by $\xi_\varphi \in H$ the canonical unit vector that implements φ. For every $x \in M$, denote by μ_x^φ the unique finite Borel measure on $\mathbf{R}$ that satisfies

$$\varphi(x^* \sigma_t^\varphi(x)) = \langle \Delta_\varphi^{it}(x\xi_\varphi), x\xi_\varphi \rangle = \int_{\mathbf{R}} \exp(ist)\, \mathrm{d}\mu_x^\varphi(s) \quad \text{for all } t \in \mathbf{R}.$$

For any Borel subset $U \subset \mathbf{R}$, we have $U \in \mathcal{C}(\Delta_\varphi)$ (resp. $U \in \mathcal{C}(\Delta_\varphi|_{\mathrm{L}^2(M)\ominus \mathrm{L}^2(P)})$) if and only if $\mu_x^\varphi(U) = 0$ for all $x \in M$ (resp. for all $x \in M \ominus P$). Since $pMp \subset M$ and $p(M \ominus P)p \subset M \ominus P$, it is clear that $\mathcal{C}(\Delta_\varphi) \subset \mathcal{C}(\Delta_{\varphi_p})$ (resp. $\mathcal{C}(\Delta_\varphi|_{\mathrm{L}^2(M)\ominus \mathrm{L}^2(P)}) \subset \mathcal{C}(\Delta_{\varphi_p}|_{\mathrm{L}^2(pMp)\ominus \mathrm{L}^2(pPp)})$). It remains to prove that $\mathcal{C}(\Delta_{\varphi_p}) \subset \mathcal{C}(\Delta_\varphi)$ (resp. $C(\Delta_{\varphi_p}|_{\mathrm{L}^2(pMp)\ominus \mathrm{L}^2(pPp)}) \subset \mathcal{C}(\Delta_\varphi|_{\mathrm{L}^2(M)\ominus \mathrm{L}^2(P)})$).

Up to shrinking $p \in M^\varphi$ if necessary and since we have $p_2 M p_2 \subset p_1 M p_1$ and $p_2(M \ominus P)p_2 \subset p_1(M \ominus P)p_1$ whenever $p_2 \leq p_1$ with p_1, p_2 nonzero projections in M^φ, we may assume without loss of generality that $\varphi(p) = m^{-1}$ with $m \in \mathbf{N}$. Since M^φ is a II_1 factor, we may find partial isometries $u_1, \dots, u_m \in M^\varphi$ such that $u_1 = p$, $u_j^* u_j = p$ for all $1 \leq j \leq m$ and $\sum_{j=1}^m u_j u_j^* = 1$.

Let $x \in M$ (resp. $x \in M \ominus P$). Since $\varphi(x^* \sigma_t^\varphi(x)) = \varphi(p) \sum_{i,j=1}^m \varphi_p((u_j^* x u_i)^* \sigma_t^{\varphi_p}(u_j^* x u_i))$ for all $t \in \mathbf{R}$, we have $\mu_x^\varphi = \varphi(p) \sum_{i,j=1}^m \mu_{u_j^* x u_i}^{\varphi_p}$. If $x \in M \ominus P$, then $u_j^* x u_i \in p(M \ominus P)p$ for all $1 \leq i, j \leq m$. This implies that $\mathcal{C}(\Delta_{\varphi_p}) \subset \mathcal{C}(\Delta_\varphi)$ (resp. $\mathcal{C}(\Delta_{\varphi_p}|_{\mathrm{L}^2(pMp)\ominus \mathrm{L}^2(pPp)}) = \mathcal{C}(\Delta_\varphi|_{\mathrm{L}^2(M)\ominus \mathrm{L}^2(P)})$) and finishes the proof. ∎

Popa's intertwining-by-bimodules. Popa introduced his *intertwining-by-bimodules* theory in [Po02, Po03]. In the present work, we make use of this theory in the context of semifinite von Neumann algebras. We introduce the following terminology. Let M be any σ-finite semifinite von Neumann algebra endowed with a fixed faithful normal semifinite trace Tr. Let 1_A and 1_B be any nonzero projections in M and let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be any von Neumann subalgebras. Assume that $\mathrm{Tr}(1_A) < +\infty$ and that $\mathrm{Tr}|_B$ is semifinite.

We say that A *embeds into* B *inside* M and write $A \prec_M B$ if there exist a projection $e \in A$, a finite trace projection $f \in B$, a nonzero partial isometry $v \in eMf$ and a unital normal homomorphism $\theta : eAe \to fBf$ such that $av = v\theta(a)$ for all $a \in eAe$. We use the following useful characterization [Po02, Po03] (see also [HR10, Lemma 2.2]).

Theorem 2.2. *Keep the same notation as above. Denote by* $\mathrm{E}_B : 1_B M 1_B \to B$ *the unique trace preserving conditional expectation. Then the following conditions are equivalent.*

(i) $A \prec_M B$.
(ii) *There exists no net* $(w_i)_{i\in I}$ *of unitaries in* $\mathcal{U}(A)$ *such that* $\lim_i \|\mathrm{E}_B(y^* w_i x)\|_2 = 0$ *for all* $x, y \in 1_A M 1_B$.

3. A criterion for the unitary conjugacy of states

Recall that for any von Neumann algebra N, any $\theta \in N_*$ and any $a, b \in N$, we define $(a\theta b)(y) := \theta(bya)$ for every $y \in N$.

Theorem 3.1. *Let* M *be a von Neumann algebra with a faithful normal state* $\varphi \in M_*$ *and* $P \subset M$ *a von Neumann subalgebra that is the range of a* φ*-preserving conditional expectation* $\mathrm{E}_P : M \to P$. *Let* $\psi \in M_*$ *be another faithful normal state and* $q \in M^\psi$ *a nonzero projection. Then the following statements are equivalent.*

(i) *There exists a nonzero finite trace projection* $r \in \mathrm{L}_\psi(\mathbf{R})$ *such that*
$$\Pi_{\varphi,\psi}(\mathrm{L}_\psi(\mathbf{R})qr) \prec_{\mathrm{c}_\varphi(M)} \mathrm{c}_\varphi(P) \; .$$
(ii) *There exist a faithful normal positive functional* $\theta \in P_*$ *and a nonzero partial isometry* $v \in M$ *such that* $p = vv^* \in M^{\theta\circ \mathrm{E}_P}$, $q_0 = v^*v \in qM^\psi q$ *and* $\psi q_0 = v^*(\theta \circ \mathrm{E}_P)v$.

Proof. Assume that (i) holds. Write $w_t = [\mathrm{D}\psi : \mathrm{D}\varphi]_t$. We claim that there exists a $\delta > 0$ and $x_1, \dots, x_k \in qM$ such that

$$\sum_{i,j=1}^{k} \varphi\big(\, \mathrm{E}_P(x_i^* \, w_t \, \sigma_t^\varphi(x_j)) \, \mathrm{E}_P(x_i^* \, w_t \, \sigma_t^\varphi(x_j))^* \,\big) \geq \delta \tag{3.1}$$

for all $t \in \mathbf{R}$. Assuming that the claim is false, we prove that (i) does not hold. Take a net $t_i \in \mathbf{R}$ such that
$$\lim_i \varphi\big(\, \mathrm{E}_P(x^* \, w_{t_i} \, \sigma_{t_i}^\varphi(y)) \, \mathrm{E}_P(x^* \, w_{t_i} \, \sigma_{t_i}^\varphi(y))^* \,\big) = 0$$
for all $x, y \in qM$. Using the 2-norm $\| \cdot \|_2$ w.r.t. the canonical trace on $\mathrm{c}_\varphi(M)$, we get that for all finite trace projections $p, p' \in \mathrm{L}_\varphi(\mathbf{R})$ and all $x, y \in qM$,
$$\begin{aligned}\big\|p \, \mathrm{E}_P(x^* \, w_{t_i} \, \sigma_{t_i}^\varphi(y)) \, p'\big\|_2^2 &\leq \big\|p \, \mathrm{E}_P(x^* \, w_{t_i} \, \sigma_{t_i}^\varphi(y))\big\|_2^2 \\ &= \mathrm{Tr}(p) \, \varphi\big(\, \mathrm{E}_P(x^* \, w_{t_i} \, \sigma_{t_i}^\varphi(y)) \, \mathrm{E}_P(x^* \, w_{t_i} \, \sigma_{t_i}^\varphi(y))^* \,\big) \to 0 \; .\end{aligned}$$

We then also get for all finite trace projections $p, p' \in \mathrm{L}_\varphi(\mathbf{R})$, all $s, s' \in \mathbf{R}$ and all $x, y \in M$ that

$$\begin{aligned}\big\|\mathrm{E}_{\mathrm{c}_\varphi(P)}\big(p\,\lambda_\varphi(s)^*\,x^*\,\Pi_{\varphi,\psi}(\lambda_\psi(t_i)q)\,y\,\lambda_\varphi(s')\,p'\big)\big\|_2 \\ &= \big\|\lambda_\varphi(s)^*\,p\,\mathrm{E}_P\big((qx)^* w_{t_i}\sigma^\varphi_{t_i}(qy)\big)\,p'\,\lambda_\varphi(t_i+s')\|_2 \\ &= \big\|p\,\mathrm{E}_P\big((qx)^* w_{t_i}\sigma^\varphi_{t_i}(qy)\big)\,p'\|_2 \to 0\;.\end{aligned}$$

The linear span of such elements $x\lambda_\varphi(s)p$ is dense in $\mathrm{L}^2(\mathrm{c}_\varphi(M), \mathrm{Tr})$. So whenever $r \in \mathrm{L}_\psi(\mathbf{R})$ is a finite trace projection and $a, b \in \mathrm{c}_\varphi(M)$, we first approximate $\Pi_{\varphi,\psi}(r)\,b$ in $\|\cdot\|_2$ by a linear combination of $y\lambda_\varphi(s')p'$ and conclude that

$$\big\|\mathrm{E}_{\mathrm{c}_\varphi(P)}\big(p\,\lambda_\varphi(s)^*\,x^*\,\Pi_{\varphi,\psi}(\lambda_\psi(t_i)qr)\,b\big)\big\|_2 \to 0$$

for all finite trace projections $p \in \mathrm{L}_\varphi(\mathbf{R})$ and all $s \in \mathbf{R}$, $x \in M$. We then approximate $\Pi_{\varphi,\psi}(r)\,a$ in $\|\cdot\|_2$ by a linear combination of $x\lambda_\varphi(s)p$ and conclude that

$$\big\|\mathrm{E}_{\mathrm{c}_\varphi(P)}\big(a^*\,\Pi_{\varphi,\psi}(\lambda_\psi(t_i)qr)\,b\big)\big\|_2 \to 0\;.$$

Applying Theorem 2.2, we conclude that (i) does not hold. This concludes the proof of the claim.

Fix $\delta > 0$ and $x_1, \ldots, x_k$ such that (3.1) holds for all $t \in \mathbf{R}$. Denote by $\langle M, e_P\rangle$ the basic construction for $P \subset M$, i.e. the von Neumann algebra acting on $\mathrm{L}^2(M, \varphi)$ generated by M acting by left multiplication and the orthogonal projection $e_P : \mathrm{L}^2(M, \varphi) \to \mathrm{L}^2(P, \varphi)$. As in [ILP96, Section 2.1], denote by $\hat\varphi$ the canonical faithful normal semifinite weight on $\langle M, e_P\rangle$ characterized by

$$\sigma^{\hat\varphi}_t(xe_Py) = \sigma^\varphi_t(x)e_P\sigma^\varphi_t(y) \quad \text{and} \quad \hat\varphi(xe_Py) = \varphi(xy)$$

for all $x, y \in \mathbf{R}$. We have $\sigma^{\hat\varphi}_t(T) = \Delta^{\mathrm{i}t}_\varphi T \Delta^{-\mathrm{i}t}_\varphi$ for all $T \in \langle M, e_P\rangle$ and all $t \in \mathbf{R}$. In particular, $\sigma^{\hat\varphi}_t(x) = \sigma^\varphi_t(x)$ for all $x \in M$. Denote by T_M the unique faithful normal semifinite operator valued weight from $\langle M, e_P\rangle$ to M such that $\hat\varphi = \varphi \circ \mathrm{T}_M$. Note that $\mathrm{T}_M(e_P) = 1$.

Define $\hat\psi = \psi \circ \mathrm{T}_M$. So, $\hat\psi$ is a faithful normal semifinite weight on $\langle M, e_P\rangle$ and by [Ha77, Theorem 4.7], we have

$$[\mathrm{D}\hat\psi : \mathrm{D}\hat\varphi]_t = [\mathrm{D}\psi : \mathrm{D}\varphi]_t = w_t$$

for all $t \in \mathbf{R}$. In particular, we get that

$$\sigma^{\hat\psi}_t(xe_Py) = w_t\sigma^\varphi_t(x)e_P\sigma^\varphi_t(y)w_t^* \tag{3.2}$$

for all $x, y \in M$ and $t \in \mathbf{R}$.

Define the positive element $X \in q\langle M, e_P\rangle q$ by

$$X = \sum_{j=1}^k x_j e_P x_j^*\;.$$

Also define the normal positive functional $\Omega \in \langle M, e_P\rangle_*$ given by

$$\Omega(T) = \sum_{i=1}^k \hat\varphi(e_P x_i^* T x_i e_P)$$

for all $T \in \langle M, e_P\rangle$. Using (3.2) and (3.1), we get for every $t \in \mathbf{R}$,

$$\begin{aligned}\Omega(\sigma^{\hat\psi}_t(X)) &= \sum_{i,j=1}^k \hat\varphi\big(e_P x_i^* w_t\sigma^\varphi_t(x_j)e_P\sigma^\varphi_t(x_j)^* w_t^* x_i e_P\big) \\ &= \sum_{i,j=1}^k \varphi\big(\mathrm{E}_P(x_i^* w_t\sigma^\varphi_t(x_j))\,\mathrm{E}_P(x_i^* w_t\sigma^\varphi_t(x_j))^*\big) \geq \delta\;.\end{aligned}$$

Define K as the σ-weakly closed convex hull of $\{\sigma_t^{\hat\psi}(X) : t \in \mathbf{R}\}$ inside $q\langle M, e_P\rangle q$. Note that $\|Y\| \leq \|X\|$ for all $Y \in K$. Also, every $Y \in K$ is positive and satisfies $\hat\psi(Y) \leq \hat\psi(X) < +\infty$, by the $\sigma_t^{\hat\psi}$-invariance and σ-weak lower semicontinuity of $\hat\psi$. We then also have

$$\hat\psi(Y^*Y) = \hat\psi(Y^2) \leq \|Y\|\,\hat\psi(Y) \leq \|X\|\,\hat\psi(X)$$

for all $Y \in K$. By [HI15, Lemma 4.4], the image of K in $\mathrm{L}^2(\langle M, e_P\rangle, \hat\psi)$ is norm closed. So, there is a unique element $X_0 \in K$ where the function $Y \mapsto \hat\psi(Y^*Y)$ attains its minimal value. Since this function is $\sigma_t^{\hat\psi}$-invariant, it follows that $\sigma_t^{\hat\psi}(X_0) = X_0$ for all T. Since $\Omega(\sigma_t^{\hat\psi}(X)) \geq \delta$ for all $t \in \mathbf{R}$, also $\Omega(X_0) \geq \delta$, so that $X_0 \neq 0$. Since $\mathrm{T}_M \circ \sigma_t^{\hat\psi} = \sigma_t^{\psi} \circ \mathrm{T}_M$ and since T_M is σ-weakly lower semicontinuous, we get that $\|\mathrm{T}_M(Y)\| \leq \|\mathrm{T}_M(X)\| < +\infty$ for all $Y \in K$. In particular, $\|\mathrm{T}_M(X_0)\| < +\infty$.

Take $\varepsilon > 0$ small enough such that the spectral projection $e = 1_{[\varepsilon,+\infty)}(X_0)$ is nonzero. It follows that e is a projection in $q\langle M, e_P\rangle q$ satisfying $\sigma_t^{\hat\psi}(e) = e$ for all $t \in \mathbf{R}$ and $\|\mathrm{T}_M(e)\| < +\infty$. By Lemma 3.3 below, we may assume that $e \prec e_P$ inside $\langle M, e_P\rangle$. Take a partial isometry $V \in \langle M, e_P\rangle$ such that $V^*V = e$ and $VV^* \leq e_P$. Let $p_0 \in P$ be the unique projection such that $VV^* = p_0 e_P$. We get that $V = p_0 V$. Since $e \leq q$, we also have $V = Vq$.

Since $\|\mathrm{T}_M(V^*V)\| = \|\mathrm{T}_M(e)\| < +\infty$, it follows from the push down lemma [ILP96, Proposition 2.2] (where the factoriality assumption is unnecessary) that

$$V = e_P V = e_P\, \mathrm{T}_M(e_P V) = e_P\, \mathrm{T}_M(V)\ .$$

Write $v = \mathrm{T}_M(V)$. Then, $v \in M$ and $V = e_P v$. By construction, $v \in p_0 M q$.

Since $e_P\langle M, e_P\rangle e_P = P e_P$, we can uniquely define $u_t \in P$ such that

$$u_t e_P = V w_t \sigma_t^{\hat\varphi}(V^*)$$

for all $t \in \mathbf{R}$. Since $V^*V = e$ and $w_t\sigma_t^{\hat\varphi}(V^*V)w_t^* = \sigma_t^{\hat\psi}(e) = e$, we get that $u_t u_t^* = p_0$ and $u_t^* u_t = \sigma_t^{\varphi}(p_0)$ for all $t \in \mathbf{R}$. Also, $t \mapsto u_t$ is strongly continuous and

$$\begin{aligned} u_t\sigma_t^{\varphi}(u_s)\, e_P &= u_t e_P\, \sigma_t^{\hat\varphi}(u_s e_P) = V w_t \sigma_t^{\hat\varphi}(V^*)\, \sigma_t^{\hat\varphi}(V w_s \sigma_s^{\hat\varphi}(V^*)) \\ &= V w_t\, \sigma_t^{\hat\varphi}(V^*V)\, \sigma_t^{\varphi}(w_s)\, \sigma_{t+s}^{\hat\varphi}(V^*) = V\, \sigma_t^{\hat\psi}(e)\, w_t \sigma_t^{\varphi}(w_s)\, \sigma_{t+s}^{\hat\varphi}(V^*) \\ &= V\, w_{t+s}\, \sigma_{t+s}^{\hat\varphi}(V^*) = u_{t+s} e_P \end{aligned}$$

for all $s, t \in \mathbf{R}$. So, $(u_t)_{t\in\mathbf{R}}$ is a 1-cocycle for $\varphi|_P$. By [Co72, Théorème 1.2.4] (see also [Ta03, Theorem VIII.3.21] for a formulation adapted to non faithful states), there is a unique faithful normal semifinite weight θ on $p_0 P p_0$ such that $[\mathrm{D}\theta : \mathrm{D}\varphi|_P]_t = u_t$ for all $t \in \mathbf{R}$. Define the faithful normal semifinite weight θ_1 on $p_0 M p_0$ by $\theta_1 = \theta \circ \mathrm{E}_P$. By [Ha77, Theorem 4.7], we have that $[\mathrm{D}\theta_1 : \mathrm{D}\varphi]_t = u_t$ for all $t \in \mathbf{R}$.

Since $u_t \in P$, we get that

$$\begin{aligned} e_P u_t \sigma_t^{\varphi}(v) &= u_t e_P \sigma_t^{\varphi}(v) = u_t e_P \sigma_t^{\hat\varphi}(V) \\ &= V w_t \sigma_t^{\hat\varphi}(V^*V) = V\, \sigma_t^{\hat\psi}(V^*V)\, w_t = V w_t = e_P v w_t\ . \end{aligned}$$

Applying T_M, we conclude that $u_t\sigma_t^{\varphi}(v) = v w_t$ for all $t \in \mathbf{R}$. Replacing v by its polar part, we may assume that $v \in M$ is a partial isometry such that $p_1 = vv^* \in (p_0 M p_0)^{\theta_1}$ and $q_1 = v^*v \in qM^{\psi}q$. We then get that

$$[\mathrm{D}(v^*\theta_1 v) : \mathrm{D}\varphi]_t = v^*[\mathrm{D}\theta_1 : \mathrm{D}\varphi]_t \sigma_t^{\varphi}(v) = v^* u_t \sigma_t^{\varphi}(v) = q_1[\mathrm{D}\psi : \mathrm{D}\varphi]_t = [\mathrm{D}(\psi q_1) : \mathrm{D}\varphi]_t$$

for all $t \in \mathbf{R}$. We conclude that $\psi q_1 = v^*\theta_1 v$.

In particular, $\theta(\mathrm{E}_P(p_1)) = \theta_1(p_1) = \psi(q_1) < +\infty$. Also, $\sigma_t^{\theta}(\mathrm{E}_P(p_1)) = \mathrm{E}_P(\sigma_t^{\theta_1}(p_1)) = \mathrm{E}_P(p_1)$, for all $t \in \mathbf{R}$. We can thus find a nonzero spectral projection f of $\mathrm{E}_P(p_1)$ such that $fv \neq 0$,

$\theta(f) < +\infty$ and $f \in (p_0 P p_0)^\theta$. Since $u_t \sigma_t^\varphi(v) = v w_t$, $[\mathrm{D}\theta : \mathrm{D}\varphi|_P]_t = u_t$ and $f \in (p_0 P p_0)^\theta$, we get that

$$u_t \sigma_t^\varphi(fv) = f u_t \sigma_t^\varphi(v) = f v w_t \; .$$

We then replace θ by θf and v by the polar part of fv. Then, θ is a faithful normal positive functional on fPf, the projection $p = vv^*$ belongs to $(fMf)^{\theta \circ \mathrm{E}_P}$, the projection $q_0 = v^* v$ belongs to $qM^\psi q$ and $\psi q_0 = v^*(\theta \circ \mathrm{E}_P) v$. Adding to θ an arbitrary faithful normal state on $(1-f)P(1-f)$, it follows that (ii) holds.

Conversely, assume that (ii) holds. Take θ, q_0, p and v as in the statement of (ii). Define $w_t = [\mathrm{D}\psi : \mathrm{D}\varphi]_t$ and $u_t = [\mathrm{D}\theta : \mathrm{D}\varphi|_P]_t$. Since $\psi q_0 = v^*(\theta \circ \mathrm{E}_P)v$, we get that $v w_t = u_t \sigma_t^\varphi(v)$ for all $t \in \mathbf{R}$. This means that

$$v \, \Pi_{\varphi,\psi}(\lambda_\psi(t)) = \Pi_{\varphi|_P,\theta}(\lambda_\theta(t)) \, v$$

for all $t \in \mathbf{R}$. Also, $v \, \Pi_{\varphi,\psi}(qr) = v \, \Pi_{\varphi,\psi}(r) \neq 0$ for every nonzero finite trace projection $r \in \mathrm{L}_\psi(\mathbf{R})$. We conclude that

$$\Pi_{\varphi,\psi}(\mathrm{L}_\psi(\mathbf{R})qr) \prec_{\mathrm{c}_\varphi(M)} \mathrm{c}_\varphi(P)$$

for every nonzero finite trace projection $r \in \mathrm{L}_\psi(\mathbf{R})$, so that (i) holds. □

Applying Theorem 3.1 to the case $P = \mathbf{C}1$, we get the following result.

Corollary 3.2. *Let M be a von Neumann algebra with faithful normal states $\psi, \varphi \in M_*$ and let $q \in M^\psi$ be a nonzero projection. Then the following statements are equivalent.*

(i) *There exists a nonzero finite trace projection $r \in \mathrm{L}_\psi(\mathbf{R})$ such that*

$$\Pi_{\varphi,\psi}(\mathrm{L}_\psi(\mathbf{R})qr) \prec_{\mathrm{c}_\varphi(M)} \mathrm{L}_\varphi(\mathbf{R}) \; .$$

(ii) *There exists a nonzero partial isometry $v \in M$ such that $p = vv^*$ belongs to M^φ, $q_0 = v^*v$ belongs to $qM^\psi q$, and*

$$\frac{1}{\psi(q_0)} \, \psi q_0 = \frac{1}{\varphi(p)} \, v^* \varphi v \; .$$

Lemma 3.3. *Let ψ be a faithful normal semifinite weight on a von Neumann algebra N and $e \in N^\psi$ a projection satisfying $0 < \psi(e) < +\infty$. Let $e_1 \in N$ be any projection with central support equal to 1. Then there exists a nonzero projection $e_0 \in eN^\psi e$ satisfying $e_0 \prec e_1$ inside N.*

Proof. Since the central support of e_1 equals 1, we can find a nonzero projection $f \in N$ such that $f \leq e$ and $f \prec e_1$. Define the faithful normal state θ on eNe given by $\theta(x) = \psi(e)^{-1}\psi(x)$ for all $x \in eNe$. By [HU15, Lemma 2.1], there exists a projection $e_0 \in (eNe)^\theta$ such that $e_0 \sim f$ inside eNe. Then, e_0 is a nonzero projection in $eN^\psi e$ and $e_0 \prec e_1$ inside N. □

4. Isomorphisms of free Araki–Woods factors

The isomorphism part of Corollary B follows from the following result that we deduce from [Sh96].

Theorem 4.1. *Let μ be any finite symmetric Borel measure on $\mathbf{R}$ and $m : \mathbf{R} \to \mathbf{N} \cup \{+\infty\}$ any symmetric Borel multiplicity function. Denote by Λ the subgroup of $\mathbf{R}$ generated by the atoms of μ and assume that $\Lambda \neq \{0\}$. There is an isomorphism*

$$\Gamma(\mu, m)'' \cong \Gamma(\mu * \delta_\Lambda, +\infty)''$$

preserving the free quasi-free states, where δ_Λ denotes any atomic finite symmetric Borel measure on $\mathbf{R}$ with set of atoms equal to Λ.

Proof. For every $0 < a < 1$, we denote by B_a the von Neumann algebra $\mathrm{B}(\ell^2(\mathbf{N}))$ equipped with the faithful normal state θ_a given by

$$\theta_a(T) = (1-a)\sum_{k=0}^{\infty} a^k \langle T(\delta_k), \delta_k\rangle \; ,$$

where $(\delta_k)_{k\in\mathbf{N}}$ is the standard orthonormal basis of $\ell^2(\mathbf{N})$. Throughout the proof of the theorem, we always assume that a free Araki–Woods factor comes with its free quasi-free state and that all free products are taken w.r.t. the canonical states that we fixed. We always equip a free product with the free product state and a tensor product with the tensor product state.

We first prove that for every $0 < a < 1$, for all finite symmetric Borel measures μ on $\mathbf{R}$ and all symmetric Borel multiplicity functions $m : \mathbf{R} \to \mathbf{N} \cup \{+\infty\}$, there exists a state preserving isomorphism

$$\Gamma(\mu, m)'' * B_a \cong \Gamma(\mu * \delta_{\mathbf{Z}\log(a)}, +\infty)'' \mathbin{\overline{\otimes}} B_a \; . \tag{4.1}$$

To prove (4.1), fix an orthogonal representation $(U_t)_{t\in\mathbf{R}}$ of $\mathbf{R}$ on a real Hilbert space $H_\mathbf{R}$ having (μ, m) as its spectral invariant. Denote by $H = H_\mathbf{R} + \mathrm{i}H_\mathbf{R}$ the complexification of $H_\mathbf{R}$. Define the positive operator Δ on H such that $U_t = \Delta^{\mathrm{i}t}$ and denote by $J : H \to H$ the anti-unitary operator given by $J(\xi + \mathrm{i}\eta) = \xi - \mathrm{i}\eta$ for all $\xi, \eta \in H_\mathbf{R}$. Define $H_1 = H \otimes \ell^2(\mathbf{N}^2)$. On H_1, we consider the positive operator Δ_1 and anti-unitary operator J_1 given by

$$\Delta_1(\xi \otimes \delta_{ij}) = a^{j-i}\,\Delta(\xi) \otimes \delta_{ij} \quad \text{and} \quad J_1(\xi \otimes \delta_{ij}) = J(\xi) \otimes \delta_{ji}$$

for all $i, j \in \mathbf{N}$ and $\xi \in D(\Delta)$. Here, (δ_{ij}) denotes the standard orthonormal basis of $\ell^2(\mathbf{N}^2)$. Note that $J_1\Delta_1 J_1 = \Delta_1^{-1}$.

Denote by $\mathcal{F}(H_1)$ the full Fock space of H_1 and by θ_1 the vector state on $\mathrm{B}(\mathcal{F}(H_1))$ implemented by the vacuum vector. For every $\xi \in H$, define the element $L(\xi) \in \mathrm{B}(\mathcal{F}(H_1)) \mathbin{\overline{\otimes}} B_a$ given by

$$L(\xi) = \sum_{i,j=0}^{\infty} \ell(\xi \otimes \delta_{ij}) \otimes e_{ij}\sqrt{(1-a)a^i} \; .$$

By [Sh96, Theorem 5.2], we can realize $\Gamma(\mu, m)'' * B_a$ as the von Neumann algebra $\mathcal{M}$ generated by

$$\{L(\xi) + L(J\Delta^{1/2}\xi)^* : \xi \in D(\Delta^{1/2})\} \quad \text{and} \quad 1 \otimes B_a \; .$$

Moreover, the free product state on $\Gamma(\mu, m)'' * B_a$ is given by the restriction of $\theta_1 \otimes \theta_a$ to $\mathcal{M}$.

To conclude the proof of (4.1), it thus suffices to show that there is a state preserving isomorphism

$$\big((1 \otimes e_{00})\mathcal{M}(1 \otimes e_{00}), \theta_1\big) \cong \big(\Gamma(\mu * \delta_{\mathbf{Z}\log(a)}, +\infty)'', \varphi_{\mu*\delta_{\mathbf{Z}\log(a)},+\infty}\big) \; . \tag{4.2}$$

The left hand side of (4.2) is generated by the operators

$$(1 \otimes e_{0i})\,(L(\xi) + L(J\Delta^{1/2}\xi)^*)\,(1 \otimes e_{j0}) = \sqrt{(1-a)a^i}\,\Big(\ell\big(\xi \otimes \delta_{ij}\big) + \ell\big(J_1\Delta_1^{1/2}(\xi \otimes \delta_{ij})\big)^*\Big) \; .$$

So, the left hand side of (4.2) equals $(\Gamma(\mu_1, m_1)'', \varphi_{\mu_1,m_1})$ where μ_1 and m_1 are chosen so that the measure class and multiplicity function of $\log(\Delta_1)$ equal $\mathcal{C}(\mu_1)$ and m_1. One checks that $\mathcal{C}(\mu_1) = \mathcal{C}(\mu * \delta_{\mathbf{Z}\log(a)})$ and $m_1 = +\infty$ a.e. So we have proved the existence of the state preserving isomorphism (4.1).

We next prove that for every $t > 0$, there is a state preserving isomorphism

$$\Gamma(\mu, m)'' * \Gamma(\delta_t + \delta_{-t}, 1)'' \cong \Gamma(\mu * \delta_{\mathbf{Z}t} \vee \delta_{\mathbf{Z}t}, +\infty)'' \; . \tag{4.3}$$

By [Sh96, Theorem 4.8], there is a state preserving isomorphism

$$\Gamma(\delta_t + \delta_{-t}, 1)'' \cong \mathrm{L}^\infty([0,1]) * B_{\exp(-t)} \; .$$

By [Sh96, Theorem 2.11], the free Araki–Woods functor Γ turns direct sums into free products. Writing $\mu_1 = \mu + \delta_0$, $m_1 = m + \delta_0$ and using (4.1), we obtain the state preserving isomorphisms

$$\Gamma(\mu,m)'' * \Gamma(\delta_t + \delta_{-t},1)'' \cong \Gamma(\mu_1,m_1)'' * B_{\exp(-t)} \cong \Gamma(\mu_1 * \delta_{\mathbf{Z}t},+\infty)'' \mathbin{\overline{\otimes}} B_{\exp(-t)} \; .$$

Applying this to $(\mu,m) = (\mu_1 * \delta_{\mathbf{Z}t},+\infty)$, we also have the state preserving isomorphisms

$$\Gamma(\mu_1 * \delta_{\mathbf{Z}t},+\infty)'' \cong \Gamma(\mu_1 * \delta_{\mathbf{Z}t},+\infty)'' * \Gamma(\delta_t + \delta_{-t},1)'' \cong \Gamma(\mu_1 * \delta_{\mathbf{Z}t},+\infty)'' \mathbin{\overline{\otimes}} B_{\exp(-t)} \; .$$

Combining both, it follows that (4.3) holds.

We are now ready to prove the theorem. Fix an atom $t > 0$ of μ. Writing $(\mu,m) = (\mu_0,m_0) + (\delta_t + \delta_{-t},1)$, we get from (4.3) the state preserving isomorphism

$$\Gamma(\mu,m)'' \cong \Gamma(\mu_0,m_0)'' * \Gamma(\delta_t + \delta_{-t},1)'' \cong \Gamma(\mu_0 * \delta_{\mathbf{Z}t} \vee \delta_{\mathbf{Z}t},+\infty)'' \cong \Gamma(\mu * \delta_{\mathbf{Z}t},+\infty)'' \; . \tag{4.4}$$

Let $\{t_n : n \geq 0\}$ be the positive atoms of μ, with repetitions if there are only finitely many of them. For every $n \geq 0$, define

$$\mu_n = \mu * \delta_{\mathbf{Z}t_0} * \cdots * \delta_{\mathbf{Z}t_n} \; .$$

For every n, t_{n+1} is an atom of μ_n. Repeatedly applying (4.4), we find the state preserving isomorphisms

$$\Gamma(\mu,m)'' \cong \Gamma(\mu_0,+\infty)'' \cong \Gamma(\mu_n,+\infty)'' \; .$$

So, we also get state preserving isomorphisms

$$\Gamma(\mu,m)'' \cong \Gamma(\mu_0,+\infty)'' \cong \mathop{\ast}_{n\in\mathbf{N}} \Gamma(\mu_0,+\infty)'' \cong \mathop{\ast}_{n\in\mathbf{N}} \Gamma(\mu_n,+\infty)'' \cong \Gamma(\vee_{n\in\mathbf{N}}\mu_n,+\infty) \; .$$

Since $\vee_{n\in\mathbf{N}}\mu_n$ is equivalent with $\mu * \delta_\Lambda$, the theorem follows. □

We deduce the isomorphism part of Theorem F from the following result, generalizing [Sh97a, Theorem 5.1] and proved using the same methods. For every faithful normal state ψ on a von Neumann algebra A and for every nonzero projection $p \in A$, we denote by ψ_p the faithful normal state on pAp given by $\psi_p(a) = \psi(p)^{-1}\psi(a)$ for all $a \in pAp$.

Proposition 4.2. *Let μ be a finite symmetric Borel measure on $\mathbf{R}$ and fix the free Araki–Woods factor $(M,\varphi) = (\Gamma(\mu,+\infty)'',\varphi_{\mu,+\infty})$. Let A be a von Neumann algebra with a faithful normal state ψ having a factorial centralizer A^ψ. For every nonzero projection $p \in A^\psi$, there is a state preserving isomorphism*

$$\Big(p\big((M,\varphi) * (A,\psi)\big)p, (\varphi * \psi)_p\Big) \cong \Big((M,\varphi) * (pAp,\psi_p), \varphi * \psi_p\Big) \; .$$

To prove Proposition 4.2, we need the following lemma. It is a direct consequence of [Sh97a, Corollary 2.5]. To formulate the lemma, we use yet another convention for the construction of free Araki–Woods factors. We call *involution* on a Hilbert space H any closed densely defined antilinear operator S satisfying $S(\xi) \in \mathrm{D}(S)$ and $S(S(\xi)) = \xi$ for all $\xi \in \mathrm{D}(S)$. Taking the polar decomposition $S = J\Delta^{1/2}$ of such an involution, we obtain an anti-unitary operator J and a nonsingular positive selfadjoint operator Δ satisfying $J\Delta J = \Delta^{-1}$. Denoting by U_t the restriction of Δ^{it} to the real Hilbert space $H_{\mathbf{R}} = \{\xi \in H : J(\xi) = \xi\}$, we obtain an orthogonal representation $(U_t)_{t\in\mathbf{R}}$. Every orthogonal representation of $\mathbf{R}$ arises in this way. The associated free Araki–Woods factor can be realized on the full Fock space $\mathcal{F}(H)$ as the von Neumann algebra generated by the operators $\ell(\xi) + \ell(S(\xi))^*$, $\xi \in \mathrm{D}(S)$. We denote this realization of the free Araki–Woods factor as $\Gamma(H,S)''$.

Lemma 4.3. *Let K be a Hilbert space and $\Omega \in K$ a unit vector. Let H be a Hilbert space and $H_0 \subset H$ a total subset. Assume that*

- *$A \subset \mathrm{B}(K)$ is a von Neumann subalgebra and $\langle \cdot\,\Omega,\Omega\rangle$ defines a faithful state ψ on A,*
- *for every $\xi \in H_0$, we are given an operator $L(\xi) \in \mathrm{B}(K)$,*

such that the following conditions hold:

(i) $L(\xi_1)^* a L(\xi_2) = \psi(a) \langle \xi_2, \xi_1 \rangle 1$ *for all* $\xi_1, \xi_2 \in H_0$ *and* $a \in A$,
(ii) $L(\xi)^* a\Omega = 0$ *for all* $\xi \in H_0$ *and* $a \in A$,
(iii) *denoting by* $\mathcal{A}$ *the* $*$*-algebra generated by* A *and* $\{L(\xi) : \xi \in H_0\}$, *we have that* $\mathcal{A}\Omega$ *is dense in* K.

Then, L *can be uniquely extended to a linear map* $L : H \to \mathrm{B}(K)$ *such that the above properties remain valid. For every involution* S *on* H *with associated free Araki–Woods factor* $\Gamma(H,S)''$, *there is a unique normal homomorphism*

$$\pi : \big(\Gamma(H,S)'', \varphi_{(H,S)}\big) * (A,\psi) \to \mathrm{B}(K)$$

satisfying $\pi\big(\ell(\xi) + \ell(S(\xi))^*\big) = L(\xi) + L(S(\xi))^*$ *for all* $\xi \in \mathrm{D}(S)$ *and* $\pi(a) = a$ *for all* $a \in A$. *Also,* $\langle \pi(\,\cdot\,)\Omega, \Omega\rangle$ *equals the free product state* $\varphi_{(H,S)} * \psi$.

Using Lemma 4.3, we can prove Proposition 4.2.

Proof of Proposition 4.2. Since A^ψ is a factor, we can choose partial isometries $v_i \in A^\psi$, $i \geq 1$, such that $v_i^* v_i \leq p$, $\sum_{i=1}^\infty v_i v_i^* = 1$ and $\psi(v_i^* v_i) = \psi(p)/n_i$ for some integers $n_i \geq 1$. We can then also choose partial isometries $w_{is} \in pA^\psi p$, $s = 1, \ldots, n_i$, such that $w_{is} w_{is}^* = v_i^* v_i$ for all s and $\sum_{s=1}^{n_i} w_{is}^* w_{is} = p$.

Since (M,φ) is a free Araki–Woods factor with infinite multiplicity, we can choose an involution S_0 on a Hilbert space H_0 and realize (M,φ) as $\Gamma(H,S)''$, where $H = H_0 \otimes \ell^2(\mathbf{N}^2)$ and S is given by $S(\xi \otimes \delta_{kl}) = S_0(\xi) \otimes \delta_{lk}$ for all $\xi \in \mathrm{D}(S_0)$ and all $k,l \geq 1$. We then consider the standard free product representation for $\Gamma(H,S)'' * A$ on the Hilbert space K with vacuum vector Ω. Note that $p\big(\Gamma(H,S)'' * A\big)p$ is generated by

$$pAp \cup \Big\{ v_i^* \big(\ell(\xi \otimes \delta_{kl}) + \ell(S_0(\xi) \otimes \delta_{lk})^*\big) v_j \;\Big|\; i,j,k,l \geq 1, \xi \in \mathrm{D}(S_0) \Big\} . \tag{4.5}$$

For all $k,l \geq 0$, $i,j \geq 1$ and $\xi \in H$, define

$$L_{ijkl}(\xi) = \psi(p)^{-1/2} \sum_{s=1}^{n_i} \sum_{t=1}^{n_j} w_{is}^* v_i^* \, \ell(\xi \otimes \delta_{n_i k + s, n_j l + t})\, v_j w_{jt} .$$

A direct computation shows that

$$L_{i'j'k'l'}(\xi')^* \, a \, L_{ijkl}(\xi) = \delta_{ijkl,i'j'k'l'} \, \langle \xi, \xi' \rangle \, \psi_p(a)\, p$$

for all $i,j,i',j' \geq 1$, $k,l,k',l' \geq 0$, $\xi, \xi' \in H$ and $a \in pAp$.

Applying Lemma 4.3 to the Hilbert space $H_1 = H \otimes \ell^2(\mathbf{N}^2 \times \mathbf{N}_0^2)$ with involution $S_1(\xi \otimes \delta_{ijkl}) = S_0(\xi) \otimes \delta_{jilk}$, it follows that $\Gamma(H_1,S_1)'' * pAp$ can be realized as the von Neumann algebra N generated by

$$pAp \cup \Big\{ L_{ijkl}(\xi) + L_{jilk}(S_0(\xi))^* \;\Big|\; i,j \geq 1, k,l \geq 0, \xi \in \mathrm{D}(S_0) \Big\} ,$$

with the free product state being implemented by $\psi(p)^{-1/2} p\Omega$.

Note that

$$w_{is} \, \big(L_{ijkl}(\xi) + L_{jilk}(S_0(\xi))^*\big) \, w_{jt}^* = \psi(p)^{-1/2} \, v_i^* \, \big(\ell(\xi \otimes \delta_{n_i k+s, n_j l+t}) + \ell(S_0(\xi) \otimes \delta_{n_j l+t, n_i k+s})^*\big) \, v_j .$$

For fixed $i,j \geq 1$, the parameters $n_i k + s$ and $n_j l + t$ with $k,l \geq 0$, $s = 1,\ldots,n_i$ and $t = 1,\ldots,n_j$ exactly run through $\mathbf{N}^2$. So, we find back the generating set of (4.5) and conclude that $p\big(\Gamma(H,S)'' * A\big)p$ equals $\Gamma(H_1,S_1)'' * pAp$ in a state preserving way. Since also $\Gamma(H_1,S_1)'' \cong (M,\varphi)$ in a state preserving way, this concludes the proof of the proposition. ◻

5. Proofs of Theorem A and Corollaries B, C, D

Combining Corollary 3.2 with the deformation/rigidity theorems for free Araki–Woods factors and for free product factors obtained in [HR10, HU15], we get the following theorem.

Theorem 5.1. *Let (M, φ) be either a free Araki–Woods factor with its free quasi-free state or a free product $*_n(M_n, \varphi_n)$ of amenable von Neumann algebras equipped with the free product state. Let $\psi \in M_*$ be any faithful normal state on M and denote by $[\mathrm{D}\psi : \mathrm{D}\varphi]_t$ Connes' Radon–Nikodym 1-cocycle between ψ and φ. Let $z \in \mathcal{Z}(M^\psi)$ be the central projection such that $M^\psi(1-z)$ is amenable and $M^\psi z$ has no amenable direct summand.*

There exists a sequence of partial isometries $v_n \in M$ such that the projection $q_n = v_n v_n^*$ belongs to M^ψ, the projection $p_n = v_n^* v_n$ belongs to M^φ, $\sum_n q_n = z$,

$$q_n\, [\mathrm{D}\psi : \mathrm{D}\varphi]_t = \lambda_n^{\mathrm{i}t}\, v_n\, \sigma_t^\varphi(v_n^*) \quad \text{and} \quad \psi q_n = \lambda_n\, v_n \varphi v_n^* \,,$$

with $\lambda_n = \psi(q_n)/\varphi(p_n)$.

Proof. Let $q \in M^\psi$ be a nonzero projection such that $qM^\psi q$ has no amenable direct summand. Let $r_0 \in \mathrm{L}_\psi(\mathbf{R})$ be a nonzero finite trace projection. Put $r = \Pi_{\varphi,\psi}(qr_0)$. Then r is a nonzero finite trace projection in the core $c_\varphi(M)$ and $\Pi_{\varphi,\psi}(\mathrm{L}_\psi(\mathbf{R}))r$ commutes with $qM^\psi qr$. Since $qM^\psi qr$ has no amenable direct summand, it follows from [HR10, Theorem 5.2] (in the case where M is a free Araki–Woods factor) and [HU15, Theorem 4.3] (in the case where M is a free product of amenable von Neumann algebras) that $\Pi_{\varphi,\psi}(\mathrm{L}_\psi(\mathbf{R}))r \prec_{c_\varphi(M)} \mathrm{L}_\varphi(\mathbf{R})$.

By Theorem 3.1, we find a nonzero partial isometry $v \in qM$ such that the projection $q_0 = vv^*$ belongs to M^ψ, the projection $p = v^*v$ belongs to M^φ and $\psi q = \lambda\, v\varphi v^*$. In particular, $\lambda = \psi(q_0)/\varphi(p)$ and $q_0\, [\mathrm{D}\psi : \mathrm{D}\varphi]_t = \lambda^{\mathrm{i}t}\, v\sigma_t^\varphi(v^*)$.

Since $q \in M^\psi$ was an arbitrary nonzero projection such that $qM^\psi q$ has no amenable direct summand, the theorem follows by a maximality argument. □

In order to apply Theorem 5.1 to the classification of free Araki–Woods factors, we need the following description of the centralizer of the free quasi-free state.

Remark 5.2. When $M = \Gamma(\mu, m)''$ is an arbitrary free Araki–Woods factor with free quasi-free state $\varphi = \varphi_{\mu,m}$, the centralizer M^φ can be described as follows. Denote by $M_a = \Gamma(\mu_a, m)''$ the almost periodic part of M. First note that $M^\varphi = M_a^\varphi$. So if $\mu_a = 0$, we have $M^\varphi = \mathbf{C}1$. If μ_a is concentrated on $\{0\}$, we conclude that $M^\varphi = M_a = \mathrm{L}(\mathbf{F}_{m(0)})$, where the last isomorphism follows because the free Araki–Woods factor associated with the m-dimensional trivial representation, i.e. $\Gamma(\delta_0, m)''$, is isomorphic with $\mathrm{L}(\mathbf{F}_m)$. When $\mu_a(\log \lambda) > 0$ for some $0 < \lambda < 1$, there is a state preserving inclusion $T_\lambda \subset M_a$, where T_λ is the unique free Araki–Woods factor of type III_λ (see [Sh96, Section 4]). It then follows from [Sh96, Corollary 6.8] that M^φ is a factor that contains a copy of the free group factor $\mathrm{L}(\mathbf{F}_\infty)$, so that M^φ is nonamenable. Actually, using [Dy96], we get that $M^\varphi \cong \mathrm{L}(\mathbf{F}_\infty)$ in this case.

Theorem A is a particular case of the following more general result.

Theorem 5.3. *Let μ, ν be finite symmetric Borel measures on $\mathbf{R}$ and $m, n : \mathbf{R} \to \mathbf{N} \cup \{+\infty\}$ symmetric Borel multiplicity functions. Assume that $\nu_a \neq 0$ and either $\mathrm{supp}(\nu_a) \neq \{0\}$ or $\mathrm{supp}(\nu_a) = \{0\}$ with $n(0) \geq 2$.*

If the free Araki–Woods factors $\Gamma(\mu, m)''$ and $\Gamma(\nu, n)''$ are isomorphic then there exist nonzero projections $p \in (\Gamma(\mu, m)'')^{\varphi_{\mu,m}}$ and $q \in (\Gamma(\nu, n)'')^{\varphi_{\nu,n}}$ and a state preserving isomorphism

$$\big(p\,\Gamma(\mu, m)''\,p, (\varphi_{\mu,m})_p\big) \cong \big(q\,\Gamma(\nu, n)''\,q, (\varphi_{\nu,n})_q\big)$$

where $(\varphi_{\mu,m})_p = \frac{\varphi_{\mu,m}(p\,\cdot\,p)}{\varphi_{\mu,m}(p)}$ and $(\varphi_{\nu,n})_q = \frac{\varphi_{\nu,n}(q\,\cdot\,q)}{\varphi_{\nu,n}(q)}$.

In particular, the joint measure classes $\mathcal{C}(\bigvee_{k\geq 1}\mu^{*k})$ *and* $\mathcal{C}(\bigvee_{k\geq 1}\nu^{*k})$ *are equal.*

Moreover, in case $\operatorname{supp}(\nu_a)\neq\{0\}$ *or* $\operatorname{supp}(\nu_a)=\{0\}$ *with* $n(0)=+\infty$*, there exists a state preserving isomorphism* $(\Gamma(\mu,m)'',\varphi_{\mu,m})\cong(\Gamma(\nu,n)'',\varphi_{\nu,n})$*.*

Proof. Put $(M,\varphi):=(\Gamma(\mu,m)'',\varphi_{\mu,m})$ and $(N,\theta):=(\Gamma(\nu,n)'',\varphi_{\nu,n})$. Let $\pi:M\to N$ be any isomorphism between M and N. Put $\psi:=\theta\circ\pi$. By our assumptions on ν and Remark 5.2, the centralizer M^ψ is nonamenable. By Theorem 5.1, we find a nonzero partial isometry $v\in M$ such that $p=v^*v\in M^\varphi$, $q=vv^*\in M^\psi$ and $\operatorname{Ad}(v):(pMp,\varphi_p)\to(qMq,\psi_q)$ is state preserving. It follows in particular that $pM^\varphi p=(pMp)^{\varphi_p}\cong(qMq)^{\psi_q}=qM^\psi q$ is a nonamenable II_1 factor. So M^φ cannot be abelian and Remark 5.2 implies that M^φ is a II_1 factor. Applying Lemma 2.1 twice, we have

$$\mathcal{C}(\bigvee_{k\in\mathbf{N}}\mu^{*k})=\mathcal{C}(\Delta_\varphi)=\mathcal{C}(\Delta_{\varphi_p})=\mathcal{C}(\Delta_{\psi_q})=\mathcal{C}(\Delta_\psi)=\mathcal{C}(\bigvee_{k\in\mathbf{N}}\nu^{*k}).$$

This implies that $\mathcal{C}(\bigvee_{k\geq 1}\mu^{*k})=\mathcal{C}(\bigvee_{k\geq 1}\nu^{*k})$.

Assume now that either $\operatorname{supp}(\nu_a)\neq\{0\}$ or $\operatorname{supp}(\nu_a)=\{0\}$ with $n(0)=+\infty$. In the latter case where $\nu(\{0\})>0$ and $n(0)=+\infty$, we use that the free Araki–Woods functor Γ turns direct sums into free products (see [Sh96, Theorem 2.11]) and conclude that there exists a state preserving isomorphism

$$(N,\theta)\cong(N,\theta)*(\mathrm{L}(\mathbf{F}_\infty),\tau)\ . \tag{5.1}$$

In the case where ν has at least one atom different from 0, it follows similarly from the classification of almost periodic free Araki–Woods factors (see [Sh96]) that (5.1) holds.

Put $q_0=\pi(q)$. Above, we have proved that there exists a state preserving isomorphism $(pMp,\varphi_p)\cong(q_0Nq_0,\theta_{q_0})$. Taking a smaller p if needed, we may assume that $\varphi(p)=1/k$ for some integer $k\geq 1$. Combining (5.1) with Proposition 4.2 and the fact that the fundamental group of $\mathrm{L}(\mathbf{F}_\infty)$ equals $\mathbf{R}^*_+$ (see [Ra91]), it follows that there exists a state preserving isomorphism $(q_0Nq_0,\theta_{q_0})\cong(q_1Nq_1,\theta_{q_1})$ whenever $q_1\in N^\theta$ is a nonzero projection.

Choose a projection $q_1\in N^\theta$ with $\theta(q_1)=1/k$. So, there exists a state preserving isomorphism $(pMp,\varphi_p)\cong(q_1Nq_1,\theta_{q_1})$. Since $\varphi(p)=1/k=\theta(q_1)$ and since both M^φ and N^θ are factors, taking $k\times k$ matrices, we find a state preserving isomorphism $(M,\varphi)\cong(N,\theta)$. □

Proof of Corollary B. Let $\mu,\nu\in\mathcal{S}(\mathbf{R})$ such that $\Lambda(\mu_a)=\Lambda(\nu_a)=:\Lambda$ and $\mathcal{C}(\mu_c*\delta_\Lambda)=\mathcal{C}(\nu_c*\delta_\Lambda)$. Let $m,n:\mathbf{R}\to\mathbf{N}\cup\{+\infty\}$ be any symmetric Borel multiplicity functions. Then we have $\mathcal{C}(\mu*\delta_\Lambda)=\mathcal{C}(\nu*\delta_\Lambda)$. By Theorem 4.1, there is a state preserving isomorphism $\Gamma(\mu,m)''\cong\Gamma(\nu,n)''$.

Conversely, let $\mu,\nu\in\mathcal{S}(\mathbf{R})$ and $m,n:\mathbf{R}\to\mathbf{N}\cup\{+\infty\}$ be any symmetric Borel multiplicity functions such that $\Gamma(\mu,m)''\cong\Gamma(\nu,n)''$. By Theorem A, we have that $\mathcal{C}(\bigvee_{k\geq 1}\mu^{*k})=\mathcal{C}(\bigvee_{k\geq 1}\nu^{*k})$. Since for every $k\geq 1$, we have $\mu_c^{*k}\prec\mu_c$ and $\nu_c^{*k}\prec\nu_c$, it follows that

$$\begin{aligned}\mathcal{C}(\mu_c*\delta_{\Lambda(\mu_a)}\vee\delta_{\Lambda(\mu_a)})&=\mathcal{C}(\bigvee_{k\geq 1}\mu^{*k})\\&=\mathcal{C}(\bigvee_{k\geq 1}\nu^{*k})\\&=\mathcal{C}(\nu_c*\delta_{\Lambda(\nu_a)}\vee\delta_{\Lambda(\nu_a)}).\end{aligned}$$

This implies that $\Lambda(\mu_a)=\Lambda(\nu_a)$ and $\mathcal{C}(\mu_c*\delta_{\Lambda(\mu_a)})=\mathcal{C}(\nu_c*\delta_{\Lambda(\nu_a)})$. □

Proof of Corollary C. Put $(M,\varphi):=(\Gamma(\mu,m)'',\varphi_{\mu,m})$. Let $\psi\in M_*$ be a faithful normal state such that M^ψ is nonamenable. By Theorem 5.1, we find a nonzero partial isometry $v\in M$ such that $q = vv^* \in M^\psi$, $p = v^*v \in M^\varphi$, $qM^\psi q$ has no amenable direct summand and $\psi q = \lambda\, v\varphi v^*$ with $\lambda = \psi(q)/\varphi(p)$. It follows that $pM^\varphi p \cong qM^\psi q$ has no amenable direct summand. By Remark 5.2, this means that either μ_a has an atom different from 0 or μ_a is concentrated on $\{0\}$ with $m(0) \geq 2$. Conversely, if μ_a satisfies these properties, it follows from Remark 5.2 that the centralizer of the free quasi-free state is nonamenable. □

Proof of Corollary D. By Corollary C, the von Neumann algebra $\Gamma(\lambda + \delta_0, 1)''$ has amenable centralizers while $\Gamma(\lambda + \delta_0, 2)''$ does not. □

Example 5.4. Many different measures in the family $\mathcal{S}(\mathbf{R})$ of Corollary B can be constructed as follows. Let $K \subset \mathbf{R}$ be an *independent* Borel set, meaning that every n-tuple of distinct elements in K generates a free abelian group of rank n. By [Ru62, Theorems 5.1.4 and 5.2.2], there exist compact independent $K \subset \mathbf{R}$ such that K is homeomorphic to a Cantor set. Fix such a $K \subset \mathbf{R}$ and put $L = K \cup (-K)$. Also fix a countable subgroup $\Lambda < \mathbf{R}$.

For every continuous symmetric probability measure μ on $\mathbf{R}$ that is concentrated on L, define the measure class $\widetilde{\mu}$ on $\mathbf{R}$ given by

$$\widetilde{\mu} = \bigvee_{x \in \Lambda, n \geq 1} (x + \mu^{*n}) .$$

By construction, each $\widetilde{\mu}$ is a continuous symmetric measure class on $\mathbf{R}$ that is invariant under translation by Λ and that satisfies $\widetilde{\mu} * \widetilde{\mu} \prec \widetilde{\mu}$.

Given continuous symmetric probability measures μ_1 and μ_2 that are concentrated on L, we claim that $\mathcal{C}(\widetilde{\mu_1}) = \mathcal{C}(\widetilde{\mu_2})$ if and only if $\mathcal{C}(\mu_1) = \mathcal{C}(\mu_2)$. One implication is obvious. The other implication is a consequence of the following result contained in [LP97, Corollary 1] : if η_1 and η_2 are concentrated on L and $\eta_1 \perp \eta_2$, then also $\eta_1 \perp (x + \eta_2^{*k})$ for all $x \in \mathbf{R}$ and all $k \geq 1$.

Choosing Λ to be a nontrivial subgroup of $\mathbf{R}$ and applying Corollary B, for all continuous symmetric probability measures μ_1 and μ_2 concentrated on the Cantor set L, we find that

$$\Gamma(\widetilde{\mu_1} \vee \delta_\Lambda, m_1)'' \cong \Gamma(\widetilde{\mu_2} \vee \delta_\Lambda, m_2)'' \quad \text{iff} \quad \mathcal{C}(\mu_1) = \mathcal{C}(\mu_2) .$$

Adding the Lebesgue measure to $\widetilde{\mu}$, we claim that we also have

$$\Gamma(\lambda \vee \widetilde{\mu_1} \vee \delta_\Lambda, m_1)'' \cong \Gamma(\lambda \vee \widetilde{\mu_2} \vee \delta_\Lambda, m_2)'' \quad \text{iff} \quad \mathcal{C}(\mu_1) = \mathcal{C}(\mu_2) .$$

By [Sh97b, Corollary 8.6], for all these free Araki–Woods factors, the τ-invariant equals the usual topology on $\mathbf{R}$, so that they cannot be distinguished by Connes' invariants.

To prove the claim, define L_n as the n-fold sum $L_n = L + \cdots + L$ and put $S = \bigcup_{n\geq 1} L_n$. Below we prove that $\lambda(S) = 0$. The claim then follows from Corollary B : if $\mathcal{C}(\lambda \vee \widetilde{\mu_1}) = \mathcal{C}(\lambda \vee \widetilde{\mu_2})$, restricting to S, we get that $\mathcal{C}(\widetilde{\mu_1}) = \mathcal{C}(\widetilde{\mu_2})$. As proven above, this implies that $\mathcal{C}(\mu_1) = \mathcal{C}(\mu_2)$.

It remains to prove that $\lambda(L_n) = 0$ for all n. If for some $n \geq 1$, we have $\lambda(L_n) > 0$, then $L_{2n} = L_n - L_n$ contains a neighborhood of 0. Every nonzero $x \in L_{2n}$ can be uniquely written as $x = \alpha_1 y_1 + \cdots + \alpha_k y_k$ with $k \geq 1$, $y_1, \ldots, y_k$ distinct elements in K and $\alpha_i \in \mathbf{Z} \setminus \{0\}$ with $|\alpha_i| \leq 2n$ for all i. So if $x \in L_{2n}$ is nonzero, we have that $(2n+1)x \notin L_{2n}$. Therefore, L_{2n} does not contain a neighborhood of 0 and it follows that $\lambda(L_n) = 0$ for all $n \geq 1$.

6. Proof of Theorem E

To prove Theorem E, we combine [HU15, Theorem 4.3] and Theorem 3.1 with the following lemma. Whenever θ is a faithful normal state on a von Neumann algebra M, we denote by $M_{\mathrm{ap},\theta}$ the von Neumann subalgebra of M generated by the almost periodic part of (σ_t^θ).

Lemma 6.1. *For $i = 1, 2$, let (M_i, φ_i) be von Neumann algebras with a faithful normal state. Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ their free product. Denote by $\mathrm{E}_{M_1} : M \to M_1$ the unique φ-preserving conditional expectation. Let θ_1 be a faithful normal state on M_1 and define $\theta = \theta_1 \circ \mathrm{E}_{M_1}$. Let $q \in M^\theta$ be a projection.*

There exist projections $q_0, q_1, \ldots$ with $q_0 \in M_1^{\theta_1}$ and $q_i \in M^\theta$ for all $i \geq 1$ such that

(i) $\sum_{i=0}^\infty q_i = q$,
(ii) $q_0 M_{\mathrm{ap},\theta} q_0 = q_0 M_{1,\mathrm{ap},\theta_1} q_0$ *and* $q_0 M^\theta q_0 = q_0 M_1^{\theta_1} q_0$,
(iii) *for every $i \geq 1$, there exists a partial isometry $v_i \in M$ with $v_i v_i^* = q_i$, $v_i^* v_i \in M^\varphi$ and*

$$\frac{1}{\theta(q_i)}\, \theta q_i = \frac{1}{\varphi(v_i^* v_i)}\, v_i \varphi v_i^* \; .$$

Proof. Fix standard representations $M_i \subset \mathrm{B}(H_i)$. For every faithful normal state μ on M_i, denote by $\xi_\mu \in H_i$ the canonical unit vector that implements μ.

Define $u_t = [\mathrm{D}\theta_1 : \mathrm{D}\varphi_1]_t \in \mathcal{U}(M_1)$. Note that also $[\mathrm{D}\theta : \mathrm{D}\varphi]_t = [\mathrm{D}\theta_1 \circ \mathrm{E} : \mathrm{D}\varphi_1 \circ \mathrm{E}]_t = u_t$ for all $t \in \mathbf{R}$. Let $e_1, e_2, \ldots$ be a maximal sequence of nonzero projections in $M_1^{\theta_1}$ such that $e_i e_j = 0$ whenever $i \neq j$ and such that for every $i \geq 1$, there exists a partial isometry $w_i \in M_1$ and a $\lambda_i > 0$ with $w_i w_i^* = e_i$ and $u_t \sigma_t^{\varphi_1}(w_i) = \lambda_i^{\mathrm{i}t} w_i$ for all $t \in \mathbf{R}$. Define $e_0 = 1 - \sum_{i=1}^\infty e_i$. Then $e_0 \in M_1^{\theta_1}$. By construction, the unitary representation $(U_t)_{t \in \mathbf{R}}$ on H_1 given by $U_t(x \xi_{\varphi_1}) = u_t \sigma_t^{\varphi_1}(x) \xi_{\varphi_1}$ for all $x \in M_1$ is weakly mixing on $e_0 H_1$.

For $i = 1, 2$, define $\overset{\circ}{H}_i = H_i \ominus \mathbf{C}\xi_{\varphi_i}$. For every $k \geq 1$, define the Hilbert space

$$K_k = H_1 \otimes \underbrace{\overset{\circ}{H}_2 \otimes \overset{\circ}{H}_1 \otimes \quad \cdots \quad \otimes \overset{\circ}{H}_1 \otimes \overset{\circ}{H}_2}_{k \text{ times } \overset{\circ}{H}_2 \text{ and } k-1 \text{ times } \overset{\circ}{H}_1, \text{ alternatingly}} \otimes H_1 \; .$$

We can then identify the standard Hilbert space H for M with

$$H = H_1 \oplus \bigoplus_{k=1}^\infty K_k \; .$$

Under this identification, $\xi_\varphi = \xi_{\varphi_1} \in H_1$ and $\xi_\theta = \xi_{\theta_1} \in H_1$. Denote by $(V_t)_{t \in \mathbf{R}}$ the unitary representation on H_1 given by $V_t(x \xi_{\theta_1}) = \sigma_t^{\varphi_1}(x) u_t^* \xi_{\theta_1}$ for all $x \in M_1$. Under the above identification of H, we get that

$$\Delta_\theta^{\mathrm{i}t} = \Delta_{\theta_1}^{\mathrm{i}t} \oplus \bigoplus_{k=1}^\infty \Bigl(U_t \otimes \Delta_{\varphi_2}^{\mathrm{i}t} \otimes \Delta_{\varphi_1}^{\mathrm{i}t} \otimes \cdots \otimes \Delta_{\varphi_1}^{\mathrm{i}t} \otimes \Delta_{\varphi_2}^{\mathrm{i}t} \otimes V_t \Bigr) \; .$$

Since $(U_t)_{t \in \mathbf{R}}$ is weakly mixing on $e_0 H_1$, we conclude that $(\Delta_\theta^{\mathrm{i}t})_{t \in \mathbf{R}}$ is weakly mixing on $e_0 H \ominus e_0 H_1$. It follows that

$$e_0 M_{\mathrm{ap},\theta} e_0 = e_0 M_{1,\mathrm{ap},\theta_1} e_0 \quad \text{and} \quad e_0 M^\theta e_0 = e_0 M_1^{\theta_1} e_0 \; . \tag{6.1}$$

Let $q_1, q_2, \ldots$ be a maximal sequence of nonzero projections in $q M^\theta q$ such that $q_i q_j = 0$ if $i \neq j$ and such that statement (iii) in the lemma holds for every $i \geq 1$. Define $q_0 = q - \sum_{i=1}^\infty q_i$. Then $q_0 \in M^\theta$. We prove that $q_0 \leq e_0$. Once this is proven, it follows from (6.1) that $q_0 \in M_1^{\theta_1}$ and that $q_0 M_{\mathrm{ap},\theta} q_0 = q_0 M_{1,\mathrm{ap},\theta_1} q_0$, so that the lemma follows.

If $q_0 \not\leq e_0$, we find $j \geq 1$ such that $q_0 e_j \neq 0$. Then the polar part v of $q_0 w_j$ is a nonzero partial isometry in M satisfying $v v^* \leq q_0$ and $u_t \sigma_t^{\varphi_1}(v) = \lambda_j^{\mathrm{i}t} v$ for all $t \in \mathbf{R}$. So, the projection $v v^*$ could be added to the sequence $q_1, q_2, \ldots$, contradicting its maximality. Therefore, $q_0 \leq e_0$ and the lemma is proved. ☐

Theorem E will be an immediate consequence of the following more technical proposition that will also be used in Section 7 below.

Proposition 6.2. *For $i = 1, 2$, let (M_i, φ_i) be von Neumann algebras with a faithful normal state. Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ their free product and by $\mathrm{E}_{M_i} : M \to M_i$ the unique φ-preserving conditional expectation. Let ψ be a faithful normal state on M. Define the set of projections $\mathcal{P} \subset M^\psi$ given by $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ where*

- *for $i = 1, 2$, $\mathcal{P}_i$ consists of the projections $q \in M^\psi$ for which there exists a partial isometry $v \in M$ and a faithful normal state θ_i on M_i with $v^*v = q$, $e = vv^* \in M_i^{\theta_i}$,*
$$\frac{1}{\psi(q)}\, v\psi v^* = \frac{1}{\theta_i(e)}\, (\theta_i \circ \mathrm{E}_{M_i})e \quad ,$$
$$vM_{\mathrm{ap},\psi}v^* = eM_{i,\mathrm{ap},\theta_i}e \quad \text{and} \quad vM^\psi v^* = eM_i^{\theta_i}e\ ,$$
- *$\mathcal{P}_3$ consists of the projections $q \in M^\psi$ for which there exists a partial isometry $v \in M$ with $v^*v = q$, $e = vv^* \in M^\varphi$,*
$$\frac{1}{\psi(q)}\, v\psi v^* = \frac{1}{\varphi(e)}\, \varphi e \quad \text{and} \quad vM^\psi v = eM^\varphi e\ .$$

If $q \in M^\psi$ is a projection such that $qM^\psi q$ has no amenable direct summand, then q can be written as a sum of projections in $\mathcal{P}$.

Proof. Let ψ be a faithful normal state on M and $q \in M^\psi$ a projection such that $qM^\psi q$ has no amenable direct summand. It suffices to prove that q dominates a nonzero projection in $\mathcal{P}$, since then a maximality argument can be applied.

Fix any nonzero finite trace projection $r_0 \in \mathrm{L}_\psi(\mathbf{R})$ and put $r = \Pi_{\varphi,\psi}(qr_0)$. Define the von Neumann subalgebra $Q \subset r\mathrm{c}_\varphi(M)r$ given by
$$Q = \Pi_{\varphi,\psi}\big(\mathrm{L}_\psi(\mathbf{R})qr_0 \vee qM^\psi qr_0\big)\ .$$
Note that Q has no amenable direct summand. By [HU15, Theorem 4.3],
$$\text{either} \quad Q' \cap r\mathrm{c}_\varphi(M)r \prec_{\mathrm{c}_\varphi(M)} \mathrm{L}_\varphi(\mathbf{R}) \quad \text{or} \quad Q \prec_{\mathrm{c}_\varphi(M)} \mathrm{c}_\varphi(M_i) \ \text{ for } i = 1 \text{ or } i = 2.$$
Since $\Pi_{\varphi,\psi}(\mathrm{L}_\psi(\mathbf{R})qr_0)$ belongs to both Q and $Q' \cap r\mathrm{c}_\varphi(M)r$, it follows that
$$\Pi_{\varphi,\psi}(\mathrm{L}_\psi(\mathbf{R})qr_0) \prec_{\mathrm{c}_\varphi(M)} \mathrm{c}_\varphi(M_i)$$
for $i = 1$ or $i = 2$.

By Theorem 3.1, we find a faithful normal state θ_i on M_i and a partial isometry $v \in M$ such that $q_0 = v^*v$ is a nonzero projection in $qM^\psi q$, $p = vv^*$ belongs to $M^{\theta_i \circ \mathrm{E}_{M_i}}$ and
$$\frac{1}{\psi(q_0)}\, v\psi v^* = \frac{1}{\theta_i(\mathrm{E}_{M_i}(p))}\, (\theta_i \circ \mathrm{E}_{M_i})p\ .$$
Write $\theta = \theta_i \circ \mathrm{E}_{M_i}$. By Lemma 6.1, we either find a nonzero projection $e \leq p$ such that $e \in M_i^{\theta_i}$ and $eM_{\mathrm{ap},\theta}e = eM_{i,\mathrm{ap},\theta_i}e$ and $eM^\theta e = eM_i^{\theta_i}e$, or we find a nonzero projection $p_0 \in pM^\theta p$ and a partial isometry $w \in M$ such that $ww^* = p_0$, $e = w^*w$ belongs to M^φ and
$$\frac{1}{\theta(p_0)}\, w^*\theta w = \frac{1}{\varphi(e)}\, \varphi e\ .$$
In the first case, we get that the projection $v^*ev \leq q$ belongs to $\mathcal{P}_i$, while in the second case, the projection $v^*p_0v \leq q$ belongs to $\mathcal{P}_3$. □

Proof of Theorem E. Denote by $\mathrm{E}_{M_i} : M \to M_i$ the unique φ-preserving conditional expectation. If M^φ is nonamenable, then obviously, M does not have all its centralizers amenable. If M_i admits a faithful normal state θ_i such that $M_i^{\theta_i}$ is nonamenable, then $\theta_i \circ \mathrm{E}_{M_i}$ is a faithful normal state on M with $M_i^{\theta_i} \subset M^{\theta_i \circ \mathrm{E}_{M_i}}$, so that again, M does not have all its centralizers amenable.

Conversely, assume that ψ is a faithful normal state on M such that M^ψ is nonamenable. Take a nonzero projection $q \in M^\psi$ such that $qM^\psi q$ has no amenable direct summand. By 6.2, we either find $i \in \{1, 2\}$ and a faithful normal state θ_i on M_i such that $M_i^{\theta_i}$ is nonamenable, or we find that M^φ is nonamenable. ☐

7. Further structural results and proof of Theorem F

We start by showing that the invariant of Theorem A is not a complete invariant for the family of free Araki–Woods factors $\Gamma(\mu, m)''$ arising from finite symmetric Borel measures μ on $\mathbf{R}$ whose atomic part μ_a is nonzero and not supported on $\{0\}$.

Theorem 7.1. *Let $\Lambda < \mathbf{R}$ be any countable subgroup such that $\Lambda \neq \{0\}$ and denote by δ_Λ a finite atomic measure on $\mathbf{R}$ whose set of atoms equals Λ. Let η be any continuous finite symmetric Borel measure on $\mathbf{R}$ such that $\mathcal{C}(\eta) = \mathcal{C}(\eta * \delta_\Lambda)$ and such that the measures $(\eta^{*k})_{k\geq 1}$ are pairwise singular.*

Put $\mu = \delta_\Lambda + \eta$ and $\nu = \delta_\Lambda + \eta + \eta * \eta$. Then,

$$\Gamma(\mu, 1)'' \ncong \Gamma(\nu, 1)'' \quad \text{and} \quad \mathcal{C}\Big(\bigvee_{k\geq 1} \mu^{*k}\Big) = \mathcal{C}\Big(\bigvee_{k\geq 1} \nu^{*k}\Big).$$

Proof. By construction, we have $\mathcal{C}(\bigvee_{k\geq 1} \mu^{*k}) = \mathcal{C}(\bigvee_{k\geq 1} \nu^{*k})$. We denote $M := \Gamma(\mu, 1)''$ and $N := \Gamma(\nu, 1)''$. We denote by $Q \subset N$ the canonical von Neumann subalgebra given by $Q := \Gamma(\delta_\Lambda + \eta, 1)''$. Put $\varphi := \varphi_{\mu,1}$ and $\psi := \varphi_{\nu,1}$. Observe that the inclusion $Q \subset N$ is globally invariant under the modular automorphism group σ^ψ.

Assume by contradiction that $M \cong N$. By Theorem A, there exists a state preserving isomorphism $\pi : (M, \varphi) \to (N, \psi)$ of M onto N. Then, π extends to a unitary operator $U : \mathrm{L}^2(M, \varphi) \to \mathrm{L}^2(N, \psi)$ satisfying $U\Delta_\varphi U^* = \Delta_\psi$. Define the real Hilbert space

$$H^\mu_{\mathbf{R}} := \left\{ f \in \mathrm{L}^2_{\mathbf{C}}(\mathbf{R}, \mu) : f(-s) = \overline{f(s)} \text{ for } \mu\text{-almost every } s \in \mathbf{R} \right\}$$

and the orthogonal representation

$$U^\mu : \mathbf{R} \curvearrowright H^\mu_{\mathbf{R}} : U^\mu_s(f)(t) = \exp(\mathrm{i}st) f(t) \ .$$

Denote by $s(\xi) := \ell(\xi) + \ell(\xi)^*$, $\xi \in H^\mu_{\mathbf{R}}$, the canonical semicircular elements that generate M and satisfy $\sigma^\varphi_t(s(\xi)) = s(U^\mu_t \xi)$. By construction, $\mathcal{C}(\Delta_\psi|_{\mathrm{L}^2(N)\ominus \mathrm{L}^2(Q)}) = \bigcap_{k\geq 2} \mathcal{C}(\eta^{*k})$. Since μ is singular w.r.t. η^{*k} for all $k \geq 2$, it follows that $\pi(s(\xi)) \in Q$ for all $\xi \in H^\mu_{\mathbf{R}}$. But then, $\pi(M) \subset Q$, which is impossible because π is surjective. ☐

Note that Example 5.4 provides many measures η satisfying the assumptions of Theorem 7.1.

We could not prove or disprove that the measure class $\mathcal{C}(\mu_c * \delta_\Lambda(\mu_a))$ is an invariant for the family of free Araki–Woods factors $\Gamma(\mu, m)''$ arising from finite symmetric Borel measures μ on $\mathbf{R}$ whose atomic part μ_a is nonzero and not supported on $\{0\}$.

Using Theorem 5.1 in combination with the results of [BH16], we can also clarify the relation between free Araki–Woods factors and free products of amenable von Neumann algebras. Combining [Sh96, Theorems 2.11 and 4.8], it follows that every almost periodic free Araki–Woods factor is isomorphic with a free product of von Neumann algebras of type I. Conversely, by [Ho06], many free products of type I von Neumann algebras are isomorphic with free Araki–Woods factors. In [Dy92, Theorem 4.6], it is proved that a free product of two amenable von Neumann algebras w.r.t. faithful normal traces is always isomorphic to the direct sum of an interpolated free group factor and a finite dimensional algebra. It is therefore tempting to believe that every type III factor arising as a free product of amenable von Neumann algebras w.r.t. faithful normal states is a free Araki–Woods factor. The following example shows however that this is almost never the case if one of the states fails to be almost periodic.

Theorem 7.2. *Let $(P, \theta) = *_n(P_n, \theta_n)$ be a free product of amenable von Neumann algebras. Assume that the centralizer P^θ has no amenable direct summand and that at least one of the θ_n is not almost periodic.*

Then P is not isomorphic to a free Araki–Woods factor. Even more: there is no faithful normal homomorphism π of P into a free Araki–Woods factor M such that $\pi(P) \subset M$ is with expectation.

The same conclusions hold if (P, θ) is any von Neumann algebra with a faithful normal state satisfying the following three properties: the centralizer P^θ has no amenable direct summand, θ is not almost periodic and P is generated by a family of amenable von Neumann subalgebras $P_n \subset P$ that are globally invariant under the modular automorphism group (σ_t^θ).

Proof. Let (M, φ) be a free Araki–Woods factor with its free quasi-free state. Let (P, θ) be a von Neumann algebra with a faithful normal state θ such that the centralizer P^θ has no amenable direct summand and such that P is generated by a family of amenable von Neumann subalgebras $P_n \subset P$ that are globally invariant under the modular automorphism group (σ_t^θ). Let $\pi : P \to M$ be a normal homomorphism and $\mathrm{E} : M \to \pi(P)$ a faithful normal conditional expectation. We prove that θ is almost periodic.

Define the faithful normal state $\psi \in M_*$ given by $\psi = \theta \circ \pi^{-1} \circ \mathrm{E}$. Since $\pi(P^\theta) \subset M^\psi$, we get that M^ψ has no amenable direct summand. By Theorem 5.1, we find partial isometries $v_n \in M$ such that $q_n = v_n v_n^* \in M^\psi$, $p_n = v_n^* v_n \in M^\varphi$, $\sum_n q_n = 1$ and

$$q_n \, [\mathrm{D}\psi : \mathrm{D}\varphi]_t = \lambda_n^{\mathrm{i}t} \, v_n \, \sigma_t^\varphi(v_n^*)$$

where $\lambda_n = \psi(q_n)/\varphi(p_n)$.

Replacing (M, φ) by the free product of (M, φ) and the appropriate almost periodic free Araki–Woods factor, we still get a free Araki–Woods factor and we may assume that M^φ is a factor and that each λ_n is an eigenvalue for the free quasi-free state. We can then choose partial isometries $w_n \in M$ such that $\sigma_t^\varphi(w_n) = \lambda_n^{-\mathrm{i}t} w_n$, $w_n w_n^* = p_n$ and such that $e_n = w_n^* w_n$ belongs to M^φ with $\sum_n e_n = 1$. So, we find that

$$q_n \, [\mathrm{D}\psi : \mathrm{D}\varphi]_t = v_n w_n \, \sigma_t^\varphi(w_n^* v_n^*)$$

for all n and all $t \in \mathbf{R}$. We conclude that $v = \sum_n v_n w_n$ is a unitary in M satisfying $[\mathrm{D}\psi : \mathrm{D}\varphi]_t = v \, \sigma_t^\varphi(v^*)$. This means that $\varphi = \psi \circ \mathrm{Ad}\, v$ and that the homomorphism $\eta = \mathrm{Ad}\, v^* \circ \pi$ satisfies $\sigma_t^\varphi \circ \eta = \eta \circ \sigma_t^\theta$.

So for every n, the subalgebra $\eta(P_n) \subset M$ is amenable and globally invariant under the modular automorphism group (σ_t^φ). By [BH16, Theorem 4.1], it follows that $\eta(P)$ lies in the almost periodic part of (M, φ). This implies that the restriction of (σ_t^θ) to P_n is almost periodic. Since this holds for every n, we conclude that θ is almost periodic. □

From Proposition 6.2, we get the following rigidity results for free product von Neumann algebras. Roughly, the result says that an arbitrary free product of a "very much non almost periodic" M_1 with an almost periodic M_2 remembers the almost periodic part M_2 up to amplification.

Recall that a faithful normal positive functional φ on a von Neumann algebra M is said to be *weakly mixing* if the unitary representation $\sigma_t^\varphi(\,\cdot\,)$ on $\mathrm{L}^2(M,\varphi) \ominus \mathbf{C}1$ is weakly mixing, and that φ is said to be *almost periodic* if the unitary representation $\sigma_t^\varphi(\,\cdot\,)$ on $\mathrm{L}^2(M,\varphi)$ is almost periodic.

Proposition 7.3. *For $i = 1, 2$, let (M_i, φ_i) and (N_i, ψ_i) be von Neumann algebras with faithful normal states. Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ and $(N, \psi) = (N_1, \psi_1) * (N_2, \psi_2)$ their free products. Assume that*

- *M_1 and N_1 have all their centralizers amenable and φ_1, ψ_1 are weakly mixing states,*
- *$M_2^{\varphi_2}$ and $N_2^{\psi_2}$ have no amenable direct summand and φ_2, ψ_2 are almost periodic.*

If $M \cong N$, there exist nonzero projections $e \in M_2$ and $q \in N_2$ such that $eM_2e \cong qN_2q$.

Proof. Whenever μ is a faithful normal state on M and $q \in M^\mu$ is a projection such that $qM^\mu q$ has no amenable direct summand, we can apply Proposition 6.2. Since M_1 has all its centralizers amenable, the set $\mathcal{P}_1$ in Proposition 6.2 equals $\{0\}$. Since φ_1 is weakly mixing, the almost periodic part of a state of the form $\theta_2 \circ \mathrm{E}_{M_2}$ (and, in particular, of φ) is contained in M_2. It thus follows from Proposition 6.2 that there exist sequences of projections $q_i \in M^\mu$ and $e_i \in M_2$, as well as partial isometries $v_i \in M$ and faithful normal positive functionals θ_i on $e_iM_2e_i$ such that $v_i^*v_i = q_i$, $v_iv_i^* = e_i$, $\sum_{i=0}^\infty q_i = q$ and $v_i\mu v_i^*$ equals $\theta_i \circ \mathrm{E}_{M_2}$ on e_iMe_i for all $i \geq 0$.

Let $\pi : M \to N$ be an isomorphism of M onto N. We first apply the result in the first paragraph to $\mu = \psi \circ \pi$. Since ψ_1 is weakly mixing, $M^\mu = \pi^{-1}(N_2^{\psi_2})$. We find nonzero projections $q \in N_2^{\psi_2}$ and $e \in M_2$, a partial isometry $v \in M$ and a faithful normal positive functional θ on eM_2e such that $v^*v = \pi^{-1}(q)$, $vv^* = e$ and $v\mu v^* = \theta \circ \mathrm{E}_{M_2}$ on eMe. Since φ_1 is weakly mixing, the almost periodic part of $\theta \circ \mathrm{E}_{M_2}$ equals $eM_{2,\mathrm{ap},\theta}e$. Since ψ_1 is weakly mixing and ψ_2 is almost periodic, the almost periodic part of μ equals $\pi^{-1}(N_2)$. It follows that

$$v\pi^{-1}(N_2)v^* = eM_{2,\mathrm{ap},\theta}e \; . \tag{7.1}$$

By [HU15, Lemma 2.1], every projection in M_2 is equivalent, inside M_2, with a projection in $M_2^{\varphi_2}$. So conjugating e and θ, we may assume that $e \in M_2^{\varphi_2}$. We then apply the result of the first paragraph of the proof to the free product $N = N_1 * N_2$, the faithful normal state $\mu' = \varphi \circ \pi^{-1}$ and the projection $f = \pi(e)$ in $N^{\mu'}$. Since φ_1 is weakly mixing, we have $N^{\mu'} = \pi(M_2^{\varphi_2})$. We thus find projections $e_i \in eM_2^{\varphi_2}e$ summing up to e, projections $p_i \in N_2$, partial isometries $w_i \in N$ and faithful normal positive functionals Ω_i on $p_iN_2p_i$ such that $w_i^*w_i = \pi(e_i)$, $w_iw_i^* = p_i$ and $w_i\mu'w_i^* = \Omega_i \circ \mathrm{E}_{N_2}$ on p_iNp_i for all i. In particular, $\Omega(p_i) = \varphi_2(e_i)$ and $\sum_i \Omega(p_i) = \varphi_2(e)$.

Define the projection $p \in \mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} N_2$ given by $p = \sum_i e_{ii} \otimes p_i$. Define the faithful normal positive functional Ω on $p(\mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} N_2)p$ given by

$$\Omega(T) = \sum_i \Omega_i(T_{ii}) \; .$$

Finally define $W \in \ell^2(\mathbf{N}) \otimes N$ given by $W = \sum_i e_i \otimes w_i$. It follows that $W^*W = \pi(e)$, $WW^* = p$ and

$$W\mu'W^* = \Omega \circ \mathrm{E}_{\mathrm{B}(\ell^2(\mathbf{N}))\overline{\otimes} N_2} \quad \text{on} \;\; p(\mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} N)p \; .$$

As above, $N_{\mathrm{ap},\mu'} = \pi(M_{\mathrm{ap},\varphi}) = \pi(M_2)$. Since ψ_1 is weakly mixing, the almost periodic part of the functional $\Omega \circ \mathrm{E}_{\mathrm{B}(\ell^2(\mathbf{N}))\overline{\otimes} N_2}$ on $p(\mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} N)p$ is contained in $p(\mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} N_2)p$. It follows that

$$W\pi(eM_2e)W^* \subset p(\mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} N_2)p \; . \tag{7.2}$$

Write $V = W\pi(v)$. Then $V \in \ell^2(\mathbf{N}) \otimes N$ is a partial isometry with $VV^* = p \in \mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} N_2$ and $V^*V = q \in N_2$. Using (7.1) and (7.2), we find that

$$VqN_2qV^* = W\pi(eM_{2,\mathrm{ap},\theta}e)W^* \subset W\pi(eM_2e)W^* \subset p(\mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} N_2)p \,. \tag{7.3}$$

Since $N_2^{\psi_2}$ has no amenable direct summand, it follows in particular that N_2 is diffuse. Then (7.3) implies that $V \in \ell^2(\mathbf{N}) \otimes N_2$ and therefore, all the inclusions in (7.3) are equalities. In particular, $eM_{2,\mathrm{ap},\theta}e = eM_2e$ so that (7.1) implies that $qN_2q \cong eM_2e$. This concludes the proof of the proposition. □

Combining Propositions 4.2 and 7.3, we can easily prove Theorem F.

Proof of Theorem F. Put $(M,\varphi) = (\Gamma(\mu,+\infty)'', \varphi_{\mu,+\infty})$ as in the formulation of the Proposition. Then, φ is weakly mixing and by Corollary C, the free Araki–Woods factor M has all its centralizers amenable. For $i = 1, 2$, let (A_i, ψ_i) be von Neumann algebras with almost periodic faithful normal states having a nonamenable factorial centralizer $A_i^{\psi_i}$.

If the free products $(M_i, \varphi_i) = (M,\varphi) * (A_i, \psi_i)$ satisfy $M_1 \cong M_2$, it follows from Proposition 7.3 that there exist nonzero projections $p_i \in A_i$ such that $p_1A_1p_1 \cong p_2A_2p_2$. In the first case, where the A_i are II$_1$ factors, this implies that $A_1 \cong A_2^t$ for some $t > 0$. In the second case, where the A_i are type III factors, this implies that $A_1 \cong A_2$.

For the converse, first assume that the (A_i, ψ_i) are II$_1$ factors with their tracial states and $A_1 \cong A_2^t$ for some $t > 0$. Take nonzero projections $p_i \in A_i$ such that $p_1A_1p_1 \cong p_2A_2p_2$. By the uniqueness of the trace, we have $(p_1A_1p_1, (\psi_1)_{p_1}) \cong (p_2A_2p_2, (\psi_2)_{p_2})$. It then follows from Proposition 4.2 that $p_1M_1p_1 \cong p_2M_2p_2$. Since the M_i are type III factors, this further implies that $M_1 \cong M_2$.

Finally assume that the (A_i, ψ_i) are full type III factors with almost periodic states having a factorial centralizer $A_i^{\psi_i}$ and that $\pi : A_1 \to A_2$ is an isomorphism of A_1 onto A_2. Denote by $\Gamma = \mathrm{Sd}(A_1) = \mathrm{Sd}(A_2)$ the Sd-invariant of $A_1 \cong A_2$. Define $(B_i, \theta_i) = (\mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} A_i, \mathrm{Tr} \otimes \psi_i)$. By [Co74, Lemma 4.8], the weight θ_i on B_i is a Γ-almost periodic weight. By [Co74, Theorem 4.7], there exists a unitary $U \in B_2$ and a constant $\alpha > 0$ such that $\theta_2 \circ \mathrm{Ad}(U) \circ (\mathrm{id} \otimes \pi) = \alpha\, \theta_1$. Since A^{ψ_2} is a factor, after a unitary conjugacy of π, we find nonzero projections $p_i \in A_i^{\psi_i}$ such that $\pi(p_1) = p_2$ and $(\psi_2)_{p_2} \circ \pi = (\psi_1)_{p_1}$ on $p_1A_1p_1$. As in the previous paragraph, we can use Proposition 4.2 to conclude that $M_1 \cong M_2$. □

We finally consider two further structural properties of free Araki–Woods factors: the free absorption property and the structure of its continuous core. We say that a von Neumann algebra M with a faithful normal state φ has the *free absorption property* if the free product $(N,\psi) = (M,\varphi) * (\mathrm{L}(\mathbf{F}_\infty), \tau)$ satisfies $N \cong M$. One of the key results in [Sh96] is the free absorption property for the almost periodic free Araki–Woods factors. In general, we get the following result.

Proposition 7.4. *Let $(M,\varphi) = (\Gamma(\mu,+\infty)'', \varphi_{\mu,+\infty})$ be a free Araki–Woods factor with infinite multiplicity. Then (M,φ) has the free absorption property if and only if the atomic part μ_a is nonzero.*

Proof. If $\mu(\{0\}) > 0$, then (M,φ) freely splits off $(\mathrm{L}(\mathbf{F}_\infty), \tau)$ and the free absorption property immediately holds. If $\mu(\{a\}) > 0$ for some $a \neq 0$, then (M,φ) freely splits off an almost periodic free Araki–Woods factor of type III and the free absorption property follows from [Sh96, Theorem 5.4]. Conversely, if $\mu_a = 0$, it follows from Corollary C that M has all its centralizers amenable. But then M cannot have the free absorption property. □

One of the most intriguing isomorphism questions for free Araki–Woods factors, well outside the scope of our methods, is whether $\Gamma(\lambda,1)'' \cong \Gamma(\lambda+\delta_0,1)''$? In [Sh97a, Theorem 4.8], it was shown that the continuous core of $\Gamma(\lambda,1)''$ is isomorphic with $\mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_\infty)$. We prove that the same holds for $\Gamma(\lambda+\delta_0,1)''$. Note here that in [Ha15, Corollary 1.10], it is proved that if μ is singular w.r.t. the Lebesgue measure λ, then the continuous core of $\Gamma(\mu,m)''$ is never isomorphic with $\mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_\infty)$. Under the stronger assumption that all convolution powers μ^{*n} are singular w.r.t. the Lebesgue measure, this was already shown in [Sh02].

Proposition 7.5. *The continuous core of* $\Gamma(\lambda+\delta_0,1)''$ *is isomorphic with* $\mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_\infty)$.

Proof. In [Sh97a, Sh97b], von Neumann algebras generated by A-valued semicircular elements are introduced. In the special case where A is semifinite and equipped with a fixed faithful normal semifinite trace Tr, this construction can be summarized as follows.

Let H be a Hilbert A-bimodule, meaning that H is a Hilbert space equipped with a normal homomorphism $A \to \mathrm{B}(H)$ and a normal anti-homomorphism $A \to \mathrm{B}(H)$ having commuting images. We denote the left and right action of A on H as $a \cdot \xi \cdot b$ for all $a,b \in A$, $\xi \in H$. Further assume that S is an A-anti-bimodular involution on H. More precisely, S is a closed, densely defined operator on H such that $S(\xi) \in \mathrm{D}(S)$ with $S(S(\xi)) = \xi$ for all $\xi \in \mathrm{D}(S)$ and such that for all $\xi \in \mathrm{D}(S)$ and all $a,b \in A$, we have $a \cdot \xi \cdot b \in \mathrm{D}(S)$ and $S(a\cdot\xi\cdot b) = b^* \cdot S(\xi) \cdot a^*$. Define

$$\mathcal{F}_A(H) = \mathrm{L}^2(A,\mathrm{Tr}) \oplus \bigoplus_{k=1}^\infty \big(\underbrace{H \otimes_A H \otimes_A \cdots \otimes_A H}_{k \text{ factors}}\big) \; .$$

A vector $\xi \in H$ is called right bounded if there exists a $\kappa > 0$ such that $\|\xi \cdot a\| \leq \kappa \|a\|_{2,\mathrm{Tr}}$ for all $a \in \mathfrak{n}_{\mathrm{Tr}}$. For every $\xi \in H$, there exists an increasing sequence of projections $p_n \in A$ such that $p_n \to 1$ strongly and $\xi \cdot p_n$ is right bounded for all n. So, the subspace $H_0 \subset H$ defined as

$$H_0 = \{\xi \in \mathrm{D}(S) : \xi \;\text{ and }\; S(\xi) \;\text{ are right bounded }\}$$

is dense. For every right bounded vector $\xi \in H$, we have a natural left creation operator $\ell(\xi) \in \mathrm{B}(\mathcal{F}_A(H))$. Then we define

$$\Phi(A,\mathrm{Tr},H,S) = \big(A \cup \{\ell(\xi) + \ell(S(\xi))^* : \xi \in H_0\}\big)'' \; .$$

There is a normal conditional expectation $\mathrm{E} : \Phi(A,\mathrm{Tr},H,S) \to A$ given by $\mathrm{E}(x)P = PxP$, where $P : \mathcal{F}_A(H) \to \mathrm{L}^2(A,\mathrm{Tr})$ is the orthogonal projection. By [Sh97b, Proposition 5.2], we have that E is faithful. By construction, if $A = \mathbf{C}1$ and $\mathrm{Tr}(1) = 1$, and using the notation introduced before Lemma 4.3, we find the free Araki–Woods factor $\Phi(\mathbf{C}1,\mathrm{Tr},H,S) \cong \Gamma(H,S)''$.

The above construction can be applied to a normal completely positive map $\varphi : A \to A$ satisfying $\mathrm{Tr}(\varphi(a)b) = \mathrm{Tr}(a\varphi(b))$ for all $a,b \in \mathfrak{m}_{\mathrm{Tr}}$. To such a map φ, we associate the Hilbert A-bimodule H_φ by separation and completion of $A \otimes \mathfrak{n}_{\mathrm{Tr}}$ with inner product

$$\langle a \otimes_\varphi b, c \otimes_\varphi d \rangle = \mathrm{Tr}(b^* \varphi(a^* c) d) \; .$$

We also define the anti-unitary involution $S(a \otimes_\varphi b) = b^* \otimes_\varphi a^*$. We denote the resulting von Neumann algebra $\Phi(A,\mathrm{Tr},H,S)$ as $\Phi(A,\mathrm{Tr},\varphi)$.

Given a trace preserving inclusion $(A,\mathrm{Tr}) \subset (D,\mathrm{Tr})$, we denote by $\mathrm{E} : D \to A$ the unique Tr-preserving conditional expectation and define $\psi : D \to D : \psi(d) = \varphi(\mathrm{E}(d))$ for all $d \in D$. Then, the functor Φ satisfies

$$\Phi(A,\mathrm{Tr},\varphi) *_A D \cong \Phi(D,\mathrm{Tr},\psi) \tag{7.4}$$

where the amalgamated free product is taken w.r.t. the canonical conditional expectations.

Let $(M,\varphi) = (\Gamma(\lambda,1)'', \varphi_{\lambda,1})$. We can then reformulate [Sh97a, Theorem 4.1] as

$$\mathrm{c}_\varphi(M) \cong \Phi(A,\mathrm{Tr},\varphi) \tag{7.5}$$

where $A = \mathrm{L}^\infty(\mathbf{R})$, $\mathrm{Tr}(f) = \int_{\mathbf{R}} f(x)\exp(-x)\,\mathrm{d}x$ and $\varphi : A \to A$ is such that the associated A-bimodule H is isomorphic with the coarse A-bimodule $\mathrm{L}^2(\mathbf{R}^2)$ with anti-unitary involution $(S\xi)(x,y) = \overline{\xi(y,x)}$. Under the identification $\mathrm{L}_\varphi(\mathbf{R}) \cong \mathrm{L}^\infty(\mathbf{R})$, the isomorphism in (7.5) respects the canonical conditional expectations $\mathrm{c}_\varphi(M) \to \mathrm{L}_\varphi(\mathbf{R})$ and $\Phi(A, \mathrm{Tr}, \varphi) \to A$.

By [Sh97a, Theorem 4.8], we have in this particular case that $\Phi(A, \mathrm{Tr}, \varphi) \cong \mathrm{B}(\ell^2(\mathbf{N})) \overline{\otimes} \mathrm{L}(\mathbf{F}_\infty)$. Let now (D, Tr_D) be an arbitrary diffuse abelian von Neumann algebra with a faithful normal semifinite trace satisfying $\mathrm{Tr}(1) = +\infty$ and let $\psi : D \to D$ be any normal completely positive map satisfying $\mathrm{Tr}_D(\psi(c)d) = \mathrm{Tr}_D(c\psi(d))$ for all $c, d \in \mathfrak{m}_{\mathrm{Tr}_D}$ such that the associated D-bimodule H and anti-unitary involution S are isomorphic with $\mathrm{L}^2(D, \mathrm{Tr}_D) \otimes \mathrm{L}^2(D, \mathrm{Tr}_D)$ with $S(c \otimes d) = d^* \otimes c^*$. Since there exists an isomorphism $\alpha : D \to A$ of D onto A satisfying $\mathrm{Tr} \circ \alpha = \mathrm{Tr}_D$, it follows that

$$
\begin{aligned}
\Phi(D, \mathrm{Tr}_D, \psi) &\cong \Phi(D, \mathrm{Tr}_D, \mathrm{L}^2(D, \mathrm{Tr}_D) \otimes \mathrm{L}^2(D, \mathrm{Tr}_D), S) \\
&\cong \Phi(A, \mathrm{Tr}, \mathrm{L}^2(A, \mathrm{Tr}) \otimes \mathrm{L}^2(A, \mathrm{Tr}), S) \\
&\cong \Phi(A, \mathrm{Tr}, \varphi) \cong \mathrm{B}(\ell^2(\mathbf{N})) \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_\infty)\ .
\end{aligned} \tag{7.6}
$$

Write $(M_1, \varphi_1) = (\Gamma(\lambda + \delta_0, 1)'', \varphi_{\lambda+\delta_0,1})$. Since $(M_1, \varphi_1) \cong (M, \varphi) * (B, \tau)$ for some diffuse abelian von Neumann algebra B with faithful normal state τ, it follows that

$$\mathrm{c}_{\varphi_1}(M_1) \cong \mathrm{c}_\varphi(M) *_{\mathrm{L}_\varphi(\mathbf{R})} (\mathrm{L}_\varphi(\mathbf{R}) \mathbin{\overline{\otimes}} B)\ .$$

Since the isomorphism in (7.5) respects the conditional expectations, we conclude that

$$\mathrm{c}_{\varphi_1}(M_1) \cong \Phi(A, \mathrm{Tr}, \varphi) *_A (A \mathbin{\overline{\otimes}} B)\ ,$$

where the conditional expectation $A \mathbin{\overline{\otimes}} B \to A$ is given by $\mathrm{id} \otimes \tau$. Write $D = A \mathbin{\overline{\otimes}} B$ and define $\psi : D \to D : \psi(d) = \varphi((\mathrm{id} \otimes \tau)(d)) \otimes 1$. By (7.4), we get that

$$\Phi(A, \mathrm{Tr}, \varphi) *_A (A \mathbin{\overline{\otimes}} B) \cong \Phi(D, \mathrm{Tr} \otimes \tau, \psi)\ .$$

Since D is diffuse abelian and the D-bimodule associated with ψ is isomorphic with the coarse D-bimodule, it follows from (7.6) that $\Phi(D, \mathrm{Tr} \otimes \tau, \psi) \cong \mathrm{B}(\ell^2(\mathbf{N})) \overline{\otimes} \mathrm{L}(\mathbf{F}_\infty)$. So, the proposition is proved. □

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