Citation	<p>Florian Maurin, Claus Claeys, Lucas Van Belle, Wim Desmet, (2017), Title: Bloch theorem with revised boundary conditions applied to glide, screw and rotational symmetric structures Computer Methods in Applied Mechanics and Engineering, Volume 318, Pages 497–513</p>
Archived version	<p>Author manuscript: the content is identical to the content of the published paper, but without the final typesetting by the publisher</p>
Published version	<p>10.1016/j.cma.2017.01.034</p>
Journal homepage	<p>http://www.sciencedirect.com/science/article/pii/S0045782516310933</p>
Author contact	<p>florian.maurin@kuleuven.be + 32 (0)16 37 23 91</p>
IR	<p>https://lirias.kuleuven.be/handle/123456789/572087</p>

(article begins on next page)

Bloch theorem with revised boundary conditions applied to glide, screw and rotational symmetric structures

Florian Maurin^{a,b,*}, Claus Claeys^{a,b}, Lucas Van Belle^{a,b}, Wim Desmet^{a,b}

^a*KU Leuven, Department of Mechanical Engineering, Division PMA
Celestijnenlaan 300B - box 2420, B-3001 Leuven, Belgium.*

^b*Member of Flanders Make.*

Abstract

Wave propagation in complex periodic systems is often addressed with the Bloch theorem, and consists in applying periodic boundary conditions to a discretized unit cell. While this method has been developed for structures periodic by translation, in a recent work, for quasi-one-dimensional wave propagation, it has been shown that screw (translation plus rotation) and glide (translation plus reflection) periodicities can be accounted for as well, keeping the Cartesian coordinate system but revisiting the periodic boundary conditions. The goal of the present paper is to generalize this concept to quasi-two-dimensional wave propagation (two dimensional waves propagating in three dimensional structures). Dispersion relations for a set of reduced problems are then compared to results from the classical method, when available. By considering a smaller periodicity, the computational cost is decreased and the number of folding curves and non-interacting intersecting curves is reduced, improving their interpretability. While the size of a unit cell is divided by a factor two when glide symmetries are considered, this ratio is significantly increased for screw or rotational symmetries. Moreover, the proposed revisited Bloch method is applicable to screw symmetric structures that do not possess purely translational symmetries, and for which the classical method cannot be used (e.g. chiral nanotubes, longitudinally wrinkled helicoids).

Keywords: Bloch theorem, periodic boundary conditions, glide, screw and rotational symmetry, quasi-two-dimensional wave propagation, dispersion analysis

*Corresponding author: florian.maurin@kuleuven.be

1. Introduction

Wave propagation in infinite periodic structures is of interest in a large range of applications such as acoustic metamaterials [1, 2] and non-destructive evaluation for structural health monitoring [3]. For complex structures, these challenges are often being addressed via the Bloch theorem, restricting the analysis to a single unit cell to which periodic boundary conditions are applied. For discretized unit cells, Bloch theorem can be implemented following two different approaches: the transfer-matrix method [4–6] and the inverse method [7–10] (see a review of both methods in [11]). Contrary to the inverse approach, the transfer-matrix method has the advantage of providing evanescent modes, but its implementation in multiple dimensions requires more efforts [12, 13]. Besides the Bloch theorem, other approaches can be used to investigate wave propagation in infinite periodic structures, such as for example the homogenization methods for low [14, 15] and moderated [16] frequencies. However, this paper focuses on the use of the Bloch theorem.

While the Bloch theorem applies to structures periodic by translation, it has been recently shown that for quasi-one-dimensional wave propagation, glide (translation plus reflection) and screw (translation plus rotation) symmetries can be accounted for as well with both the transfer-matrix and the inverse approaches [17]. Advantages are that the considered unit cell is smaller, reducing the computational cost and error, while improving the dispersion curve readability. By quasi-one-dimensional wave propagation, we mean that the wave propagates in one direction, whereas the structure is at least two dimensional (the second dimension being finite and small). The goal of the present paper is to extend this concept to quasi-two-dimensional wave propagation: two-dimensional waves propagating in three-dimensional structures.

However, in quasi-two dimensions, some restrictions on the symmetries exist, which are presented in Table 1. For quasi-one-dimensional structures, while translated, a unit cell can be rotated (screw) or reflected (glide) by an angle θ and the periodicity is along the translational axis. Since these transformations apply around an axis, the periodicity cannot be extended to a second straight direction, except for one specific case, a glide symmetry with a plane of reflection coinciding with the two directions of periodicity ($\theta = 0$). To get a screw symmetric structure with two directions of periodicity, the circumferential direction has to be considered. Indeed, in some specific case as a helix, the structure can be parametrized from two different directions: along the helix axis, or along the circumferential direction, and these two possibilities are respectively referred to as t^* and θ^* in Table 1. This justifies also the presence of rotational symmetries which are screw symmetries without translational components.

These symmetries are illustrated in Fig. 1. Quasi-one-dimensional wave prop-

Table 1: The different period types considered in Bloch theorem, depending on the wave and structure dimensions. The symbol * indicates if the wave propagates along the translational (t^*) or rotational (θ^*) direction, where t is the spacial period by translation and θ is the angle of rotation/reflection.

Structure Periodicity	1D	2D	3D
1D	Translation(t^*)	Translation(t^*) Glide($t^*, 0$)=Screw(t^*, π) Rotation(θ^*)	Translation(t^*) Glide(t^*, θ) Screw(t^*, θ) or (t, θ^*) Rotation(θ^*)
2D		$2 \times$ Translations(t^*)	$2 \times$ Translations(t^*) $2 \times$ Glides($t^*, 0$) Translation(t^*)+Glide($t^*, 0$) Translation(t^*)+Screw(t, θ^*) Translation(t^*)+Rotation(θ^*)
3D			$3 \times$ Translations(t^*)

agation in glide symmetric undulated or buckled beams is investigated in [18–21] (Fig. 1a), and accounting for the glide divides the size of the problem by a factor two [17]. Twisted cables (Fig. 1b) are good examples for which computation gains are even more important. Indeed, considering the classical Bloch theorem, the period by translation is a helix step (2π), whereas with appropriate boundary conditions, the height of the unit cell can be arbitrarily reduced to the length of one finite element [17, 22]. Otherwise, the cable slice can be defined with an infinitesimally small thickness, leading to the semi-analytical finite element (SAFE) method [23, 24]. Another interesting example is the tetrahelix (Fig. 1c) built by stacking tetrahedra, such that two consecutive tetrahedra are shifted by an irrational angle: there is no period by pure translation. Consequently, it is not possible to apply the classical Bloch theorem, whereas the screw symmetry can be used [17]. The same holds for longitudinally wrinkled helicoids [25–27], and is detailed in Sec. 3.1.1.

Examples of structures possessing one translational periodicity in one direction and one glide plane symmetry in a second direction are the undulated metal sheet commonly used as materials for roofs (Fig. 1d) and some corrugated cores as the one shown in Fig. 1e. Structures with double-glide periodicity can be found also in corrugated cores as the pyramidal one (Fig. 1f and Sec. 3.2.2), and in composite and fabric materials (e.g. the plane weave pattern, Fig. 1g). An example of quasi-two-dimensional wave propagation with one wave propagating along the circumferential direction is the cylindrically curved metamaterial panel shown in Fig. 1h [28]. In this

Figure 1: Periodic structures in one (a-c) and two (d-i) directions, and with the minimum periodicity surrounded by dashed lines. Glide symmetric undulated buckled beam [19] (a), a multi-wire cable (b), a screw symmetric tetrahelix [17] (c), a glide symmetric undulated sheet (d) and triangular corrugate core [30] (d), a double-glide symmetric pyramidal core [31] (f) and plain weave fabric [32] (g), a rotational symmetric metamaterial panel [28] (h), and a screw symmetric pipe (i). (b) is under CC0 license and (e-g) are reprinted with permission from [30–32], respectively.

last work, Bloch theorem is used to get the dispersion characteristics, discretizing the unit cell in the cylindrical coordinate system. This differs from the present approach where the Cartesian coordinate system is used, providing a simplified formulation and a natural choice for computer implementation. Other examples with rotations are the screw symmetric pipes (Fig. 1i) and the chiral nanotubes [29] investigated in Sec. 3.2.4.

This paper completes the initial work on quasi-one-dimensional wave propagation [17], and is organized as follows. In Sec. 2, appropriate boundary conditions are proposed to account for the additional symmetries for the inverse method of the Bloch theorem. In Sec. 3, the applicability of the proposed method is demonstrated in a set of problems, followed by the conclusions.

2. Bloch Theorem For Glide And Screw Symmetric Problems

Wave propagation in quasi-two-dimensional periodic structures is investigated with the Bloch theorem, restricting the analysis to a single unit cell, and taking advantage of periodic boundary conditions [33]. We denote $\mathbf{r}_P$ the position of the point P within the reference cell, and by $\boldsymbol{\rho}_P = \mathbf{r}_P + n_1\mathbf{e}_1 + n_2\mathbf{e}_2$ the same position P relative to the $\{n_1, n_2\}$ -th unit cell, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the basis vectors of the direct lattice. Assuming a harmonic wave with angular frequency ω and amplitude $\hat{\mathbf{u}}$, the wave displacement $\mathbf{u}(\mathbf{r}_P, t)$ at time t and position $\mathbf{r}_P$ is given by

$$\mathbf{u}(\mathbf{r}_P, t) = \hat{\mathbf{u}}e^{\mathbf{k} \cdot \mathbf{r}_P - i\omega t}, \quad (1)$$

where $i^2 = -1$ and $\mathbf{k}$ is the wave vector. The wave at the point $\boldsymbol{\rho}_P$ yields

$$\mathbf{u}(\boldsymbol{\rho}_P, t) = \mathbf{u}(\mathbf{r}_P, t)e^{(n_1\mu_1 + n_2\mu_2)}, \quad (2)$$

where the propagation constant components $\mu_1 = \mathbf{k} \cdot \mathbf{e}_1$ and $\mu_2 = \mathbf{k} \cdot \mathbf{e}_2$ are complex numbers such that their real and imaginary parts represent the attenuation and phase constants, respectively. For glide or screw symmetries we have

$$\mathbf{u}(\boldsymbol{\rho}_P, t) = (\mathbf{R}_1)^{n_1}(\mathbf{R}_2)^{n_2}\mathbf{u}(\mathbf{r}_P, t)e^{(n_1\mu_1 + n_2\mu_2)}, \quad (3)$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are reflection ($\mathbf{R}_{\text{ef}\{\mathbf{v}, \theta\}}$) or rotation ($\mathbf{R}_{\text{ot}\{\mathbf{v}, \theta\}}$) matrices, $\mathbf{v}$ being the axis of rotation and θ its angle. For example, for three-dimensional elements with 3 rotation-free degrees of freedom (dof) and an axis of symmetry in the direction $\mathbf{e}_x$, $\mathbf{R}_1$ or $\mathbf{R}_2$ are defined as

$$\mathbf{R}_{\text{ef}\{\mathbf{e}_x, \theta\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & \sin 2\theta & -\cos 2\theta \end{bmatrix}, \quad (4a)$$

$$\mathbf{R}_{\text{ot}\{\mathbf{e}_x, \theta\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}. \quad (4b)$$

Figure 2: Periodic unit cell.

Note that in Eq. (3), there should be no order between the application of the transformation in the $\mathbf{e}_1$ or the $\mathbf{e}_2$ directions, resulting to the constraint

$$\mathbf{R}_1 \mathbf{R}_2 = \mathbf{R}_2 \mathbf{R}_1. \quad (5)$$

In general two rotation/reflection matrices of different axis are not commutative except if one of them is the identity matrix or if they are both diagonals [34]. This is the reason why Bloch theorem can be applied to axisymmetric geometries but not spherical ones.

The equation of motion for the discretized unit cell reads

$$[\mathbf{K} - \omega^2 \mathbf{M}] \mathbf{u} = \mathbf{f}, \quad (6)$$

where $\mathbf{u}$ and $\mathbf{f}$ are the nodal displacement and force vectors, and $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices, respectively. The dofs are reorganized into left, right, bottom, top, left-bottom, left-top, right-bottom, right-top, and internal regions, denoted respectively $(\)_L$, $(\)_R$, $(\)_B$, $(\)_T$, $(\)_{LB}$, $(\)_{LT}$, $(\)_{RB}$, $(\)_{RT}$, and $(\)_I$ (Fig. 2), such that the generalized displacement and force vectors can be written under the form $\mathbf{u} = \{\mathbf{u}_L^T, \mathbf{u}_R^T, \mathbf{u}_B^T, \mathbf{u}_T^T, \mathbf{u}_{LB}^T, \mathbf{u}_{LT}^T, \mathbf{u}_{RB}^T, \mathbf{u}_{RT}^T, \mathbf{u}_I^T\}^T$ and $\mathbf{f} = \{\mathbf{f}_L^T, \mathbf{f}_R^T, \mathbf{f}_B^T, \mathbf{f}_T^T, \mathbf{f}_{LB}^T, \mathbf{f}_{LT}^T, \mathbf{f}_{RB}^T, \mathbf{f}_{RT}^T, \mathbf{f}_I^T\}^T$.

From Eq. (3), the nodal displacements between extremities of the unit cell re-

spectively in the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions are given by:

$$\begin{bmatrix} \mathbf{u}_R \\ \mathbf{u}_{RB} \\ \mathbf{u}_{RT} \end{bmatrix} = e^{\mu_1} \begin{bmatrix} \mathbf{I}_l \otimes \mathbf{R}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{lb} \otimes \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{lb} \otimes \mathbf{R}_1 \end{bmatrix} \begin{bmatrix} \mathbf{u}_L \\ \mathbf{u}_{LB} \\ \mathbf{u}_{LT} \end{bmatrix}, \quad (7a)$$

$$\begin{bmatrix} \mathbf{u}_T \\ \mathbf{u}_{LT} \\ \mathbf{u}_{RT} \end{bmatrix} = e^{\mu_2} \begin{bmatrix} \mathbf{I}_b \otimes \mathbf{R}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{lb} \otimes \mathbf{R}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{lb} \otimes \mathbf{R}_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_B \\ \mathbf{u}_{LB} \\ \mathbf{u}_{RB} \end{bmatrix}, \quad (7b)$$

where $\otimes$ is the Kronecker product and $\mathbf{I}$ the identity matrix of size defined by subscripts; l , b and lb being the number of nodes at the left, bottom and left-bottom sides, respectively. Noting that among the 6 equations composing the system (7), only 5 are independent, this system is valid only if Eq. (5) is verified. Among the 9 subgroups composing the unit cell, only 4 are independent, such that $\mathbf{u}$ can be expressed as:

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_L \\ \mathbf{u}_R \\ \mathbf{u}_B \\ \mathbf{u}_T \\ \mathbf{u}_{LB} \\ \mathbf{u}_{LT} \\ \mathbf{u}_{RB} \\ \mathbf{u}_{RT} \\ \mathbf{u}_I \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ e^{\mu_1} \mathbf{I}_l \otimes \mathbf{R}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & e^{\mu_2} \mathbf{I}_b \otimes \mathbf{R}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & e^{\mu_1} \mathbf{I}_{lb} \otimes \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & e^{\mu_2} \mathbf{I}_{lb} \otimes \mathbf{R}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & e^{\mu_1 + \mu_2} \mathbf{I}_{lb} \otimes \mathbf{R}_1 \mathbf{R}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u}_L \\ \mathbf{u}_B \\ \mathbf{u}_{LB} \\ \mathbf{u}_I \end{bmatrix} = \mathbf{A}(\boldsymbol{\mu}) \mathbf{u}^{(r)}. \quad (8)$$

From equilibrium conditions, the force field $\mathbf{f}$ reads [35]:

$$\begin{aligned} \mathbf{f}_R + e^{\mu_1} (\mathbf{I}_l \otimes \mathbf{R}_1) \mathbf{f}_L &= \mathbf{0}, \\ \mathbf{f}_T + e^{\mu_2} (\mathbf{I}_b \otimes \mathbf{R}_2) \mathbf{f}_B &= \mathbf{0}, \\ \mathbf{f}_{RT} + e^{\mu_1} (\mathbf{I}_{lb} \otimes \mathbf{R}_1) \mathbf{f}_{LT} + e^{\mu_2} (\mathbf{I}_{lb} \otimes \mathbf{R}_2) \mathbf{f}_{RB} + e^{\mu_1 + \mu_2} (\mathbf{I}_{lb} \otimes \mathbf{R}_1 \mathbf{R}_2) \mathbf{f}_{LB} &= \mathbf{0}. \end{aligned} \quad (9)$$

Assuming no internal forces ($\mathbf{f}_I = \mathbf{0}$), a purely propagating wave ($\text{Re}(\boldsymbol{\mu}) = \mathbf{0}$), and substituting Eqs. (8) and (9) into Eq. (6), then pre-multiplying both sides by $\mathbf{A}^H(\boldsymbol{\mu})$, where H denotes the conjugate transpose, gives:

$$[\mathbf{K}^{(r)}(\boldsymbol{\mu}) - \omega^2 \mathbf{M}^{(r)}(\boldsymbol{\mu})] \mathbf{u}^{(r)} = \mathbf{0}, \quad (10)$$

where $\mathbf{K}^{(r)}(\boldsymbol{\mu}) = \mathbf{A}^H(\boldsymbol{\mu}) \mathbf{K} \mathbf{A}(\boldsymbol{\mu})$ and $\mathbf{M}^{(r)}(\boldsymbol{\mu}) = \mathbf{A}^H(\boldsymbol{\mu}) \mathbf{M} \mathbf{A}(\boldsymbol{\mu})$ are the reduced stiffness and mass matrices, respectively. Eq. (10) is an eigenvalue problem which gives eigenvalues ω corresponding to the propagation constants $\boldsymbol{\mu}$.

3. Numerical Results

This section demonstrates the advantages of the proposed method in a set of problems. All numerical results are computed in *MATLAB* with in-house code.

3.1. Quasi-one-dimensional wave propagation

Before investigating two-dimensional wave propagation, we recall the effects of additional symmetries on one-dimensional wave propagation by investigating a structure not mentioned in [17] but not less interesting: the longitudinally-wrinkled helicoid.

3.1.1. Longitudinally-wrinkled helicoid

Twisted and stretched elastic ribbons handle multiple morphological instabilities [25, 26], and we focus here on the longitudinally-wrinkled helicoid (Fig. 3a). For some specific configurations, twisting the stretched ribbon compresses regions close to the centroid both in the longitudinal and radial directions, resulting in the formation of periodic buckled patterns. In [27], long-wave propagation in this structure is investigated and solitary waves are observed experimentally. Solitary waves result from the equilibrium between nonlinear and dispersive effects, and details on how the dispersion curves are obtained are given here.

A steel ribbon from *Brütsch & Rüegger* is used and has the following characteristics: width $W = 12.7$ mm, thickness $t = 0.04$ mm, length $L = 25$ cm, Young's Modulus $E = 210$ GPa, Poisson ratio $\nu = 0.3$, and density $\rho = 7800$ kg m⁻³. In order to get the first longitudinally-wrinkled mode, the strip is stretched by a tension $T = 67$ N then twisted by an angle $\Theta = \pi$ (Fig. 3b). The thin ribbon is modeled by isogeometric Kirchhoff-Love shell elements [36] capable of handling large geometrically-nonlinear deformations. Quadratic NURBS basis functions are used and the domain is discretized by 30 and 300 elements in the radial and longitudinal directions, respectively. Since this shell formulation is rotation-free, the $\mathcal{C}^1$ -continuity between two adjacent unit cells has to be enforced to lock the hinge connection, as shown in [37]. Simulation results are shown in Figs. 3c-g and we refer the reader to [27] for more details.

Similarly to the tetrahelix studied in [17], the longitudinally-wrinkled helicoid does not possess purely translational symmetries meaning that the classical Bloch theorem cannot be used (the helix step over the buckling period depends on the configuration, and is in general an irrational number). However, the screw period can be used as unit cell with rotational boundary conditions ($\mathbf{R}_{\text{ot}\{e_x, \theta\}}$, Eq. (4b)). The twisting angle is $\theta = 2\Theta\lambda_{lon}/L$ with $2\lambda_{lon}$ the translational buckling period (Fig. 3).

Figure 3: Experimental deformed geometry (a), numerical model (b), numerical deformed geometry (c), numerical Von Mises bending stresses (d), radial bending stresses (e) and its sign (f-g), of the longitudinally-wrinkled helicoid. Only half of the longitudinal direction is represented.

Dispersion curves are shown in full lines in the first Brillouin zone (FBZ) in Fig. 4a. However, using this period, the curves fold at $\text{Im}(\widetilde{\mu}_x) = \pm\pi$ (see dashed circles), presaging that the considered period is not the minimum one [17]. Indeed, from the Von Mises stress representation (Fig. 3d), the period seems to be reducible by two and the question is to know which type of symmetry is involved. To this aim, it is necessary to investigate the symmetries in the transverse direction. The problem with the Von Mises stress is that it is an absolute quantity, and therefore, the curvature direction cannot be obtained. Instead, we consider directly the bending

Figure 4: Dispersion curves for the longitudinally-wrinkled helicoid normalized by $\omega_0 = 2\pi/l\sqrt{E/\rho}$, and obtained using the period $2\lambda_{lon}$ (a) and λ_{lon} (b) in full lines in the FBZ and extended to the adjacent BZ in dashed lines. Colorization is subjective. Dashed circles emphasize curve folding in the FBZ.

stress in the radial direction (σ_{rr} , Fig. 3e) proportional to the radial curvature. From the sign of σ_{rr} (Fig. 3f), a detailed observation (see dashed circles in Fig. 3g) shows that once the unit cell is translated by λ_{lon} , the plane ($\mathbf{e}_x - \mathbf{e}_n$) is not a simple plane of mirror symmetry, but a mirror which also shift the sign of the $\mathbf{e}_n$ direction: this

is a screw symmetry of angle π . While Figs. 3d-g are projected on the undeformed configuration, the twisted angle of the total screw is $\theta = \Theta\lambda_{lon}/L + \pi$.

Resulting dispersion curves are shown in Fig. 4b in full lines in the FBZ, and are found unfolded in this representation. To show that the information displayed in full lines in Figs. 4a and 4b is equivalent, results are extended to the adjacent Brillouin zones (BZ) by shifting the full lines of Figs. 4a and 4b by $\pm 2iq\pi$ and $\pm iq\pi$, respectively ($q \in \mathbb{N}^*$). Indeed, if we denote $\widetilde{\mu}_x$ and $\widehat{\mu}_x$ the propagation constants and noting that the periods are $2\lambda_{lon}$ and λ_{lon} , respectively, $\widehat{\mu}_x = \widetilde{\mu}_x/2$ justifying the fact that the shift is not the same. When dashed and full lines are considered together, Figs. 4a and 4b are identical. Dashed lines however do not provide any additional information given the periodicity of the dispersion relations, and depending on the application, there is no interest to return to this representation.

To highlight this last statement, the full lines of the dispersion curves shown in Fig. 4 are isolated in Fig. 5 in the irreducible Brillouin zone (IBZ) ($\text{Im}(\mu_x) = [0 \ \pi]$). If the focus of analysis is to look for intersections between acoustic and dispersion branches [38], Fig. 5a would be more appropriate. However, if the focus is to get the physical interpretation of the different branches, the representation of Fig. 5b is more clear. Indeed, the number of folding curves is considerably decreased, meaning that a same branch (a same phenomenon) is represented by a continuous line. Moreover, when the minimum unit cell is not used, it is difficult to see when two intersecting dispersion curves interact, whereas using the minimum symmetry, most of the non-interacting intersections are not present anymore (see dashed rectangles in Fig. 5).

Due to the fact that both radial and normal directions are shifted while translated, a phase shift π is present for wavemodes possessing radial or normal components, and the consequence is that their dispersion branches originate at $\text{Im}(\widehat{\mu}_x) = \pi$ instead of 0. Indeed, this is the case for example for the radial and normal flexural branches which possess radial and normal components, respectively, as illustrated in Figs. 6b-c. For the first shear mode (Fig. 6d) both radial and normal components are present, such that the corresponding branch is two times shifted by π , and so comes back to $\text{Im}(\widehat{\mu}_x) = 0$. For the second shear mode (Fig. 6e), since the plane $(\mathbf{e}_x - \mathbf{e}_n)$ is mirror symmetric, effects on the radial direction annihilate, and the only remaining wave component is $\mathbf{e}_n$; this branch initiates at $\text{Im}(\widehat{\mu}_x) = \pi$. Since all the even shear modes are symmetric, all their branches start at $\text{Im}(\widehat{\mu}_x) = \pi$ whereas the odd modes initiate at $\text{Im}(\widehat{\mu}_x) = 0$.

In conclusion, considering the smallest unit cell in the Bloch theorem drastically reduces the number of folding curves and the non-interacting intersections, producing band structures which are easier to interpret. Moreover, the mirror symmetry shifts

Figure 5: Dispersion curves for the longitudinally-wrinkled helicoid normalized by $\omega_0 = 2\pi/l\sqrt{E/\rho}$, and obtained using the period $2\lambda_{lon}$ (a) and λ_{lon} (b) in the IBZ. Longitudinal (L), normal/radial flexural (Fn/Fr), and shear (S_i) branches (i denotes the mode number) [27] are subjectively colored. Dashed circles and rectangles emphasize curve folding and non-intersecting intersections, respectively.

Figure 6: Mode shapes at $\text{Im}(\widetilde{\mu}_x) = 0.2$ of a flat ribbon (the use of a flat ribbon is justified by the desire to simplify the interpretations). The scale represents the displacement in the $\mathbf{e}_x$ (a), $\mathbf{e}_r$ (b), and $\mathbf{e}_n$ (c-f) directions normalized by its extrema, whereas the displacements in the other directions are negligible.

by π the origin of the dispersion branches in function of their wavemode components, facilitating the mode classification, as emphasized also in the next examples.

Figure 7: Undulated shell composed of 9 unit cells (a), with detailed view of one profile (b), and the topology in (c). The dashed square and diamond indicate the minimum period by translation and by glide, respectively (c).

3.2. Quasi-two-dimensional wave propagation

In this section, two double-glide symmetric problems, the pyramidal truss and the undulated plate, are investigated first, followed by the dispersion analysis of structures involving rotational symmetries: a cylinder and a nanotube. The following material notation and numerical values are used in this section: $E = 70$ GPa is the Young's modulus, $\nu = 0.33$ the Poisson ratio, $G = E/(2(1 - \nu))$ the shear modulus, and $\rho = 2700$ kg m⁻³ the density.

3.2.1. Undulated shell

In this example, the undulated thin shell analyzed in [21, 37] is revisited to account for the double-glide symmetry (Fig. 7). The surface is parametrized by

$$z(x, y) = \frac{z_0}{2} \left(\cos \left(2\pi \frac{x}{l} \right) + \cos \left(2\pi \frac{y}{l} \right) \right) \quad (11a)$$

$$= sz_0 \left| \cos \left(\pi \frac{\sqrt{2}}{l} u \right) \right| \left| \cos \left(\pi \frac{\sqrt{2}}{l} v \right) \right|, \quad (11b)$$

where

$$s = \text{sgn} \left\{ \cos \left(\pi \frac{\sqrt{2}}{l} u \right) \right\} \text{sgn} \left\{ \cos \left(\pi \frac{\sqrt{2}}{l} v \right) \right\},$$

Figure 8: Undulated shell unit cells defined by the minimum period by translation (a) and by glide (b), and resulting dispersion curves in the IBZ contour normalized by $\omega_0 = 2\pi/l\sqrt{E/(1-\nu^2)}/\rho$ (c,d), respectively. Colorization is subjective. Modal band gap in gray. Dashed circles emphasize curve folding.

and l and z_0 are the undulation period and magnitude, respectively. $u = \frac{x+y}{\sqrt{2}}$ and $v = \frac{-x+y}{\sqrt{2}}$ are the coordinates in the orthonormal system $\{\mathbf{e}_u = \frac{\mathbf{e}_x+\mathbf{e}_y}{\sqrt{2}}, \mathbf{e}_v = \frac{-\mathbf{e}_x+\mathbf{e}_y}{\sqrt{2}}\}$. From Eq. (11), the smallest period by translation is l whereas a smaller period of $l/\sqrt{2}$ in the $\mathbf{e}_u$ and $\mathbf{e}_v$ directions exists when considering the double-glide symmetry ($\mathbf{R}_1 = \mathbf{R}_2 = \mathbf{R}_{\text{ef}\{e_x \pm e_y, 0\}}$). Results are given for $l = 0.1$ m, $\zeta = z_0/l = 0.05$ and $\gamma = t/l = 0.05$, and the geometry is modeled by isogeometric Kirchhoff-Love shells [36] discretized with quadratic B-splines and refined by 32 elements in each directions. Note that using this shell formulation, the $\mathcal{C}^1$ -continuity has to be enforced, as shown in [37].

Dispersion curves are shown in Fig. 8 for both unit cells. Similarly to what observed for quasi-one-dimensional wave propagation, when the selected period is not the minimum one, dispersion curves fold (dashed circles in Fig. 8c) and dispersion curves are more clear when the smallest unit cell is used. Another advantage is that the bending mode is easily identified since it starts at $\widehat{M}$ in the IBZ contour (Fig. 8d).

Similarly to Fig. 4, it is possible to recover the result obtained with the classical

Figure 9: FBZ considering the minimum translational (a) and glide (b) period, with the IBZ in gray. Dispersion along arrows of the same color are identical. Both representations in (b) are equivalent.

Bloch from the revisited one accounting for the double-glide symmetry. Indeed, using the fact that the propagation constant is $\mu = 2\pi k/\text{Period}$ (k being the wavenumber), one gets

$$\begin{aligned}\widetilde{\mu}_x &= \widetilde{k}_x l = \frac{\widehat{k}_u - \widehat{k}_v}{\sqrt{2}} l = \frac{\widehat{\mu}_u - \widehat{\mu}_v \pm 2iq\pi}{2} = \frac{\widehat{\mu}_u - \widehat{\mu}_v}{2} \pm iq\pi, \\ \widetilde{\mu}_y &= \widetilde{k}_y l = \frac{\widehat{k}_u + \widehat{k}_v}{\sqrt{2}} l = \frac{\widehat{\mu}_u + \widehat{\mu}_v \pm 2iq\pi}{2} = \frac{\widehat{\mu}_u + \widehat{\mu}_v}{2} \pm iq\pi,\end{aligned}\tag{12}$$

where $q \in \mathbb{N}^*$. The different operations required to recover the result in the IBZ contour of the minimum period by translation from the glide period are shown in Fig. 9.

One can note that when the dispersion curves are plotted only in the contour of the IBZ for the glide period ($\widehat{\Gamma} - \widehat{X} - \widehat{M} - \widehat{\Gamma}$), the branch on the interval $(\widetilde{X} - \widetilde{M})$ of the IBZ for the translational period is missing (red arrows of Fig. 9). But this path for the glide period is inside the IBZ, and so, does not bring any additional information (note that this is not true for non-symmetric unit cells [13, 39] as shown in Sec. 3.2.4). As an illustration that the information does not depend on the representation, the frequency range of the modal band gap is found identical for both unit cells (in a modal band gap, a single mode propagates by direction; we refer the reader to [21] for more details).

While in this shell model there is only one layer of nodes in the $\mathbf{e}_z$ direction at the unit cell boundary, the interface can be more complex requiring some caution, as illustrated next with the double-glide symmetric pyramidal truss.

3.2.2. Pyramidal truss

In this example, quasi-two-dimensional wave propagation is investigated in a pyramidal truss structure shown in Fig. 10. The green shade unit cells are exactly the

Figure 10: Isometric (a) and top (b) views of a pyramidal truss constructed from the brick (c). Green-shade unit cells are translated red-shade ones with reversed $\mathbf{e}_z$ direction. Dashed large square, diamond, and small square in (b) represent the minimum period by translation in the $\mathbf{e}_u/\mathbf{e}_v$ directions, by translation in the $\mathbf{e}_x/\mathbf{e}_y$ directions, and by glide in the $\mathbf{e}_u/\mathbf{e}_v$ directions, respectively.

same as the red ones except that they are translated by a period l in the $\mathbf{e}_u$ or $\mathbf{e}_v$ directions, and that the transverse direction $\mathbf{e}_z$ is flipped: this structure is double-glide symmetric. When viewed from the top (Fig. 10b), the most obvious period by translation is a large square of side $2l$, but a smaller period ($\sqrt{2}l$) exists on the diagonal direction. However, considering glide symmetries, the minimum period is a square of length l . While wave propagation in Kirigami auxetic pyramidal cores has been already investigated in [40, 41] using the classical Bloch theorem, the goal of the present study is to show advantages of accounting for the double-glide symmetry.

Each unit cell is carefully discretized at its boundaries, such that the mesh is mirror symmetric with respect to the plane $\mathbf{e}_x - \mathbf{e}_y$ placed in the middle of the structure. To this end, the structure is constructed from the elementary brick shown in Fig. 10c and composed of 398 tetrahedral elements [42]. Numerical simulations are performed for $l = 6/\sqrt{2}$ cm, $2b = t = 5$ mm, and elements linking the top to the bottom are inclined by 45 degrees.

Dispersion curves for the three unit cells mentioned previously (Figs. 10b and 11a-c) are shown in their respective FBZ in Figs. 11d-f, and in the contour of their IBZ in Figs. 11g-i. If the minimum period is not used, curves or surfaces folding are observed, whereas dispersion curves are more clear when the smallest unit cell is

Figure 11: Pyramidal truss unit cells defined by the minimum period by translation in the $\mathbf{e}_u/\mathbf{e}_v$ directions, by translation in the $\mathbf{e}_x/\mathbf{e}_y$ directions, and by glide in the $\mathbf{e}_u/\mathbf{e}_v$ directions, respectively. Resulting dispersion curves in the FBZ (d-f) and in the IBZ contour (g-i), normalized by $\omega_0 = 2\pi/l\sqrt{E/(1-\nu^2)/\rho}$. Colorization is subjective and dashed circles emphasize curve folding (g-i).

used.

Computation time for the three different periods is shown in Table 2, and although the size of the problem is each time reduced by a factor of two, the computation gain is more important due to the fact that the stiffness and mass matrices grow quadratically.

While the size of the problem is divided by 2 when taking advantage of glide symmetries, for rotational symmetries the gain is even more important, as shown next.

Table 2: Computation time for the dispersion surfaces shown in Figs. 11d-f (each direction is discretized by 100 points) with an *Intel® Core i7 4720HQ* and taking advantage of sparse matrices.

	Fig. 11d	Fig. 11e	Fig. 11f
Computation time (min)	120	50	20

3.2.3. Homogeneous cylinder

To show advantages of using rotational symmetries into Bloch theorem, a homogeneous cylinder of radius $r = 10t = 0.01$ m and axis $\mathbf{e}_z$ is investigated. The geometry is modeled by isogeometric cubic Kirchhoff-Love shell elements [36] taking care of the $\mathcal{C}^1$ -continuity at the unit cell interface [37]. The circumferential section of this cylinder is arbitrarily divided into 16 elements.

Whereas the classical Bloch theorem in one dimension along the cylinder axis considers the full circumference as unit cell (16 blocks), taking advantage of the revisited boundary conditions allows the reduction of the unit cell to an arbitrary small surface, and we choose to take 1 block ($\mathbf{R}_1 = \mathbf{I}$ and $\mathbf{R}_2 = \mathbf{R}_{\text{ot}\{e_z, \pi/8\}}$, Fig. 12a). Dispersion curves are shown in Fig. 12b, and are compared to an analytical solution given by the eigenvalue problem [43]

$$[\mathbf{K}_{\text{eff}} - \omega^2 \mathbf{M}_{\text{eff}}] \mathbf{u} = \mathbf{0}, \quad (13)$$

where

$$\mathbf{M}_{\text{eff}} = \rho t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\mathbf{K}_{\text{eff}} = - \begin{bmatrix} -\bar{E} \left(k_1^2 + \frac{1}{2}(1-\nu)k_2^2 \right) - \frac{D}{r^2} (k_1^2 + (1-\nu)k_2^2) & -\frac{i}{2}(1+\nu)\bar{E}k_1k_2 & i\frac{k_1}{r} (\bar{E} + D(k_1^2 + k_2^2)) \\ \frac{i}{2}(1+\nu)\bar{E}k_1k_2 & -\bar{E} \left(k_2^2 + \frac{1}{2}(1-\nu)k_1^2 \right) & \nu\bar{E}\frac{k_2}{r} \\ i\frac{k_1}{r} (\bar{E} + D(k_1^2 + k_2^2)) & -\nu\bar{E}\frac{k_2}{r} & D(k_1^2 + k_2^2)^2 + \frac{\bar{E}}{r^2} \end{bmatrix},$$

with $\bar{E} = Et/(1-\nu^2)$, $D = Et^3/(12(1-\nu^2))$, and k_1 and k_2 are the complex wavenumber in the longitudinal and circumferential directions, respectively. Noting that this system is 3×3 , the comparison to numerical results is restricted to the first three branches and is found in perfect agreement. Note also that dispersion curves are still folding in the given frequency range since a smaller unit cell can be considered.

While these results have been obtained assuming an infinite periodicity both in the circumferential and axial directions, the cylinder is in reality closed along its circumference and interactions between circumferential waves have to be considered.

Figure 12: Dispersion curves for a cylinder considering the unit cell in dark gray (a), in the IBZ contour and normalized by $\omega_0 = \pi^2 \sqrt{E/(1-\nu^2)}/\rho$ (b). Analytical Eq. (13) in dotted lines (d). Dashed circles emphasize curve folding.

Consequently, the phase change of a wave propagating around the circumference has to be a multiple of 2π , restricting the circumferential complex wavenumber to some discrete values ($k_2 = iN$, $N \in \mathbb{N}_0$). While the closed cylinder is investigated in [44], results are not reproduced here for the cylinder, but this procedure will be detailed and applied in Sec. 3.2.4 to the more complex nanotube structure.

This approach of using a small circumferential portion is already considered for the wave finite element (WFE) approach [44–46], but is slightly different in the sense that the rotation matrix is directly applied to the stiffness and mass matrices, whereas in the current approach it is applied to the periodic boundary conditions. While the approach used in [44] works fine for rotational symmetries where only one rotation matrix is required, for screw symmetric structures, each direction has its own angle of rotation. The approach presented in [44] cannot be used whereas the current approach allows the use of two different rotation matrices, as illustrated next.

3.2.4. Nanotube with a chiral angle

In this example, a structure similar to a graphene nanotube is investigated, and is constructed as follows. A hexagonal lattice of side length l is tilted by a chiral angle ψ , and then rolled up around a virtual cylinder of radius r (Fig. 13). This chiral angle and the radius are linked, and they can only take some discrete values. Indeed, taking the flat dashed rectangular ribbon shown in Fig. 13b, once wrapped, the top-left corner has to coincide with the bottom-right corner. Defining the unit cell in Fig. 13c, it means that the dashed rectangular ribbon is composed by $n/2$ unit cells

Figure 13: Nanotube like structure (a), partially constructed from the ribbon (b) itself constructed from the unit cell (c).

(n being a positive odd number equal to 15 in the present study). The chiral angle is defined by

$$\psi = \text{atan} \left(\frac{2h}{nw} \right) = \text{atan} \left(\frac{\sqrt{3}}{n} \right), \quad (14)$$

where $h = 3/2l$ and $w = \sqrt{3}l$ are the height and width of the unit cell. The radius of the cylinder is defined by the diagonal of the dashed rectangular ribbon such that

$$r = \frac{\sqrt{(nw/2)^2 + h^2}}{2\pi} = \frac{\sqrt{3}\sqrt{n^2 + 3}}{4\pi}l. \quad (15)$$

The structure is screw symmetric, with periodicity in two directions such that the unit cell can be rotated by an angle θ_{LR} and translated along $\mathbf{e}_z$ by Δz_{LR} or rotated

by θ_{BT} and translated along $\mathbf{e}_z$ by Δz_{BT} , where

$$\begin{aligned}\theta_{LR} &= l/r\sqrt{3}\cos\psi, \\ \theta_{BT} &= l/(2r)\left(\sqrt{3}\cos\psi - 3\sin\psi\right), \\ \Delta z_{LR} &= l\sqrt{3}\sin\psi, \\ \Delta z_{BT} &= l/2\left(\sqrt{3}\sin\psi + 3\cos\psi\right).\end{aligned}\tag{16}$$

While both screw angles are irrationals, two different unit cells will never be aligned, meaning that there is no pure periodicity by translation, and that the classical Bloch theorem cannot be used. Note that the literature on band structure of nanotubes is abundant, but is restricted to Hamiltonian systems with analytical potentials [47]. It is the reason why we consider here a nanotube structure at a larger scale.

The length of each beam is $l = 1$ m and the beam cross section is square with thickness $t = 2\sqrt{3}l/50$. Each beam is discretized by 10 three-dimensional Timoshenko beam elements [48], involving 6 dofs per node (3 displacements and 3 rotations). Accounting for the screw symmetry, Bloch theorem can be used considering the unit cell shown in Fig. 14a and with the rotations matrices $\mathbf{R}_1 = \mathbf{I}_2 \otimes \mathbf{R}_{\text{ot}\{\mathbf{e}_z, \theta_{LR}\}}$ and $\mathbf{R}_2 = \mathbf{I}_2 \otimes \mathbf{R}_{\text{ot}\{\mathbf{e}_z, \theta_{BT}\}}$ (where $\mathbf{I}_2 \otimes$ is used to account for the 6 dofs per node). Although $\mathbf{R}_1$ and $\mathbf{R}_2$ are both non-diagonals, they share the same rotation axis and so, Eq. (5) is verified.

The first Brillouin zone in the reciprocal space is shown in Fig. 14b, and coordinates of this area in terms of the propagation constant are given in Fig. 14c [49]. For a flat hexagonal lattice, due to the hexagonal symmetry, the IBZ is the triangle $\Gamma - M - X$ [9]. However, in the present study, the curvature annihilates the $p6m$ group symmetry of the flat hexagonal lattice, and the unit cell has no internal symmetry anymore. Since the FBZ is always point symmetric, the IBZ is then only half of the FBZ, and is illustrated in gray in Fig. 14c (for a given direction and its opposite one, the dispersion characteristics are identical).

Dispersion surfaces and its isolines in the FBZ are shown in Fig. 14d-f for the first two bands. Note that from the dispersion isolines, the extrema do not belong to any contour. These are the consequences of the absence of internal symmetry, as discussed in [13, 39].

Figs. 14d-f present the dispersion surfaces and isolines obtained assuming an infinite periodicity both in the circumferential and the axial direction of the nanotube, but without considering wave interactions induced by the closed structure. Indeed, the circumferential direction cannot be considered infinite and the nanotube has to

Figure 14: Nanotube unit cell (a) and FBZ on the reciprocal space (b) with coordinates of this area in terms of the propagation constant (c). IBZ in gray (b-c). Dispersion surfaces normalized by $\omega_0 = \pi^2 \sqrt{E/\rho}$ (d), and dispersion isolines (e-f) of the two first bands in the FBZ. Maximum and minimum of the bands are in yellow and blue circles, respectively (e-f).

be closed, enforcing the phase change of a wave to be multiple of 2π as the wave propagates around the circumference ($k_\theta = iN / N \in \mathbb{N}_0$ and $k_z \in \mathbb{C}$ being respectively the circumferential complex wavenumber and the complex wavenumber in direction of the nanotube axis) [44]. Noting that the circumference of the nanotube is $(n+1)/2 \mathbf{e}_1 - \mathbf{e}_2$ (see Fig. 13b), the circumference is now expressed in terms of both $\mathbf{e}_1$ and $\mathbf{e}_2$ vectors, contrary to the cylinder. The goal is to retrieve μ_1 and μ_2 after imposing the constraints holding on k_θ and k_z . Recalling that $\mu_1 = \mathbf{k} \cdot \mathbf{e}_1$ and $\mu_2 = \mathbf{k} \cdot \mathbf{e}_2$, the wavevector in the reciprocal basis and orthonormal basis is:

$$\mathbf{k} = \mu_1 \mathbf{e}^1 + \mu_2 \mathbf{e}^2 = \frac{k_\theta}{r} \mathbf{e}_\theta + k_z \mathbf{e}_z. \quad (17)$$

Using the relation $\mathbf{e}_i \mathbf{e}^j = \delta_{ij}$, the reciprocal lattice basis ($\mathbf{e}^1 \mathbf{e}^2$) is linked to the direct

lattice basis $(\mathbf{e}_1 \ \mathbf{e}_2)$ by the transform

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{e}_2 \\ \mathbf{e}_1 \cdot \mathbf{e}_2 & \mathbf{e}_2 \cdot \mathbf{e}_2 \end{bmatrix} \begin{bmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \end{bmatrix} = 3l^2 \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \end{bmatrix}. \quad (18)$$

Starting from the definition of the circumference, the orthonormal basis is expressed in terms of the reciprocal lattice basis as

$$\begin{aligned} \mathbf{e}_\theta &= \frac{1}{2\pi r^2} \left(\frac{n+1}{2} \mathbf{e}_1 - \mathbf{e}_2 \right) = \frac{3l^2}{2\pi r^2} \left(\frac{n}{2} \mathbf{e}^1 + \frac{n-3}{4} \mathbf{e}^2 \right), \\ \mathbf{e}_z &= \frac{\mathbf{e}^1 + \frac{n+1}{2} \mathbf{e}^2}{a}, \end{aligned} \quad (19)$$

where $a = \|\mathbf{e}^1 + \frac{n+1}{2} \mathbf{e}^2\|$. Plugging Eq. (19) into Eq. (17) gives the components of the propagation constant in the direct lattice basis such that

$$\begin{aligned} \mu_1 &= \frac{3l^2}{2\pi r^2} \frac{n}{2} k_\theta + \frac{1}{a} k_z, \\ \mu_2 &= \frac{3l^2}{2\pi r^2} \frac{n-3}{4} k_\theta + \frac{n+1}{2} \frac{1}{a} k_z. \end{aligned} \quad (20)$$

Since k_z can take any arbitrary value whereas $k_\theta = iN$, k_z is removed from Eq. (20), linking μ_1 to μ_2 :

$$\mu_1 = \frac{3l^2}{2\pi r^2} \frac{n^2 + 3}{2(n+1)} iN + \frac{2}{n+1} \mu_2 = 2\pi \frac{2}{n+1} iN + \frac{2}{n+1} \mu_2. \quad (21)$$

The resulting dispersion curves are shown in Fig. 15a for the different values of $N = \{0, 1, \dots, (n-1)/2\}$, and are compared to the ones obtained using the one-dimensional Bloch method considering as unit cell the nanotube circumference. There are at least two possible choices for the circumferential unit cell as shown in Fig. 15b-c. A slice of nanotube orthogonal to its axis and of height Δz_{BT} , or a set of $(n+1)/2$ consecutive unit cells defined in Fig. 14c. In order to apply the Bloch method in one dimension, similarly to the twisted ribbon (Sec. 3.1.1), boundary conditions have to account for the rotation θ_{BT} between each circumferential unit cell. Note also that it has been verified that both unit cells give identical dispersion curves. Considering together the $(n+1)/2$ solutions generated from Eq. (21) (Fig. 15a), the results from the one dimensional wave propagation are exactly recovered, whereas the computation time is decreased by a factor 16.

Finally, note that in Fig. 15a, two dispersion curves start at $\text{Im}(\mu_1) = \theta_{BT}$, where θ_{BT} corresponds to the rotation angle between two consecutive unit cells. The

Figure 15: (a) Dispersion curves normalized by $\omega_0 = \pi^2 \sqrt{E/\rho}$ obtained from Eq. (21) with the different values of N emphasize in multicolored full lines, and compared to results from the one-dimensional Bloch method in dashed lines using the circumferential unit cell (b) or (c) (the choice of (b) or (c) leads to identical results). Band gap in gray (a).

explanation is the same as the dispersion branches starting at $\text{Im}(\mu_1) = \pi$ in the wrinkled helicoid (where the rotation between each unit cell is π), and allows to identify modes with transversal components. This was also observed in dispersion curves of a tetrahelix (staging of tetrahedra where the rotation angle between each tetrahedron is an irrational number θ_{Tetra}), and the two modes starting at $\text{Im}(\mu_1) = \theta_{\text{Tetra}}$ were identified as flexural modes propagating and turning along the axis (one clockwise and one counterclockwise) [17].

4. Conclusions

In the presence of glide and screw symmetries, periodic boundary conditions in Bloch theorem are revisited to take advantage of a reduced unit cell. While in the case of glide symmetries the size of the problem is divided by two, for rotational and screw ones, the gain is significantly increased. Using the minimum period, the number of curve folding and intersections is reduced, simplifying the interpretation of the different branches. Finally, the revisited Bloch method is applicable to structures that do not possess purely-translational symmetries for which the classical method is not applicable.

Acknowledgments

The Research Fund KU Leuven is gratefully acknowledged for its support. This research was also partially supported by Flanders Make, the strategic research centre for the manufacturing industry. The research of F. Maurin is funded by an Experienced Researcher grant within the European ANTARES Project, under the FP7 Marie Curie Programme (GA606817). The European Commission also supports the research of C. Claeys through the ENLIGHT-project. The research of L. Van Belle is funded by a grant from the Research Foundation - Flanders (FWO).

- [1] C. Claeys, E. Deckers, B. Pluymers, W. Desmet, A lightweight vibro-acoustic metamaterial demonstrator: Numerical and experimental investigation, *Mechanical Systems and Signal Processing* 70-71 (2016) 853–880.
- [2] V. Dorodnitsyn, B. Van Damme, Analytical analysis of slow and fast pressure waves in a two-dimensional cellular solid with fluid-filled cells, *The Journal of the Acoustical Society of America* 139 (6) (2016) 3332–3340.
- [3] P. E. Mix, *Introduction to Nondestructive Testing: A Training Guide*, John Wiley & Sons, 2005.
- [4] D. J. Mead, A new method of analyzing wave propagation in periodic structures; Applications to periodic Timoshenko beams and stiffened plates, *Journal of Sound and Vibration* 104 (1) (1986) 9–27.
- [5] A. Luongo, F. Romeo, Real wave vectors for dynamic analysis of periodic structures, *Journal of Sound and Vibration* 279 (1-2) (2005) 309–325.
- [6] B. R. Mace, D. Duhamel, M. J. Brennan, L. Hinke, Finite element prediction of wave motion in structural waveguides., *The Journal of the Acoustical Society of America* 117 (5) (2005) 2835–2843.
- [7] M. Aberg, The usage of standard finite element codes for computation of dispersion relations in materials with periodic microstructure, *The Journal of the Acoustical Society of America* 102 (4) (1997) 2007.
- [8] M. Ruzzene, F. Scarpa, F. Soranna, Wave beaming effects in two-dimensional cellular structures, *Smart Materials and Structures* 12 (3) (2003) 363–372.
- [9] A. S. Phani, J. Woodhouse, N. A. Fleck, Wave propagation in two-dimensional periodic lattices., *The Journal of the Acoustical Society of America* 119 (4) (2006) 1995–2005.

- [10] M. J. Leamy, Exact wave-based Bloch analysis procedure for investigating wave propagation in two-dimensional periodic lattices, *Journal of Sound and Vibration* 331 (7) (2012) 1580–1596.
- [11] M. I. Hussein, M. J. Leamy, M. Ruzzene, Dynamics of phononic materials and structures: Historical origins, recent progress, and future outlook, *Applied Mechanics Reviews* 66 (4) (2014) 040802.
- [12] M. Collet, M. Ouisse, M. Ruzzene, M. N. Ichchou, Floquet-Bloch decomposition for the computation of dispersion of two-dimensional periodic, damped mechanical systems, *International Journal of Solids and Structures* 48 (20) (2011) 2837–2848.
- [13] F. Farzbod, M. J. Leamy, Analysis of Bloch’s Method and the Propagation Technique in Periodic Structures, *Journal of Vibration and Acoustics* 133 (3) (2011) 031010.
- [14] V. I. Erofeyev, Wave Processes in Solids with Microstructure, Vol. 8 of Series on Stability, Vibration and Control of Systems, Series A, World Scientific, 2003.
- [15] F. Maurin, A. Spadoni, Wave dispersion in periodic post-buckled structures, *Journal of Sound and Vibration* 333 (19) (2014) 4562–4578.
- [16] V. Filonova, D. Fafalis, J. Fish, Dispersive computational continua, *Computer Methods in Applied Mechanics and Engineering* 298 (2016) 58–79.
- [17] F. Maurin, Bloch theorem with revised boundary conditions applied to glide and screw symmetric, quasi-one-dimensional structures, *Wave Motion* 61 (2016) 20–39.
- [18] F. Maurin, A. Spadoni, Low-frequency wave propagation in post-buckled structures, *Wave Motion* 51 (2) (2014) 323–334.
- [19] F. Maurin, A. Spadoni, Wave propagation in periodic buckled beams. Part I: Analytical models and numerical simulations, *Wave Motion* 66 (2016) 190–209.
- [20] F. Maurin, A. Spadoni, Wave propagation in periodic buckled beams. Part II: Experiments, *Wave Motion* 66 (2016) 210–219.
- [21] G. Trainiti, J. J. Rimoli, M. Ruzzene, Wave propagation in periodically undulated beams and plates, *International Journal of Solids and Structures* 75-76 (2015) 260–276.

- [22] F. Treyssède, Numerical investigation of elastic modes of propagation in helical waveguides., *The Journal of the Acoustical Society of America* 121 (6) (2007) 3398–3408.
- [23] F. Treyssède, Elastic waves in helical waveguides, *Wave Motion* 45 (4) (2008) 457–470.
- [24] F. Treyssède, L. Laguerre, Investigation of elastic modes propagating in multi-wire helical waveguides, *Journal of Sound and Vibration* 329 (10) (2010) 1702–1716.
- [25] A. P. Korte, E. L. Starostin, G. H. M. van der Heijden, Triangular buckling patterns of twisted inextensible strips, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 467 (2125) (2010) 285–303.
- [26] J. Chopin, V. Démery, B. Davidovitch, Roadmap to the Morphological Instabilities of a Stretched Twisted Ribbon, *Journal of Elasticity* 119 (1-2) (2014) 137–189.
- [27] F. Maurin, Solitary waves in longitudinally wrinkled and creased helicoids, *International Journal of Non-Linear Mechanics* 89 (2017) 133–141.
- [28] A. Nateghi, L. Van Belle, C. Claeys, E. Deckers, B. Pluymers, W. Desmet, Wave propagation in locally resonant cylindrically curved metamaterial panels, *International Journal of Mechanical Sciences*.
- [29] T. Chang, J. Geng, X. Guo, Chirality- and size-dependent elastic properties of single-walled carbon nanotubes, *Applied Physics Letters* 87 (25) (2005) 1–3.
- [30] F. Côté, V. Deshpande, N. Fleck, A. Evans, The compressive and shear responses of corrugated and diamond lattice materials, *International Journal of Solids and Structures* 43 (20) (2006) 6220–6242.
- [31] D. T. Queheillalt, H. N. Wadley, Pyramidal lattice truss structures with hollow trusses, *Materials Science and Engineering: A* 397 (1-2) (2005) 132–137.
- [32] Y. Duan, M. Keefe, T. A. Bogetti, B. A. Cheeseman, Modeling the role of friction during ballistic impact of a high-strength plain-weave fabric, *Composite Structures* 68 (3) (2005) 331–337.
- [33] L. Brillouin, *Wave propagation in periodic structures; electric filters and crystal lattices*, Courier Dover Publications, 1946.

- [34] J. H. Kwak, S. Hong, Linear Algebra, Birkhäuser Boston, Boston, MA, 2004.
- [35] R. S. Langley, A note on the force boundary conditions for two-dimensional periodic structures with corner freedoms, *Journal of Sound and Vibration* 167 (2) (1993) 377–381.
- [36] J. Kiendl, K. U. Bletzinger, J. Linhard, R. Wüchner, Isogeometric shell analysis with Kirchhoff-Love elements, *Computer Methods in Applied Mechanics and Engineering* 198 (49-52) (2009) 3902–3914.
- [37] F. Maurin, L. Coox, F. Greco, E. Deckers, C. Claeys, W. Desmet, Bloch theorem for isogeometric analysis of periodic problems governed by high-order partial differential equations, *Computer Methods in Applied Mechanics and Engineering* 311 (2016) 743–763.
- [38] C. Claeys, P. Sas, W. Desmet, On the acoustic radiation efficiency of local resonance based stop band materials, *Journal of Sound and Vibration* 333 (14) (2014) 3203–3213.
- [39] S. Nojima, Y. Kamakura, Irreducible first Brillouin-zone for two-dimensional binary-compound photonic crystals, *Journal of the Physical Society of Japan* 77 (3) (2008) 034403.
- [40] M. Ouisse, M. Collet, F. Scarpa, A piezo-shunted Kirigami auxetic lattice for adaptive elastic wave filtering, *Smart Materials and Structures*, In press.
- [41] F. Scarpa, M. Ouisse, M. Collet, K. Saito, Kirigami Auxetic Pyramidal Core: Mechanical Properties and Wave Propagation Analysis in Damped Lattice, *Journal of Vibration and Acoustics* 135 (4) (2013) 041001.
- [42] W. R. D. Cook, D. S. Malkus, Concepts and applications of finite element analysis, Wiley India Pvt. Limited, 2002.
- [43] J. F. Doyle, Wave Propagation in Structures-Spectral Analysis Using Fast Discrete Fourier Transforms, Springer Verlag, 1997.
- [44] E. Manconi, B. R. Mace, Wave characterization of cylindrical and curved panels using a finite element method, *The Journal of the Acoustical Society of America* 125 (1) (2009) 154.
- [45] Y. Waki, B. R. Mace, M. J. Brennan, Free and forced vibrations of a tyre using a wave/finite element approach, *Journal of Sound and Vibration* 323 (3-5) (2009) 737–756.

- [46] J. M. Renno, B. R. Mace, Calculating the forced response of cylinders and cylindrical shells using the wave and finite element method, *Journal of Sound and Vibration* 333 (21) (2014) 5340–5355.
- [47] S. Reich, C. Thomsen, J. Maultzsch, *Carbon Nanotubes: Basic Concepts and Physical Properties*, Wiley-VCH, 2004.
- [48] J. N. Reddy, *An Introduction to Nonlinear Finite Element Analysis*, OUP Oxford, 2004.
- [49] A. Spadoni, M. Ruzzene, S. Gonella, F. Scarpa, Phononic properties of hexagonal chiral lattices, *Wave Motion* 46 (7) (2009) 435–450.