

HOMOTOPY MOMENT MAPS

MARTIN CALLIES, YAËL FRÉGIER, CHRISTOPHER L. ROGERS, AND MARCO ZAMBON

ABSTRACT. Associated to any manifold equipped with a closed form of degree > 1 is an ‘ L_∞ -algebra of observables’ which acts as a higher/homotopy analog of the Poisson algebra of functions on a symplectic manifold. In order to study Lie group actions on these manifolds, we introduce a theory of homotopy moment maps. Such a map is a L_∞ -morphism from the Lie algebra of the group into the observables which lifts the infinitesimal action. We establish the relationship between homotopy moment maps and equivariant de Rham cohomology, and analyze the obstruction theory for the existence of such maps. This allows us to easily and explicitly construct a large number of examples. These include results concerning group actions on loop spaces and moduli spaces of flat connections. Relationships are also established with previous work by others in classical field theory, algebroid theory, and dg geometry. Furthermore, we use our theory to geometrically construct various L_∞ -algebras as higher central extensions of Lie algebras, in analogy with Kostant’s quantization theory. In particular, the so-called ‘string Lie 2-algebra’ arises this way.

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1. INTRODUCTION

This paper represents part of a larger project which involves studying the symmetries of manifolds equipped with a closed differential form. The motivation for this work stems from the desire to have a more conceptual understanding of the role these manifolds play in differential cohomology, generalized geometry, and field theory. In our approach, we view such manifolds as generalizations of symplectic manifolds.

As a first step, we consider symmetries arising from a Lie group acting on a manifold by diffeomorphisms which preserve a closed differential form. A key component of our formalism is the

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‘homotopy moment map’. This is a natural generalization of the moment map used to study symmetries in symplectic geometry. However, unlike their symplectic counterparts, our moment maps do not correspond to morphisms between Lie algebras. Instead, they are morphisms between objects called ‘ L_∞ -algebras’, which can be thought of as homotopy-theoretic upgrades of Lie algebras. More precisely, an L_∞ -algebra is a graded vector space equipped with a skew-symmetric bracket which satisfies the Jacobi identity up to coherent homotopy. The coherent homotopy is given as part of the data by an infinite sequence of higher degree multi-linear brackets which satisfy additional Jacobi-like identities. L_∞ -algebras with underlying vector spaces concentrated in the first non-positive n -degrees are often called ‘Lie n -algebras’. In particular, a Lie 1-algebra is an ordinary Lie algebra.

Morphisms between L_∞ -algebras are not just linear maps which preserve the brackets. This definition is too strict. Rather, a morphism is an infinite collection of multi-linear maps which preserve the brackets up to, again, coherent homotopy. We emphasize that this notion of morphism between L_∞ -algebras plays a crucial role.

Perhaps it seems strange that these higher homotopical structures should appear when studying something as classical as actions of Lie groups on manifolds. To understand why they are needed, we have to first recall some facts concerning symmetries in symplectic geometry.

1.1. Symplectic geometry. The important infinitesimal symmetries of a symplectic manifold correspond to the Hamiltonian vector fields. These form a Lie algebra whose bracket is the usual commutator of vector fields. The space of smooth functions is also a Lie algebra, whose bracket is specified by the symplectic 2-form. This is the underlying Lie algebra of the Poisson algebra. If the manifold is connected (we always assume this is the case), then Kostant [23] showed that the Poisson algebra is characterized as a particular extension of the Lie algebra of Hamiltonian vector fields by \mathbb{R} . The 2-cocycle representing this central extension is determined by the symplectic form.

Now suppose we have a Lie group G , with Lie algebra \mathfrak{g} , acting on the manifold via diffeomorphisms which preserve the symplectic form. Assume further that the associated infinitesimal action is given by a Lie algebra morphism from \mathfrak{g} to the Hamiltonian vector fields. A ‘moment map’ for the action corresponds to a lift of this Lie algebra morphism to the central extension given by the Poisson algebra¹. Whether a moment map exists or not for a particular G -action is an important question in symplectic geometry. It can be thought of as the symplectic analog of determining when a projective representation of G lifts to a linear one.

The relationship between symmetries in symplectic geometry and representation theory is made explicit via ‘geometric quantization’. If the symplectic form represents an integral cohomology class, then it corresponds to the curvature of a principal $U(1)$ -bundle equipped with a connection. In this case, the Poisson algebra is isomorphic to a Lie algebra consisting of the $U(1)$ -invariant vector fields on the bundle whose flows preserve the connection. This Lie algebra acts naturally as differential operators on sections of the associated Hermitian line bundle. Hence, if there is a G -action on the symplectic manifold, then a moment map for this action gives a representation of the Lie algebra \mathfrak{g} on the space of sections. In certain cases, this action integrates to a global G -action.

If no moment map exists, then Kostant’s construction produces non-trivial central extensions of both \mathfrak{g} and G which naturally act on the space of sections of the Hermitian line bundle. Many important Lie groups can be constructed this way e.g. central extensions of loop groups, as well as the Heisenberg and Bott-Virasoro groups [5, Sec. 2.4].

1.2. “Higher” symplectic geometry. Let us return to the more general case, and consider a manifold equipped with a closed form of degree $n + 1 > 2$. Such a manifold also has Hamiltonian vector fields, and these form a Lie algebra just as they do in symplectic geometry. To pursue the analogy further, one might try to construct a central extension of the Hamiltonian vector fields using

¹Technically, this lift is not the moment map, but rather the co-moment map.

the closed $(n + 1)$ -form. Unlike the symplectic case, the form does not induce a skew-symmetric bracket on functions. But it does on a particular subspace of $(n - 1)$ -forms, called Hamiltonian $(n - 1)$ -forms. This bracket, however, fails to satisfy the Jacobi identity.

This lack of a genuine Lie bracket is the reason why L_∞ -algebras appear. In previous work [33], the third author considered such manifolds when the closed $(n + 1)$ -form satisfied a mild non-degeneracy condition. These are called ‘ n -plectic’ or ‘multisymplectic’ manifolds. He associated to such a manifold a Lie n -algebra whose underlying vector space consists of the Hamiltonian $(n - 1)$ -forms and all other forms of lower degree. Its brackets are completely determined by the $(n + 1)$ -form and the de Rham differential. Later, the fourth author showed that the non-degeneracy assumption is not necessary for the construction, and therefore any ‘pre- n -plectic’ manifold has such a Lie n -algebra [43].

In this work, we slightly generalize these previous constructions and associate to any manifold equipped with a closed $(n + 1)$ -form its ‘Lie n -algebra of observables’. When the form is non-degenerate, this Lie n -algebra is isomorphic to the one constructed in [33], and in particular, we recover the underlying Lie algebra of the usual Poisson algebra when $n = 1$. In analogy with Kostant’s central extension for a symplectic manifold, this Lie n -algebra of observables can be characterized uniquely, up to homotopy, as a L_∞ -extension of the Lie algebra of Hamiltonian vector fields whose classifying cocycle is determined by the closed form. (See Thm. 3.4.1 in [15].)

If a Lie group G acts on a pre- n -plectic manifold (M, ω) and the infinitesimal action induces a Lie algebra morphism between \mathfrak{g} and the Hamiltonian vector fields, then we define a homotopy moment map, or just ‘moment map’ for short, to be a lift of this Lie algebra morphism to an L_∞ -morphism from \mathfrak{g} to the Lie n -algebra of observables. The precise definition is given in Def. 5.1. For $n = 1$, we recover the usual notion of a (co)-moment map in pre-symplectic geometry. At first sight, this definition may seem too abstract or technical to be useful. However, thanks to some of the tools developed here, we can easily and systematically construct such maps and therefore produce a large variety of interesting examples. In almost all of these, the moment map is not a “strict” L_∞ -morphism.

The ultimate goal is to complete the analogy with the symplectic case by understanding the role homotopy moment maps play in quantization and representation theory. Indeed, the results in [15] imply that a homotopy moment map will lift a \mathfrak{g} -action on (M, ω) to an action on a ‘higher bundle gerbe’ over M whose curvature is ω . (See also Remark 4.9 below). Along with this, we are also interested in pursuing the geometric relationship between these moment maps and conserved quantities, in the sense of Hamiltonian dynamics and also classical field theory. (See Sec. 13 for further details.)

1.3. Summary of results. Our exposition throughout is aimed at a broad audience of geometers and topologists. We assume the reader has essentially no expertise in homotopical algebra or familiarity with higher geometric structures.

We begin with a quick introduction to L_∞ -algebras in Sec. 3 and leave the more technical aspects to the appendix. We review the necessary background on n -plectic geometry, Hamiltonian vector fields, and the Lie n -algebra of observables in Sec. 4. We introduce the homotopy moment map in Sec. 5.

Equivariant de Rham cohomology. In Sec. 6, we present our first main result: If we have a compact Lie group acting on a manifold, then from any $(n + 1)$ -cocycle in equivariant de Rham cohomology we can naturally and explicitly produce a G -invariant pre- n -plectic structure and a homotopy moment map (Thm. 6.6 and Thm. 6.8). The formula for the moment map generalizes the important relationship between moment maps in symplectic geometry and degree 2 cocycles in equivariant cohomology. We find this result particularly interesting since it suggests a geometric interpretation of higher degree equivariant cocycles.

Closed 3-forms. The first truly new (i.e. non-symplectic) examples will arise on manifolds equipped with a closed 3-form. Such manifolds also play an important role in generalized geometry and the theory of gerbes. So in Sec. 7, we focus on aspects specific to this case.

Basic examples. We then present some basic examples in Sec. 8. These include:

- Exact pre-plectic forms: This generalizes familiar results in symplectic geometry involving G -invariant symplectic potentials. Special cases include G -actions on exterior powers of cotangent bundles, and the action of $\mathrm{SO}(n)$ on \mathbb{R}^n equipped with the usual volume form.
- Compact Lie groups: The Cartan 3-form on such a group is invariant under conjugation and can be uniquely extended to an equivariant closed 3-form. This gives a moment map for the adjoint action. We point out a relationship between this moment map and certain quasi-Hamiltonian G -spaces.
- $\mathrm{SO}(n)$ -action on the n -sphere: This generalizes the Hamiltonian circle action on S^2 whose moment map corresponds to the “height function” along the z -axis.

Obstructions and higher central extensions. In order to produce more examples, we study in Sec. 9 the obstructions to lifting a G -action to a homotopy moment map. The results we present here are natural generalizations of the symplectic ones. The existence of a moment map for a G -action on a (connected) pre- n -plectic manifold implies that a degree $(n+1)$ class $[c]$ in Lie algebra cohomology is trivial. Conversely, if $[c] = 0$ and M satisfies certain topological assumptions, then we can always construct a moment map lifting the action (Thm. 9.6).

If $[c] \neq 0$, then in Sec. 9.3 we show how to construct a L_∞ -morphism not from \mathfrak{g} , but rather from a Lie n -algebra $\widehat{\mathfrak{g}}$. The Lie n -algebra $\widehat{\mathfrak{g}}$ is built using a representative of $[c]$ and plays the role of a (non-trivial) higher central extension of \mathfrak{g} . This gives a new way to geometrically construct Lie n -algebras. For example, via this construction we recover the string Lie 2-algebra $\mathbf{string}(\mathfrak{g})$, which plays an interesting role in elliptic cohomology, and ‘string structures’.

Moduli spaces and loop spaces. In Sec. 10, we use the results of Sec. 9 to produce a more sophisticated example of a homotopy moment map on an infinite-dimensional manifold. If P is a principal G -bundle on a $(n+1)$ -dimensional compact oriented manifold, then a degree $(n+1)$ invariant polynomial on \mathfrak{g} gives a pre- n -plectic structure on the space of connections of P . We show that this $(n+1)$ -form is invariant under the action of the gauge group, and that this action admits a moment map. If the $(n+1)$ -form is actually n -plectic, then we can perform a Marsden-Weinstein reduction procedure to obtain the moduli space of flat connections, endowed with a pre- n -plectic form. This generalizes the well-known Atiyah-Bott construction in symplectic geometry for G -bundles over Riemann surfaces.

We continue to focus on infinite-dimensional examples in Sec. 11. There we show that a homotopy moment map for a G -action on a pre-2-plectic manifold (M, ω) can be transgressed to an ordinary moment map on the pre-symplectic loop space $(LM, L\omega)$, where $L\omega$ is transgression of ω , and the action of G on LM is “point-wise”. This gives an example of how the higher geometry on M interacts with the classical geometry on LM .

Comparisons with other work. Numerous generalizations of moment maps already exist in the literature. In Sec. 12, we describe some relationships between homotopy moment maps and related work done by others. In particular, we consider: multi-momentum maps studied by a variety of authors in multisymplectic field theory [9, 21], the multi-moment maps of Madsen and Swann [26, 27], Bursztyn, Cavalcanti, and Gualtieri’s work on group actions and Courant algebroids [7], and Uribe’s work on group actions and dg-manifolds [41]. Finally, we conclude in Sec. 13 with some open questions.

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2. PRELIMINARIES

Here we list the notation and conventions used throughout the paper. We also give a brief review of the Cartan calculus for multi-vector fields.

2.1. Graded linear algebra. Let V be a \mathbb{Z} graded vector space. For any $k \in \mathbb{Z}$, $V[k]$ is the graded vector space

$$V[k]^i = V^{i+k}.$$

Let x_1, \dots, x_n be elements of V and $\sigma \in \mathcal{S}_n$ a permutation. The **Koszul sign** $\epsilon(\sigma) = \epsilon(\sigma; x_1, \dots, x_n)$ is defined by the equality

$$x_1 \cdots x_n = \epsilon(\sigma; x_1, \dots, x_n) x_{\sigma(1)} \cdots x_{\sigma(n)},$$

which holds in the free graded commutative algebra generated by V , with product denoted by concatenation of elements. Given $\sigma \in \mathcal{S}_n$, let $(-1)^\sigma$ denote the usual sign of a permutation. Note that $\epsilon(\sigma)$ does not include the sign $(-1)^\sigma$.

We say $\sigma \in \mathcal{S}_{p+q}$ is a **(p, q)-unshuffle** iff $\sigma(i) < \sigma(i+1)$ whenever $i \neq p$. The set of (p, q) -unshuffles is denoted by $\text{Sh}(p, q)$. For example, $\text{Sh}(2, 1)$ is the set of cycles $\{(1), (23), (123)\}$. The notion of a (i_1, \dots, i_k) -unshuffle and the set $\text{Sh}(i_1, \dots, i_k)$ are defined similarly.

If V and W are graded vector spaces, a linear map $f: V^{\otimes n} \rightarrow W$ is **skew-symmetric** iff

$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (-1)^\sigma \epsilon(\sigma) f(v_1, \dots, v_n),$$

for all $\sigma \in \mathcal{S}_n$. The degree of an element $x_1 \otimes \cdots \otimes x_n \in V^{\otimes \bullet}$ of the graded tensor algebra generated by V is defined to be $|x_1 \otimes \cdots \otimes x_n| = \sum_{i=1}^n |x_i|$.

Finally, the following sign occurs frequently, so we give its own notation. For an integer k define:

$$\zeta(k) = -(-1)^{\frac{k(k+1)}{2}}.$$

So for $k = 1, 2, 3, 4, 5, \dots$ we have $\zeta(k) = 1, 1, -1, -1, 1, \dots$. Notice that $\zeta(k-1)\zeta(k) = (-1)^k$ for all k .

2.2. Group actions. Throughout this paper G denotes a Lie group and \mathfrak{g} its Lie algebra. A G action on a manifold M is always from the left, unless stated otherwise. Our convention for the induced action on forms is the one given in [22, Sec. 2.1]. Namely, G acts on $\Omega^\bullet(M)$ from the left via inverse pullback

$$g \cdot \omega \mapsto \phi_{g^{-1}}^* \omega,$$

where ϕ_g is the diffeomorphism corresponding to g . We denote the corresponding infinitesimal action of the Lie algebra \mathfrak{g} by the map

$$(1) \quad v_- : \mathfrak{g} \rightarrow \mathfrak{X}(M), \quad x \mapsto v_x,$$

where

$$v_x|_p = \frac{d}{dt} \exp(-tx) \cdot p|_{t=0} \quad \forall p \in M.$$

We call v_- the **fundamental vector field** associated to the G -action. Note that it is *minus* the infinitesimal generator associated to the G -action, and hence it is a morphism of Lie algebras.

If G is finite-dimensional, then we denote by \mathfrak{g}^\vee the dual of the Lie algebra \mathfrak{g} . Recall that the Chevalley-Eilenberg differential δ_{CE} on $\Lambda^*(\mathfrak{g}^\vee)$ is

$$(2) \quad \begin{aligned} & \delta_{\text{CE}}: \Lambda^n(\mathfrak{g}^\vee) \rightarrow \Lambda^{n+1}(\mathfrak{g}^\vee), \\ & \delta_{\text{CE}}(c)(x_1, \dots, x_{n+1}) := \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} c([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}). \end{aligned}$$

If M is a G -manifold, then let $\iota_{\mathfrak{g}}^k: \Omega^*(M) \rightarrow \Lambda^k(\mathfrak{g}^\vee) \otimes \Omega^{*-k}(M)$ be the insertion operations

$$(3) \quad \begin{aligned} & \iota_{\mathfrak{g}}^k: \Omega^*(M) \rightarrow \Lambda^k(\mathfrak{g}^\vee) \otimes \Omega^{*-k}(M), \\ & \iota_{\mathfrak{g}}^k \alpha(x_1, \dots, x_k) := \iota_{v_{x_k}} \cdots \iota_{v_{x_1}} \alpha \in \Omega^{*-k}(M) \end{aligned}$$

where $\alpha \in \Omega^*(M)$, and $x_1, \dots, x_k \in \mathfrak{g}$. We will also use $\iota_{\mathfrak{g}}$ to denote $\iota_{\mathfrak{g}}^1$.

2.3. Cartan calculus. The **Schouten bracket** of two decomposable multivector fields $u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_n \in \mathfrak{X}^{\wedge \bullet}(M)$ is

$$(4) \quad [u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_n] = \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j} [u_i, v_j] \wedge u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_m \wedge v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n,$$

where $[u_i, v_j]$ is the usual Lie bracket of vector fields.

The **interior product** of a decomposable multivector field $v_1 \wedge \cdots \wedge v_n$ with $\alpha \in \Omega^\bullet(M)$ is

$$(5) \quad \iota(v_1 \wedge \cdots \wedge v_n) \alpha = \iota_{v_n} \cdots \iota_{v_1} \alpha,$$

where $\iota_{v_i} \alpha$ is the usual interior product of vector fields and differential forms.

The **Lie derivative** \mathcal{L}_v of a differential form along a multivector field $v \in \mathfrak{X}^{\wedge \bullet}(M)$ is the graded commutator of d and $\iota(v)$:

$$(6) \quad \mathcal{L}_v \alpha = d\iota(v)\alpha - (-1)^{|v|} \iota(v)d\alpha,$$

where $\iota(v)$ is considered as a degree $-|v|$ operator.

The last identity we will need is for the graded commutator of the Lie derivative and the interior product. Given $u, v \in \mathfrak{X}^{\wedge \bullet}(M)$, it follows from [16, Proposition A3] that

$$(7) \quad \iota([u, v])\alpha = (-1)^{(|u|-1)|v|} \mathcal{L}_u \iota(v)\alpha - \iota(v)\mathcal{L}_u \alpha.$$

3. L_∞ -ALGEBRAS

In this section we briefly review L_∞ -algebras and explicitly describe L_∞ -morphisms for the special cases considered in this paper.

Definition 3.1 ([24]). An **L_∞ -algebra** is a graded vector space L equipped with a collection

$$\left\{ l_k: L^{\otimes k} \rightarrow L \mid 1 \leq k < \infty \right\}$$

of graded skew-symmetric linear maps with $|l_k| = 2 - k$ such that the following identity holds for $1 \leq m < \infty$:

$$(8) \quad \sum_{\substack{i+j=m+1, \\ \sigma \in \text{Sh}(i, m-i)}} (-1)^\sigma \epsilon(\sigma) (-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(m)}) = 0.$$

In the appendix (Sec. A.3), we recall how any L_∞ -algebra (L, l_k) corresponds to a certain kind of graded coalgebra $C(L)$ equipped with a coderivation Q which satisfies the identity

$$Q \circ Q = 0.$$

This identity is the origin of Eq. (8). But it is easy to see that for small values of m that Eq. (8) is a ‘‘generalized Jacobi identity’’ for the multi-brackets $\{l_k\}$. For $m = 1$, it implies that the degree 1 linear map l_1 satisfies

$$l_1 \circ l_1 = 0$$

and hence every L_∞ -algebra (L, l_k) has an underlying cochain complex (L, l_1) .

Definition 3.2. An L_∞ -algebra $(L, \{l_k\})$ is a **Lie n -algebra** iff the underlying graded vector space L is concentrated in degrees $0, -1, \dots, 1 - n$.

Note that if $(L, \{l_k\})$ is a Lie n -algebra, then by degree counting $l_k = 0$ for $k > n + 1$. An ordinary Lie algebra is the same as a Lie 1-algebra.

3.1. **L_∞ -morphisms.** The following definition may at first seem satisfactory:

Definition 3.3 ([24]). Let (L, l_k) and (L', l'_k) be L_∞ -algebras. A degree 0 linear map $f: L \rightarrow L'$ is a **strict L_∞ -morphism** iff

$$(9) \quad l'_k \circ f^{\otimes k} = f \circ l_k \quad \forall k \geq 1.$$

However, this definition of L_∞ -morphism does not reflect the higher structure naturally residing within the theory. Indeed, the better definition [24, Remark 5.3] uses the aforementioned relationship between L_∞ -algebras and dg-coalgebras. We emphasize that the flexibility provided by this higher structure is what allows us to produce the many explicit examples of homotopy moment maps considered in this paper.

Definition 3.4. An **L_∞ -morphism** between L_∞ -algebras (L, l_k) and (L', l'_k) is a morphism $F: (C(L), Q) \rightarrow (C(L'), Q')$ between their corresponding differential graded (dg) coalgebras. That is, F is a morphism between the graded coalgebras $C(L)$ and $C(L')$ such that

$$(10) \quad F \circ Q = Q' \circ F.$$

It turns out that an L_∞ -morphism between (L, l_k) and (L', l'_k) corresponds to an infinite collection of graded skew-symmetric ‘structure maps’

$$f_k: L^{\otimes k} \rightarrow L' \quad 1 \leq k < \infty,$$

where $|f_k| = 1 - k$, and such that a complicated compatibility relation with the multi-brackets is satisfied. In particular, the degree zero map f_1 must be a morphism between the underlying complexes (L, l_1) and (L', l'_1) :

$$f_1 l_1 = l'_1 f_1.$$

The compatibility relation, in the language of coalgebras, corresponds exactly to Eq. (10). Strict morphisms in the sense of Def. 3.3 correspond to the special case when $f_k = 0$ for $k \geq 2$. (See Prop. A.5 for more details.) Outside of Sec. A.4, we shall mildly abuse notation and denote a L_∞ -morphism via its structure maps as

$$(f_k): (L, l_k) \rightarrow (L', l'_k).$$

L_∞ -morphisms are composable in the usual sense, and hence one can speak of the category of L_∞ -algebras without explicitly describing the higher structure mentioned above.

Definition 3.5. We denote by Lie_∞ the category whose objects are L_∞ -algebras (Def. 3.1) and whose morphisms are L_∞ -morphisms (Def. 3.4).

The following is the correct notion of equivalence between L_∞ -algebras which reflects the aforementioned homotopical structure between morphisms.

Definition 3.6. A morphism $(f_k): (L, l_k) \rightarrow (L', l'_k)$ of L_∞ -algebras is a **\mathbf{L}_∞ -quasi-isomorphism** iff the morphism of complexes

$$f_1: (L, l_1) \rightarrow (L', l'_1)$$

induces an isomorphism on the cohomology:

$$H^\bullet(f_1): H^\bullet(L) \xrightarrow{\cong} H^\bullet(L').$$

Remark 3.7. L_∞ -quasi-isomorphisms induce an equivalence relation on the category of L_∞ -algebras. Indeed, such a morphism between two L_∞ -algebras exists if and only if the algebras are homotopy equivalent (e.g., as objects of a simplicial category [12, Sec. 3.2]). Roughly speaking, the situation here is analogous to the Whitehead theorem for weak homotopy equivalences between CW complexes.

3.2. Morphisms from Lie algebras to L_∞ -algebras. In this paper, we will be particularly interested in L_∞ -morphisms from a Lie algebra to a Lie n -algebra (L', l'_k) with the following property:

$$(P) \quad \forall k \geq 2 \quad l'_k(x_1, \dots, x_k) = 0 \quad \text{whenever} \quad \sum_{i=1}^k |x_i| < 0.$$

The following characterization is proven in the appendix (Cor. A.8).

Proposition 3.8. *If $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra and (L', l'_k) is a Lie n -algebra satisfying property (P), then the graded skew-symmetric maps*

$$f_k: \mathfrak{g}^{\otimes k} \rightarrow L', \quad |f_k| = 1 - k, \quad 1 \leq k \leq n$$

are the components of an L_∞ -morphism $\mathfrak{g} \rightarrow L'$ if and only if $\forall x_i \in \mathfrak{g}$

$$(11) \quad \sum_{1 \leq i < j \leq k} (-1)^{i+j+1} f_{k-1}([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_k) \\ = l'_1 f_k(x_1, \dots, x_k) + l'_k(f_1(x_1), \dots, f_1(x_k)).$$

for $2 \leq k \leq n$ and

$$(12) \quad \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} f_n([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_{n+1}) = l'_{n+1}(f_1(x_1), \dots, f_1(x_{n+1})).$$

4. L_∞ -ALGEBRAS FROM CLOSED DIFFERENTIAL FORMS

Here we recall various definitions and results about closed differential forms from previous work [33] [43], and introduce the Lie n -algebra of observables associated to any manifold equipped with a closed $(n+1)$ -form.

Definition 4.1. An $(n+1)$ -form ω on a smooth manifold M is **n -plectic**, or more specifically an **n -plectic structure**, if it is both closed:

$$d\omega = 0,$$

and non-degenerate:

$$\forall x \in M \quad \forall v \in T_x M, \quad \iota_v \omega = 0 \Rightarrow v = 0.$$

If ω is an n -plectic form on M , then we call the pair (M, ω) an **n -plectic manifold**. More generally, if ω is closed, but not necessarily non-degenerate, then we call (M, ω) a **pre- n -plectic manifold**

Obviously, a (pre-) 1-plectic manifold is a (pre-) symplectic manifold.

Definition 4.2. Let (M, ω) be a pre- n -plectic manifold. An $(n-1)$ -form α is **Hamiltonian** iff there exists a vector field $v_\alpha \in \mathfrak{X}(M)$ such that

$$d\alpha = -\iota_{v_\alpha}\omega.$$

We say v_α is a **Hamiltonian vector field** corresponding to α . The set of Hamiltonian $(n-1)$ -forms and the set of Hamiltonian vector fields on an pre- n -plectic manifold are both vector spaces and are denoted as $\Omega_{\text{Ham}}^{n-1}(M)$ and $\mathfrak{X}_{\text{Ham}}(M)$, respectively. Note that if ω is n -plectic, then associated to every Hamiltonian form is a unique Hamiltonian vector field.

Definition 4.3. A vector field v on a pre- n -plectic manifold (M, ω) is a **local Hamiltonian vector field** iff

$$\mathcal{L}_v\omega = 0.$$

The set of local Hamiltonian vector fields is a vector space and is denoted as $\mathfrak{X}_{\text{LHam}}(M)$.

Definition 4.4. Let (M, ω) be a pre- n -plectic manifold. Given $\alpha, \beta \in \Omega_{\text{Ham}}^{n-1}(M)$, the **bracket** $\{\alpha, \beta\}$ is the $(n-1)$ -form given by

$$\{\alpha, \beta\} = \iota_{v_\beta}\iota_{v_\alpha}\omega,$$

where v_α and v_β are any Hamiltonian vector fields for α and β respectively.

The bracket is well-defined, for if both v_α and v'_α are Hamiltonian for $\alpha \in \Omega_{\text{Ham}}^{n-1}(M)$, then both $\iota_{v_\beta}\iota_{v_\alpha}\omega$ and $\iota_{v_\beta}\iota_{v'_\alpha}\omega$ are equal to $-\iota_{v_\beta}d\alpha$.

Proposition 4.5. *If (M, ω) is a pre- n -plectic manifold and $v_1, v_2 \in \mathfrak{X}_{\text{LHam}}(M)$ are local Hamiltonian vector fields, then $[v_1, v_2]$ is a global Hamiltonian vector field with*

$$d\iota([v_1, v_2])\omega = -\iota_{[v_1, v_2]}\omega,$$

and $\mathfrak{X}_{\text{LHam}}(M)$ and $\mathfrak{X}_{\text{Ham}}(M)$ are Lie subalgebras of $\mathfrak{X}(M)$.

Proof. if v_1, v_2 are locally Hamiltonian, then by Eq. (7),

$$\mathcal{L}_{v_1}\iota_{v_2}\omega = \iota_{[v_1, v_2]}\omega.$$

On the other hand, by Eq. (6),

$$\mathcal{L}_{v_1}\iota_{v_2}\omega = \iota_{v_1}d\iota_{v_2}\omega + d\iota_{v_1}\iota_{v_2}\omega.$$

But $\iota_{v_1}d\iota_{v_2}\omega = 0$, since $d\iota_{v_2} = \mathcal{L}_{v_2} - \iota_{v_2}d$. □

Prop. 4.5 implies in particular that if v_α and v_β are Hamiltonian vector fields for α and β respectively, then $[v_\alpha, v_\beta]$ is a Hamiltonian vector field for $\{\alpha, \beta\}$.

The next theorem gives a natural L_∞ -structure on differential forms, which extends the bracket $\{\cdot, \cdot\}$ on $\Omega_{\text{Ham}}^{n-1}(M)$. The theorem is essentially Thm. 5.2 in [33], together with its generalization Thm. 6.7 in [43].

Theorem 4.6. *Given a pre- n -plectic manifold (M, ω) , there is a Lie n -algebra $L_\infty(M, \omega) = (L, \{l_k\})$ with underlying graded vector space*

$$L^i = \begin{cases} \Omega_{\text{Ham}}^{n-1}(M) & i = 0, \\ \Omega^{n-1+i}(M) & 1 - n \leq i < 0, \end{cases}$$

and maps $\{l_k: L^{\otimes k} \rightarrow L \mid 1 \leq k < \infty\}$ defined as

$$l_1(\alpha) = d\alpha,$$

if $|\alpha| < 0$ and

$$l_k(\alpha_1, \dots, \alpha_k) = \begin{cases} \varsigma(k)\iota(v_{\alpha_1} \wedge \dots \wedge v_{\alpha_k})\omega & \text{if } |\alpha_1 \otimes \dots \otimes \alpha_k| = 0, \\ 0 & \text{if } |\alpha_1 \otimes \dots \otimes \alpha_k| < 0, \end{cases}$$

for $k > 1$, where v_{α_i} is any Hamiltonian vector field associated to $\alpha_i \in \Omega_{\text{Ham}}^{n-1}(M)$.

4.1. The Lie n -algebra of observables. Note that for the $n = 1$ case, the underlying complex of $L_\infty(M, \omega)$ is just the vector space of Hamiltonian functions $C^\infty(M)_{\text{Ham}} \subseteq C^\infty(M)$. The only non-trivial bracket is $l_2 = \{\cdot, \cdot\}$, which is a Lie bracket. Hence, we recover the the underlying Lie algebra of the usual Poisson algebra associated to a pre-symplectic manifold. In the symplectic case, $C^\infty(M)_{\text{Ham}} = C^\infty(M)$, and there is a well-defined surjective Lie algebra morphism

$$\pi: C^\infty(M) \rightarrow \mathfrak{X}_{\text{Ham}}(M)$$

sending a function to its unique Hamiltonian vector field. If M is connected, then we see π fits in the short exact sequence

$$(13) \quad 0 \rightarrow \mathbb{R} \rightarrow C^\infty(M) \xrightarrow{\pi} \mathfrak{X}_{\text{Ham}}(M) \rightarrow 0.$$

This is the Kostant-Souriau central extension [23, 39]. It characterizes the underlying Lie algebra of $C^\infty(M)$, up to isomorphism, as the unique central extension determined by the symplectic form (evaluated at a point $p \in M$).

For the pre-symplectic case, Hamiltonian functions can have more than one corresponding Hamiltonian vector field, and so a map $C^\infty(M)_{\text{Ham}} \rightarrow \mathfrak{X}_{\text{Ham}}(M)$ may not exist. Therefore one instead considers the Lie algebra

$$\begin{aligned} C^\infty(\widetilde{M})_{\text{Ham}} &= \{(v, f) \in \mathfrak{X}_{\text{Ham}}(M) \oplus C^\infty(M)_{\text{Ham}} \mid df = -\iota_v \omega\} \\ [(v_1, f_1), (v_2, f_2)]_L &= ([v_1, v_2], \{f_1, f_2\}). \end{aligned}$$

The projection $(v, f) \mapsto v$ then gives a central extension

$$(14) \quad 0 \rightarrow \mathbb{R} \rightarrow C^\infty(\widetilde{M})_{\text{Ham}} \xrightarrow{\pi} \mathfrak{X}_{\text{Ham}}(M) \rightarrow 0$$

which generalizes (13) to any connected pre-symplectic manifold [5, Prop 2.3]. If (M, ω) is symplectic, then it is easy to see that $C^\infty(\widetilde{M})_{\text{Ham}}$ is isomorphic to $C^\infty(M)_{\text{Ham}} = C^\infty(M)$ as Lie algebras.

The higher analog of the central extension (14) for a pre- n -plectic manifold is obtained by slightly modifying the construction of $L_\infty(M, \omega)$.

Theorem 4.7. *Given a pre- n -plectic manifold (M, ω) , there is a Lie n -algebra $\text{Ham}_\infty(M, \omega)$ with underlying graded vector space*

$$\begin{aligned} L^0 &= \Omega_{\text{Ham}}^{n-1}(M) = \{(v, \alpha) \in \mathfrak{X}_{\text{Ham}}(M) \oplus \Omega_{\text{Ham}}^{n-1}(M) \mid d\alpha = -\iota_v \omega\} \\ L^i &= \Omega^{n-1+i}(M) \quad 1 - n \leq i < 0, \end{aligned}$$

and structure maps:

$$\begin{aligned} \tilde{l}_1(\alpha) &= \begin{cases} (0, d\alpha) & \text{if } |\alpha| = -1, \\ d\alpha & \text{if } |\alpha| < -1, \end{cases} \\ \tilde{l}_2(x_1, x_2) &= \begin{cases} ([v_1, v_2], \iota(v_1 \wedge v_2)\omega) = ([v_1, v_2], \{\alpha_1, \alpha_2\}) & \text{if } |x_1 \otimes x_2| = 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and, for $k > 2$:

$$\tilde{l}_k(x_1, \dots, x_k) = \begin{cases} \varsigma(k)\iota(v_{\alpha_1} \wedge \dots \wedge v_{\alpha_k})\omega & \text{if } |x_1 \otimes \dots \otimes x_k| < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Eq. (8) is satisfied since the bracket of vector fields satisfies the Jacobi identity while the higher structure maps are identical to those of $L_\infty(M, \omega)$. \square

We call $\text{Ham}_\infty(M, \omega)$ the **Lie n -algebra of observables** associated to (M, ω) .

Proposition 4.8. *Let (M, ω) be a pre- n -plectic manifold.*

(1) *The cochain map*

$$\pi: \text{Ham}_\infty(M, \omega) \rightarrow \mathfrak{X}_{\text{Ham}}(M)$$

defined to be the projection $(v, \alpha) \mapsto v$ in degree 0, and trivial in all lower degrees lifts to a strict morphism of L_∞ -algebras.

(2) *If (M, ω) is n -plectic, then the cochain map*

$$\begin{array}{ccccccc} C^\infty(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{n-2}(M) & \xrightarrow{d} & \Omega_{\text{Ham}}^{n-1}(M) \\ \downarrow \text{id} & & \downarrow \text{id} & & & & \downarrow \text{id} & & \downarrow \phi \\ C^\infty(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{n-2}(M) & \xrightarrow{0 \oplus d} & \underbrace{\Omega_{\text{Ham}}^{n-1}(M)} \end{array}$$

with $\phi(\alpha) = (v_\alpha, \alpha)$, where v_α is the unique Hamiltonian vector field associated to α , lifts to a strict L_∞ -quasi-isomorphism $L_\infty(M, \omega) \xrightarrow{\sim} \text{Ham}_\infty(M, \omega)$.

Proof. To prove (1), note we have

$$\pi \tilde{l}_2((v_1, \alpha_1), (v_2, \alpha_2)) = [\pi(v_1, \alpha_1), \pi(v_2, \alpha_2)].$$

Hence, Eq. (9) is satisfied. For (2), note that we have the equalities

$$\begin{aligned} \tilde{l}_2 \phi^{\otimes 2} &= \phi l_2 \\ \tilde{l}_k \phi^{\otimes k} &= l_k \quad \forall k > 2. \end{aligned}$$

Hence, the cochain map lifts to a strict L_∞ -morphism. The map ϕ is an isomorphism, hence the corresponding L_∞ -morphism is a quasi-isomorphism. \square

Remark 4.9. There is a nice conceptual interpretation of the Lie n -algebra $\text{Ham}_\infty(M, \omega)$ within the context of differential cohomology whenever ω represents an integral cohomology class i.e.

$$[\omega] \in \text{im}(H^{n+1}(M, \mathbb{Z}) \rightarrow H^{n+1}(M, \mathbb{R})).$$

For any manifold M there is a short exact sequence

$$0 \rightarrow H^n(M, \text{U}(1)) \rightarrow H_{\text{Del}}^n(M) \xrightarrow{\text{curv}} \Omega_{\text{cl/int}}^{n+1}(M) \rightarrow 0,$$

where $H^\bullet(M, \text{U}(1))$ is ordinary $\text{U}(1)$ -valued cohomology, $\Omega_{\text{cl/int}}^{n+1}(M)$ is the group of closed and integral $(n+1)$ -forms, and $H_{\text{Del}}^\bullet(M)$ is ‘smooth Deligne cohomology’ [5, Sec. 1.5]. The group $H_{\text{Del}}^n(M)$ classifies certain higher geometric objects which one could call ‘principal $\text{U}(1)$ n -bundles’ equipped with an ‘ n -connection’. The curvature of such an n -connection is given by the surjection in the above sequence, and therefore is an integral pre- n -plectic form on M .

For example, $H_{\text{Del}}^1(M)$ is in bijection with isomorphism classes of principal $\text{U}(1)$ bundles with connection over M . The surjection in the above sequence sends such a bundle to its curvature 2-form. The group $H_{\text{Del}}^2(M)$ classifies geometric objects called $\text{U}(1)$ -gerbes equipped a ‘2-connection’ (also called a ‘connective structure’ and its ‘curving’), whose curvature is a closed integral 3-form on M .

It is known that the infinitesimal symmetries of gerbes form a Lie 2-algebra (for example, see [11]), and this pattern continues for $n > 2$. In [15], it is shown for any $n > 1$ that the Lie n -algebra of connection-preserving infinitesimal symmetries of a principal $\text{U}(1)$ n -bundle whose curvature is ω is quasi-isomorphic to the Lie n -algebra $\text{Ham}_\infty(M, \omega)$.

5. HOMOTOPY MOMENT MAPS

In symplectic geometry, a moment map $M \rightarrow \mathfrak{g}^\vee$ can be equivalently expressed as a co-moment map i.e. a Lie algebra morphism $\mathfrak{g} \rightarrow C^\infty(M)$. In this section, we describe the natural L_∞ analog of this co-moment map. We call this a homotopy moment map.

Definition/Proposition 5.1. *Let G be a Lie group with Lie algebra \mathfrak{g} . Let (M, ω) be a pre- n -plectic manifold equipped with a G -action which preserves ω , and such that the infinitesimal \mathfrak{g} -action $x \mapsto v_x$ is via Hamiltonian vector fields. A **homotopy moment map** (or **moment map** for short) is a lift*

$$(15) \quad \begin{array}{ccc} & & \text{Ham}_\infty(M, \omega) \\ & \nearrow \text{dashed} & \downarrow \pi \\ \mathfrak{g} & \xrightarrow{v_-} & \mathfrak{X}_{\text{Ham}}(M). \end{array}$$

of the Lie algebra morphism v_- through the L_∞ -morphism π (defined in Prop. 4.8) in the category of L_∞ -algebras.

Such a lift corresponds to an L_∞ -morphism

$$(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$$

such that

$$-\iota_{v_x} \omega = d(f_1(x)) \quad \text{for all } x \in \mathfrak{g}.$$

Before we give a proof of the above correspondence between lifts and morphisms into $L_\infty(M, \omega)$, we explain in more detail what a homotopy moment map actually is:

- The condition $-\iota_{v_x} \omega = d(f_1(x))$ implies that v_x is a Hamiltonian vector field for $f_1(x) \in \Omega_{\text{Ham}}^{n-1}(M)$.
- By Prop. 3.8, an L_∞ -morphism $\mathfrak{g} \rightarrow L_\infty(M, \omega)$ consists of a collection of n skew-symmetric maps

$$f_k: \mathfrak{g}^{\otimes k} \rightarrow L, \quad 1 \leq k \leq n,$$

where L is the underlying vector space of $L_\infty(M, \omega)$ and $|f_k| = 1 - k$, which satisfy

$$(16) \quad \sum_{1 \leq i < j \leq k} (-1)^{i+j+1} f_{k-1}([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k) = df_k(x_1, \dots, x_k) + \varsigma(k) \iota(v_1 \wedge \dots \wedge v_k) \omega$$

for $2 \leq k \leq n$ and

$$(17) \quad \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} f_n([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}) = \varsigma(n+1) \iota(v_1 \wedge \dots \wedge v_{n+1}) \omega,$$

where v_i is the vector field associated to x_i via the \mathfrak{g} -action. (These equalities are obtained from Eqs. (11) and (12) via substitution using the definition of f_1 and the maps l_k defined in Thm. 4.6. Notice that Thm. 4.6 implies that $L_\infty(M, \omega)$ has property (P).)

- Finally, note that Prop. 4.5 imply that $v_{[x,y]} = [v_x, v_y]$ is a Hamiltonian vector field for

$$\{f_1(x), f_1(y)\} = l_2(f_1(x), f_1(y)).$$

Of course, $f_1: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$ need not preserve the bracket on \mathfrak{g} i.e. in general $f_1([x, y]) \neq \{f_1(x), f_1(y)\}$. This is a good property, in view of the facts that the Lie bracket of \mathfrak{g} satisfies the Jacobi identity but $\{\cdot, \cdot\}$ does not.

Proof of Def./Prop. 5.1. Suppose we have an L_∞ -morphism

$$(\tilde{f}_k): \mathfrak{g} \rightarrow \mathbf{Ham}_\infty(M, \omega)$$

corresponding to a lift (15). Note that $\mathbf{Ham}_\infty(M, \omega)$ satisfies Property (P). Therefore the morphisms (\tilde{f}_k) are trivial for $k \geq n + 1$ and satisfy the compatibility equations given in Prop. 3.8. By the definition of the projection π , the degree 0 map can be written as

$$\tilde{f}_1(x) = (v_x, f_1(x)) \in \widetilde{\Omega_{\mathbf{Ham}}^{n-1}}(M)$$

where $f_1: \mathfrak{g} \rightarrow \Omega_{\mathbf{Ham}}^{n-1}(M)$ is a linear map satisfying $-\iota_{v_x}\omega = d(f_1(x))$. Moreover, π is a strict L_∞ -morphism and therefore

$$\tilde{f}_1([x, y]) = (v_{[x, y]}, f_1([x, y])) = ([v_x, v_y], f_1([x, y])).$$

Combining these observations with the fact that structure maps \tilde{l}_k of $\mathbf{Ham}_\infty(M, \omega)$ agree with those of $L_\infty(M, \omega)$ for $k \geq 3$, we obtain an L_∞ morphism

$$(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$$

with $f_k = \tilde{f}_k$ for $k > 1$. The fact that every such morphism gives a lift is now obvious. \square

Remark 5.2. Note we have a generalization of the fact that, in symplectic geometry, the image of a moment map is a Lie subalgebra of the Poisson algebra of functions on the symplectic manifold.

More precisely, given a moment map with components $f_k: \mathfrak{g}^{\otimes k} \rightarrow L$ (for $1 \leq k \leq n$), its image is not an L_∞ -subalgebra of $L_\infty(M, \omega)$ in general (unless $n = 1$). However, the subcomplex generated by its image, which we denote by I and which is given by

$$I^k = \begin{cases} \text{im}(f_n) & k = -n + 1 \\ \text{im}(f_{-k+1}) + d(\text{im}(f_{-k+2})) & -n + 2 \leq k \leq 0 \end{cases}$$

is an L_∞ -subalgebra of $L_\infty(M, \omega)$. Indeed I is closed w.r.t. the differential d by construction. It is seen to be closed w.r.t. the higher brackets using the definition of the latter (Thm. 4.6) together with Eq. (16) and (17), and because the Hamiltonian vector field of an exact element of $\Omega_{\mathbf{Ham}}^{n-1}(M)$ is zero.

6. EQUIVARIANT COHOMOLOGY

In this section, we establish the relationship between equivariant cohomology and moment maps. We interpret the defining equations for a moment map within the context of the Bott-Shulman-Stasheff de Rham model for equivariant cohomology. This model is related to the more computationally tractable Cartan model via a natural chain map (a generalization of the ‘‘Cartan’’ map), which is a quasi-isomorphism if the group acting is compact. In particular, given any $(n+1)$ -cocycle in the Cartan model extending a G -invariant pre- n -plectic form, we obtain an explicit formula for a corresponding moment map. This correspondence is natural in the sense that the moment map for the pullback of a cocycle along a G -equivariant map is the pullback of the moment map. As a special case, this formula recovers the well known relationship between moment maps in symplectic geometry and 2-cocycles in the Cartan model.

6.1. The Bott-Shulman-Stasheff model. Let G be a Lie group and M a G -manifold. Let $G \times M_\bullet$ denote the simplicial manifold

$$G \times M_n = G^n \times M$$

with the face maps $d_i: G^n \times M \rightarrow G^{n-1} \times M$ given by

$$(g_1, \dots, g_n, p) \mapsto \begin{cases} (g_2, \dots, g_n, p) & i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n, p) & 0 < i < n, \\ (g_1, \dots, g_{n-1}, g_n p) & i = n. \end{cases}$$

The Bott-Shulman-Stasheff complex is the total complex of the double complex of differential forms on $G \times M_\bullet$:

$$(18) \quad \begin{aligned} \Omega^{j,k}(G \times M_\bullet) &:= \Omega^k(G^j \times M), \\ \Omega^*(G \times M_\bullet) &:= \text{Tot}(\Omega^{*,*}(G \times M_\bullet)), \\ D &:= \partial + (-1)^j d, \end{aligned}$$

where ∂ is the simplicial differential and d is the de Rham differential. The de Rham theorem of Bott-Shulman-Stasheff [6] implies that the cohomology of $(\Omega^*(G \times M_\bullet), D)$ is the equivariant cohomology of M with real coefficients.

6.2. Moment maps from equivariant cohomology. We assume that G is finite-dimensional, and denote by \mathfrak{g}^\vee the dual of the Lie algebra \mathfrak{g} . If (M, ω) is a pre- n -plectic manifold equipped with a G -action preserving ω , then it follows directly from Def./Prop. 5.1 that a moment map corresponds to a sum $f = \sum_{k=1}^n f_k$ with

$$f_k \in \Lambda^k(\mathfrak{g}^\vee) \otimes \Omega^{n-k}(M)$$

such that

$$(\delta_{\text{CE}} \otimes \text{id})f + (\text{id} \otimes d)f = - \sum_{k=1}^{n+1} \varsigma(k) \iota_{\mathfrak{g}}^k \omega.$$

Above, $\delta_{\text{CE}}: \Lambda^n(\mathfrak{g}^\vee) \rightarrow \Lambda^{n+1}(\mathfrak{g}^\vee)$ is the Chevalley-Eilenberg differential defined in Eq. (2), and $\iota_{\mathfrak{g}}^k: \Omega^*(M) \rightarrow \Lambda^k(\mathfrak{g}^\vee) \otimes \Omega^{*-k}(M)$ is the insertion operation defined in Eq. (3). At this point, it is convenient to define the following complex:

$$(19) \quad \begin{aligned} C_{\mathfrak{g}}^{k,m}(M) &:= \Lambda^k(\mathfrak{g}^\vee) \otimes \Omega^m(M) \\ C_{\mathfrak{g}}^*(M) &:= \text{Tot}(C_{\mathfrak{g}}^{*,*}(M)) \\ \mathbf{d} &:= \delta_{\text{CE}} + (-1)^k d. \end{aligned}$$

Given a vector in the total complex $\alpha \in C_{\mathfrak{g}}^i(M)$, we denote by α_k the summand of α belonging to $\Lambda^k(\mathfrak{g}^\vee) \otimes \Omega^{i-k}(M)$.

The complex (19) leads us to a proposition whose proof is straightforward.

Proposition 6.1. *If (M, ω) is a pre- n -plectic manifold equipped with a G -action preserving ω , then $f = \sum_{k=1}^n f_k \in C_{\mathfrak{g}}^n(M)$ is a moment map if and only if*

$$(20) \quad \mathbf{d}f^\varsigma = \sum_{k=1}^{n+1} (-1)^{k+1} \iota_{\mathfrak{g}}^k \omega$$

where $f^\varsigma = \sum_{i=1}^n f_i^\varsigma$ with $f_i^\varsigma = \varsigma(i) f_i$.

Remark 6.2. The homotopy theory of L_∞ -algebras provides a nice conceptual explanation for the appearance of the complex $C_{\mathfrak{g}}^*(M)$. The multilinear maps $v_1 \wedge v_2 \wedge \cdots \wedge v_k \mapsto \varsigma(k) \iota_{v_k} \iota_{v_{k-1}} \cdots \iota_{v_1} \omega$ define an L_∞ -morphism F^ω from the Lie algebra of Hamiltonian vector fields $\mathfrak{X}_{\text{Ham}}(M)$ to an abelian L_∞ -algebra $B^n A$ corresponding to the cochain complex $\Omega^0(M) \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} d\Omega^{n-1}(M)$, where $d\Omega^{n-1}(M)$ is in degree 0 [15, Prop. 3.8]. (The notation $B^n A$ is used to remind the reader of the analogous situation for classifying spaces of bundles.) The homotopy fiber of F^ω is the Lie n -algebra of observables $\text{Ham}_\infty(M, \omega)$, i.e. there is a homotopy pullback diagram

$$\begin{array}{ccc} \text{Ham}_\infty(M, \omega) & \longrightarrow & 0 \\ \pi \downarrow & \lrcorner^h & \downarrow \\ \mathfrak{X}_{\text{Ham}}(M) & \xrightarrow{F^\omega} & B^n A \end{array}$$

As in Def. 5.1, we have the Lie algebra morphism $v_- : \mathfrak{g} \rightarrow \mathfrak{X}_{\text{Ham}}(M)$ encoding the infinitesimal action. The structure maps of the composition $F^\omega \circ v_-$ are precisely the multilinear maps $\iota_{\mathfrak{g}}^k \omega$. There is, in general, a “mapping space” for L_∞ -algebras, i.e., a simplicial set $\text{Map}_\bullet(L, L')$ whose 0-simplicies are L_∞ -morphisms between L and L' [12, Sec. 3.2]. In the present case, one can show a moment map, or lift, $\mathfrak{g} \rightarrow \text{Ham}_\infty(M, \omega)$ exists if and only if $F^\omega \circ v_-$ is null-homotopic, i.e., $[F^\omega \circ v_-] = [0] \in \pi_0 \text{Map}_\bullet(\mathfrak{g}, B^n A)$. This is in complete analogy with how isomorphism classes of principal bundles correspond to homotopy classes of maps into classifying spaces. In the present case, since $B^n A$ is an abelian L_∞ -algebra (a chain complex), the simplicial set $\text{Map}_\bullet(\mathfrak{g}, B^n A)$ is a simplicial vector space. Via the Dold–Kan correspondence, the normalized chain complex of $\text{Map}_\bullet(\mathfrak{g}, B^n A)$ is homotopy equivalent to the complex $C_{\mathfrak{g}}^*(M)$. Thus, a homotopy moment map exists if and only if $F^\omega \circ v_-$ is null-homotopic if and only if it is a trivial n -cocycle in $C_{\mathfrak{g}}^*(M)$. This last statement is precisely the content of Prop. 6.1.

Applications of the complex $C_{\mathfrak{g}}^*(M)$ as a computational tool are presented in [17].

The relationship between moment maps and equivariant cohomology stems from the fact that the complex $C_{\mathfrak{g}}^*(M)$ can be identified with the subcomplex

$$\Omega^*(G \times M)^G \subseteq \Omega^{1,*}(G \times M_\bullet)$$

of the double complex (18). This is the subcomplex of forms on $G \times M$ invariant with respect to the G -action $g' \cdot (g, p) = (g'g, p)$. We view $\Omega^*(G \times M)^G$ as the total complex of the double complex

$$(21) \quad \begin{aligned} \Omega^{k,m}(G \times M)^G &:= \Gamma(G \times M, \Lambda^k T^* G \otimes \Lambda^m T^* M)^G \subseteq \Omega^{k+m}(G \times M)^G, \\ \Omega^*(G \times M)^G &= \text{Tot}(\Omega^{*,*}(G \times M)^G), \\ d &= d^G + (-1)^k d^M \end{aligned}$$

where d^G and d^M denote the de Rham differentials in the G and M directions, respectively.

Lemma 6.3. *Restriction to $M = \{e\} \times M \xrightarrow{i} G \times M$ induces an isomorphism of double complexes*

$$r : (\Omega^{*,*}(G \times M)^G, d^G, d^M) \xrightarrow{\cong} (C_{\mathfrak{g}}^{*,*}(M), \delta_{\text{CE}}, d),$$

and hence, an isomorphism of total complexes

$$r : (\Omega^*(G \times M)^G, d) \xrightarrow{\cong} (C_{\mathfrak{g}}^*(M), \mathbf{d}).$$

Proof. We use the following observation: if $E \rightarrow B$ is a G -equivariant vector bundle and $i : M \hookrightarrow B$ an embedding such that the action $G \times M \rightarrow B$ is a diffeomorphism, then the restriction of sections $\Gamma(B, E)^G \rightarrow \Gamma(M, i^* E)$ is an isomorphism. Hence, the restriction of sections of $\Lambda^m T^*(G \times M)$ to $M = \{e\} \times M$ induces an isomorphism

$$\Omega^m(G \times M)^G = \Gamma(G \times M, \Lambda^m(T^*(G \times M)))^G \xrightarrow{\cong} \Gamma(M, i^* \Lambda^m T^*(G \times M)) = \Gamma(M, \Lambda^m(\mathfrak{g}^\vee \oplus T^* M)).$$

Composing this with the natural isomorphism

$$(22) \quad \begin{aligned} \Psi : \Lambda^m(\mathfrak{g}^\vee \oplus T_p^* M) &\xrightarrow{\cong} \bigoplus_{k+\ell=m} \Lambda^k(\mathfrak{g}^\vee) \otimes \Lambda^\ell T_p^* M \\ \Psi(\alpha)((x_1, \dots, x_k) \otimes (w_1, \dots, w_\ell)) &= \alpha((x_1, 0), \dots, (x_k, 0), (0, w_1), \dots, (0, w_\ell)) \end{aligned}$$

gives the isomorphism $r : (\Omega^{*,*}(G \times M)^G) \xrightarrow{\cong} (C_{\mathfrak{g}}^{*,*}(M))$. It remains to show r respects the differentials. Indeed, since the Chevalley–Eilenberg differential is the restriction of the de Rham differential to left-invariant differential forms on G (e.g. [18, Chapter IV, Prop. 3]), we have $rd^G = \delta_{\text{CE}} r$. And finally, $rd^M = dr$ follows immediately from the definition of the differentials d^M and d . \square

Now that we identified $C_{\mathfrak{g}}^*(M)$ as sitting inside $\Omega^{1,*}(G \times M_{\bullet})$, we can reinterpret the term on the right hand side of the moment map condition in Prop. 6.1 in terms of the Bott-Shulman-Stasheff complex.

Proposition 6.4. *Let $\omega \in \Omega^{n+1}(M)^G$. Then*

$$r(\partial\omega) = \sum_{k=1}^{n+1} (-1)^{k+1} \iota_{\mathfrak{g}}^k \omega.$$

Proof. The face map $d_1: G \times M \rightarrow M$ is the G -action. Therefore, it is G -equivariant and hence $d_1^* \omega \in \Omega^{n+1}(G \times M)^G$. Since d_0 is the projection $G \times M \rightarrow M$, we also have $d_0^* \omega \in \Omega^{n+1}(G \times M)^G$, and hence

$$\partial\omega = d_0^* \omega - d_1^* \omega \in \Omega^{n+1}(G \times M)^G.$$

The differential of the map d_1 at the point (e, p) is given by

$$d_{1*}|_{(e,p)}(x, u) = u - v_x, \quad \text{for } x \in T_e G, u \in T_p M.$$

If $x_1, \dots, x_{n+1} \in \mathfrak{g}$ and $u_1, \dots, u_{n+1} \in T_p M$, then

$$\begin{aligned} r(d_0^* \omega - d_1^* \omega)(u_1, \dots, u_{n+1}) &= 0, \\ r(d_0^* \omega - d_1^* \omega)(x_1, \dots, x_k, u_1, \dots, u_{n-k+1}) &= (-1)^{k+1} (\iota_{\mathfrak{g}}^k \omega(x_1, \dots, x_k))(u_1, \dots, u_{n-k+1}). \end{aligned}$$

Thus, $r(\partial\omega) = \sum_{k=1}^{n+1} (-1)^{k+1} \iota_{\mathfrak{g}}^k \omega$. □

Corollary 6.5. *An element $f = \sum_{k=1}^n f_k \in C_{\mathfrak{g}}^n(M)$ with $f_k \in C_{\mathfrak{g}}^{k, n-k}(M)$ is a homotopy moment map for the pre- n -plectic form $\omega \in \Omega^{n+1}(M)^G$ if and only if $f^{\mathfrak{c}}$ satisfies $\mathbf{d}f^{\mathfrak{c}} = r(\partial\omega)$.*

If G is compact and $\omega \in \Omega^{n+1}(M)$, then let $\omega^G \in \Omega^{n+1}(M)^G$ denote the G -invariant form obtained by averaging. Similarly, if $\beta \in \Omega^n(G \times M)$, then let β^G denote the G -invariant form on $G \times M$ obtained by averaging with respect to the action $g' \cdot (g, p) = (g'g, p)$.

We now state our first main result which relates equivariant cohomology to moment maps. In particular, if G is compact, then any cocycle in the Bott-Shulman-Stasheff complex gives a moment map.

Theorem 6.6. *Let M be a G -manifold and $\alpha = \sum_{i=0}^{n+1} \alpha_i \in \Omega^{n+1}(G \times M_{\bullet})$ a degree $(n+1)$ -cocycle in the Bott-Shulman-Stasheff complex with $\alpha_i \in \Omega^{i, n-i+1}(G \times M_{\bullet}) = \Omega^{n-i+1}(G^i \times M)$.*

- (1) *If $\alpha_0 \in \Omega^{n+1}(M)^G$ and $\alpha_1 \in \Omega^n(G \times M)^G$, then $f^{\mathfrak{c}} = \sum_{k=1}^n r(\alpha_1)_k$ defines a homotopy moment map f for the G -invariant pre- n -plectic form $\omega = \alpha_0$.*
- (2) *If G is compact, then $f^{\mathfrak{c}} = \sum_{k=1}^n r(\alpha_1^G)_k$ defines a homotopy moment map f for the G -invariant pre- n -plectic form $\omega = \alpha_0^G \in \Omega^{n+1}(M)^G$.*

Proof. The cocycle condition $D\alpha = 0$ implies that

$$d\alpha_0 = 0, \quad \partial\alpha_0 = d\alpha_1.$$

Therefore, $\omega = \alpha_0$ is indeed a pre- n -plectic form. Since $r(\partial\omega)_0 = 0$, we have $\mathbf{d}r(\alpha_1) = \mathbf{d} \sum_{k=1}^n r(\alpha_1)_k$. The first claim then follows from Cor. 6.5.

For the second claim, note that the following equalities hold:

$$\mathbf{d}r(\alpha_1^G) = r(d\alpha_1^G) = r((d\alpha_1)^G) = r((\partial\alpha_0)^G) = r(\partial\alpha_0^G),$$

and, like before, we have $\mathbf{d}r(\alpha_1^G) = \sum_{k=1}^n \mathbf{d}r(\alpha_1^G)_k$. □

6.3. Moment maps from the Cartan model. When G is compact, the Cartan model for the equivariant de Rham cohomology of M is better suited for computations than the Bott-Shulman-Stasheff model. Indeed, in symplectic geometry the relationship between equivariant cohomology and moment maps is typically described using this model. Analogously, the main result in this section is that any cocycle in the Cartan model gives a homotopy moment map.

Recall that the Cartan model [22, Sec. 4.2] is the complex

$$C_G^*(M) = (S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))^G$$

in which \mathfrak{g}^\vee is implicitly placed in degree 2, and with differential

$$d_G(\alpha)(x) = d(\alpha(x)) - \iota_{v_x}(\alpha(x)) \quad \forall x \in \mathfrak{g},$$

where $\alpha \in C_G(M)$ is regarded as a polynomial map $\alpha: \mathfrak{g} \rightarrow \Omega^\bullet(M)$. For any k , denote by

$$C_G^k(M) = \bigoplus_{2j \leq k} (S^j(\mathfrak{g}^\vee) \otimes \Omega^{k-2j}(M))^G$$

the degree k component of the Cartan model $C_G(M)$. Given an element $\alpha \in C_G^k(M)$, we denote by α_j its component in $(S^j(\mathfrak{g}^\vee) \otimes \Omega^{k-2j}(M))^G$, for all $j = 1, \dots, \lfloor \frac{k}{2} \rfloor$. Hence,

$$\alpha = \alpha_0 + \dots + \alpha_{\lfloor \frac{k}{2} \rfloor}.$$

Definition 6.7. An **extension** of an invariant closed differential form $\omega \in \Omega^k(M)$ is a cocycle $\alpha \in C_G^k(M)$ such that

$$\alpha_0 = \omega.$$

A **j -step extension** is an extension of the form

$$\alpha = \alpha_0 + \dots + \alpha_j.$$

Theorem 6.8. *Given a degree $n+1$ cocycle $\omega + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} P_i \in C_G^{n+1}(M)$ in the Cartan complex, with $\omega \in \Omega^{n+1}(M)^G$ and $P_i \in (S^i(\mathfrak{g}^\vee) \otimes \Omega^{n-2i+1}(M))^G$, there is a natural homotopy moment map $\{f_k\}$ for the G -action on the pre- n -plectic manifold (M, ω) . More precisely, for $k = 1, \dots, n$ we have*

$$(23) \quad f_k = \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{(-1)^i i! (k-i)!}{2^{i-1} (k-2i+1)!} \text{Alt}_k(\iota_{\mathfrak{g}}^{k-2i+1} P_i(\cdot, \underbrace{[\cdot, \cdot], \dots, [\cdot, \cdot]}_{i-1})),$$

where $\text{Alt}_k: (\mathfrak{g}^\vee)^{\otimes k} \rightarrow \Lambda^k(\mathfrak{g}^\vee)$ is the (ungraded) skew-symmetrization

$$q_1 \otimes q_2 \otimes \dots \otimes q_k \mapsto \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} (-1)^\sigma q_{\sigma(1)} \otimes q_{\sigma(2)} \otimes \dots \otimes q_{\sigma(k)}.$$

In particular, the homotopy moment map f is G -equivariant, i.e., $f_k \in (\Lambda^k(\mathfrak{g}^\vee) \otimes \Omega^{n-k}(M))^G$.

Proof. The complete proof requires some technical prerequisites, and so it is given in the appendix (see B.6). Here we provide a sketch of the key points. In Appendix B, we first construct a natural map

$$\Phi: C_G^*(M) \rightarrow \Omega^*(G \ltimes M_\bullet)$$

from the Cartan complex to the Bott-Shulman-Stasheff complex. This map is often mentioned in the literature, but we were unable to find a reference which is explicit enough for our needs. We then consider the $(n+1)$ -cocycle $\alpha = \sum_{k=0}^{n+1} \alpha_k = \Phi(\omega + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} P_i)$ in the Bott-Shulman-Stasheff complex and show that the components α_0 and α_1 are G -invariant, even if G is not compact. The formula (23) for the moment map f then follows from applying the first part of Thm. (6.6) to α . As the formula shows, f is constructed by applying the insertion operator $\iota_{\mathfrak{g}}^k$ to invariant elements in $(S^i(\mathfrak{g}^\vee) \otimes \Omega^{n-2i+1}(M))^G$. Hence, f is equivariant. \square

For the applications and examples ahead, it is worthwhile to write out the first few terms of the moment map f . If $\omega + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} P_i \in C_G^{n+1}(M)$ is a degree $n+1$ cocycle as in Thm. 6.8, then

$$\begin{aligned}
 (24) \quad & f_1 = -P_1, \\
 & f_2 = -\text{Alt}_2 \iota_{\mathfrak{g}} P_1, \\
 & f_3 = \text{Alt}_3 \iota_{\mathfrak{g}}^2 P_1 - \text{Alt}_3 P_2(\cdot, [\cdot, \cdot]), \\
 & f_4 = \text{Alt}_4 \iota_{\mathfrak{g}}^3 P_1 - 2 \text{Alt}_4 \iota_{\mathfrak{g}} P_2(\cdot, [\cdot, \cdot]), \\
 & f_5 = -\text{Alt}_5 \iota_{\mathfrak{g}}^4 P_1 + 3 \text{Alt}_4 \iota_{\mathfrak{g}}^2 P_2(\cdot, [\cdot, \cdot]) - 3 \text{Alt}_5 P_3(\cdot, [\cdot, \cdot], [\cdot, \cdot]), \\
 & \vdots
 \end{aligned}$$

Also, we remark that if $M = \text{pt}$, then $C_G^*(M) = C_G^*(\text{pt}) = (S^*(\mathfrak{g}^\vee))^G$, and Thm. 6.8 gives a map $(S^i(\mathfrak{g}^\vee))^G \rightarrow (\Lambda^{2i-1}(\mathfrak{g}^\vee))^G$ which sends P_i to

$$f_{2i-1}^\zeta = (-1)^i \frac{i!(i-1)!}{2^{i-1}} \text{Alt}_{2i-1} \left(P_i(\cdot, \underbrace{[\cdot, \cdot], \dots, [\cdot, \cdot]}_{i-1}) \right).$$

This assignment differs by an additional factor of $-i!$ from the ‘‘Cartan map’’ defined in [10, Section 2] (cf. [19, Chapter VI Prop. IV]).

Applying Thm. 6.8 to the special case when the degree $(n+1)$ Cartan cocycle is just a 1-step extension is particularly interesting. Indeed, it recovers the classical correspondence in symplectic geometry between moment maps and equivariant cohomology.

Corollary 6.9. *If (M, ω) is a pre- n -plectic manifold equipped with a G -action, and $\omega - \mu$ is a 1-step extension of ω , then the maps*

$$\begin{aligned}
 f_k &: \mathfrak{g}^{\otimes k} \rightarrow \Omega^{n-k}(M), \\
 f_k(x_1, \dots, x_k) &= \zeta(k) \iota(v_{x_1} \wedge \dots \wedge v_{x_{k-1}}) \mu(x_k),
 \end{aligned}$$

for $1 \leq k \leq n$, are the components of a G -equivariant moment map $\mathfrak{g} \rightarrow L_\infty(M, \omega)$.

Proof. Applying Thm. 6.8 to the cocycle $\omega + P_1$, where $P_1 = -\mu$, we obtain

$$f_k = -\zeta(k) \text{Alt}_k(\iota_{\mathfrak{g}}^{k-1} P_1) = \zeta(k) \text{Alt}_k(\iota_{\mathfrak{g}}^{k-1} \mu).$$

Let $x_1, \dots, x_k \in \mathfrak{g}$. The skew-symmetrization of $\iota_{\mathfrak{g}}^{k-1} \mu$ implies that $f_k(x_1, \dots, x_k)$ is the sum of terms of the form

$$(25) \quad \zeta(k) (-1)^\sigma \frac{1}{k!} \iota(v_{x_{\sigma(1)}} \wedge \dots \wedge v_{x_{\sigma(k-1)}}) \mu(x_{\sigma(k)})$$

where $\sigma \in \mathcal{S}_k$.

Since $d_G(\omega - \mu) = 0$, we have the equalities

$$(26) \quad d\omega = 0, \quad d\mu(x) = -\iota_{v_x} \omega, \quad \iota_{v_x} \mu(x) = 0$$

for all $x \in \mathfrak{g}$. The latter equality implies that

$$\iota_{v_x} \mu(y) = -\iota_{v_y} \mu(x) \quad \forall x, y \in \mathfrak{g},$$

which further implies

$$\iota_{v_{k-1}} \iota_{v_{k-2}} \dots \iota_{v_1} \mu(x_k) = -\iota_{v_{k-1}} \iota_{v_{k-2}} \dots \iota_{v_k} \mu(x_1),$$

where we write v_i for the vector field v_{x_i} . Hence, each term (25) is skew-symmetric, and therefore $f_k(x_1, \dots, x_k) = \zeta(k) \iota(v_{x_1} \wedge \dots \wedge v_{x_{k-1}}) \mu(x_k)$. □

Remark 6.10. Note that in the proof of Thm. 6.8 and Cor. 6.9, we do not require that G be compact. We did not need to use the fact that the cohomology of $C_G(M)$ is isomorphic to the G -equivariant cohomology of M . In particular, only the algebraic properties of the complexes $C_G(M)$ and $\Omega^*(G \times M_\bullet)$ are used to show that 1-step extensions give equivariant moment maps. Hence, Cor. 6.9 also implies that we can produce equivariant moment maps from 1-step extensions in $C_G(M)$ where G is any *non-compact* Lie group as well. This provides a useful algebraic tool to build examples with.

7. CLOSED 3-FORMS

In this section, we analyze the simplest case in which non-strict moment maps appear, namely the case of closed 3-forms. Under the assumption that the Lie group G is compact semisimple, we prove the existence of moment maps provided the G -action has a fixed point, and the uniqueness of moment maps up to a certain equivalence. Furthermore, in the general setting, we show that not all equivariant moment maps arise from 1-step extensions.

An important example is the Lie group itself equipped with its Cartan 3-form. We consider this as a special case later on in Sec. 8.2.

7.1. Review of symplectic case. Let us first briefly discuss the case of 2-forms. In this case, moment maps as in Def./Prop. 5.1 are necessarily strict by degree reasons. Let G be a Lie group acting on a symplectic manifold (M, ω) . The classical notion of equivariant moment map is the following [8]: a map $J: M \rightarrow \mathfrak{g}^\vee$ such that v_x is the Hamiltonian vector field of $J^*(x)$ for all $x \in \mathfrak{g}$ and so that J is equivariant w.r.t. the G -action on M and the coadjoint action of G on \mathfrak{g}^\vee . In terms of the co-moment map, i.e. the pullback of functions $f = J^*: \mathfrak{g} \rightarrow C^\infty(M)$, this means

$$\begin{aligned} v_x &\text{ is the Hamiltonian vector field of } f(x), \text{ for all } x \in \mathfrak{g}, \\ f: (\mathfrak{g}, [\cdot, \cdot]) &\rightarrow (C^\infty(M), \{\cdot, \cdot\}) \text{ is a Lie algebra morphism,} \end{aligned}$$

where $\{\cdot, \cdot\}$ the Poisson bracket on M . Hence, in the symplectic case, our Def. 5.1 agrees with classical notion of moment map.

An equivalent characterization of moment map is that $\omega - f$ is a closed degree 2 element of the Cartan model for equivariant cohomology. In other words: in the symplectic case, all homotopy moment maps arise from equivariant extensions of ω .

7.2. Notation. In the remainder of this section, let ω be a closed 3-form on M which is invariant under the action of a Lie group G . In this case, a moment map (Def. 5.1) consists of two components

$$f_1: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^1(M), \quad f_2: \mathfrak{g} \otimes \mathfrak{g} \rightarrow C^\infty(M),$$

where the second is skew-symmetric, such that

- A. v_x is the Hamiltonian vector field of $f_1(x)$, for all $x \in \mathfrak{g}$,
- B. the following two equations are satisfied:

$$(27) \quad f_1([x, y]) - \underbrace{\{f_1(x), f_1(y)\}}_{\omega(v_x, v_y, \cdot)} = df_2(x, y)$$

$$(28) \quad -\underbrace{l_3(f_1(x), f_1(y), f_1(z))}_{\omega(v_x, v_y, v_z)} = f_2(x, [y, z]) - f_2(y, [x, z]) + f_2(z, [x, y]).$$

7.3. Existence of moment maps. If G is also connected and semisimple, then it is well-known for the symplectic case that a symplectic G -action admits a moment map [8, Chapter X]. However, for the present case, we now need an additional condition on the action.

Proposition 7.1. *If G is compact, connected, and semisimple and for all $x \in \mathfrak{g}$ there is point $p \in M$ such that $v_x(p) = 0$, then there exists an equivariant homotopy moment map.*

Notice that the assumption on the action in Prop. 7.1 is satisfied when M is oriented, compact, and has non-zero Euler characteristic; since in this case the Poincaré-Hopf theorem implies that every vector field on M has a zero.

The proof of Prop. 7.1 is based on the following lemmas, which are straightforward analogs of well-known statements in symplectic geometry. We phrase these more generally for ω any invariant closed form $\omega \in \Omega^{n+1}(M)$ with $n \geq 2$.

Lemma 7.2. *If \mathfrak{g} satisfies $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, then there exists a linear map $\mu: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$ such that v_x is a Hamiltonian vector field for $\mu(x)$, that is, $d\mu(x) = -\iota_{v_x}\omega$ for all $x \in \mathfrak{g}$.*

Proof. Let $x \in \mathfrak{g}$. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, we can write $x = \sum_i [x_i, x'_i]$. The locally Hamiltonian vector field v_x can hence be written as $v_{\sum_i [x_i, x'_i]} = \sum_i [v_{x_i}, v_{x'_i}]$, and by Prop. 4.5 it is the Hamiltonian vector field of

$$\mu(x) = \sum_i \iota(v_{x_i} \wedge v_{x'_i})\omega.$$

Now let x range through a basis of \mathfrak{g} , and extend μ to obtain a linear map $\mu: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$. \square

Lemma 7.3. *If G is compact and the linear map $\mu: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$ satisfies $d\mu(x) = -\iota_{v_x}\omega$ for all $x \in \mathfrak{g}$, then*

$$\mu'(x) = \int_G g^*(\mu(\text{Ad}_g x))$$

is equivariant and satisfies $d\mu'(x) = -\iota_{v_x}\omega$.

Proof. For every $g \in G$ and $x \in \mathfrak{g}$, one computes that $d(g^*(\mu(\text{Ad}_g x))) = -\iota_{v_x}\omega$ using the fact that ω is G -invariant. Integrating over G we obtain a map μ' which is equivariant. \square

Lemma 7.4. *If the linear map $\mu: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$ is equivariant and $d\mu(x) = -\iota_{v_x}\omega$, then $\iota_{v_x}\mu(x)$ is automatically a closed $n-2$ form for all $x \in \mathfrak{g}$.*

Proof. The equivariance of μ is equivalent to $\mathcal{L}_{v_x}\mu(y) = \mu([x, y])$ for all $x, y \in \mathfrak{g}$. Now $d\iota_{v_x}\mu(x) = -\iota_{v_x}d\mu(x) + \mathcal{L}_{v_x}\mu(x) = 0$. \square

Proof of Prop. 7.1. Since G is semisimple, Lemma 7.2 produces $\mu: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$ satisfying $d\mu(x) = -\iota_{v_x}\omega$. The compactness of G allows to apply Lemma 7.3, and therefore we may assume that μ is equivariant. For every $x \in \mathfrak{g}$, the function $\iota_{v_x}\mu(x)$ is a constant function by Lemma 7.4. We conclude that it must be identically zero since it vanishes at p . Hence $\omega - \mu$ is a 1-step extension, so Thm. 6.9 produces an equivariant moment map with components $f_1 = \mu$ and $f_2(x, y) = \iota_{v_x}\mu(y)$. \square

7.4. Uniqueness of moment maps. Next, we comment briefly on uniqueness issues. Recall that a moment map in symplectic geometry is unique if $H^1(\mathfrak{g}, \mathbb{R}) = 0$ [8, Section 26]. Similarly, in the pre-2-plectic case, cohomological constraints on \mathfrak{g} will ensure uniqueness, but only up to a certain equivalence given by an action of $C^\infty(M)$.

Lemma 7.5.

- a) *If $\omega - \mu$ is a 1-step extension of ω and $\psi: \mathfrak{g} \rightarrow C^\infty(M)$ is any G -equivariant linear map, then $\omega - (\mu + d\psi)$ is also a 1-step extension.*
- b) *If $(f_1, f_2): \mathfrak{g} \rightarrow L_\infty(M, \omega)$ is a moment map and $\psi: \mathfrak{g} \rightarrow C^\infty(M)$ is any linear map, then we obtain a new moment map with components*

$$\begin{aligned} \tilde{f}_1 &= f_1 + d\psi \\ \tilde{f}_2(x, y) &= f_2(x, y) + \psi([x, y]). \end{aligned}$$

Proof. A straightforward calculation left for the reader. \square

Remark 7.6. In [17], a notion of equivalence for moment maps is considered whose equivalence classes are strictly larger than those arising in Lemma 7.5 b.

Proposition 7.7. *If G is compact and semisimple or, more generally, $H^1(\mathfrak{g}, \mathbb{R}) = H^2(\mathfrak{g}, \mathbb{R}) = 0$, then:*

- a) *Any two equivariant extensions are related as in Lemma 7.5 a,*
- b) *Any two moment maps are related as in Lemma 7.5 b.*

Proof. a) We have to show that given two 1-step extensions $\omega - \mu, \omega - \mu'$

there is a G -equivariant $\tilde{\psi}: \mathfrak{g} \rightarrow C^\infty(M)$ such that $\mu' = \mu + d\tilde{\psi}$. Recall that by Thm. 6.9, $f_1 = \mu$ and $f_2(x, y) = \iota_{v_x}\mu(y)$ are the components of a moment map. Similarly, we obtain a moment map (f'_1, f'_2) using μ' . Hence from Eq. (27), we see that $(\mu' - \mu)([x, y])$ is an exact 1-form for all $x, y \in \mathfrak{g}$. From $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, which is equivalent to $H^1(\mathfrak{g}, \mathbb{R}) = 0$, we deduce that there exists a map $\psi: \mathfrak{g} \rightarrow C^\infty(M)$ such that $\mu' = \mu + d\psi$.

The map ψ will not be equivariant in general. We now modify it suitably to obtain an equivariant map. Since μ and μ' are 1-step extensions of ω , their equivariance and Eq. (26) imply that we have for all $x, y \in \mathfrak{g}$:

$$(29) \quad \mathcal{L}_{v_x}d\psi(y) = d\psi([x, y]), \quad \iota_{v_x}d\psi(x) = \mathcal{L}_{v_x}\psi(x) = 0.$$

Define

$$c(x, y) = \mathcal{L}_{v_x}\psi(y) - \psi([x, y]).$$

The map c is clearly the obstruction to ψ being equivariant, and Eq. (29) implies that it is \mathbb{R} -valued and skew-symmetric.

We claim that c is a Lie algebra cocycle. We have

$$(\delta_{\text{CEC}})(x, y, z) = -(c([x, y], z) + c.p.) = (\mathcal{L}_{v_z}\psi([x, y]) - \psi([z, [x, y]])) + c.p.$$

This is zero by the Jacobi identity of \mathfrak{g} and because

$$\mathcal{L}_{v_z}\psi([x, y]) + c.p. = \iota_{v_z}(\mu' - \mu)([x, y]) + c.p. = (f_2(z, [x, y]) - f'_2(z, [x, y])) + c.p. = 0,$$

where the last equality uses Eq. (28).

From $H^2(\mathfrak{g}, \mathbb{R}) = 0$ we know that there exists $b: \mathfrak{g} \rightarrow \mathbb{R}$ such that $c = d_{\mathfrak{g}}b$, that is, $c(x, y) = -b([x, y])$ for all $x, y \in \mathfrak{g}$. The map

$$\tilde{\psi} = \psi - b: \mathfrak{g} \rightarrow C^\infty(M)$$

clearly satisfies $\mu' = \mu + d\tilde{\psi}$, and it is equivariant since $\mathcal{L}_{v_x}\tilde{\psi}(y) = \mathcal{L}_{v_x}\psi(y) = \tilde{\psi}([x, y])$.

b) We have to show that given two moment maps with components (f_1, f_2) and (f'_1, f'_2) respectively, there is $\psi: \mathfrak{g} \rightarrow C^\infty(M)$ such that $f'_1 - f_1 = d\psi$ and $(f'_2 - f_2)(x, y) = \psi([x, y])$.

Notice that $f'_2 - f_2: \Lambda^2\mathfrak{g} \rightarrow C^\infty(M)$ is a Lie algebra cocycle w.r.t. the *trivial* representation of \mathfrak{g} on $C^\infty(M)$, by Eq. (28). From $H^2(\mathfrak{g}, C^\infty(M)) = H^2(\mathfrak{g}, \mathbb{R}) \otimes C^\infty(M) = 0$ we obtain a map $\psi: \mathfrak{g} \rightarrow C^\infty(M)$ such that $(f'_2 - f_2)(x, y) = \psi([x, y])$ for all $x, y \in \mathfrak{g}$. To conclude we just need to assure that the identity $f'_1 - f_1 = d\psi$ holds: it does when both sides are applied to elements of the form $[x, y] \in \mathfrak{g}$, by Eq. (27) and the above. As $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, it holds for all elements of \mathfrak{g} . \square

7.5. Equivariant moment maps not arising from equivariant cocycles. Let G act on the pre-2-plectic manifold (M, ω) . Let $\omega - \mu$ be an equivariant 1-step extension, and let $(f_1, f_2): \mathfrak{g} \rightarrow L_\infty(M, \omega)$ be the corresponding equivariant moment map (Thm. 6.9). Here we explain how to modify this moment map to obtain a new *equivariant* moment map that does *not* arise from any equivariant cocycle.

If a linear map $\tilde{f}_1: \mathfrak{g} \rightarrow \Omega^1(M)$

- takes values in closed 1-forms
- is equivariant: $d(\iota_{v_x}\tilde{f}_1(y)) = \mathcal{L}_{v_x}\tilde{f}_1(y) = \tilde{f}_1([x, y])$ for all $x, y \in \mathfrak{g}$,

then $f_1 + \tilde{f}_1$ satisfies condition (A) at the beginning of Sec. 7.2 and is equivariant. If a skew-symmetric map $\tilde{f}_2: \mathfrak{g} \otimes \mathfrak{g} \rightarrow C^\infty(M)$

- is equivariant

- satisfies $\tilde{f}_1([x, y]) = d(\tilde{f}_2(x, y))$ for all $x, y \in \mathfrak{g}$
- satisfies $\tilde{f}_2(x, [y, z]) + c.p. = 0$ for all $x, y, z \in \mathfrak{g}$,

then $f_1 + \tilde{f}_1$ and $f_2 + \tilde{f}_2$ are the components of a new equivariant moment map. (The last two conditions above guarantee Eq. (27) and (28) are satisfied). Furthermore, if we require that the constant function $\iota_{v_x} \tilde{f}_1(x)$ is non-zero for some $x \in \mathfrak{g}$, then the last condition in Eq. (26) cannot be satisfied, and hence the new moment map can *not* arise from any equivariant cocycle. When \mathfrak{g} is an abelian Lie algebra and M is connected, the equivariance of \tilde{f}_1 boils down to the condition that $\iota_{v_x} \tilde{f}_1(y)$ is a constant function for all $x, y \in \mathfrak{g}$, and the three conditions on \tilde{f}_2 simply imply that \tilde{f}_2 takes values in the constant functions. Below we present a concrete instance of this construction:

Example 7.8. Let G be the abelian group $S^1 \times S^1$, and $(M, \omega) = (S^1 \times S^1 \times \mathbb{R}, d\theta_1 \wedge d\theta_2 \wedge dz)$. We take the infinitesimal action of \mathfrak{g} on M to be $(1, 0) \in \mathfrak{g} \mapsto \partial_{\theta_1}$, $(0, 1) \mapsto \partial_{\theta_2}$. It is easily checked that $\omega - \mu$ is a 1-step extension (see Eq. (26)), where

$$\mu: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^1(M), \quad (1, 0) \mapsto zd\theta_2, (0, 1) \mapsto -zd\theta_1.$$

We can take \tilde{f}_1 to be

$$\tilde{f}_1: \mathfrak{g} \rightarrow \Omega_{\text{closed}}^1(M), \quad (1, 0) \mapsto d\theta_1, (0, 1) \mapsto d\theta_2,$$

and any arbitrary skew-symmetric $\tilde{f}_2: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$. Then, as seen above, $f_1 + \tilde{f}_1$ and $f_2 + \tilde{f}_2$ are the components of an equivariant moment map, which can not arise from any equivariant cocycle since $\iota_{v_x} \tilde{f}_1(x) = 1 \neq 0$ for $x = (1, 0)$.

The discussion in this subsection proves:

Proposition 7.9. *Not all equivariant moment maps for actions on 2-plectic manifolds arise from cocycles in the Cartan complex via the formula given in Thm. (6.8).*

Remark 7.10. In [17] Example 7.8 is used to obtain an equivariant moment map which is *not equivalent* (in the sense of [17], see Remark 7.6) to any moment map arising from a 1-step extension of ω . This provides a statement stronger than Prop. 7.9 above.

8. EXAMPLES

In this section we present more examples of moment maps, many of which are generalizations of interesting examples from symplectic geometry. In light of Remark 6.10, all of them can be understood as arising from extensions as in Thm. 6.9, even if G is not compact.

In Sec. 10 we will give one more (infinite dimensional) example.

8.1. Exact pre- n -plectic forms. Let M be a manifold with a G -action, let $\alpha \in \Omega^n(M)^G$ be a G -invariant n -form and consider $\omega = d\alpha$.

Lemma 8.1. *If $\mu \in (\Omega^{n-1}(M) \otimes \mathfrak{g}^\vee[-2])^G$ is defined by $\mu(x) = \iota_{v_x} \alpha$, then $\omega - \mu$ is a 1-step extension of ω .*

Proof. It suffices to check that μ is G -equivariant and satisfies (26). The equivariance is encoded by the condition $\mathcal{L}_{v_x} \mu(y) = \mu([x, y])$ for all $x, y \in \mathfrak{g}$, and follows from the Cartan relation $\iota_{[v, w]} = [\mathcal{L}_v, \iota_w] = \mathcal{L}_v \circ \iota_w - \iota_w \circ \mathcal{L}_v$, together with the fact that $\mathcal{L}_{v_x} \alpha = 0$ for all x in \mathfrak{g} . Concerning (26), clearly $d\omega = dd\alpha = 0$. Moreover, the Cartan relation $d \circ \iota_{v_x} + \iota_{v_x} \circ d = \mathcal{L}_{v_x}$, together with the invariance of α gives $d\mu(x) = -\iota_{v_x} \omega$. Finally the antisymmetry of α gives $\iota_{v_x} \mu(x) = 0$. \square

Hence by Thm 6.9 we obtain a homotopy moment map $\mathfrak{g} \rightarrow L_\infty(M, \omega)$, given by

$$f_k: \mathfrak{g}^{\otimes k} \rightarrow \Omega^{n-k}(M), \quad 1 \leq k \leq n$$

$$f_k(x_1, \dots, x_k) = (-1)^{k-1} \zeta(k) \iota(v_{x_1} \wedge \dots \wedge v_{x_k}) \alpha$$

We present two concrete examples. The first one generalizes actions on cotangent bundles by cotangent lifts in symplectic geometry.

Example 8.2 (Cotangent lifts). If G acts on a manifold N and n is an integer, take $M = \Lambda^n T^*N$ and the G -action induced by the cotangent lift. Take $\alpha \in \Omega^n(M)$ to be the canonical form defined by

$$\alpha(w_1, \dots, w_n)|_\xi = \xi(\pi_* w_1, \dots, \pi_* w_n) \quad \forall \xi \in M$$

where $\pi: M \rightarrow N$ is the projection and $w_1, \dots, w_n \in T_\xi M$. The n -form α is invariant, and it is known that $d\alpha$ is an n -plectic form. (In the case $n = 1$ it is, up to sign, the canonical symplectic form on T^*N). From Lemma 8.1 we know that an equivariant extension μ of ω is given by $\mu(x) = \iota_{v_x} \alpha$, i.e.

$$\mu(x)(w_2, \dots, w_n)|_\xi = \xi((v_x)_{\pi(\xi)}, \pi_* w_2, \dots, \pi_* w_n),$$

where v_x denotes the fundamental vector field for the action on $\Lambda^n T^*N$ (which restricts to the fundamental vector field for the action on N).

In other words, $(\mu(x))_\xi = \pi^*(\iota_{v_x} \xi)$.

Example 8.3 (Linear actions on vector spaces). It is well-known that any symplectic representation $G \rightarrow \text{GL}(V)$ on a symplectic vector space (V, ω) is Hamiltonian. More precisely, if we denote the action of $\xi \in \mathfrak{g}$ on V by $\xi \cdot p$, and consider the unique moment map $J: V \rightarrow \mathfrak{g}^\vee$ vanishing at the origin, then its **components** $J^\xi: V \rightarrow \mathbb{R} \quad \forall \xi \in \mathfrak{g}$ are the quadratic functions

$$J^\xi(p) = J(p)(\xi) = -\frac{1}{2} \omega(p, \xi \cdot p).$$

Below we generalize such actions to higher degree forms, and will see that the the analogs of the components for the moment map are no longer quadratic.

Let V be any vector space and $\omega \in \Lambda^{n+1} V^*$, giving rise to a constant pre- n -plectic form on V (which is obviously closed and hence exact). Consider a linear action of a Lie group $G \rightarrow \text{GL}(V)$ preserving ω . We claim that a G -invariant primitive of ω is

$$\alpha = \frac{\iota_E \omega}{n+1}$$

where E is the Euler vector field on V (in coordinates, $E = \sum_j x_j \frac{\partial}{\partial x_j}$). Indeed it is straightforward to check that $d\iota_E \omega = \mathcal{L}_E \omega = (n+1)\omega$. Also we have the equality $\mathcal{L}_{v_\xi} \iota_E \omega = \iota_E \mathcal{L}_{v_\xi} \omega + \iota_{[v_\xi, E]} \omega = 0$ for all $\xi \in \mathfrak{g}$, since ω is G -invariant and E commutes with all linear vector fields.

Hence, we have a moment map induced by the 1-step extension of ω given in Lemma 8.1. Let us study this map in more detail. We denote by $\phi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ the Lie algebra morphism associated to the linear action. The infinitesimal generator of the action given by any $\xi \in \mathfrak{g}$ is the linear vector field $v_\xi|_p = -\phi(\xi)p$ (matrix multiplication). For all $p \in V$ we have $E|_p = p$, so we obtain the following expression for the k -th component of moment map:

$$\begin{aligned} f_k(\xi_1, \dots, \xi_k)|_p &= (-1)^{k-1} \zeta(k) \iota(v_{\xi_1} \wedge \dots \wedge v_{\xi_k}) \alpha|_p \\ &= -\zeta(k) \frac{1}{n+1} \iota(p \wedge \phi(\xi_1)p \wedge \dots \wedge \phi(\xi_k)p) \omega. \end{aligned}$$

Notice that the coefficients of the $(n-k)$ -form $f_k(\xi_1, \dots, \xi_k)$ are polynomials of degree $k+1$.

The next example is a special case of Example 8.3 and generalizes the following simple case of Hamiltonian action on a symplectic manifold: the action of the circle on \mathbb{R}^2 by rotations, with moment map $(x_1, x_2) \mapsto -\frac{1}{2}(x_1^2 + x_2^2)$.

Example 8.4 ($\text{SO}(n)$ -action on \mathbb{R}^n). We consider the canonical action of $G = \text{SO}(n)$ on \mathbb{R}^n , the latter endowed with the constant volume form

$$\omega = dx_1 \dots dx_n.$$

It is G -invariant, and by Example 8.3 an invariant primitive is

$$\alpha = \frac{1}{n} \sum_{k=1}^n (-1)^{k+1} x_k dx_1 \dots \widehat{dx_k} \dots dx_n.$$

A basis of the Lie algebra $\mathfrak{so}(n)$ is $\{e_{ij} : 1 \leq i < j \leq n\}$, where e_{ij} denotes the matrix with -1 in the (i, j) -th position, 1 in the (j, i) -th position and zeros elsewhere. The corresponding generators of the action are given by

$$v_{ij} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}.$$

By Lemma 8.1 we know that an equivariant 1-step extension μ of ω is given by

$$\begin{aligned} \mu(e_{ij}) &= \iota_{v_{ij}} \alpha \\ &= \frac{1}{n} \left(\sum_{k=1}^{i-1} - \sum_{k=i+1}^n \right) (-1)^{k+1+i} x_j x_k dx_1 \dots \widehat{dx_i} \dots \widehat{dx_k} \dots dx_n \\ &\quad - \frac{1}{n} \left(\sum_{k=1}^{j-1} - \sum_{k=j+1}^n \right) (-1)^{k+1+j} x_i x_k dx_1 \dots \widehat{dx_j} \dots \widehat{dx_k} \dots dx_n, \end{aligned}$$

and that we can build a moment map out of μ . Notice that, in the symplectic case ($n = 2$), one recovers $\mu(e_{12}) = -\frac{1}{2}(x_1^2 + x_2^2)$.

8.2. Adjoint action and conjugacy classes. Let G be a compact Lie group whose Lie algebra \mathfrak{g} is equipped with an Ad-invariant inner-product $\langle \cdot, \cdot \rangle$. Then G equipped with the bi-invariant Cartan 3-form

$$\omega = \frac{1}{12} \langle \theta_L, [\theta_L, \theta_L] \rangle = \frac{1}{12} \langle \theta_R, [\theta_R, \theta_R] \rangle$$

is a pre-2-plectic manifold. Here θ_L and θ_R are, respectively, the left and right invariant Maurer-Cartan 1-forms on G satisfying

$$d\theta_L + \frac{1}{2}[\theta_L, \theta_L] = 0 \quad d\theta_R - \frac{1}{2}[\theta_R, \theta_R] = 0.$$

If G is, for example, also semi-simple, then ω is non-degenerate and hence (G, ω) is 2-plectic.

Clearly, the action of G on itself via conjugation preserves ω , and this action gives rise to a homotopy moment map. The Hamiltonian vector field associated to $x \in \mathfrak{g}$ is

$$v_x = v_x^L - v_x^R,$$

where v^L and v^R are, respectively, the left and right invariant vector fields on G associated to x . A straightforward calculation using the above Maurer-Cartan equations and the identity $\text{Ad}_g \theta_L = \theta_R$ gives:

$$\frac{1}{2} d \langle \theta_L + \theta_R, x \rangle = -\iota(v_x) \omega.$$

The fact that this action lifts to a moment map follows from Thm. 6.9 and the well-known fact that the \mathfrak{g}^V -valued 1-form

$$\mu(x) = \frac{1}{2} \langle \theta_L + \theta_R, x \rangle \quad \forall x \in \mathfrak{g}$$

gives an equivariant extension $\omega - \mu$ of the Cartan 3-form.

Let us write out the structure map $f_2: \mathfrak{g} \otimes \mathfrak{g} \rightarrow C^\infty(G)$ explicitly for this case. By definition, at a point $g \in G$, we have

$$\begin{aligned} f_2(x, y)(g) &= \iota(v_x) \mu(y)|_g = \frac{1}{2} \langle \theta_L(v_x) + \theta_R(v_x), y \rangle|_g \\ &= \frac{1}{2} \langle (\text{Ad}_g - \text{Ad}_{g^{-1}})x, y \rangle. \end{aligned}$$

This piece of the moment map is related to an interesting invariant 2-form defined on the conjugacy classes of G . The Hamiltonian vector fields v_x are minus the fundamental vector fields associated to the conjugation action, and therefore span the tangent spaces of the conjugacy classes. It follows from Proposition 3.1 in [1] that if $\iota_C: C \hookrightarrow G$ is the inclusion of a conjugacy class, then

$$dB = -\iota_C^* \omega$$

where $B \in \Omega^2(C)^G$ is

$$B_g(v_x, v_y) = f_2(x, y)(g) \quad \forall g \in C \quad \forall x, y \in \mathfrak{g}.$$

Conjugacy classes are important examples of “quasi-Hamiltonian G -spaces” [1], just as coadjoint orbits are examples of Hamiltonian G -spaces in symplectic geometry. As the above example suggests, it may be interesting to investigate further the relationship between quasi-Hamiltonian G -spaces and homotopy moment maps.

8.3. Examples from 2-step extensions. The previous examples of moment maps arose from 1-step extensions. Here we give examples in which 2-step extensions arise naturally. The second example, the $\mathrm{SO}(n)$ -actions on the n -sphere, generalizes the well-known Hamiltonian action of S^1 on S^2 .

8.3.1. Products. Let G_i act on the manifold M_i and α_i be an equivariant cocycle in the Cartan model for this action, i.e. $d_{G_i} \alpha_i = 0$ ($i = 1, 2$). Consider the product $\alpha_1 \alpha_2$ (obtained simply wedge-multiplying the differential form components). Then $\alpha_1 \alpha_2$ is an equivariant cocycle in the Cartan model for the product action of $G_1 \times G_2$ on $M_1 \times M_2$, since

$$d_{G_1 \times G_2}(\alpha_1 \alpha_2) = (d_{G_1 \times G_2} \alpha_1) \alpha_2 \pm \alpha_1 (d_{G_1 \times G_2} \alpha_2) = (d_{G_1} \alpha_1) \alpha_2 \pm \alpha_1 (d_{G_2} \alpha_2) = 0.$$

We spell this out when the equivariant cocycles are of the kind considered in Cor. 6.9:

Proposition 8.5. *Let G_i act on (M_i, ω_i) with $\omega_i \in \Omega^{n_i+1}(M_i)$ a closed form, for $i = 1, 2$. Let $\omega_i - \mu_i$ be 1-step extensions, and regard μ_i as maps $\mathfrak{g}_i \rightarrow \Omega^{n_i-1}(M_i)$. Then the product action of $G_1 \times G_2$ on the pre- $(n_1 + n_2 + 1)$ -plectic manifold $(M_1 \times M_2, \omega_1 \omega_2)$ admits an equivariant extension given by*

$$\omega_1 \omega_2 - \eta + p$$

where

$$\begin{aligned} \eta: \mathfrak{g}_1 \oplus \mathfrak{g}_2 &\rightarrow \Omega^{n_1+n_2}(M_1 \times M_2), & x_1 + x_2 &\mapsto \mu_1^{x_1} \omega_2 + \omega_1 \mu_2^{x_2} \\ p: S^2(\mathfrak{g}_1 \oplus \mathfrak{g}_2) &\rightarrow \Omega^{n_1+n_2-2}(M_1 \times M_2), & (x_1 + x_2, y_1 + y_2) &\mapsto \frac{1}{2} (\mu_1^{x_1} \mu_2^{y_2} + \mu_1^{y_1} \mu_2^{x_2}). \end{aligned}$$

(Here we denote by x_i, y_i, \dots elements of \mathfrak{g}_i , by v_{x_i} the corresponding vector fields on M_i , and for the sake of readability we omit wedge products and write μ^x for $\mu(x)$.)

In particular, when ω_1 and ω_2 are symplectic, the action of $G_1 \times G_2$ on the 3-plectic manifold $(M_1 \times M_2, \omega_1 \omega_2)$ admits a moment map.

8.3.2. $\mathrm{SO}(n)$ -action on the n -sphere. A classical example of a Hamiltonian action in symplectic geometry is the action of $S^1 = \mathrm{SO}(2)$ on S^2 by rotations about the z -axis. The “height function” $-z: S^2 \rightarrow \mathbb{R}$ provides a moment map.

We now extend this to S^n for $n \leq 5$, by means of a computation that makes clear how to generalize this to higher values of n as well. More precisely, we assume the following set-up: the unit sphere $M = S^n \subset \mathbb{R}^{n+1}$ is endowed with the $(n-1)$ -plectic volume form ω obtained restricting

$$\sum_{k=1}^{n+1} (-1)^{k+1} x_k dx_1 \dots \widehat{dx_k} \dots dx_{n+1} = \alpha \wedge dx_{n+1} + \frac{(-1)^n}{n} x_{n+1} \cdot d\alpha,$$

where

$$\alpha = \sum_{k=1}^n (-1)^{k+1} x_k dx_1 \dots \widehat{dx}_k \dots dx_n.$$

View $G = \mathrm{SO}(n)$ as a subgroup of $\mathrm{SO}(n+1)$ (embedded as matrices with a “1” in the lower right corner), and consider the obvious action on $S^n \subset \mathbb{R}^{n+1}$ by matrix multiplication.

It is sufficient to find a 2-step extension of ω in the Cartan complex, since then applying Thm. 6.8 one obtains an equivariant moment map. To this aim, we first spell out what it means to have such a cocycle in the Cartan complex. Consider an element $\omega + P + Q$ of the Cartan complex for the action of some Lie group G , where the degrees as polynomials on \mathfrak{g} are 0, 1, 2 respectively. Choose a basis of \mathfrak{g} , giving rise to infinitesimal generators of the action v_I and a basis $\{\xi_I\}$ of \mathfrak{g}^\vee . Write $P = \sum_{I,J} \xi_I \otimes P^I$, and $Q = \sum_I \xi_I \xi_J \otimes Q^{IJ}$, where $Q^{IJ} = Q^{JI}$. Then $\omega + P + Q$ is a Cartan cocycle iff $d\omega = 0$ and

$$(30) \quad -\iota_{v_I} \omega + dP^I = 0 \text{ for all } I,$$

$$(31) \quad -\mathrm{Sym}(\iota_{v_J} P^I) + dQ^{IJ} = 0 \text{ for all } I \leq J,$$

$$(32) \quad \mathrm{Sym}(\iota_{v_K} Q^{IJ}) = 0 \text{ for all } I \leq J \leq K.$$

Here $\mathrm{Sym}(\iota_{v_J} P^I) = \frac{1}{2}(\iota_{v_J} P^I + \iota_{v_I} P^J)$ denotes the symmetrization, and similarly for $\mathrm{Sym}(\iota_{v_K} Q^{IJ})$.

We now proceed to find a 2-step extension of ω in the Cartan complex.

Step 1: find P solving equation (30). This $\mathrm{SO}(n)$ -action on S^n preserves ω , for in Ex. 8.4 we saw that it preserves α . With v_{ij} as in Ex. 8.4, we have

$$\iota_{v_{ij}} \omega = d \left(\frac{(-1)^{n+1}}{n} \iota_{v_{ij}} \alpha \cdot x_{n+1} \right) + \frac{n+1}{n} \iota_{v_{ij}} \alpha \wedge dx_{n+1},$$

using the fact that the action preserves α . To write out the right-most term we consider $\iota_{v_{ij}} \alpha$. Using the fact that the function $\sum_{k=1}^{n+1} x_k^2$ equals one on S^n , and therefore the pullback to S^n of its differential $2 \sum_{k=1}^{n+1} x_k dx_k$ vanishes, a lengthy computation shows that

$$\iota_{v_{ij}} \alpha = (-1)^{i+j} (1 - x_{n+1}^2) dx_1 \dots \widehat{dx}_i \dots \widehat{dx}_j \dots dx_n + (\text{terms containing } dx_{n+1}).$$

Hence

$$\iota_{v_{ij}} \alpha \wedge dx_{n+1} = (-1)^{i+j+n} d \left(\left(x_{n+1} - \frac{1}{3} x_{n+1}^3 \right) dx_1 \dots \widehat{dx}_i \dots \widehat{dx}_j \dots dx_n \right).$$

A primitive for $\iota_{v_{ij}} \omega$ is therefore

$$(33) \quad P(e_{ij}) = \frac{(-1)^{n+1}}{n} x_{n+1} \cdot \iota_{v_{ij}} \alpha + (-1)^{i+j+n} \frac{n+1}{n} \left(x_{n+1} - \frac{1}{3} x_{n+1}^3 \right) dx_1 \dots \widehat{dx}_i \dots \widehat{dx}_j \dots dx_n.$$

Notice that $P \in \mathfrak{so}(n)^\vee \otimes \Omega^{n-1}(M)$ is $\mathrm{SO}(n)$ -invariant: the condition

$$\mathcal{L}_{v_{ij}} P(e_{i'j'}) = P([e_{ij}, e_{i'j'}]),$$

for all $i < j$ and $i' < j'$, follows from a computation that uses the identity $[\mathcal{L}_v, \iota_w] = \iota_{[v,w]}$ and a careful care of signs.

Step 2: find Q solving equation (31). We now look for Q so that eq. (31) is satisfied. Fix $i < j$ and $l < m$, where all four indices lie in $\{1, \dots, n\}$. Assume first that all of i, j, l, m are distinct. Then, writing d_i as a short-form for dx_i , we have

$$\iota_{v_{lm}}(d_1 \dots \widehat{d}_i \dots \widehat{d}_j \dots d_n) = (-1)^{l+m+1} B_{l,m}^{i,j} (x_l d_l + x_m d_m) (d_1 \dots \widehat{d}_i \dots \widehat{d}_j \dots \widehat{d}_l \dots \widehat{d}_m \dots d_n)$$

where we define

$$B_{l,m}^{i,j} := (-1)^{\mathrm{card}\{x \in \{i,j\}: l < x < m\}}.$$

From eq. (33), using the obvious fact that $\iota_{v_{lm}}\iota_{v_{ij}}\alpha = -\iota_{v_{ij}}\iota_{v_{lm}}\alpha$, we obtain

$$\begin{aligned} & \frac{1}{2}(\iota_{v_{lm}}P^{ij} + \iota_{v_{ij}}P^{lm}) = \\ & \frac{n+1}{2n} \left(x_{n+1} - \frac{1}{3}x_{n+1}^3 \right) (-1)^{n+1}(-1)^{i+j+l+m} B_{l,m}^{i,j}(x_i d_i + x_j d_j + x_l d_l + x_m d_m)(d_1 \dots \widehat{d}_i \dots \widehat{d}_j \dots \widehat{d}_l \dots \widehat{d}_m \dots d_n). \end{aligned}$$

Using that $\sum_{k=1}^{n+1} x_k dx_k$ vanishes on S^n , we can replace the sum $x_i d_i + x_j d_j + x_l d_l + x_m d_m$ above by $-x_{n+1} d_{n+1}$, and we see that the above expression is an exact form, with primitive

$$Q^{(ij)(lm)} := \frac{n+1}{2n} \left(\frac{1}{3}x_{n+1}^3 - \frac{1}{15}x_{n+1}^5 \right) (-1)^n (-1)^{i+j+l+m} B_{l,m}^{i,j}(d_1 \dots \widehat{d}_i \dots \widehat{d}_j \dots \widehat{d}_l \dots \widehat{d}_m \dots d_n).$$

When not all of i, j, l, m are distinct, one can prove that $\iota_{v_{lm}}P^{ij} + \iota_{v_{ij}}P^{lm} = 0$, hence even in that case a primitive is given by the above formula for $Q^{(ij)(lm)}$, which equals zero since we are removing twice the same one-form. Notice that $Q^{(ij)(lm)} = Q^{(lm)(ij)}$, since $B_{l,m}^{i,j} = B_{i,j}^{l,m}$. Further $Q \in S^2(\mathfrak{so}(n))^\vee \otimes \Omega^{n-4}(S^n)$ is $\text{SO}(n)$ -invariant.

Step 3: check that equation (32) is satisfied. We have to check that

$$\iota_{v_{pq}}Q^{(ij)(lm)} + \iota_{v_{lm}}Q^{(pq)(ij)} + \iota_{v_{ij}}Q^{(lm)(pq)} = 0.$$

Since $n \leq 5$, two of the six indices appearing above will agree, so we may assume that $i = p$. In that case the middle term vanishes, while the first and last one cancel each other out, as one can check keeping track carefully of the signs.

Remark 8.6. 1) Writing out explicitly Eq. (33) we obtain

$$\begin{aligned} P(e_{ij}) &= (-1)^{i+j+n} \left(x_{n+1} - \frac{n-2}{3n}x_{n+1}^3 \right) dx_1 \dots \widehat{dx}_i \dots \widehat{dx}_j \dots dx_n \\ &+ \frac{(-1)^{i+j}}{n} \left(\sum_{k=1}^{i-1} - \sum_{k=i+1}^{j-1} + \sum_{k=j+1}^n \right) (-1)^{k-1} x_k x_{n+1}^2 \cdot dx_1 \dots \widehat{dx}_i \dots \widehat{dx}_j \dots \widehat{dx}_k \dots dx_{n+1}. \end{aligned}$$

Notice that the cubic terms disappear only in the case $n = 2$.

2) The above proof suggests that, for arbitrary values of n , an extension of the volume form ω on S^n to a cocycle in the Cartan complex is given by $\omega + P_1 + \dots + P_{\lfloor \frac{n}{2} \rfloor}$ where P_1 is given in eq. (33) and, for $k \geq 2$:

$$P_k((i_1, j_1) \otimes \dots \otimes (i_k, j_k)) = \pm \frac{1}{k!} \frac{n+1}{n} \left(\frac{x_{n+1}^{2k-1}}{(2k-1)!!} - \frac{x_{n+1}^{2k+1}}{(2k+1)!!} \right) (d_1 \dots \widehat{d}_{i_1} \dots \widehat{d}_{j_1} \dots \widehat{d}_{i_k} \dots \widehat{d}_{j_k} \dots d_n),$$

where $i_1 < j_1, \dots, i_k < j_k$ are integers between 1 and n , and $N!! := N \cdot (N-2) \cdot \dots \cdot 5 \cdot 3$.

3) If n is even, it is easy to see that the volume form ω on the n -sphere has a $\frac{n}{2}$ -step extension. Indeed, the cocycle condition for a $\frac{n}{2}$ -step extension $\omega + \sum_{i=1}^{\frac{n}{2}} P_i$ with $P_i \in (S^i(\mathfrak{so}(n))^\vee \otimes \Omega^{n-2i}(M))^{\text{SO}(n)}$ reads

$$\begin{aligned} dP_1 &= \iota_{\mathfrak{so}(n)}\omega, \\ dP_2 &= \text{Sym}(\iota_{\mathfrak{so}(n)}P_1), \\ &\dots \\ dP_{\frac{n}{2}} &= \text{Sym}(\iota_{\mathfrak{so}(n)}P_{\frac{n-2}{2}}), \\ 0 &= \text{Sym}(\iota_{\mathfrak{so}(n)}P_{\frac{n}{2}}) \end{aligned}$$

Using $H^{2i-1}(S^n) = 0$ and the fact that we can average to obtain $\mathrm{SO}(n)$ -invariant forms, we can solve these one after the other for $P_1, P_2, \dots, P_{\frac{n}{2}}$. Note that since n is even, $P_{\frac{n}{2}}$ has differential form degree zero and hence the last equation is automatically satisfied.

9. OBSTRUCTIONS AND CENTRAL EXTENSIONS

Here we describe an obstruction to the existence of moment maps characterized by a class in Lie algebra cohomology (Thm. 9.5). Conversely, we show that if both the class and certain de Rham cohomology groups vanish, then a moment map always exists (Thm. 9.6). If the obstruction does not vanish, we obtain a L_∞ -morphism into $\mathrm{Ham}_\infty(M, \omega)$ not from the Lie algebra, but from a Lie n -algebra which can be described a ‘higher central extension’ (Prop. 9.10).

Throughout this section, we assume we have a Lie group G acting on a pre- n -plectic manifold (M, ω) such that ω is preserved via infinitesimal diffeomorphisms i.e. the Lie algebra \mathfrak{g} acts via local Hamiltonian vector fields:

$$\mathcal{L}_{v_x} \omega = 0,$$

giving us the usual Lie algebra morphism

$$\mathfrak{g} \rightarrow \mathfrak{X}_{\mathrm{LHam}}(M), \quad x \mapsto v_x.$$

9.1. Lie algebra cohomology. For any Lie algebra \mathfrak{k} (possibly infinite-dimensional), we have the cochain complex $\mathrm{CE}(\mathfrak{k}) = \mathrm{Hom}(\Lambda^\bullet \mathfrak{k}, \mathbb{R})$ equipped with the usual Chevalley-Eilenberg differential δ_{CE} as in Eq. (2)

$$\delta_{\mathrm{CE}}(c)(x_1, \dots, x_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} c([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}).$$

Before considering G -actions, we make an observation about the Lie algebra of local Hamiltonian vector fields. The following proposition says that ω determines a class in $H_{\mathrm{CE}}^{n+1}(\mathfrak{X}_{\mathrm{LHam}}(M))$.

Proposition 9.1. *If (M, ω) is a pre- n -plectic manifold, then $\forall p \in M$ the linear map*

$$\begin{aligned} c_p: \Lambda^{n+1} \mathfrak{X}_{\mathrm{LHam}}(M) &\rightarrow \mathbb{R} \\ v_1 \wedge \dots \wedge v_{n+1} &\mapsto (-1)^n \zeta(n+1) \iota(v_1 \wedge \dots \wedge v_{n+1}) \omega|_p, \end{aligned}$$

is a degree $(n+1)$ -cocycle in $\mathrm{CE}(\mathfrak{X}_{\mathrm{LHam}}(M))$. Moreover, if M is connected, then the cohomology class $[c_p]$ is independent of $p \in M$.

To prove the above proposition, we need the following technical lemma. It is a special case of [26, Lemma 3.1], to which we refer the reader for a proof. It also generalizes [33, Lem. 3.7][43, Lem. 6.8].

Lemma 9.2. *If (M, ω) is a pre- n -plectic manifold and $v_1, \dots, v_m \in \mathfrak{X}_{\mathrm{LHam}}(M)$ with $m \geq 2$ then*

$$(34) \quad d\iota(v_1 \wedge \dots \wedge v_m) \omega = (-1)^m \sum_{1 \leq i < j \leq m} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \dots \wedge \hat{v}_i \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_m) \omega.$$

Now we have what we need to prove Prop. 9.1.

Proof of Prop. 9.1. Clearly $c_p \in \mathrm{Hom}(\Lambda^{n+1} \mathfrak{X}_{\mathrm{LHam}}(M), \mathbb{R})$. We compute:

$$(35) \quad \delta_{\mathrm{CE}}(c_p)(x_1, \dots, x_{n+2}) = \zeta(n+1) \sum_{1 \leq i < j \leq n+2} (-1)^{n+i+j} \iota([v_i, v_j] \wedge v_1 \wedge \dots \wedge \hat{v}_i \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_{n+2}) \omega|_p.$$

We use Lemma 9.2 for $m = n + 2$. The right-hand side of of Eq. (35) above is equal to plus or minus the right-hand side of Eq. 34 evaluated at the point p . However, the left-hand side of Eq. (34) vanishes because $\omega \in \Omega^{n+1}(M)$. Hence, $\delta_{\text{CE}}(c_p) = 0$.

Now, assume M is connected, and let $p' \in M$. There exists a path $\gamma: [0, 1] \rightarrow M$ such that $\gamma(0) = p$ and $\gamma(1) = p'$. We define a map $b: \Lambda^n \mathfrak{X}_{\text{LHam}}(M) \rightarrow \mathbb{R}$ by

$$b(v_1, \dots, v_n) = -\zeta(n+1) \int_{\gamma} \iota(v_1 \wedge \dots \wedge v_n) \omega.$$

It follows from Lemma 9.2 that

$$\begin{aligned} d\iota(v_1 \wedge \dots \wedge v_{n+1})\omega = \\ (-1)^{n+1} \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \dots \wedge \hat{v}_i \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_{n+1})\omega. \end{aligned}$$

Integrating both sides of the above equation over γ gives

$$\iota(v_1 \wedge \dots \wedge v_{n+1})\omega|_{p'} - \iota(v_1 \wedge \dots \wedge v_{n+1})\omega|_p = (-1)^n \zeta(n+1) \delta_{\text{CE}}(b)(v_1, \dots, v_{n+1}),$$

and, hence, $c_{p'} - c_p = \delta_{\text{CE}} b$. \square

If G is acting on (M, ω) , then Prop. 9.1 gives an important corollary.

Corollary 9.3. *If (M, ω) is a pre- n -plectic manifold equipped with a G -action such that \mathfrak{g} preserves ω then $\forall p \in M$ the linear map*

$$\begin{aligned} c_p^{\mathfrak{g}}: \Lambda^{n+1} \mathfrak{g} \rightarrow \mathbb{R} \\ x_1 \wedge \dots \wedge x_{n+1} \mapsto (-1)^n \zeta(n+1) \iota(v_1 \wedge \dots \wedge v_{n+1})\omega|_p, \end{aligned}$$

where v_i is the vector field associated to $x_i \in \mathfrak{g}$, is a degree $(n+1)$ -cocycle in $\text{CE}(\mathfrak{g})$. Moreover, if M is connected, then the cohomology class $[c_p^{\mathfrak{g}}]$ is independent of $p \in M$.

Proof. By assumption, \mathfrak{g} acts via local Hamiltonian vector fields, and $c_p^{\mathfrak{g}}$ is the pullback of the cocycle defined in Prop. 9.1 along the Lie algebra morphism v_- . \square

Remark 9.4. Note that if the G -orbit of a point p is of dimension smaller than $n+1$, then $[c_p^{\mathfrak{g}}] = 0$. In particular, this holds if p is a fixed point of the G -action.

The next proposition shows that the class $[c_p^{\mathfrak{g}}] \in H_{\text{CE}}^{n+1}(\mathfrak{g})$ is an obstruction to having a homotopy moment map.

Proposition 9.5. *If (M, ω) is a connected pre- n -plectic manifold, and M is equipped with a G -action which induces a homotopy moment map $\mathfrak{g} \rightarrow L_{\infty}(M, \omega)$, then*

$$[c_p^{\mathfrak{g}}] = 0$$

where $[c_p^{\mathfrak{g}}] \in H_{\text{CE}}^{n+1}(\mathfrak{g})$ is the cohomology class defined in Cor. 9.3.

Proof. By Def. 5.1 the homotopy moment map corresponds to structure maps f_1, \dots, f_n satisfying Eqs. (16) and (17). Since $|f_n| = 1 - n$, the map f_n takes values in $C^{\infty}(M)$. Let $p \in M$, and define

$$b(x_1, \dots, x_n) = (-1)^{n+1} f_n(x_1, \dots, x_n)|_p.$$

Clearly, $b \in \text{Hom}(\Lambda^n \mathfrak{g}, \mathbb{R})$. Eq. (17) then implies

$$(c_p^{\mathfrak{g}})(x_1, \dots, x_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} b([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}).$$

Hence, $c_p^{\mathfrak{g}} = \delta_{\text{CE}} b$. \square

9.2. Lifting \mathfrak{g} -actions to moment maps. Recall from Prop. 4.8 that there is a surjective (and strict) L_∞ -morphism

$$\pi: \text{Ham}_\infty(M, \omega) \rightarrow \mathfrak{X}_{\text{Ham}}(M)$$

which is simply the projection $(v, \alpha) \mapsto v$ in degree 0. Suppose we have a Lie group G acting on (M, ω) , such that the infinitesimal action of \mathfrak{g} is via Hamiltonian vector fields. Exhibiting a moment map for such an action means finding a lift

$$(36) \quad \begin{array}{ccc} & & \text{Ham}_\infty(M, \omega) \\ & \nearrow & \downarrow \pi \\ \mathfrak{g} & \xrightarrow{v_-} & \mathfrak{X}_{\text{Ham}}(M) \end{array}$$

in the category of L_∞ -algebras. Since \mathfrak{g} acts by Hamiltonian vector fields there always exists a (non-unique) degree zero *linear map*

$$\begin{aligned} \mathfrak{g} &\rightarrow \text{Ham}_\infty(M, \omega) \\ x &\mapsto (v_x, \phi(x)) \in \mathfrak{X}_{\text{Ham}}(M) \oplus \Omega_{\text{Ham}}^{n-1}(M) \end{aligned}$$

such that $d\phi(x) = -\iota_{v_x}\omega$. When does such a linear map lift to an L_∞ -morphism? Thm. 9.5 implies that it is necessary that the cohomology class $[c_p^{\mathfrak{g}}]$ vanish. The next theorem shows that when certain topological assumptions are satisfied, this is also sufficient.

Theorem 9.6. *Let (M, ω) be a connected pre- n -plectic manifold equipped with a G -action such that \mathfrak{g} acts via Hamiltonian vector fields. Let*

$$\phi: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$$

be any linear map such that $d\phi(x) = -\iota_{v_x}\omega$ for all $x \in \mathfrak{g}$. If $H_{\text{dR}}^i(M) = 0$ for $1 \leq i \leq n-1$ and $[c_p^{\mathfrak{g}}] = 0$, where $[c_p^{\mathfrak{g}}] \in H_{\text{CE}}^{n+1}(\mathfrak{g})$ is the cohomology class defined in Cor. 9.3, then there exists a homotopy moment map

$$(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$$

such that

$$f_1 = \phi.$$

Proof. Let $f_1 = \phi$, implying that $d(f_1(x)) = -\iota_{v_x}\omega$ for all $x \in \mathfrak{g}$. Notice that this equation is what is obtained allowing $k = 1$ in Eq. (16) (taking $f_0 = 0$). We now find recursively solutions for the equations appearing in (16).

Claim 1: *For every $2 \leq k \leq n+1$, if f_{k-1} satisfies Eq. (16) for $k-1$, then*

$$(37) \quad \sum_{1 \leq i < j \leq k} (-1)^{i+j+1} f_{k-1}([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k) - \varsigma(k) \iota(v_1 \wedge \dots \wedge v_k) \omega$$

is a closed $n+1-k$ -form for all $x_1, \dots, x_k \in \mathfrak{g}$.

To prove the claim we proceed as follows. We have

$$\begin{aligned}
 & d\iota(v_1 \wedge \cdots \wedge v_k)\omega \\
 &= (-1)^k \sum_{1 \leq i < j \leq k} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k)\omega \\
 &= (-1)^k \sum_{1 \leq i < j \leq k} (-1)^{i+j} \varsigma(k-1) \left(-(\delta_{\text{CE}} f_{k-2})([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k) \right. \\
 & \qquad \qquad \qquad \left. - df_{k-1}([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k) \right) \\
 &= \underbrace{(-1)^k \varsigma(k-1)}_{=\varsigma(k)} \left((-\delta_{\text{CE}}^2 f_{k-2})(x_1, \dots, x_k) \right. \\
 & \qquad \qquad \qquad \left. + d \sum_{1 \leq i < j \leq k} (-1)^{i+j+1} f_{k-1}([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k) \right)
 \end{aligned}$$

using Lemma 9.2 in the first equality, and in the second the fact that f_{k-1} satisfies Eq. (16) for $k-1$ as well as the definition of δ_{CE} . Since the Chevalley-Eilenberg differential δ_{CE} squares to zero, the claim follows.

Claim 2: For all $2 \leq k \leq n$, there exist $f_k: \Lambda^k \mathfrak{g} \rightarrow \Omega^{n-k}(M)$ satisfying Eq. (16) for k .

We prove Claim 2 by induction on k . The case $k=1$ holds, as seen earlier, with $f_1 = \phi$. We fix $2 \leq k \leq n$. By the induction assumption we are allowed to apply Claim 1 for k . The assumption $H^{n+1-k}(M) = 0$ implies that there exists $f_k: \Lambda^k \mathfrak{g} \rightarrow \Omega^{n-k}(M)$ such that $f_k(x_1, \dots, x_k)$ is a primitive for the $n+1-k$ -form (37), for all $x_1, \dots, x_k \in \mathfrak{g}$. Equivalently, f_k satisfies Eq. (16) for k , proving Claim 2.

In general, f_n will not satisfy Eq. (17). It will iff $h \in \text{Hom}(\Lambda^{n+1} \mathfrak{g}, C^\infty(M))$ vanishes, where

$$h(x_1, \dots, x_{n+1}) = (\delta_{\text{CE}} f_n)(x_1, \dots, x_{n+1}) + \varsigma(n+1) \iota(v_1 \wedge \cdots \wedge v_{n+1})\omega.$$

Now fix $p \in M$. We evaluate both summands of h at p , and obtain two elements of $\text{Hom}(\Lambda^{n+1} \mathfrak{g}, \mathbb{R})$: the first one is δ_{CE} -exact by construction, the second is equal to $\pm c_p^{\mathfrak{g}}$, hence it is δ_{CE} -exact by assumption. This means that there exists $b \in \text{Hom}(\Lambda^n \mathfrak{g}, \mathbb{R})$ such that $h|_p = \delta_{\text{CE}} b$. However by Claim 1 (for $k=n+1$) we know that $h(x_1, \dots, x_{n+1})$ is a closed zero form for all $x_1, \dots, x_{n+1} \in \mathfrak{g}$, and since M is connected this means that h lies in $\text{Hom}(\Lambda^{n+1} \mathfrak{g}, \mathbb{R})$. Hence

$$h = \delta_{\text{CE}} b \in \text{Hom}(\Lambda^{n+1} \mathfrak{g}, \mathbb{R}).$$

Replacing f_n by $f_n - b$ we therefore obtain a solution of Eq. (17), which still satisfies Eq. (16) for n as b takes values in the constants. We conclude that $f_1, \dots, f_{n-1}, f_n - b$ are the components of a homotopy moment map. \square

Remark 9.7. Note that G need not be finite-dimensional here; the theorem also applies to actions by locally exponential infinite-dimensional Lie groups. In Sec. 10, we consider a case in which G is such a group acting on a pre- n -plectic locally convex topological vector space.

Remark 9.8. The assumptions of Thm. 9.6 can be weakened; in fact, only particular components of $H_{\text{CE}}^\bullet(\mathfrak{g}) \otimes H_{\text{dR}}^\bullet(M)$ need to vanish [17].

9.3. Central n -extensions. If (M, ω) is a connected symplectic manifold, then Kostant's construction [23] gives a morphism of central extensions

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \widehat{\mathfrak{g}} & \longrightarrow & C^\infty(M) \\ \downarrow & & \downarrow \pi \\ \mathfrak{g} & \xrightarrow{v_-} & \mathfrak{X}_{\text{Ham}}(M) \end{array}$$

where $\widehat{\mathfrak{g}}$ is the central extension corresponding to the 2-cocycle $c_p^{\mathfrak{g}}$. This central extension is non-trivial iff there is no moment map which lifts the \mathfrak{g} -action.

Now we describe how these ideas generalize to higher cases. First we recall a theorem of Baez and Crans [3, Thm. 55]: Given a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ and a degree $(n+1)$ -cocycle $c: \Lambda^{n+1}\mathfrak{g} \rightarrow \mathbb{R}$, there exists a Lie n -algebra whose underlying complex is \mathfrak{g} in degree 0, \mathbb{R} in degree $1-n$ and 0 in all other degrees. The structure maps are trivial except in degree zero where we have:

$$\begin{aligned} l_2(x_1, x_2) &= [x_1, x_2] \\ l_{n+1}(x_1, \dots, x_{n+1}) &= c(x_1, \dots, x_{n+1}) \\ l_k &= 0 \quad \text{if } k \neq 2, k \neq n+1. \end{aligned}$$

We call this Lie n -algebra a **central n -extension** of \mathfrak{g} and denote it by $\widehat{\mathfrak{g}}_c$. If c and c' are two such cocycles which differ by a coboundary, then the corresponding Lie n -algebras are quasi-isomorphic (Cor. A.10). If $n=1$, then we recover the usual notion of central extension by setting $l_2 = [\cdot, \cdot] + c$. Let $\pi_{\mathfrak{g}}: \widehat{\mathfrak{g}}_c \rightarrow \mathfrak{g}$ denote the projection. It clearly lifts to a strict L_∞ -morphism.

Proposition 9.9. *The short exact sequence of complexes*

$$\mathbb{R}[n-1] \rightarrow \widehat{\mathfrak{g}}_c \xrightarrow{\pi_{\mathfrak{g}}} \mathfrak{g}$$

lifts to a strict exact sequence [34, Def. 9.3] in the category of L_∞ -algebras.

The following proposition is the higher analog of Kostant's construction in symplectic geometry for central extensions such as the Heisenberg Lie algebra.

Proposition 9.10. *Let (M, ω) be a connected pre- n -plectic manifold equipped with a G -action such that \mathfrak{g} acts via Hamiltonian vector fields and let $p \in M$. Assume $H_{\text{dR}}^k(M) = 0$ for $1 \leq k \leq n-1$. If $\widehat{\mathfrak{g}}$ is the central n -extension constructed from the $(n+1)$ -cocycle $c_p^{\mathfrak{g}}$ defined in Cor. 9.3, then there exists an L_∞ -morphism*

$$(f_i): \widehat{\mathfrak{g}} \rightarrow \text{Ham}_\infty(M, \omega)$$

such that the following diagram (strictly) commutes

$$(38) \quad \begin{array}{ccc} \widehat{\mathfrak{g}} & \xrightarrow{(f_i)} & \text{Ham}_\infty(M, \omega) \\ \pi_{\mathfrak{g}} \downarrow & & \downarrow \pi \\ \mathfrak{g} & \xrightarrow{v_-} & \mathfrak{X}_{\text{Ham}}(M) \end{array}$$

Proof. We shall produce maps f_1, \dots, f_n such that equalities given in Prop. A.9 are satisfied. Since \mathfrak{g} acts by Hamiltonian vector fields, there exists a linear map $\phi: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$ with $\phi(x) = (v_x, \alpha_x)$ such that $d\alpha_x = -\iota_{v_x}\omega$. The map

$$\begin{aligned} f_1(x) &= \phi(x) \quad \forall x \in \mathfrak{g} \\ f_1(r) &= (-1)^n r \in C^\infty(M) \quad \forall r \in \mathbb{R} \end{aligned}$$

gives a degree 0 chain map f_1 from the underlying complex of $\widehat{\mathfrak{g}}$ to that of $\text{Ham}_\infty(M, \omega)$.

We then proceed as we did in the first part of the proof of Thm. 9.6. Namely, since all closed k -forms have a primitive for $0 \leq k \leq n-1$, we inductively obtain maps $f_i: \Lambda^i \mathfrak{g} \rightarrow \Omega^{n-i}(M)$ for $i = 2, \dots, n$ such that Eq. (72) is satisfied. Let $b: \Lambda^n \mathfrak{g} \rightarrow \mathbb{R}$ be

$$b(x_1, \dots, x_n) = f_n(x_1, \dots, x_n)|_p,$$

and let $\tilde{f}_n = f_n - b$.

Since $db(x_1, \dots, x_n) = 0$ for all x_i , the map \tilde{f}_n also satisfies (72). It remains to show that Eq. (73) holds i.e. given $x_1, \dots, x_{n+1} \in \mathfrak{g}$, the function

$$\begin{aligned} C = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} \tilde{f}_n([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_{n+1}) \\ + (-1)^n c_p^{\mathfrak{g}}(x_1, \dots, x_{n+1}) - \varsigma(n+1) \iota(v_1 \wedge \dots \wedge v_{n+1}) \omega \end{aligned}$$

vanishes. Using Lemma 9.2 and Eq. (72) for the case $m = n$, we conclude that C is closed and therefore:

$$C = C(p) = (-1)^n c_p^{\mathfrak{g}}(x_1, \dots, x_{n+1}) - \varsigma(n+1) \iota(v_1 \wedge \dots \wedge v_{n+1}) \omega|_p = 0.$$

Hence, the collection $f_1, \dots, f_{n-1}, \tilde{f}_n$ gives the desired morphism, and it follows from the definition of f_1 that the diagram (38) commutes. \square

A more homotopy-theoretic and conceptual interpretation of the above proposition appears in Sec. 3.5 of [15].

Below we give some examples of what kinds of Lie n -algebras can be constructed in this way.

Example 9.11 (Heisenberg n -algebra). Let V be a finite-dimensional real vector space. A linear non-zero skew-symmetric form $\omega \in \Lambda^{n+1} V^*$ of degree $n+1$ induces a translation-invariant closed differential form on V . Therefore, (V, ω) is a pre- n -plectic manifold and V (seen as an abelian Lie algebra) acts on itself via translations. This gives a Lie algebra morphism $v_-: V \rightarrow \mathfrak{X}_{\text{Ham}}(V)$. Since ω is non-zero, the degree $(n+1)$ class $[c_p^V]$ is non-trivial. Hence, there is no homotopy moment map lifting the action of V . Let \widehat{V} be the associated central n -extension. Prop. 9.10 implies that \widehat{V} sits in a commuting diagram of L_∞ -algebras of the form (38).

Compare with Ex. 8.3, for which any linear action on (V, ω) admits a moment map.

Example 9.12 (String Lie 2-algebra). Let G be a compact connected simple Lie group, and let $\omega = \frac{1}{12} \langle \theta_L, [\theta_L, \theta_L] \rangle$ be the Cartan 3-form. As previously mentioned in Sec. 8.2, (G, ω) is a 2-plectic manifold, and the action of G on itself via conjugation lifts to a homotopy moment map. Clearly, ω is also preserved by the action of G on itself via left-translation, but the corresponding degree 3 class $[c_p^{\text{gl}}]$ is not trivial. (Indeed, $\langle \cdot, [\cdot, \cdot] \rangle$ is a generator of $H_{\text{CE}}^3(\mathfrak{g})$). The corresponding central 2-extension is the **string Lie 2-algebra** $\text{string}(\mathfrak{g})$. When $G = \text{Spin}(n)$, this Lie 2-algebra (or rather its integration) plays a very interesting role in a certain elliptic cohomology theory and in the theory of “spin structures” on loop spaces. (See, for example, Sec. 1 of [37] for a review.)

Since G is compact and simple we have $H_{\text{dR}}^1(G) \cong H_{\text{CE}}^1(\mathfrak{g}) = 0$. Hence, Prop. 9.10 implies that there is a commuting diagram of L_∞ -algebras:

$$\begin{array}{ccc} \text{string}(\mathfrak{g}) & \longrightarrow & \text{Ham}_\infty(G, \omega) \\ \pi^{\mathfrak{g}} \downarrow & & \downarrow \pi \\ \mathfrak{g} & \xrightarrow{v_-^{\text{left}}} & \mathfrak{X}_{\text{Ham}}(G) \end{array}$$

This result gives a nice conceptual interpretation to the relationship previously established in [4] between $\text{string}(\mathfrak{g})$ and $L_\infty(G, \omega)$.

10. MODULI SPACES OF FLAT CONNECTIONS

Here we consider homotopy moment maps on spaces of connections over higher-dimensional manifolds (see Thm. 10.7). Currently, our motivation for this example is simply to generalize the famous Atiyah-Bott construction [2] in symplectic geometry. Since our construction begins by considering an invariant polynomial in $S(\mathfrak{g}^\vee)^G$ of higher degree ≥ 2 , it is possible that some of these ideas could find application in certain topological field theories which generalize Chern-Simons theory.

10.1. Invariant polynomials. Given an integer $n \geq 1$, we consider the following data:

- a real, finite dimensional Lie algebra \mathfrak{g} equipped with a invariant polynomial $q \in S^{n+1}(\mathfrak{g}^\vee)^G$,
- a $(n+1)$ -dimensional compact, oriented manifold M , and
- a principal G -bundle $\pi: P \rightarrow M$, where G is any Lie group integrating \mathfrak{g} . The group G acts on the right of P via diffeomorphisms R_g .

We denote by

$$\hat{\xi}(p) = \frac{d}{dt} R_{\exp(t\xi)}(p)|_{t=0}$$

the infinitesimal generators of the action of G on P , for all $\xi \in \mathfrak{g}$. We say an invariant polynomial q is **non-degenerate** iff the map

$$\begin{aligned} \mathfrak{g} &\rightarrow S^n(\mathfrak{g}^\vee) \\ x &\mapsto \iota_x q \end{aligned}$$

is injective.

Example 10.1. If G is a matrix group, then the symmetrized (real) trace gives obvious examples of invariant polynomials. In particular, for $G = \mathrm{SU}(N)$, we define:

$$q_k(x_1, \dots, x_k) = -\frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \mathrm{Re} \mathrm{Tr}(x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)}) \quad \forall x_i \in \mathfrak{su}(N).$$

It is well known that the polynomial q_2 gives a real inner product on $\mathfrak{su}(N)$, but more generally, one can show for $G = \mathrm{SU}(2)$ that every q_{2n} is non degenerate, for $n > 0$. Consider $\{e_i\}$ the basis of $G = \mathrm{SU}(2)$ given by

$$e_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad e_2 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad e_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The identity

$$(39) \quad e_i e_j = -\frac{1}{4} \delta_{ij} I + \frac{1}{2} \sum_k \varepsilon_{ijk} e_k,$$

where ε_{ijk} is totally skew and $\varepsilon_{123} = 1$, implies that $e_i e_j^{2n} = (-\frac{1}{4})^n e_i$. Therefore

$$e_i e_j^{2n+1} = (-\frac{1}{4})^{n+1} \delta_{ij} I + \frac{1}{2} (-\frac{1}{4})^n \sum_k \varepsilon_{ijk} e_k.$$

In particular

$$(40) \quad q_{2(n+1)}(e_i, e_j, \dots, e_j) = -\mathrm{Re} \mathrm{Tr}(e_i e_j^{2n+1}) = -2(-\frac{1}{4})^{n+1} \delta_{ij}.$$

This enables to show that $q_{2(n+1)}$ is non-degenerate. Indeed, suppose there exists $x \in \mathfrak{su}(2)$ such that

$$q_{2(n+1)}(x, y_1, \dots, y_{2n+1}) = 0 \quad \forall y_i \in \mathfrak{su}(2).$$

Write $x = \sum_i x^i e_i$, then (40) means that $x_i = q_{2(n+1)}(x, e_i, \dots, e_i) = 0$ for all i implies that $x = 0$. So $q_{2(n+1)}$ is non-degenerate. Note that this is not true for all q_k . In fact, Eq. (39) shows that $q_3 = 0$.

10.2. The gauge group action. A **connection** on P is a \mathfrak{g} -valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ satisfying

$$A(\hat{\xi}) = \xi, \quad R_g^* A = \text{Ad}_{g^{-1}} A$$

for all $\xi \in \mathfrak{g}$ and $g \in G$. The set \mathcal{A} of all connections on P is an affine space modeled on the vector space $(\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$, where the first factor denotes the 1-forms on P annihilated by vectors tangent to the fibers. The left action of G on $\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g}$ is

$$g \cdot (\alpha \otimes \xi) = R_g^* \alpha \otimes \text{Ad}_g \xi.$$

The **gauge group** \mathcal{G} of P is the group of smooth maps $f: P \rightarrow G$ satisfying

$$R_g^* f(p) = g^{-1} f(p) g \quad \forall g \in G.$$

Such a map f can be identified with a G -equivariant map $\phi: P \rightarrow P$ covering id_M where

$$\phi(p) = R_{f(p)}(p).$$

The gauge group acts on the space of connections \mathcal{A} from the left:

$$(41) \quad f \cdot A = \text{Ad}_f A + (f^{-1})^* \theta_L,$$

where $\theta_L \in \Omega^1(G, \mathfrak{g})$ is the left invariant Maurer-Cartan form on G and f^{-1} is the composition of f with the inversion on G . If ϕ is the bundle automorphism associated to f , then the action is simply $\phi \cdot A = (\phi^{-1})^* A$.

To obtain the infinitesimal analog of the above, we consider maps $X: P \rightarrow \mathfrak{g}$ satisfying

$$R_g^* X(p) = \text{Ad}_{g^{-1}} X(p) \quad \forall g \in G.$$

The space of all such maps forms the **Lie algebra of infinitesimal gauge transformations** $\text{Lie}(\mathcal{G})$. This plays the role of the Lie algebra associated to \mathcal{G} .

Remark 10.2. Indeed, \mathcal{G} is a locally exponential Lie group modeled on the Lie algebra $\text{Lie}(\mathcal{G})$ [42, Thm. 1.11]. This means that for each $X \in \text{Lie}(\mathcal{G})$ the initial value problem

$$\gamma(0) = e_G, \quad \gamma(t)^{-1} \cdot \gamma'(t) = X$$

has a solution $\gamma_X \in C^\infty(\mathbb{R}, \mathcal{G})$, and there exists a unique smooth exponential map

$$\exp: \text{Lie}(\mathcal{G}) \rightarrow \mathcal{G}, \quad X \mapsto \gamma_X(1)$$

and an open neighborhood $0 \in W \subset \text{Lie}(\mathcal{G})$ such that $\exp|_W$ is a diffeomorphism onto some open neighborhood of the identity e_G .

Differentiating the action (41) gives an action of $\text{Lie}(\mathcal{G})$ on \mathcal{A} . Specifically, given $X \in \text{Lie}(\mathcal{G})$ we define the fundamental vector field $V_X: \mathcal{A} \rightarrow (\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$ by

$$(42) \quad V_X(A) = \frac{d}{dt} (\exp(-tX) \cdot A) |_{t=0}.$$

Note that the assignment $X \mapsto V_X$ is a Lie algebra morphism from $\text{Lie}(\mathcal{G})$ to $\mathfrak{X}(\mathcal{A})$. Also, a simple calculation shows

$$V_X(A) = d_A X.$$

where $d_A: (\Omega_{\text{hor}}^\bullet(P) \otimes \mathfrak{g})^G \rightarrow (\Omega_{\text{hor}}^{\bullet+1}(P) \otimes \mathfrak{g})^G$ is the **(exterior) covariant derivative**

$$d_A \alpha = d\alpha + [A, \alpha].$$

10.3. Closed forms from invariant polynomials. An invariant polynomial $q \in S^{n+1}(\mathfrak{g}^\vee)^G$ gives a constant, hence closed, $(n+1)$ -form on the space of connections \mathcal{A} . To see this, first consider the following $(n+1)$ -form on P :

$$q(\alpha_1, \dots, \alpha_{n+1}) \in \Omega^{n+1}(P),$$

where each α_i is in $(\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$. This form clearly vanishes when contracted with any vertical vector on P . Moreover, the Ad invariance of q combined with the G invariance of α_i implies that $q(\alpha_1, \dots, \alpha_{n+1})$ is invariant under the action of G on P . Therefore, $q(\alpha_1, \dots, \alpha_{n+1})$ is **basic** i.e. it corresponds to the pullback of a unique $(n+1)$ -form on M along $\pi: P \rightarrow M$. We “abuse notation” by also denoting this $(n+1)$ -form on M as $q(\alpha_1, \dots, \alpha_{n+1})$. By integration, we then obtain a closed $(n+1)$ -form on \mathcal{A} :

$$(43) \quad \omega(\alpha_1, \dots, \alpha_{n+1})|_{\mathcal{A}} = \int_M q(\alpha_1, \dots, \alpha_{n+1}) \quad \forall \alpha_i \in T_A \mathcal{A} = (\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G.$$

The following proposition shows that, for some cases, ω is in fact n -plectic.

Proposition 10.3. *If the invariant polynomial $q \in S^{n+1}(\mathfrak{g}^\vee)^G$ is non-degenerate, then ω (43) is an n -plectic structure on \mathcal{A} .*

Proof. Given $\beta \in (\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$ such that

$$\int_M q(\beta, \alpha_2, \dots, \alpha_{n+1}) = 0 \quad \forall \alpha_i \in (\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G,$$

we assume, in order to lead to a contradiction, that there exists $p \in P$ such that $\beta|_p \neq 0$. Let $U \subseteq M$ be a chart containing $y = \pi(p)$ admitting a trivialization $\tau: \pi^{-1}(U) \xrightarrow{\sim} U \times G$ such that $\tau(p) = (y, e)$. Let x^1, \dots, x^{n+1} be coordinates on U . Working locally over $\pi^{-1}(U)$, and implicitly using the trivialization, we write $\beta = \sum_{i=1}^{n+1} \beta_i d\pi^* x^i$ where $\beta_i: U \times G \rightarrow \mathfrak{g}$ satisfies $\beta_i(x, g) = \text{Ad}_{g^{-1}} \beta_i(x, e)$. By our assumption, there exists an i such that $\beta_i(y, e) \neq 0$. Without loss of generality, we may further assume $i = 1$.

Since q is non-degenerate, there exists $\xi_2, \dots, \xi_{n+1} \in \mathfrak{g}$ such that $q(\beta_1(y, e), \xi_2, \dots, \xi_{n+1}) > 0$. Hence, there exists a smaller neighborhood $V \subseteq U$ containing y such that

$$q(\beta_1(x, e), \xi_2, \dots, \xi_{n+1}) > 0 \quad \forall x \in V.$$

Define \mathfrak{g} -valued maps f_2, \dots, f_{n+1} on $\pi^{-1}(V)$ by

$$f_i(x, g) = \text{Ad}_{g^{-1}} \xi_i.$$

Finally, let $\varphi: M \rightarrow [0, 1]$ be a “bump function” whose support is contained in V .

Using all of this, we can define global \mathfrak{g} -valued 1-forms $\alpha_2, \dots, \alpha_{n+1}$ on P by

$$\alpha_i = \pi^*(\varphi) d\pi^*(x^i) \otimes f_i.$$

By construction, each α_i is in $(\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$, and therefore we have a contradiction:

$$0 = \int_M q(\beta, \alpha_2, \dots, \alpha_{n+1}) = \int_{\text{supp } \varphi} q(\beta_1(x, e), \xi_2, \dots, \xi_{n+1}) dx^1 dx^2 \dots dx^{n+1} > 0$$

This implies that β is zero. Hence, ω is non-degenerate. \square

10.4. The moment map. From here on, we assume the following:

- The principal G -bundle $P \rightarrow M$ admits a flat connection.

We equip the space of connections \mathcal{A} with the closed $(n+1)$ -form ω given in Eq. (43).

We begin by considering the linear map $\mu: \text{Lie}(\mathcal{G}) \rightarrow \Omega^{n-1}(\mathcal{A})$ defined as

$$(44) \quad \mu(X)(\alpha_1, \dots, \alpha_{n-1})|_A = \int_M q(F_A, \alpha_1, \dots, \alpha_{n-1}, X),$$

for all $X \in \text{Lie}(\mathcal{G})$, $A \in \mathcal{A}$ and $\alpha_i \in T_A\mathcal{A}$. Here F_A is the curvature of the connection A :

$$F_A = dA + \frac{1}{2}[A, A] \in (\Omega_{\text{hor}}^2(P) \otimes \mathfrak{g})^G.$$

The curvature is horizontal and it satisfies $R_g^*F_A = \text{Ad}_{g^{-1}}F_A$. So, the $(n+1)$ -form $q(F_A, \alpha_1, \dots, \alpha_{n-1}, X)$ on P descends to a unique $(n+1)$ -form on M .

Proposition 10.4. *The map $\mu: \text{Lie}(\mathcal{G}) \rightarrow \Omega^{n-1}(\mathcal{A})$ defined by Eq. (44) is \mathcal{G} -equivariant, i.e.*

$$(f^{-1})^*\mu(X) = \mu(\text{Ad}_f X) \quad \forall f \in \mathcal{G}, X \in \text{Lie}(\mathcal{G}).$$

Proof. Let $A \in \mathcal{A}$ and $\alpha_1, \dots, \alpha_{n-1} \in T_A\mathcal{A}$. We have

$$(45) \quad (f^*\mu(X))(\alpha_1, \dots, \alpha_{n-1})|_A = \mu(X)(f_*\alpha_1, \dots, f_*\alpha_{n-1})|_{f \cdot A} = \int_M q(F_{f \cdot A}, f_*\alpha_1, \dots, f_*\alpha_{n-1}, X).$$

A straightforward computation shows that $F_{f \cdot A} = \text{Ad}_f F_A$. It is also not difficult to show that the differential f_* of the map $A \mapsto f \cdot A$ is $f_*\alpha_i = \text{Ad}_f \alpha_i$. Hence, from the Ad_f -invariance of q , we see that the right-hand side of Eq. (45) is

$$\mu(\text{Ad}_{f^{-1}} X)(\alpha_1, \dots, \alpha_{n-1})|_A.$$

□

Next, we show the image of μ lies in Hamiltonian forms. The associated Hamiltonian vector fields are those induced by the infinitesimal gauge transformations (42).

Proposition 10.5. *If $X \in \text{Lie}(\mathcal{G})$, then $\mu(X)$ is a Hamiltonian $(n-1)$ -form with Hamiltonian vector field V_X , where*

$$V_X|_A = d_A X = dX + [A, X] \quad \forall A \in \mathcal{A}.$$

To prove this, we will use the following lemma.

Lemma 10.6. *If $q \in S^k(\mathfrak{g}^\vee)^G$ is a degree k invariant polynomial, $A \in \mathcal{A}$ is a connection, and $\beta_1, \dots, \beta_k \in (\Omega_{\text{hor}}^\bullet(P) \otimes \mathfrak{g})^G$ are forms with $|\beta_1| + |\beta_2| + \dots + |\beta_k| = n$, then*

$$\sum_{i=1}^k (-1)^{|\beta_1| + \dots + |\beta_{i-1}|} \int_M q(\beta_1, \dots, d_A \beta_i, \dots, \beta_k) = 0,$$

where the above sign for $i=1$ is defined to be $+1$.

Proof. By replacing $d_A \beta_i$ by $d\beta_i + [A, \beta_i]$ for all i in

$$(46) \quad \sum_{i=1}^k (-1)^{|\beta_1| + \dots + |\beta_{i-1}|} q(\beta_1, \dots, d_A \beta_i, \dots, \beta_k)$$

we rewrite the above as the sum of two basic $(n+1)$ -forms on P :

$$d(q(\beta_1, \beta_2, \dots, \beta_k)) + \left(\sum_{i=1}^k (-1)^{|\beta_1| + \dots + |\beta_{i-1}|} q(\beta_1, \dots, [A, \beta_i], \dots, \beta_k) \right),$$

where the summation over i is, in fact, zero by the infinitesimal G -invariance of q . Hence, (46) descends to an exact $(n+1)$ -form on M , and so its integral vanishes by Stokes' theorem. □

Proof of Prop. 10.5. We need to show $d\mu(X) = -\iota(V_X)\omega$ for all $X \in \text{Lie}(\mathcal{G})$. Let $A \in \mathcal{A}$. Given tangent vectors $\alpha_1, \dots, \alpha_n \in T_A\mathcal{A}$, we denote by the same symbols their extension to constant vector fields on \mathcal{A} . Hence, $[\alpha_i, \alpha_j] = 0$ for all i and j , and so the de Rham differential becomes

$$d\mu(X)(\alpha_1, \dots, \alpha_n) = \sum_i (-1)^{i+1} \mathcal{L}_{\alpha_i}(\mu(X)(\alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n)).$$

The identity

$$\frac{d}{dt} F_{A+t\alpha_i} \Big|_{t=0} = d_A \alpha_i,$$

combined with Lemma 10.6 imply that

$$\begin{aligned} d\mu(X)(\alpha_1, \dots, \alpha_n) &= \sum_i (-1)^{i+1} \int_M q(d_A \alpha_i, \alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n, X) \\ &= -(-1)^n \int_M q(\alpha_1, \dots, \alpha_n, d_A X) \\ &= -(\iota(V_X)\omega)(\alpha_1, \dots, \alpha_n). \end{aligned}$$

□

The main result of this section is the following theorem. Its proof features the obstruction theory developed in Sec. 9.

Theorem 10.7. *There exists a homotopy moment map*

$$(f_i): \text{Lie}(\mathcal{G}) \rightarrow L_\infty(\mathcal{A}, \omega)$$

lifting the action of $\text{Lie}(\mathcal{G})$ on (\mathcal{A}, ω) such that

$$f_1(X)(\alpha_1, \dots, \alpha_{n-1})|_A = \mu(X)(\alpha_1, \dots, \alpha_{n-1})|_A = \int_M q(F_A, \alpha_1, \dots, \alpha_{n-1}, X)$$

for all $A \in \mathcal{A}$ and $\alpha_i \in T_A\mathcal{A}$.

Proof. By Remark 9.7, the theorem is proved if we can show that the assumptions listed in Thm. 9.6 are satisfied by taking the linear map ϕ (defined there) to be μ . Let $A_0 \in \mathcal{A}$ be a flat connection. We identify \mathcal{A} with the locally convex topological vector space $(\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$ so that A_0 corresponds to the origin. The Poincare Lemma holds [5, Lem. 1.4.1], and hence the de Rham cohomology of \mathcal{A} is \mathbb{R} in degree 0, and trivial in all higher degrees. Next, observe that the Lie algebra cocycle

$$c_{A_0}^{\text{Lie}(\mathcal{G})}(X_1, \dots, X_{n+1}) = \pm \omega(V_{X_1}, \dots, V_{X_{n+1}})|_{A_0} = \int_M q(d_{A_0} X_1, \dots, d_{A_0} X_{n+1}).$$

introduced in Cor. 9.3 is trivial. Indeed, Lemma 10.6 implies that

$$\int_M q(d_{A_0} X_1, \dots, d_{A_0} X_{n+1}) = \sum_{i=2}^{n+1} \pm \int_M q(X_1, \dots, d_{A_0}^2 X_i, \dots, d_{A_0} X_{n+1}),$$

and, since A_0 is flat, we have $d_{A_0}^2 X_i = [F_{A_0}, X_i] = 0$. □

10.5. Reduction. We equip (\mathcal{A}, ω) with a moment map given by Thm. 10.7, and now describe a type of Marsden-Weinstein reduction. As we shall see, when ω is non-degenerate, the quotient of a zero-level set gives a pre- n -plectic structure on the moduli space of flat connections.

We denote by $\mathcal{A}_{\text{flat}} \subset \mathcal{A}$ the set of flat connections. If A is a flat connection, then the tangent space $T_A\mathcal{A}_{\text{flat}}$ consists of all vectors $\alpha \in T_A\mathcal{A}$ that satisfy $d_A \alpha = d\alpha + [A, \alpha] = 0$. We also consider the following “zero level set” of μ :

$$\mathcal{C} = \{A \in \mathcal{A} \mid \mu(X)|_A = 0 \text{ for all } X \in \text{Lie}(\mathcal{G})\}.$$

The \mathcal{G} -action restricts to $\mathcal{A}_{\text{flat}}$, and by Prop. 10.4, it also restricts to \mathcal{C} . It follows from the definition of μ that

$$\mathcal{A}_{\text{flat}} \subseteq \mathcal{C}.$$

At least in certain cases, the two subspaces are equal, as the following proposition demonstrates.

Proposition 10.8. *Let $q \in S^{n+1}(\mathfrak{g}^\vee)^G$ be the invariant polynomial used in constructing $\omega \in \Omega^{n+1}(\mathcal{A})$. If q is non-degenerate, then $\mathcal{A}_{\text{flat}} = \mathcal{C}$.*

Proof. Let $A \in \mathcal{C}$ and assume, in order to lead to a contradiction, that there exists $p \in P$ such that $F_A|_p \neq 0$. We proceed as we did in the proof of Prop. 10.3, replacing there the 1-form β with the 2-form F_A . The non-degeneracy allows us to construct 1-forms $\alpha_1, \dots, \alpha_{n-1} \in (\Omega_{\text{hor}}^1(P) \otimes \mathfrak{g})^G$ and element $X \in \text{Lie}(\mathcal{G})$ such that

$$\int_M q(F_A, \alpha_1, \dots, \alpha_{n-1}, X) > 0.$$

Hence, we have $\mu(X)(\alpha_1, \dots, \alpha_{n-1})|_A > 0$, which contradicts $A \in \mathcal{C}$. \square

Next, we consider pre- n -plectic structures induced on $\mathcal{A}_{\text{flat}}$ and \mathcal{C} .

Proposition 10.9. *If $\iota: \mathcal{A}_{\text{flat}} \hookrightarrow \mathcal{A}$ is the inclusion, then*

$$V_X \in \ker \iota^* \omega$$

for all $X \in \text{Lie}(\mathcal{G})$.

Proof. Let $X \in \text{Lie}(\mathcal{G})$ and $A \in \mathcal{A}_{\text{flat}}$. Fix $\alpha_2, \dots, \alpha_{n+1} \in T_A \mathcal{A}_{\text{flat}}$. We have the equalities

$$\omega(V_X, \alpha_2, \dots, \alpha_{n+1})|_A = \omega(d_A X, \alpha_2, \dots, \alpha_{n+1}) = \int_M q(d_A X, \alpha_2, \dots, \alpha_{n+1}).$$

Hence, Lemma 10.6 implies that

$$\omega(V_X, \alpha_2, \dots, \alpha_{n+1})|_A = \sum_{i=2}^{n+1} \pm \int_M q(X, \dots, d_A \alpha_i, \dots, \alpha_{n+1}).$$

The right-hand side of the above is zero, since $d_A \alpha_i = 0$ for all α_i . \square

Recall that Prop. 10.5 implies that $\iota^* \omega$ is \mathcal{G} -invariant. This fact combined with the above two propositions gives the following:

Corollary 10.10. *If the quotient space $\mathcal{A}_{\text{flat}}/\mathcal{G}$ is a smooth manifold, then it carries a closed $(n+1)$ -form induced by ω . In addition, if ω is non-degenerate, then $\mathcal{A}_{\text{flat}}/\mathcal{G} = \mathcal{C}/\mathcal{G}$.*

Unfortunately, if ω is non-degenerate, then it does *not* follow that the $(n+1)$ -form on \mathcal{C}/\mathcal{G} is non-degenerate, as this example shows.

Example 10.11. We consider the simplest possible “higher” case. Let M be of dimension 3, $G = \mathbb{R}$, and P the trivial bundle $\mathbb{R} \times M \rightarrow M$. Let $p: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be just the multiplication of three numbers; it is clearly a non-degenerate invariant polynomial on the Lie algebra.

Since the bundle is trivial, we can write our data as

- $\mathcal{A} = \Omega^1(M)$, a vector space
- $\mathcal{G} = C^\infty(M)$
- $\text{Lie}(\mathcal{G}) = C^\infty(M)$
- $V_X = dX$ for all $X \in \text{Lie}(\mathcal{G})$
- $\omega(\alpha_1, \alpha_2, \alpha_3)|_A = \int_M \alpha_1 \wedge \alpha_2 \wedge \alpha_3$

Clearly $\mathcal{A}_{flat} = \Omega_{closed}^1(M)$, again a vector space, and $\mathcal{A}_{flat}/\mathcal{G} = H^1(M)$.

The induced 3-form on $H^1(M)$ is the just the evaluation on the fundamental class of M :

$$H^1(M) \otimes H^1(M) \otimes H^1(M) \mapsto \mathbb{R}, \quad a_1 \otimes a_2 \otimes a_3 \mapsto \langle a_1 \wedge a_2 \wedge a_3, [M] \rangle.$$

Therefore it is non-degenerate iff for all $a_1 \in H^1(M)$,

$$(47) \quad a_1 a_2 a_3 = 0 \text{ for all } a_2, a_3 \in H^1(M) \Rightarrow a_1 = 0.$$

In general, condition (47) is not satisfied. Recall that the pairing $H^1(M) \otimes H^2(M) \rightarrow H^3(M) \cong \mathbb{R}$ is non-degenerate by Poincaré duality. Hence, the non-degeneracy condition (47) is satisfied iff the product map

$$H^1(M) \otimes H^1(M) \rightarrow H^2(M)$$

is surjective.

So, for example, if $M = S^2 \times S^1$, then condition (47) is not satisfied, and indeed $H^1(S^2 \times S^1) \cong \mathbb{R}$ must carry the zero 3-form. When $M = S^1 \times S^1 \times S^1$ is the torus, condition (47) is satisfied, and $H^1(M) \cong \mathbb{R}^3$ carries a constant volume form.

11. LOOP SPACES

In this section, we show how homotopy moment maps for G -actions on pre-2-plectic manifolds (M, ω) can be transgressed to ordinary moment maps on the associated pre-symplectic loop space LM (see Thm. 11.2). Motivation for studying such actions arises, for example, in topological field theory. There one can consider a group of symmetries G acting on a “target space” M equipped with a closed form ω . The form can be transgressed to a mapping space $\text{Map}(X, M)$ i.e. the “space of fields”. The induced G -action is then defined “point-wise”. The results of this section could be interpreted as an elementary example of this process for the case $X = S^1$. Roughly speaking, this demonstrates how the higher symplectic geometry on M can interact with the ordinary geometry on $\text{Map}(X, M)$.

11.1. Actions on loop spaces. For any manifold M , the free loop space LM , i.e. the space of smooth loops

$$LM = C^\infty(S^1, M),$$

is an infinite-dimensional Fréchet manifold ([5], [40] and [30]). The tangent space at $\gamma \in LM$ can be identified with global sections of the pullback of TM along $\gamma: S^1 \rightarrow M$, i.e.

$$T_\gamma LM = C^\infty(S^1, \gamma^* TM)$$

There is a degree -1 chain map

$$\ell: \Omega^\bullet(M) \rightarrow \Omega^{\bullet-1}(LM)$$

called **transgression**. Explicitly, it sends a k -form α on M to the $(k-1)$ -form α^ℓ on LM given by the formula

$$\alpha^\ell|_\gamma(v_1, \dots, v_{k-1}) = \int_0^{2\pi} \alpha(v_1, \dots, v_{k-1}, \dot{\gamma})|_{\gamma(s)} ds \quad \forall \gamma \in LM, \quad \forall v_1, \dots, v_{k-1} \in T_\gamma LM.$$

Since transgression commutes with the de Rham differential, any pre- n -plectic structure ω on M gives a pre- $(n-1)$ -plectic structure ω^ℓ on LM .

Suppose M is a manifold equipped with a G -action $G \times M \rightarrow M$. This induces a “point-wise” action $G \times LM \rightarrow LM$ given by:

$$(g \cdot \gamma)(s) = g \cdot \gamma(s) \quad \forall g \in G, \quad \forall \gamma \in LM.$$

Given an element $x \in \mathfrak{g}$ and a loop $\gamma \in LM$, we obtain a smooth path in LM

$$\mathbb{R} \ni t \mapsto \exp(-xt) \cdot \gamma,$$

and an action of \mathfrak{g} on LM via the fundamental vector field

$$v_x^\ell|_\gamma = \frac{d}{dt} \exp(-xt) \cdot \gamma \Big|_{t=0} \quad \forall \gamma \in LM.$$

We then observe, by differentiation, that the fundamental vector field v_x^ℓ on LM evaluated at a point $\gamma \in LM$ is just γ^*v_x (a section of $\gamma^*TM \rightarrow S^1$), where v_x is the fundamental vector field on M associated to x .

More generally, restricting vector fields on M to loops in M gives us a map

$$\begin{aligned} \Gamma(TM) &\rightarrow \Gamma(TLM) \\ v &\mapsto v^\ell, \end{aligned}$$

defined by $v^\ell|_\gamma = \gamma^*v$. That v^ℓ is a smooth vector field with respect to the induced smooth structure on TLM follows from the fact that it is the composition of two smooth maps. The first of these is a map $LM \rightarrow LTM$ given by $\gamma \mapsto v \circ \gamma$ [40, Thm. 3.27]. The second map is the natural diffeomorphism of vector bundles $LTM \cong TLM$ covering the identity on LM [40, Thm. 4.2].

11.2. Actions on pre-symplectic loop spaces. Suppose (M, ω) is a pre-2-plectic manifold. Then (LM, ω^ℓ) is a pre-symplectic manifold. We have:

Proposition 11.1. *If α is a Hamiltonian 1-form with Hamiltonian vector field v , then the vector field v^ℓ is Hamiltonian for the function $\alpha^\ell: LM \rightarrow \mathbb{R}$.*

Proof. By assumption we have $d\alpha = -\iota(v)\omega$, and we wish to show

$$d\alpha^\ell = -\iota(v^\ell)\omega^\ell.$$

Let $\gamma \in LM$. For all $u \in T_\gamma LM$, we have

$$\begin{aligned} (\iota_{v^\ell}\omega^\ell)(u) &= \int_0^{2\pi} \omega(v^\ell, u, \dot{\gamma})|_{\gamma(s)} ds \\ &= - \int_0^{2\pi} d\alpha(u, \dot{\gamma})|_{\gamma(s)} ds \\ &= -(d\alpha)^\ell(u) \end{aligned}$$

□

Now suppose that (M, ω) is equipped with a G -action and with a homotopy moment map $\mathfrak{g} \rightarrow L_\infty(M, \omega)$. Let $f_1: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^1(M)$, and $f_2: \mathfrak{g} \otimes \mathfrak{g} \rightarrow C^\infty(M)$ be the corresponding structure maps for this moment map. For $x \in \mathfrak{g}$, the vector field v_x is Hamiltonian for the 1-form $\alpha_x = f_1(x)$. The 2-form ω^ℓ on LM is G -invariant, and the v_x^ℓ are Hamiltonian. The next theorem says that the homotopy moment map on (M, ω) transgresses to an ordinary moment map for the action of G on (LM, ω^ℓ)

Theorem 11.2. *If (M, ω) is a pre-2-plectic manifold equipped with a G -action and a homotopy moment map $f: \mathfrak{g} \rightarrow L_\infty(M, \omega)$, then*

$$\begin{aligned} \psi: \mathfrak{g} &\rightarrow C^\infty(LM) \\ x &\mapsto (f_1(x))^\ell \end{aligned}$$

is a moment map for the induced action of G on the pre-symplectic loop space (LM, ω^ℓ)

Proof. To prove the theorem, it is sufficient to show that ψ is a Lie algebra morphism.

The bracket of the functions $\psi(x)$ and $\psi(y)$ evaluated at $\gamma \in LM$ is:

$$\begin{aligned} \{\psi(x), \psi(y)\} |_\gamma &= \int_0^{2\pi} \omega(v_x, v_y, \dot{\gamma})|_{\gamma(s)} ds \\ &= \int_0^{2\pi} \iota_\gamma l_2(f_1(x), f_1(y))|_{\gamma(s)} ds \\ &= (l_2(f_1(x), f_1(y)))^\ell |_\gamma, \end{aligned}$$

where l_2 is the bi-linear bracket for the Lie 2-algebra $L_\infty(M, \omega)$. On the other hand, $\psi([x, y])$ is, by definition, equal to the transgression of the 1-form $f_1([x, y])$. The definition of an L_∞ -morphism implies that

$$(48) \quad l_2(f_1(x), f_1(y)) - f_1([x, y]) = df_2(x, y).$$

Since $df_2(x, y)$ is an exact 1-form, Stokes theorem implies for all $\gamma \in LM$:

$$(d(f_2(x, y)))^\ell = \int_0^{2\pi} \iota_\gamma d(f_2(x, y))|_{\gamma(s)} ds = 0$$

Hence applying the transgression operator ℓ to Eq. (48) we obtain

$$\{\psi(x), \psi(y)\} - \psi([x, y]) = 0.$$

□

Remark 11.3. The map

$$\Omega_{\text{Ham}}^1(M) \rightarrow C_{\text{Ham}}^\infty(LM), \quad \alpha \mapsto \alpha^\ell$$

is well-defined by Prop. 11.1 and preserves (binary) brackets, as shown in the proof of Thm. 11.2. Therefore there is a strict L_∞ -morphism from $L_\infty(M, \omega)$ to $L_\infty(LM, \omega^\ell)$, whose only non-vanishing component is the map $\alpha \mapsto \alpha^\ell$. Hence, given a homotopy moment map for (M, ω) , composing with the above L_∞ -morphism we get a homotopy moment map for (LM, ω^ℓ) . In [17] we extend Thm. 11.2 further.

12. RELATION TO OTHER WORK

12.1. Other notions of moment map. Let M be a manifold endowed with a closed $n+1$ -form ω , and G a Lie group acting on M preserving ω .

A **multimomentum map** in the sense of Cariñena-Crampin-Ibort [9, Sec. 4.2] is a map $f_1: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$ satisfying

$$-\iota_{v_x} \omega = d(f_1(x)) \quad \text{for all } x \in \mathfrak{g}.$$

Such maps are called **covariant momentum maps** in [21]; they are used there to study symmetries in classical field theories. Hence, if (f_k) is a homotopy moment map, then its first component is a multimomentum/covariant momentum map in the above sense.

Madsen-Swann [26, Sec. 3] consider the **n -th Lie kernel** P_n (a \mathfrak{g} -submodule of $\Lambda^n \mathfrak{g}$), and define a **multi-moment map** as an equivariant map $\nu: M \rightarrow P_n^*$ such that $\iota(v_p)\omega = d(\nu^*p)$ for all $p \in P_n$. It is clear from Eq. (16) (for $k=n$), that if $(f_i): \mathfrak{g} \rightarrow L_\infty(M, \omega)$ is an equivariant homotopy moment map, then the formula $\nu^*p = -\zeta(n)f_n(p)$ for all $p \in P_n$ defines a multi-moment map ν .

For the sake of clarity we spell out the case $n=2$, which was worked out first in [27, Sec. 2]. In that case $P_2 = \ker([\cdot, \cdot]: \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g})$. Notice that if $x, y \in \mathfrak{g}$ are commuting elements, i.e. $x \wedge y \in P_2$, then the invariance of ω under G implies that $\iota(v_x \wedge v_y)\omega$ is a closed 1-form. A multi-moment map is an equivariant map $\nu: M \rightarrow P_2^*$ such that $\iota(v_p)\omega = \sum_i \iota(v_{x_i} \wedge v_{y_i})\omega$ is exact with primitive ν^*p , for any $p = \sum_i x_i \wedge y_i \in P$.

12.2. Actions on Courant algebroids. Recall that a **Courant algebroid** consists of a vector bundle $E \rightarrow M$ with a non-degenerate symmetric pairing on the fibers, a bilinear bracket $\llbracket \cdot, \cdot \rrbracket$ on $\Gamma(E)$, and a bundle map $\rho: E \rightarrow TM$ satisfying certain conditions, see for instance [35, Def. 4.2]. Courant algebroids appear naturally in the study of Dirac and generalized complex structures.

Let (M, ω) be a pre-2-plectic manifold. There is a Courant algebroid associated to the closed 3-form ω , namely $TM \oplus T^*M$ with the natural pairing $\langle X + \xi, Y + \eta \rangle = \xi(Y) + \eta(X)$, the projection ρ onto the first factor, and bracket

$$\llbracket X + \xi, Y + \eta \rrbracket_\omega = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_Y \iota_X \omega.$$

This Courant algebroid is sometimes called the ω -twisted Courant algebroid, and we will denote it by $(TM \oplus T^*M)_\omega$.

Bursztyn-Cavalcanti-Gualtieri [7] defined the notion of extended action on a Courant algebroid. We spell out only the case of a **trivially extended action** [7, Def. 2.12] on the above Courant algebroid: it is an action of a connected Lie group G on M together with a linear map $\xi: \mathfrak{g} \rightarrow \Gamma(T^*M)$ such that $\mathfrak{g} \rightarrow \Gamma(TM \oplus T^*M)_\omega, x \mapsto v_x + \xi(x)$ is bracket-preserving and the Lie algebra morphism $x \mapsto \text{ad}_{v_x + \xi(x)} = \llbracket v_x + \xi(x), \cdot \rrbracket_\omega$ into the infinitesimal automorphisms of the Courant algebroid integrates to an action of G on $TM \oplus T^*M$.

The following lemma is essentially contained in [7, Sec. 2.2].

Lemma 12.1. *Let G act on pre-2-plectic manifold (M, ω) and $\mu \in (\mathfrak{g}^\vee \otimes \Omega^1(M))^G$. $\omega - \mu$ is an equivariant extension of ω if and only if*

$$\Psi: \mathfrak{g} \rightarrow \Gamma(TM \oplus T^*M)_\omega, \quad x \mapsto v_x - \mu(x)$$

*is a trivially extended action on the Courant algebroid $(TM \oplus T^*M)_\omega$ integrating to the action of G by tangent-cotangent lifts with isotropic image:*

$$\langle \text{im } \Psi, \text{im } \Psi \rangle = 0.$$

Proof. To simplify the notation, denote $\xi = -\mu$. Suppose $\omega + \xi$ is an equivariant extension (so Eq. (26) holds). The following two statements are contained in the text after Prop. 2.11 of [7]. First, the condition $\iota_{v_x} \omega - d(\xi(x)) = 0$ and the equivariance of ξ are equivalent to the fact that Ψ preserves brackets and $TM \oplus \{0\}$ is a \mathfrak{g} -equivariant splitting. Second, $\iota_{v_x} \omega - d(\xi(x)) = 0$ is also equivalent to $\text{ad}_{v_x + \xi(x)}$ being the Lie derivative \mathcal{L}_{v_x} , acting on vector fields and 1-forms (hence Ψ integrates to the action of G on $TM \oplus T^*M$ is by tangent-cotangent lifts). The condition $\iota_{v_x}(\xi(x)) = 0$ clearly means that the image of Ψ is isotropic.

The converse implication is proven by reversing the above argument. \square

On the other hand, we know that an equivariant extension $\omega - \mu$ delivers a moment map $(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$, by Thm. 6.9. When ω is non-degenerate, the relation between f and the extended action Ψ of Lemma 12.1 is simply

$$(49) \quad \Psi = i \circ f.$$

Here

$$(i_k): L_\infty(M, \omega) \rightarrow L_\infty((TM \oplus T^*M)_\omega)$$

is the embedding of Lie 2-algebras given in [32, Thm. 5.2]² (it is defined only when ω is non-degenerate). $L_\infty((TM \oplus T^*M)_\omega)$ denotes the Lie 2-algebra associated to the Courant algebroid $(TM \oplus T^*M)_\omega$ by Weinstein-Roytenberg [36][34, Thm. 6.5]; its underlying graded vector space is $C^\infty(M)$ in degree -1 and $\Gamma(TM \oplus T^*M)_\omega$ in degree zero, and the binary bracket in degree zero is the twisted Courant bracket, i.e. the skew-symmetrization of $\llbracket \cdot, \cdot \rrbracket_\omega$. In summary, the composition of the two Lie 2-algebra morphisms on the r.h.s. of Eq. (49) happens to be a strict morphism from \mathfrak{g} to $L_\infty((TM \oplus T^*M)_\omega)$, whose only non-trivial component is Ψ .

²The same theorem appears also as [34, Thm. 7.1] but with different sign conventions.

Remark 12.2. There is a notion of moment map for extended actions on Courant algebroids [7, Def. 2.14]. In the case of a trivially extended action, however, the only such moment map is the zero map.

Remark 12.3. Whenever the action of G on (M, ω) is by Hamiltonian vector fields, i.e. there is a linear map $f: \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^1(M)$ satisfying $\iota_{v_x} \omega = -df(x)$ for all x , one has $\text{ad}_{v_x + \xi(x)} = \mathcal{L}_{v_x}$. Hence, given an arbitrary moment map $(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$, the Lie algebra morphism into the infinitesimal automorphisms of the Courant algebroid suggested by Lemma 12.1

$$\mathfrak{g} \rightarrow \text{aut}((TM \oplus T^*M)_\omega), x \mapsto \text{ad}_{v_x - f_1(x)} = \mathcal{L}_{v_x}$$

does not encode neither ω nor the moment map. This is a clear indication the moment map (f_k) can not be encoded naturally in terms on a map of \mathfrak{g} into the infinitesimal automorphisms of the Courant algebroid.

12.3. Actions on differential graded manifolds. Let M be a manifold endowed with a closed $n+1$ -form ω . Uribe [41] considers the graded manifold $P = T[1]M \oplus \mathbb{R}[n]$, whose graded algebra of function is $\Omega(M) \otimes S[t]$, where $S[t]$ denotes polynomials in a variable t of degree n . The graded manifold P is endowed with the homological vector field $Q = d_{\text{dR}} + \omega \partial_t$, where d_{dR} is the de Rham vector field on $T[1]M$. Assume that a connected Lie group G acts on M , and assume for simplicity that ω is preserved by the action.

Denote

$$\mathfrak{sym}(P, Q) = \mathfrak{X}_{<0}(P) \oplus \{Y \in \mathfrak{X}_0(P) : [Q, Y] = 0\}$$

(the vector fields of negative degree on P , together with the degree zero vector fields commuting with Q). It is a DGLA with the usual bracket of vector fields and differential $[Q, \bullet]$. The main motivation for Uribe to study these objects is that, when $n = 2$, $\mathfrak{sym}(P, Q)$ is isomorphic to the DGLA of symmetries of the ω -twisted exact Courant algebroid over M . $\mathfrak{sym}(P, Q)$ contains a sub-DGLA $\mathfrak{gsym}(P, Q)$, whose degree zero component consists of vector fields preserving the function ω , and with degree -1 component

$$\{\iota_X + \alpha \partial_t : X \in \mathfrak{X}(M), \alpha \in \Omega^{n-1}(M), d\alpha = -\iota_X \omega\}$$

(see [41, Sec. 3.2]). Further, there is a DGLA associated to the Lie algebra \mathfrak{g} of G : it is $\mathfrak{g}[1] \oplus \mathfrak{g}$, with bracket given by the bracket and adjoint action of \mathfrak{g} , and differential $\text{id}_{\mathfrak{g}}$.

We are now ready to reproduce two statements. First, [41, Lemma 3.7] states that strict morphisms of DGLAs

$$(50) \quad \mathfrak{g}[1] \oplus \mathfrak{g} \rightarrow \mathfrak{gsym}(P, Q)$$

lifting the action of \mathfrak{g} on M are in bijective correspondence with cocycles $\omega - \mu$ in the Cartan model, where $\mu \in \Omega^{n-1}(M) \otimes S^1 \mathfrak{g}^\vee[-2]$.

Second, by [41, Prop. 2.15], L_∞ -morphisms

$$(51) \quad \mathfrak{g}[1] \oplus \mathfrak{g} \rightarrow \mathfrak{sym}(P, Q)$$

lifting the map $\mathfrak{g}[1] \oplus \mathfrak{g} \rightarrow \mathfrak{X}(M)[1] \oplus \mathfrak{X}(M)$ induced by the action of \mathfrak{g} on M are in bijective correspondence with closed extensions of ω in the BRST model $\wedge \mathfrak{g}^\vee \otimes S \mathfrak{g}^\vee \otimes \Omega(M)$.

Notice that, as $\mathfrak{gsym}(P, Q)$ is a sub-DGLA of $\mathfrak{sym}(P, Q)$, the morphisms appearing in (50) are particular cases of those appearing in (51).

The relevance of the above to our work is as follows. A cocycle $\omega - \mu$ in the Cartan model induces two kinds of objects: by Thm. 6.9 it induces a moment map of a particular form (equivariant, and determined by its unary component); by [41, Lemma 3.7] it induces an infinitesimal action of $\mathfrak{g}[1] \oplus \mathfrak{g}$ on (P, Q) of a particular form (strict, and preserving ω). Proposition 2.15 in [41] suggests that there

³It is a DGLA concentrated in degrees -1 and 0 , which corresponds to the natural structure of Lie algebra crossed module of \mathfrak{g} over itself.

is a relation between arbitrary homotopy moment maps and closed extensions of ω in the BRST model. We are pursuing such ideas in future work.

13. CONCLUDING REMARKS

This work raises many questions and suggests a variety of possible directions for future research. We mentioned some of these throughout the text. Here we give a few more.

First, several of the examples introduced in this paper deserve more thorough investigation. In particular:

- In Sec. 8.1, we consider moment maps arising from exact n -plectic forms. Manifolds equipped such n -plectic structures naturally arise in certain models of classical field theory [21]. How do homotopy moment maps relate to the conservation laws and symmetries studied within these models?

Along these lines, let us just mention that if a Hamiltonian $(n - 1)$ -form H is invariant under the G -action, the existence of a (not necessarily equivariant) moment map implies the existence of many conserved quantities, i.e., differential forms for which the Lie derivative by the Hamiltonian vector field v_H is exact. In particular, the “dynamics” given by v_H is constrained to the level sets of certain functions.

- In Sec. 8.2, we note a relationship between the homotopy moment map arising from a Lie group acting on itself via conjugation and the theory of quasi-Hamiltonian G -spaces. Moreover, if G acts on a pre-2-plectic manifold (M, ω) , with ω both G -invariant and representing a degree 3 integral cohomology class, then we also have reasons to suspect that there is a relationship between homotopy moment maps lifting the G -action, and G equivariant $U(1)$ -gerbes on M . Indeed, the results in [15] imply that a homotopy moment map lifts the \mathfrak{g} -action on (M, ω) to a \mathfrak{g} -action on any $U(1)$ -gerbe whose 3-curvature is ω . Some relationships have already been established between trivializations of G -equivariant gerbes and quasi-Hamiltonian G -spaces (e.g. [20]). What is the precise relationship which intertwines these formalisms with homotopy moment maps?

Further development of the general theory of homotopy moment maps would also be desirable. For example:

- When should two moment maps be considered equivalent? Indeed, there are well-known uniqueness results for moment maps in symplectic geometry and, for example, abstract moment maps in Hamiltonian cobordism theory. For us, the question is particularly relevant since some of our constructions (e.g. Thm. 9.6) depend on a number of choices. We briefly discussed uniqueness issues in Sec. 7.4 for the case of pre-2-plectic manifolds. Also, in Section 7 of [17], various candidates for equivalences are proposed. One of these is closely related to the notion of homotopy equivalence between L_∞ -morphisms (e.g., [12, Def. 4.7]).
- Can one perform reduction of pre- n -plectic forms using an equivariant moment map $(f_k): \mathfrak{g} \rightarrow L_\infty(M, \omega)$, analogously to the Marsden-Weinstein-Meyer reduction in symplectic geometry? It is easily checked that, if

$$S = \{p \in M : f_1(x)|_p = 0 \text{ for all } x \in \mathfrak{g}\}$$

satisfies certain regularity conditions, then it is G -invariant and the pullback of ω descends to S/G . However S is the vanishing set of a family of $(n - 1)$ -forms, and it is hard to control its smoothness properties. The only instance we are aware of where the above-mentioned reduction procedure works, apart from the one discussed in Subsection 10.5, is Ex. 8.2, namely the (free and proper) cotangent lift action of G on $\Lambda^n T^*N$: in this case S/G is

canonically isomorphic to $\Lambda^n T^*(N/G)$ with its canonical n -plectic form. This procedure is probably too naïve in general, and it only uses a small part of the information provided by the moment map.

APPENDIX A. EXPLICIT FORMULAS FOR L_∞ -MORPHISMS

Here we recall the relationship between L_∞ -morphisms and morphisms of dg coalgebras. We use this to prove Prop. 3.8 from Section 3 and other results needed throughout the text. The facts presented here are well-known to experts, but it is difficult to find explicit formulas in the literature. For more details on coalgebras, we suggest Sections 3d and 22a of [14], and Appendix B of [31]. The description of L_∞ -algebras as coalgebras is also reviewed in Section 2 of [24]. Unlike the aforementioned references, we use cohomological conventions i.e. our (co)differentials have degree +1.

A.1. The coalgebra $\mathcal{S}(V)$. Given a graded vector space V , the graded symmetric algebra

$$\begin{aligned} \mathcal{S}(V) &= \mathbb{R} \oplus V \oplus S^2(V) \oplus S^3(V) \oplus \dots \\ &= \mathbb{R} \oplus \bar{S}(V) \end{aligned}$$

is naturally a cocommutative coalgebra. This means that it is equipped with a linear map $\Delta: \mathcal{S}(V) \rightarrow \mathcal{S}(V) \otimes \mathcal{S}(V)$ (the comultiplication) such that $(\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta$ (coassociativity) and $T \circ \Delta = \Delta$ (cocommutativity), where $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$. In this case, Δ is the unique morphism of algebras such that $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$.

From Δ we also obtain a cocommutative coalgebra structure on the reduced symmetric algebra:

$$\bar{\Delta}: \bar{S}(V) \rightarrow \bar{S}(V) \otimes \bar{S}(V)$$

where $\bar{\Delta}c = \Delta c - c \otimes 1 - 1 \otimes c$ is the reduced comultiplication. Explicitly,

$$\begin{aligned} \bar{\Delta}(v_1 \odot v_2 \odot \dots \odot v_n) &= \sum_{1 \leq p \leq n-1} \sum_{\sigma \in \text{Sh}(p, n-p)} \epsilon(\sigma) (v_{\sigma(1)} \odot v_{\sigma(2)} \odot \dots \odot v_{\sigma(p)}) \\ &\quad \otimes (v_{\sigma(p+1)} \odot v_{\sigma(p+2)} \odot \dots \odot v_{\sigma(n)}). \end{aligned}$$

The reduced diagonal $\bar{\Delta}^{(n)}$ is recursively defined by the formulas:

$$\begin{aligned} \bar{\Delta}^{(0)} &= \text{id} \\ \bar{\Delta}^{(1)} &= \bar{\Delta} \\ \bar{\Delta}^{(n)} &= (\bar{\Delta} \otimes \text{id}^{\otimes(n-1)}) \circ \bar{\Delta}^{(n-1)}: \bar{S}(V) \rightarrow \bar{S}(V)^{\otimes(n+1)}. \end{aligned}$$

A simple induction argument shows that we can rewrite $\bar{\Delta}^{(n)}$ as

$$(52) \quad \bar{\Delta}^{(n)} = (\bar{\Delta}^{(n-1)} \otimes \text{id}) \circ \bar{\Delta}.$$

Using Eq. (52), it is easy to see that $\bar{S}^{\bullet \leq k}(V) \subseteq \ker \bar{\Delta}^{(k)}$. The following lemma will be useful in the proceeding sections. It follows straightforwardly via induction.

Lemma A.1. *If $v_1 \odot v_2 \odot \dots \odot v_n \in \bar{S}(V)$, and $1 \leq p \leq n-1$ then*

$$\begin{aligned} \bar{\Delta}^{(p)}(v_1 \odot \dots \odot v_n) &= \sum_{k_1, k_2, \dots, k_{p+1} \geq 1}^{k_1 + k_2 + \dots + k_{p+1} = n} \sum_{\sigma \in \text{Sh}(k_1, k_2, \dots, k_{p+1})} \epsilon(\sigma) v_{\sigma(1)} \odot \dots \odot v_{\sigma(k_1)} \\ &\quad \otimes v_{\sigma(k_1+1)} \odot \dots \odot v_{\sigma(k_1+k_2)} \otimes v_{\sigma(k_1+k_2+1)} \odot \dots \odot v_{\sigma(k_1+k_2+k_3)} \otimes \dots \\ &\quad \otimes v_{\sigma(n-k_{p+1}+1)} \odot \dots \odot v_{\sigma(n)}. \end{aligned}$$

In particular, we have

$$\bar{\Delta}^{(n-1)}(v_1 \odot \cdots \odot v_n) = \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}.$$

Note that the above lemma implies that $\ker \bar{\Delta}^{(k)} = \bar{S}^{\bullet \leq k}(V)$ for $k \geq 0$ and hence

$$(53) \quad \bar{S}(V) = \bigcup_n \ker \bar{\Delta}^{(n)}.$$

A.2. Coalgebra morphisms. A **morphism** between the reduced coalgebras $(\bar{S}(V), \bar{\Delta})$ and $(\bar{S}(V'), \bar{\Delta}')$ is a degree 0 linear map $F: \bar{S}(V) \rightarrow \bar{S}(V')$ such that

$$\bar{\Delta}' \circ F = (F \otimes F) \circ \bar{\Delta}.$$

Given such a morphism, we define the restriction-projections:

$$(54) \quad F_n^p = \text{pr}_{\bar{S}^p(V')} \circ F|_{\bar{S}^n(V)}: \bar{S}^n(V) \rightarrow \bar{S}^p(V').$$

The following proposition implies that the linear map

$$F^1: \bar{S}(V) \rightarrow V', \quad F^1 := F_1^1 + F_2^1 + \cdots$$

uniquely determines the coalgebra morphism F .

Proposition A.2. *If V and V' are graded vector spaces, and $F^1: \bar{S}(V) \rightarrow V'$ is a degree zero linear map, then there exists a unique morphism of coalgebras*

$$F: \bar{S}(V) \rightarrow \bar{S}(V')$$

lifting F^1 such that $\text{pr}_{V'} \circ F = F^1$.

Proof. The statement is a special case of Prop. 4.1 in Sec. B3 of [31] or Lemma 22.1 in [14]. Here we just recall the construction of the coalgebra morphism F . First, define for each $p > 0$:

$$\psi^{(p)}: \underbrace{\bar{S}(V) \otimes \bar{S}(V) \otimes \cdots \otimes \bar{S}(V)}_p \rightarrow \bar{S}^p(V')$$

$$\psi^{(p)}(c_1 \otimes c_2 \cdots \otimes c_p) = \frac{1}{p!} F_{k_1}^1(c_1) \odot \cdots \odot F_{k_p}^1(c_p),$$

where c_1, \dots, c_p are simple tensors with $c_i \in \bar{S}^{k_i}(V)$ and $F_{k_i}^1 = F^1|_{\bar{S}^{k_i}(V)}$. Define F to be:

$$(55) \quad F(c) = \sum_{p=0}^{\infty} \psi^{(p+1)} \circ \bar{\Delta}^{(p)}(c), \quad c \in \bar{S}(V).$$

Note the infinite sum is well-defined since Eq. (53) holds. □

We use Lemma A.1 to write out the formula for F explicitly in terms of the maps F_k^1 . Given $v_1, \dots, v_n \in V$, we have

$$(56) \quad \begin{aligned} F(v_1 \odot \cdots \odot v_n) &= F_n^1(v_1 \odot \cdots \odot v_n) + \sum_{p=1}^{n-1} \sum_{k_1, k_2, \dots, k_{p+1} \geq 1}^{k_1 + k_2 + \cdots + k_{p+1} = n} \sum_{\sigma \in \text{Sh}(k_1, k_2, \dots, k_{p+1})} \frac{\epsilon(\sigma)}{(p+1)!} \\ &\quad \times F_{k_1}^1(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k_1)}) \odot F_{k_2}^1(v_{\sigma(k_1+1)} \odot \cdots \odot v_{\sigma(k_1+k_2)}) \odot \cdots \\ &\quad \odot F_{k_{p+1}}^1(v_{\sigma(m-k_{p+1}+1)} \odot \cdots \odot v_{\sigma(n)}). \end{aligned}$$

This gives explicit formulas for the projections F_n^p defined in (54):

$$(57) \quad F_n^p(v_1 \odot \cdots \odot v_n) = \sum_{k_1, k_2, \dots, k_p \geq 1}^{k_1 + k_2 + \cdots + k_p = n} \sum_{\sigma \in \text{Sh}(k_1, k_2, \dots, k_p)} \frac{\epsilon(\sigma)}{p!} F_{k_1}^1(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k_1)}) \\ \odot F_{k_2}^1(v_{\sigma(k_1+1)} \odot \cdots \odot v_{\sigma(k_1+k_2)}) \odot \cdots \odot F_{k_p}^1(v_{\sigma(m-k_p+1)} \odot \cdots \odot v_{\sigma(n)}).$$

In particular,

$$F_n^n(v_1 \odot \cdots \odot v_n) = F_1^1(v_1) \odot F_1^1(v_2) \odot \cdots \odot F_1^1(v_n),$$

and

$$F_n^p(v_1 \odot \cdots \odot v_n) = 0 \quad \text{for } p > n.$$

A.3. L_∞ -algebras as dg coalgebras. A **codifferential** on the coalgebra $(\bar{S}(V), \bar{\Delta})$ is a degree +1 linear map $Q: \bar{S}(V) \rightarrow \bar{S}(V)$ satisfying

$$\bar{\Delta}Q = (Q \otimes \text{id})\bar{\Delta} + (\text{id} \otimes Q)\bar{\Delta}.$$

and

$$Q \circ Q = 0.$$

If V is a graded vector space, then $\mathbf{s}V$ (resp. $\mathbf{s}^{-1}V$) denotes the suspension (resp. desuspension) of V i.e.

$$(\mathbf{s}V)_i = V_{i-1}, \quad (\mathbf{s}^{-1}V)_i = V_{i+1}.$$

Recall that in Def. 3.1, we defined an L_∞ -algebra structure on L to be a collection of skew-symmetric maps $\{l_k: L^{\otimes k} \rightarrow L\}_{k=1}^\infty$ with $|l_k|_k = 2 - k$ which satisfy a rather complicated generalization of the Jacobi identity. In contrast, the following theorem provides a more elegant description.

Theorem A.3 (Thm. 2.3 [24]). *An L_∞ -structure (l_k) on a graded vector space L uniquely determines a degree 1 codifferential Q on the coalgebra*

$$C(L) = \bar{S}(\mathbf{s}^{-1}L).$$

Conversely, any such codifferential on $C(L)$ uniquely determines an L_∞ -structure on L .

We will need to briefly describe the correspondence given by the theorem. We define the restrictions

$$Q_m = Q|_{\bar{S}^m(\mathbf{s}^{-1}L)}: \bar{S}^m(\mathbf{s}^{-1}L) \rightarrow \bar{S}(\mathbf{s}^{-1}L)$$

so that $Q = Q_1 + Q_2 + Q_3 + \dots$, and the projections

$$(58) \quad Q_m^k = \text{pr}_{\bar{S}^k(\mathbf{s}^{-1}L)} \circ Q_m: \bar{S}^m(\mathbf{s}^{-1}L) \rightarrow \bar{S}^k(\mathbf{s}^{-1}L).$$

It follows from Lemma 2.4 in [24] that Q is uniquely determined by the collection of maps

$$Q_m^1 = \text{pr}_{\mathbf{s}^{-1}L} \circ Q_m: \bar{S}^m(\mathbf{s}^{-1}L) \rightarrow \mathbf{s}^{-1}L, \quad m \geq 1.$$

These are related to the skew-symmetric ‘‘structure maps’’ $l_m: L^{\otimes m} \rightarrow L$ via the formula

$$(59) \quad Q_m^1 = (-1)^{\frac{m(m-1)}{2}} \mathbf{s}^{-1} \circ l_m \circ \mathbf{s}^{\otimes m},$$

while the entire coderivation Q can be expressed as

$$(60) \quad Q_m(\mathbf{s}^{-1}x_1 \odot \cdots \odot \mathbf{s}^{-1}x_m) = Q_m^1(\mathbf{s}^{-1}x_1 \odot \cdots \odot \mathbf{s}^{-1}x_m) + \\ \sum_{i=1}^{m-1} \sum_{\sigma \in \text{Sh}(i, m-i)} \epsilon(\sigma) Q_i^1(\mathbf{s}^{-1}x_{\sigma(1)} \odot \cdots \odot \mathbf{s}^{-1}x_{\sigma(i)}) \odot \mathbf{s}^{-1}x_{\sigma(i+1)} \odot \cdots \odot \mathbf{s}^{-1}x_{\sigma(m)},$$

for all $x_i \in L$. The condition $Q \circ Q = 0$ is equivalent to the generalized Jacobi identity (8) for the collection (l_k) . In particular, it implies that l_1 is degree +1 differential on L .

A.4. L_∞ -Morphisms: General case. Thanks to Thm A.3, it is now clear what an L_∞ -morphism should be.

Definition A.4. A **morphism** between L_∞ -algebras (L, l_k) and (L', l'_k) is a coalgebra morphism

$$F: C(L) \rightarrow C(L')$$

such that

$$FQ = Q'F.$$

The following proposition says that ‘strict morphisms’ in the sense of Def. 3.3 are precisely those coalgebra morphisms that satisfy

$$\forall k \geq 2 \quad F_k^1 = 0.$$

We leave the proof to the reader.

Proposition A.5. *If (L, l_k) and (L', l'_k) are L_∞ -algebras, and $f: L \rightarrow L'$ is a degree zero linear map satisfying*

$$l'_k \circ f^{\otimes k} = f \circ l_k \quad \forall k \geq 1,$$

then the linear map $F: C(L) \rightarrow C(L')$ given by

$$F(\mathbf{s}^{-1}x_1 \odot \cdots \odot \mathbf{s}^{-1}x_k) = \mathbf{s}^{-1}f(x_1) \odot \cdots \odot \mathbf{s}^{-1}f(x_k)$$

is a strict L_∞ -morphism.

More generally, if $F: (C(L), Q) \rightarrow (C(L'), Q')$ is any L_∞ -morphism, then the projections defined in Eq. (54) and Eq. (58) allow us to write the equality $FQ = Q'F$ as

$$(61) \quad \sum_{k=1}^m F_k^1 Q_m^k = \sum_{k=1}^m Q_k^1 F_m^k \quad \forall m \geq 1.$$

Every such F is of the form (55), since by Prop. A.2 it is the unique lift of its projection $F^1 = F_1^1 + F_2^1 + F_3^1 + \cdots$. Hence, F is uniquely determined by its corresponding collection of ‘structure maps’ $\{f_k: L^{\otimes k} \rightarrow L'\}_{k \geq 1}$ which satisfy

$$(62) \quad F_k^1 = (-1)^{\frac{k(k-1)}{2}} \mathbf{s}^{-1} \circ f_k \circ \mathbf{s}^{\otimes k}.$$

Hence each f_k is graded skew-symmetric with $|f_k| = 1 - k$. Note that the equality $FQ = Q'F$ implies that the degree zero map

$$f_1: (L, l_1) \rightarrow (L', l'_1)$$

is a morphism of cochain complexes. This leads us to the notion of L_∞ -quasi-isomorphism given in Def. 3.6.

Remark A.6. The defining equations (61) for a morphism $(L, l_k) \rightarrow (L', l'_k)$ between L_∞ -algebras can be given explicitly in terms of the structure maps f_k, l_k , and l'_k . First, one uses Eqs. (57) and (60) to rewrite (61) in terms of the multi-linear maps F_k^1, Q_k^1 , and Q_k^1 . As Eqs. (59) and (62) show, these maps are nothing but the structure maps composed with the appropriate suspensions and desuspensions. (See Cor. A.8 for a special case.)

A.5. Lie algebras. Any differential graded Lie algebra (DGLA) can be thought of as a L_∞ -algebra by associating to $(\mathfrak{g}, d, [\cdot, \cdot])$ the coalgebra $\bar{S}(\mathfrak{s}^{-1}\mathfrak{g})$ with codifferential D defined by the equations

$$(63) \quad \begin{aligned} D_1(\mathfrak{s}^{-1}x) &= \mathfrak{s}^{-1}dx \\ D_2^1(\mathfrak{s}^{-1}x, \mathfrak{s}^{-1}y) &= (-1)^{|x|}\mathfrak{s}^{-1}[x, y] \\ D_k^1 &= 0, \quad k \geq 3. \end{aligned}$$

A DGLA morphism $f: \mathfrak{g} \rightarrow \mathfrak{g}'$ induces a unique strict L_∞ -morphism between $(\bar{S}(\mathfrak{s}^{-1}\mathfrak{g}), D)$ and $(\bar{S}(\mathfrak{s}^{-1}\mathfrak{g}'), D')$. We treat ordinary Lie algebras as DGLAs concentrated in degree zero with differential $d = 0$.

Now we consider L_∞ -algebra morphisms whose sources are just Lie algebras $(\mathfrak{g}, [\cdot, \cdot])$. Since the projections D_m^k are built from the structure maps D_m^1 via Eq. (60), we have

$$D_m^k = 0 \quad \text{whenever } k \neq m - 1.$$

Therefore, Eq. (61), which a coalgebra morphism $F: \bar{S}(\mathfrak{s}^{-1}\mathfrak{g}) \rightarrow \bar{S}(\mathfrak{s}^{-1}L)$ must satisfy to be an L_∞ -morphism, simplifies to

$$(64) \quad \begin{aligned} Q_1^1 F_1^1 &= 0, \\ F_{m-1}^1 D_m^{m-1} &= \sum_{k=1}^m Q_k^1 F_m^k \quad \forall m \geq 2. \end{aligned}$$

In particular, homotopy moment maps (Def. 5.1) are L_∞ -morphisms from a Lie algebra to a Lie n -algebra (L, l_k) satisfying Property (P), which we defined in Sec. 3.2 as being:

$$\forall k \geq 2 \quad l_k(x_1, \dots, x_k) = 0 \quad \text{whenever } \sum_{i=1}^k |x_i| < 0.$$

Equation (59) implies that this is equivalent to the corresponding codifferential Q on $\bar{S}(\mathfrak{s}^{-1}L)$ satisfying

$$\forall k \geq 2 \quad Q_k^1(\mathfrak{s}^{-1}x_1 \odot \dots \odot \mathfrak{s}^{-1}x_k) = 0 \quad \text{whenever } |\mathfrak{s}^{-1}x_1 \odot \dots \odot \mathfrak{s}^{-1}x_k| < k.$$

Proposition A.7. *If $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra and (L, l_k) is a Lie n -algebra satisfying Property (P), then a coalgebra morphism $F: \bar{S}(\mathfrak{s}^{-1}\mathfrak{g}) \rightarrow \bar{S}(\mathfrak{s}^{-1}L)$ is an L_∞ -algebra morphism if and only if*

$$(65) \quad F_{m-1}^1 D_m^{m-1} = Q_1^1 F_m^1 + Q_m^1 F_m^m$$

for $2 \leq m \leq n$, and

$$(66) \quad F_n^1 D_{n+1}^n = Q_{n+1}^1 F_{n+1}^{n+1},$$

where D and Q are the codifferentials determined by $[\cdot, \cdot]$, and (l_k) , respectively.

Proof. We will show the conditions given in Eqs. (65) and (66) are equivalent to those in (64). First, note that for any coalgebra morphism $F: \bar{S}(\mathfrak{s}^{-1}\mathfrak{g}) \rightarrow \bar{S}(\mathfrak{s}^{-1}L)$

$$Q_1^1 F_1^1 = 0$$

holds trivially, since F_1^1 is a degree 0 map and $\mathfrak{s}^{-1}\mathfrak{g}$ is in degree -1, while Q_1^1 has degree +1 and $\mathfrak{s}^{-1}L$ is concentrated in degrees $-n, \dots, -1$. Next, we observe that Property (P) and Eq. (57) imply that

$$\sum_{k=1}^m Q_k^1 F_m^k = Q_1^1 F_m^1 + Q_m^1 F_m^m \quad \forall m \geq 2.$$

When $m \geq n + 1$, the degree condition on $\mathfrak{s}^{-1}L$ implies that $F_m^1 = 0$ and hence

$$\begin{aligned} Q_1^1 F_m^1 &= 0 \quad \forall m \geq n + 1, \\ F_{m-1}^1 D_m^{m-1} &= 0 \quad \forall m \geq n + 2. \end{aligned}$$

For the same reason, $Q_m^1 = 0$ whenever $m \geq n + 2$. Therefore

$$Q_m^1 F_m^m = 0 \quad \forall m \geq n + 2.$$

Hence, satisfying Eqs. (65) and (66) is both necessary and sufficient for F to be an L_∞ -morphism. \square

We now prove Prop. 3.8 as a corollary of the above.

Corollary A.8 (Prop. 3.8). *If $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra and (L, l_k) is a Lie n -algebra satisfying property (P), then a collection of n skew-symmetric maps*

$$f_m: \mathfrak{g}^{\otimes m} \rightarrow L, \quad |f_m| = 1 - m, \quad 1 \leq m \leq n$$

determine an L_∞ -morphism $\bar{S}(\mathfrak{s}^{-1}\mathfrak{g}) \rightarrow \bar{S}(\mathfrak{s}^{-1}L)$ via Eq. (62) if and only if $\forall x_i \in \mathfrak{g}$

$$(67) \quad \sum_{1 \leq i < j \leq m} (-1)^{i+j+1} f_{m-1}([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_m) \\ = l_1 f_m(x_1, \dots, x_m) + l_m(f_1(x_1), \dots, f_1(x_m)).$$

for $2 \leq m \leq n$ and

$$(68) \quad \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} f_n([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_{n+1}) = l_{n+1}(f_1(x_1), \dots, f_1(x_{n+1})).$$

Proof. Assume we are given such maps f_1, \dots, f_n satisfying the above equalities. Using Eq. (62), we construct the corresponding degree 0 maps F_1^1, \dots, F_n^1 , and set $F_k^1 = 0$ for $k \geq n + 1$. By Prop. A.2, these give a unique coalgebra morphism $F: \bar{S}(\mathfrak{s}^{-1}\mathfrak{g}) \rightarrow \bar{S}(\mathfrak{s}^{-1}L)$. To show F is an L_∞ -morphism, Prop. A.7 implies it is sufficient to show Eqs. (65) and (66) hold. From Eq. (63), we have the equality $D_2^1(\mathfrak{s}^{-1}x \odot \mathfrak{s}^{-1}y) = \mathfrak{s}^{-1}[x, y]$, while Eq. (60) implies that

$$D_m^{m-1}(\mathfrak{s}^{-1}x_1 \odot \dots \odot \mathfrak{s}^{-1}x_m) = \sum_{\sigma \in \text{Sh}(2, m-2)} \epsilon(\sigma) D_2^1(\mathfrak{s}^{-1}x_{\sigma(1)} \odot \mathfrak{s}^{-1}x_{\sigma(2)}) \odot \dots \odot \mathfrak{s}^{-1}x_{\sigma(m)} \\ = \sum_{1 \leq i < j \leq m} (-1)^{i+j+1} \mathfrak{s}^{-1}[x_i, x_j] \odot \mathfrak{s}^{-1}x_1 \odot \dots \odot \widehat{\mathfrak{s}^{-1}x_i} \dots \widehat{\mathfrak{s}^{-1}x_j} \dots \odot \mathfrak{s}^{-1}x_m.$$

The signs in the last equality above are due to the fact that \mathfrak{g} is in degree 0. It follows from Eq. (62) that

$$(69) \quad F_m^1(\mathfrak{s}^{-1}x_1 \odot \dots \odot \mathfrak{s}^{-1}x_m) = \mathfrak{s}^{-1}f_m(x_1, \dots, x_m).$$

Therefore, the left-hand sides of Eqs. (65) and (66) are the desuspension of the left-hand sides of Eqs. (67) and (68), respectively.

Now we consider the right-hand sides. First, note that Eq. (69) also implies that

$$(70) \quad Q_1^1 F_m^1 = \mathfrak{s}^{-1}l_1 \circ f_m.$$

Recall Eq. (57) gives

$$F_m^m(\mathfrak{s}^{-1}x_1 \odot \dots \odot \mathfrak{s}^{-1}x_m) = F_1^1(\mathfrak{s}^{-1}x_1) \odot F_1^1(\mathfrak{s}^{-1}x_2) \odot \dots \odot F_1^1(\mathfrak{s}^{-1}x_m).$$

For each x_i , we have $|F_1^1(\mathfrak{s}^{-1}x_i)| = -1$ and $F_1^1(\mathfrak{s}^{-1}x_i) = \mathfrak{s}^{-1}f_1(x_i)$. Therefore,

$$Q_m^1 F_m^m(\mathfrak{s}^{-1}x_1 \odot \dots \odot \mathfrak{s}^{-1}x_m) = \mathfrak{s}^{-1}l_m(f_1(x_1), \dots, f_1(x_m)).$$

Combining the above equality with Eq. (70), we see that the right-hand sides of Eqs. (65) and (66) are the desuspension of the right-hand sides of Eqs. (67) and (68), respectively. Hence, F is a L_∞ -morphism.

It is easy to see that the converse follows by reversing the above arguments. \square

A.6. L_∞ -morphisms and central n -extensions. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra and $c: \Lambda^{n+1}\mathfrak{g} \rightarrow \mathbb{R}$ a degree $n+1$ cocycle in the Chevalley-Eilenberg complex associated to \mathfrak{g} . A theorem of Baez and Crans [3, Thm. 55] implies that this data gives a Lie n -algebra $\widehat{\mathfrak{g}}_c$ whose underlying vector space is

$$\begin{aligned} L_0 &= \mathfrak{g}, \\ L_i &= 0 \quad 2-n \leq i \leq -1, \\ L_{1-n} &= \mathbb{R}, \end{aligned}$$

and whose only non-trivial multibrackets are

$$\begin{aligned} l_2(x_1, x_2) &= \begin{cases} [x_1, x_2] & \text{if } x_1, x_2 \in \mathfrak{g} \\ 0 & \text{otherwise} \end{cases} \\ l_{n+1}(x_1, \dots, x_{n+1}) &= \begin{cases} c(x_1, \dots, x_{n+1}) & \text{if } x_1, \dots, x_{n+1} \in \mathfrak{g} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We call such Lie n -algebras **central n -extensions** of \mathfrak{g} .

Proposition A.9. *Let \mathfrak{g} be a Lie algebra, $c \in \text{Hom}(\Lambda^{n+1}\mathfrak{g}, \mathbb{R})$ a $(n+1)$ -cocycle, and $\widehat{\mathfrak{g}}_c$ the corresponding central n -extension. If (L, l_k) is a Lie n -algebra satisfying property (P), then a collection of n skew-symmetric maps*

$$\begin{aligned} f_1 &: \mathfrak{g} \oplus \mathbb{R}[n-1] \rightarrow L \\ f_m &: \mathfrak{g}^{\otimes m} \rightarrow L, \quad |f_m| = 1-m, \quad 2 \leq m \leq n \end{aligned}$$

determine an L_∞ -morphism $\bar{S}(\mathfrak{s}^{-1}\widehat{\mathfrak{g}}_c) \rightarrow \bar{S}(\mathfrak{s}^{-1}L)$ if and only if

$$(71) \quad l_1 f_1(r) = 0 \quad \forall r \in \mathbb{R},$$

and $\forall x_i \in \mathfrak{g}$

$$(72) \quad \sum_{1 \leq i < j \leq m} (-1)^{i+j+1} f_{m-1}([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_m) = l_1 f_m(x_1, \dots, x_m) + l_m(f_1(x_1), \dots, f_1(x_m)).$$

for $2 \leq m \leq n$ and

$$(73) \quad \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} f_n([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_{n+1}) + f_1 c(x_1, \dots, x_{n+1}) = l_{n+1}(f_1(x_1), \dots, f_1(x_{n+1})).$$

Proof. Observe the similarity between the above formulas and those given in Cor. A.8 for a L_∞ -morphism from \mathfrak{g} to (L, l_k) . Let D denote the codifferential on $\bar{S}(\mathfrak{s}^{-1}\widehat{\mathfrak{g}}_c)$. We proceed as we did in the proof of Prop. A.7 and conclude that a coalgebra morphism $F: \bar{S}(\mathfrak{s}^{-1}\widehat{\mathfrak{g}}_c) \rightarrow \bar{S}(\mathfrak{s}^{-1}L)$ is an L_∞ -morphism iff

$$\begin{aligned} Q_1 F_1(\mathfrak{s}^{-1}r) &= 0 \quad \forall r \in \mathbb{R}[n-1], \\ F_{m-1}^1 D_m^{m-1} &= Q_1^1 F_m^1 + Q_m^1 F_m^m \quad 2 \leq m \leq n, \end{aligned}$$

and

$$F_n^1 D_{n+1}^n + F_1^1 D_{n+1}^1 = Q_{n+1}^1 F_{n+1}^{n+1}.$$

Rewriting these in terms of structure maps (f_k) (cf. the proof of Cor. A.8), we obtain Eqs. (71), (72), and (73). \square

Note that a central n -extension itself satisfies Property (P), so we have the following corollary:

Corollary A.10. *If $[c] = [c'] \in H_{\text{CE}}^{n+1}(\mathfrak{g}, \mathbb{R})$, then the central n -extensions $\widehat{\mathfrak{g}}_c$ and $\widehat{\mathfrak{g}}_{c'}$ are quasi-isomorphic.*

Proof. Let $b: \Lambda^n \mathfrak{g} \rightarrow \mathbb{R}$ such that $c' = c + \delta_{\text{CE}} b$. Consider the collection of skew-symmetric maps: $f_1 = \text{id}_{\mathfrak{g} \oplus \mathbb{R}[n-1]}$, $f_k = 0$ for $2 \leq k \leq n-1$ and $f_n = b$. Using Prop. A.9, it's easy to see these give an L_∞ -morphism $\widehat{\mathfrak{g}}_c \rightarrow \widehat{\mathfrak{g}}_{c'}$. Since f_1 is the identity, it is clearly a quasi-isomorphism. \square

APPENDIX B. PROOF OF THEOREM 6.8

In this appendix, we prove Thm. 6.8, which provides an explicit formula for constructing a moment map from a cocycle in the Cartan complex. For this, we need to construct a natural chain map $\Phi: C_G^*(M) \rightarrow \Omega^*(G \times M_\bullet)$, which is a quasi-isomorphism when G is compact. This chain map seems to be well known; see for example [29, Appendix C]. However, we need an explicit formula.

The rationale for this construction of Φ is the following. We wish to identify $C_G^*(M)$ with differential forms on the base space, i.e. the (homotopy) quotient $G \times M_\bullet$. So, in essence, we require a connection. Hence, we replace M with a equivalent space $E_\bullet G \times M_\bullet$, which is the total space of a principal G bundle over $G \times M_\bullet$, and furthermore equipped with a canonical connection. The connection induces a well-known chain map (the ‘‘Cartan map’’) between the Cartan model of the total space, and forms on the base. This map is an extension of the usual Chern-Weil homomorphism, and therefore we require a workable theory of Lie algebra-valued forms in the simplicial setting. In particular, we would like a commutative product on differential forms. Unfortunately, the product on the $\Omega^*(G \times M_\bullet)$ is only homotopy commutative. So we temporarily replace this complex by an equivalent one which is strictly commutative, namely Dupont’s model for the de Rham complex of $G \times M_\bullet$.

B.1. Differential forms on simplicial manifolds. If X_\bullet is a simplicial manifold with face maps $d_i: X_n \rightarrow X_{n-1}$, $i = 0, \dots, n$, then the simplicial differential $\partial_n: \Omega^*(X_n) \rightarrow \Omega^*(X_{n+1})$ is

$$(74) \quad \partial_n = \sum_{i=0}^{n+1} (-1)^i d_i^*.$$

The Bott-Shulman-Stasheff complex is the total complex of the double complex of differential forms on X_\bullet :

$$(75) \quad \begin{aligned} \Omega^{j,k}(X_\bullet) &:= \Omega^k(X_j), \\ \Omega^*(X_\bullet) &:= (\text{Tot}(\Omega^{*,*}(X_\bullet)), D), \\ D &:= \partial + (-1)^j d, \end{aligned}$$

where d is the de Rham differential.

Example B.1. Let M be a manifold and M_\bullet the simplicial manifold $M_n = M$, whose face and degeneracy maps are id_M . Since all ∂_n are either zero or isomorphisms, the inclusion

$$(76) \quad (\Omega^*(M), d) = (\Omega^*(M_0), d) \xhookrightarrow{\ell} (\Omega^*(M_\bullet), D)$$

is an quasi-isomorphism.

Example B.2. If M is a G -manifold, let $E_\bullet G \times M$ denote the product $E_\bullet G \times M_\bullet$, i.e., the simplicial manifold

$$[n] \mapsto E_n G \times M = G^{n+1} \times M$$

with the ‘‘usual’’ face and degeneracy maps.

Proposition B.3. *The projection $\pi_M: E_\bullet G \times M \rightarrow M_\bullet$ induces a quasi-isomorphism*

$$(77) \quad \pi_M^*: \Omega^*(M_\bullet) \rightarrow \Omega^*(E_\bullet G \times M).$$

Proof. Recall, that if X_\bullet is a simplicial manifold which is paracompact in each dimension, then the de Rham theorem of Bott-Shulman-Stasheff [6] implies that there exists a natural isomorphism

$$H(\Omega^*(X_\bullet)) \xrightarrow{\cong} H(\|X_\bullet\|),$$

where $H(\|X_\bullet\|)$ is the singular cohomology with \mathbb{R} coefficients of the fat realization of X_\bullet . We denote by $|X_\bullet|$ the thin geometric realization of X_\bullet . Since $|\cdot|$ preserves products, and since both G and M are manifolds, it follows from [38, Prop. A1] that we have a commuting diagram

$$(78) \quad \begin{array}{ccccccc} H(\Omega^*(M_\bullet)) & \xrightarrow{\cong} & H(\|M_\bullet\|) & \xrightarrow{\cong} & H(|M_\bullet|) & \xrightarrow{=} & H(|M_\bullet|) \\ \pi_M^* \downarrow & & \downarrow \|\pi_M\|^* & & \downarrow |\pi_M|^* & & \downarrow (\pi_{|M_\bullet|})^* \\ H(\Omega^*(E_\bullet G \times M)) & \xrightarrow{\cong} & H(\|E_\bullet G \times M\|) & \xrightarrow{\cong} & H(|E_\bullet G \times M|) & \xrightarrow{\cong} & H(|E_\bullet G| \times |M_\bullet|). \end{array}$$

Since $|E_\bullet G|$ is contractible, the Künneth formula implies that the right vertical arrow in the diagram (78) is an isomorphism. Hence π_M^* is also an isomorphism. \square

Remark B.4. (cf. [29, Appendix C.2]) Equip $E_\bullet G \times M$ with the diagonal G -action

$$(79) \quad \begin{aligned} G \times E_n G \times M &\rightarrow E_n G \times M, \\ (h, g_0, \dots, g_n, p) &\mapsto (g_0 h^{-1}, \dots, g_n h^{-1}, hp). \end{aligned}$$

The projection

$$\pi_M: E_\bullet G \times M \rightarrow M_\bullet$$

is a morphism of simplicial G -manifolds.

Note that the map

$$\begin{aligned} G^{n+1} \times M &\rightarrow G^n \times M, \\ (g_0, \dots, g_n, p) &\mapsto (g_0 g_1^{-1}, \dots, g_{n-1} g_n^{-1}, g_n p) \end{aligned}$$

induces an isomorphism of simplicial manifolds

$$E_\bullet G \times_G M \cong G \times M_\bullet,$$

where $E_\bullet G \times_G M$ is the quotient of $E_\bullet G \times M$ by the diagonal G -action. Furthermore, the bundle $E_n G \times M \rightarrow E_n G \times_G M \cong G^n \times M$ is trivial with section s

$$(80) \quad \begin{aligned} s: G^n \times M &\rightarrow G^{n+1} \times M = E_n G \times M, \\ (g_1, \dots, g_n, p) &\mapsto (e, g_1^{-1}, \dots, (g_1 \cdots g_n)^{-1}, g_1 \cdots g_n p). \end{aligned}$$

B.2. Simplicial differential forms. We recall the notion of simplicial differential forms introduced by Dupont [13, Def 2.1]:

Let X_\bullet be a simplicial manifold with face maps $d_i: X_q \rightarrow X_{q-1}$ for $i = 0, \dots, q$. Let $\Delta^q \subset \mathbb{R}^{q+1}$ be the standard q -simplex and $\varepsilon_i: \Delta^{q-1} \rightarrow \Delta^q$ the inclusion of the i -th face.

A **simplicial differential n -form** φ on X_\bullet consists of a sequence of forms

$$\varphi^{(q)} \in \Omega^n(\Delta^q \times X_q), \quad q = 0, 1, \dots$$

satisfying

$$(\varepsilon_i \times \text{id})^* \varphi^{(q)} = (\text{id} \times d_i)^* \varphi^{(q-1)}$$

for all q and all $i = 1, \dots, q$.

The set of all simplicial n -forms on X_\bullet is denoted $\Omega_{\text{spl}}^n(X_\bullet)$. Equipped with the usual de Rham differential d , $\Omega_{\text{spl}}^*(X_\bullet)$ is a differential graded commutative algebra, which is also the total complex of the following double complex:

$$(81) \quad \Omega_{\text{spl}}^n(X_\bullet) = \bigoplus_{j+k=n} \Omega_{\text{spl}}^{j,k}(X_\bullet).$$

Here, similar to (21), $\Omega_{\text{spl}}^{j,k}(X_\bullet)$ consists of those simplicial differential n -forms $\varphi = (\varphi^{(q)})$ with the property

$$\varphi^{(q)} \in \Gamma(\Delta^q \times X_q, \Lambda^j T^* \Delta^q \otimes \Lambda^k T^* X_q) \subset \Omega^{j+k}(\Delta^q \times X_q).$$

The de Rham differential d on $\Omega_{\text{spl}}^*(X_\bullet)$ is

$$d = d^\Delta + (-1)^j d^X,$$

where d^Δ and d^X denote the de Rham differentials in the Δ^q and X_q -directions, respectively.

Dupont proved that $\Omega^*(X_\bullet)$ and $\Omega_{\text{spl}}^*(X_\bullet)$ are quasi-isomorphic.

Theorem B.5 ([13, Thm. 2.3]). *There are natural maps of doubles complexes*

$$(\Omega_{\text{spl}}^{*,*}(X_\bullet), d^\Delta, d^X) \begin{array}{c} \xrightarrow{\mathcal{J}} \\ \xleftarrow{\mathcal{C}} \end{array} (\Omega^{*,*}(X_\bullet), \partial, d),$$

which give natural chain homotopy equivalences between $(\Omega_{\text{spl}}^{*,k}(X_\bullet), d^\Delta)$ and $(\Omega^{*,k}(X_\bullet), \partial)$.

In particular, \mathcal{C} and \mathcal{J} induce quasi-isomorphisms between the total complexes $(\Omega_{\text{spl}}^*(X_\bullet), d)$ and $(\Omega^*(X_\bullet), D)$.

The map \mathcal{J} in Dupont's theorem is integration over the fiber: if $\varphi \in \Omega_{\text{spl}}^{j,k}(X_\bullet)$, then

$$(82) \quad \mathcal{J}(\varphi) := \int_{\Delta^j} \varphi^{(j)} \in \Omega^k(X_j)$$

Going the other direction, if $\beta \in \Omega^k(X_j)$, then the simplicial form $\mathcal{C}(\beta) \in \Omega_{\text{spl}}^{j,k}(X_\bullet)$ is:

$$(83) \quad \mathcal{C}(\beta)^{(q)} := \begin{cases} j! \sum_{|I|=j} \sum_{\ell=0}^j (-1)^\ell t_{i_\ell} dt_{i_0} \wedge \dots \wedge \widehat{dt_{i_\ell}} \wedge \dots \wedge dt_{i_j} \wedge \mu_I^* \beta & \text{if } q \geq j, \\ 0 & \text{if } q < j. \end{cases}$$

Here $I = (i_0, \dots, i_j)$ is a multi-index with $0 \leq i_0 < \dots < i_j \leq q$, $|I| := j$, and $\mu_I = d_{\tilde{i}_{q-j}} \circ \dots \circ d_{\tilde{i}_1} : X_q \rightarrow X_j$ is the face map corresponding to the complementary sequence $0 \leq \tilde{i}_1 < \dots < \tilde{i}_{q-j} \leq q$ of I .

Composing the inclusion ι (76), π_M^* (77) and Dupont's map \mathcal{C} (83) for $X_\bullet = E_\bullet G \times M$, we obtain a quasi-isomorphism

$$(84) \quad \Omega^*(M) \xrightarrow{\iota} \Omega^*(M_\bullet) \xrightarrow{\pi_M^*} \Omega^*(E_\bullet G \times M) \xrightarrow{\mathcal{C}} \Omega_{\text{spl}}^*(E_\bullet G \times M).$$

Note that a k -form $\alpha \in \Omega^k(M)$ is mapped to the simplicial k -form $\mathcal{C}(\pi_M^* \alpha) \in \Omega_{\text{spl}}^{0,k}(E_\bullet G \times M)$ which is given by the sequence

$$\mathcal{C}(\pi_M^* \alpha)^{(q)} = \pi_M^* \alpha \in \Omega^{0,k}(\Delta^q \times E_q G \times M).$$

The map induced by $\mathcal{C} \circ \pi_M^* \circ \iota$ on the corresponding Cartan complexes, which we consider next, provides the first step in the construction of the chain map $C_G^*(M) \rightarrow \Omega^*(G \times M)$.

B.3. Cartan complexes. If A is a G^* -module in the sense of [22, Def. 2.3.1], with differential d^A and insertion operation $\iota_{\mathfrak{g}}^A$, let

$$(85) \quad \begin{aligned} C_G(A) &:= (S(\mathfrak{g}^\vee) \otimes A)^G, \\ d_G &= \delta + d^A \end{aligned}$$

denote the Cartan complex ([22, section 6.5]), where $\delta = -\text{Sym} \circ \iota_{\mathfrak{g}}^A$ the composition of $-\iota_{\mathfrak{g}}^A$ and the symmetrization $\text{Sym}: \mathfrak{g}^\vee \otimes S^*(\mathfrak{g}^\vee) \rightarrow S^{*+1}(\mathfrak{g}^\vee)$. This is also the total complex of the double complex

$$\begin{aligned} C_G^{i,j}(A) &:= (S^i(\mathfrak{g}^\vee) \otimes A^{j-i})^G, \\ C_G^*(A) &= \text{Tot}(C_G^{*,*}(A)). \end{aligned}$$

Define the decreasing filtration on $C_G(A)$:

$$(86) \quad F_p C_G(A) := \bigoplus_{i \geq p} \bigoplus_j C_G^{i,j}(A)$$

If A is bounded below, then the associated spectral sequence clearly converges.

Lemma B.6. *Let G be a compact Lie group, A and B two G^* -modules which are bounded below as complexes and let $\phi: A \rightarrow B$ be a quasi-isomorphism of G^* -modules, i.e., a morphism of G^* -modules, which induces an isomorphism of G -modules on total cohomology. Then the induced map of Cartan complexes*

$$\text{id}_{S(\mathfrak{g}^\vee)} \otimes \phi: C_G(A) \rightarrow C_G(B)$$

is a quasi-isomorphism.

Proof. The induced map $\text{id}_{S(\mathfrak{g}^\vee)} \otimes \phi$ respects the filtrations defined in (86). Since ϕ is a quasi-isomorphism and G is compact, $\text{id}_{S(\mathfrak{g}^\vee)} \otimes \phi$ induces an isomorphism between the E_1 pages

$$E_1^{p,q}(A) = (S^p(\mathfrak{g}^\vee) \otimes H^{q-p}(A))^G \rightarrow (S^p(\mathfrak{g}^\vee) \otimes H^{q-p}(B))^G = E_1^{p,q}(B)$$

of the associated spectral sequences (e.g. [22, Thm. 6.5.1]). Since A and B are bounded below, the filtrations are bounded in each degree. Therefore, $\text{id}_{S(\mathfrak{g}^\vee)} \otimes \pi_M^*$ is a quasi-isomorphism (e.g. [28, Thm. 3.5]). \square

Example B.7. For M a G -manifold the Cartan complex of $\Omega^*(M)$ with the usual G^* -module structure is the usual Cartan complex for M :

$$(87) \quad C_G(M) = C_G(\Omega^*(M)).$$

Example B.8. For a simplicial G -manifold X_\bullet , the complex $\Omega^*(X_\bullet) = \text{Tot}(\Omega^{*,*}(X_\bullet))$ (75) with differential $D = \partial + (-1)^j d$ and the insertion operation $\iota_{\mathfrak{g}}^{\Omega^*(X_\bullet)} := (-1)^j \iota_{\mathfrak{g}}$ is a G^* -module. Note that

$$D \iota_{\mathfrak{g}}^{\Omega^*(X_\bullet)} + \iota_{\mathfrak{g}}^{\Omega^*(X_\bullet)} D = d \iota_{\mathfrak{g}} + \iota_{\mathfrak{g}} d$$

is still the usual Lie derivative. Its Cartan complex

$$(88) \quad \begin{aligned} C_G^*(X_\bullet) &:= \left(C_G(\text{Tot}(\Omega^{*,*}(X_\bullet))), D_G \right), \\ D_G &:= (-1)^j \delta + D = \partial + (-1)^j \delta + (-1)^j d \end{aligned}$$

is also the total complex of the tricomplex

$$C_G^{i,j,k}(X_\bullet) := \left(S^i(\mathfrak{g}^\vee) \otimes \Omega^{k-i}(X_j) \right)^G.$$

Note the grading is such that if $x \in C_G^{i,j,k}(X_\bullet)$, then $|x| = i + j + k$.

Example B.9. For a simplicial G -manifold X_\bullet , consider the Cartan complex of $\Omega_{\text{spl}}^*(X_\bullet)$ (81) with the usual G^* -module structure:

$$(89) \quad \begin{aligned} C_{G,\text{spl}}^*(X_\bullet) &:= C_G(\Omega_{\text{spl}}^*(X_\bullet)) \\ D_G &:= \delta + d^\Delta + (-1)^j d^X. \end{aligned}$$

Note that this is the total complex of the tricomplex

$$(90) \quad C_{G,\text{spl}}^{i,j,k}(X_\bullet) := \left(S^i(\mathfrak{g}^\vee) \otimes \Omega_{\text{spl}}^{j,k-i}(X_\bullet) \right)^G \subset \prod_{q=0}^{\infty} \left(S^i(\mathfrak{g}^\vee) \otimes \Omega^{j,k-i}(\Delta^q \times X_q) \right)^G.$$

Note that the G^* -module structures on $\Omega(M)$, $\Omega^*(E_\bullet G \times M)$ and $\Omega_{\text{spl}}^*(E_\bullet G \times M)$ are chosen in such a way, that the quasi-isomorphisms ι (76), π_M^* (77) and Dupont's map \mathcal{C} (83) are maps of G^* -modules. Therefore, Lemma B.6 now implies

Proposition B.10. *If G is compact, then the map induced by $\mathcal{C} \circ \pi_M^* \circ \iota: \Omega^*(M) \rightarrow \Omega_{\text{spl}}^*(E_\bullet G \times M)$ on the total Cartan complexes*

$$J := \text{id}_{S(\mathfrak{g}^\vee)} \otimes (\mathcal{C} \circ \pi_M^* \circ \iota): C_G^*(M) \rightarrow C_{G,\text{spl}}^*(E_\bullet G \times M).$$

is a quasi-isomorphism.

B.4. The Cartan map. Given a principal G -bundle $\pi: P \rightarrow B$ with connection $A \in \Omega^1(P, \mathfrak{g})^G$, Cartan [10] constructed a chain map $C_G^*(P) \rightarrow \Omega^*(B)$ now known as the **Cartan map**:

$$(91) \quad \text{Car}^A: C_G^*(P) \rightarrow \Omega^*(P)_{\text{hor}}^G \cong \Omega^*(B).$$

If $\beta \in (S^i(\mathfrak{g}^\vee) \otimes \Omega^*(P))^G$, then

$$\text{Car}^A(\beta) := \text{hor}_A(\langle (F_A)^i, \beta \rangle) \in \Omega^{*+2i}(P)_{\text{hor}}^G.$$

Here, $F_A = dA + \frac{1}{2}[A, A] \in \Omega^2(P, \mathfrak{g})_{\text{hor}}^G$ is the curvature of A ,

$$(F_A)^i = \underbrace{F_A \wedge \cdots \wedge F_A}_i \in \Omega^{2i}(P, \mathfrak{g}^{\otimes i})_{\text{hor}}^G,$$

$\langle \cdot, \cdot \rangle$ denotes the natural pairing induced by $\mathfrak{g} \otimes \mathfrak{g}^\vee \rightarrow \mathbb{R}$, and $\text{hor}_A: \Omega^*(P)^G \rightarrow \Omega^*(P)_{\text{hor}}^G$ denotes the projection to horizontal forms defined by A .

Remark B.11. Recall that $\pi^*: \Omega^*(B) \rightarrow \Omega^*(P)_{\text{hor}}^G$ is an isomorphism. If G is compact, then π^* composed with the inclusion

$$\Omega^*(B) \xrightarrow{\pi^*} \Omega^*(P)_{\text{hor}}^G \hookrightarrow C_G^*(P)$$

induces an isomorphism in cohomology, with homotopy inverse Car^A .

There is also a simplicial version of this construction: Let $P_\bullet \rightarrow B_\bullet$ be a simplicial principal G -bundle with a simplicial connection $A \in \Omega_{\text{spl}}^1(P_\bullet, \mathfrak{g})^G$. The connection A is given by a sequence of connections $A^{(q)} \in \Omega^1(\Delta^q \times P_q, \mathfrak{g})^G$ on the principal G -bundles $\Delta^q \times P_q \rightarrow \Delta^q \times B_p$. Applying the degree-wise Cartan maps

$$\text{Car}^{A^{(q)}}: C_G^*(\Delta^q \times P_q) \rightarrow \Omega^*(\Delta^q \times B_q)$$

gives a chain map between the total complexes (90) and (81)

$$(92) \quad \text{Car}^A: C_{G,\text{spl}}^*(P_\bullet) \rightarrow \Omega_{\text{spl}}^*(B_\bullet).$$

If G is compact, then Car^A is a quasi-isomorphism.

Example B.12. Let M be a G -manifold and $E_\bullet G \times M \rightarrow E_\bullet G \times_G M$ the simplicial principal G -bundle. For $i = 0, \dots, q$, let

$$\pi_i: E_q G \times M = G^{q+1} \times M \rightarrow G$$

denote the projections and let $\theta_L \in \Omega^1(G, \mathfrak{g})^G$ denote the left-invariant Maurer-Cartan form on G . Following Dupont [13], we consider the canonical simplicial connection

$$(93) \quad \begin{aligned} \theta &= (\theta^{(q)}) \in \Omega_{\text{spl}}^{0,1}(E_\bullet G \times M, \mathfrak{g})^G, \\ \theta^{(q)} &:= \sum_{i=0}^q t_i \pi_i^* \theta_L \in \Omega^1(\Delta^q \times E_q G \times M, \mathfrak{g})^G, \end{aligned}$$

where $t_i, i = 0, \dots, q$ are barycentric coordinates on Δ^q . The curvature of θ is

$$F_{\theta^{(q)}} = \underbrace{d\Delta\theta^{(q)}}_{\text{type 1,1}} + \underbrace{d^{E_q G \times M}\theta^{(q)} + \frac{1}{2}[\theta^{(q)}, \theta^{(q)}]}_{\text{type 0,2}} \in \Omega^{1,1}(\Delta^q \times E_q G \times M, \mathfrak{g})_{\text{hor}}^G \oplus \Omega^{0,2}(\Delta^q \times E_q G \times M, \mathfrak{g})_{\text{hor}}^G.$$

Making use of the trivializing section $s: G^q \times M \rightarrow G^{q+1} \times M$ (80), we can write the Cartan map for the connection $\theta^{(q)}$ on $\Delta^q \times E_q G \times M \rightarrow \Delta^q \times G^q \times M$ as

$$(94) \quad \begin{aligned} \text{Car}^{\theta^{(q)}} : C_G^{i,j}(\Delta^q \times E_q G \times M) &\rightarrow \Omega^{i+j}(\Delta^q \times G^q \times M), \\ \beta &\mapsto \langle s^* F_{\theta^{(q)}}^i, s^* \text{hor}_{\theta^{(q)}} \beta \rangle. \end{aligned}$$

For example, for $q = 0$, we have $\theta^{(0)} = \pi_0^* \theta_L$, $F_{\theta^{(0)}} = 0$ and $s^* \text{hor}_{\theta^{(0)}} = s^*$, and hence

$$(95) \quad \text{Car}^{\theta^{(0)}}(\beta) = \begin{cases} s^* \beta & \text{if } \beta \in C_G^{0,*}(G \times M), \\ 0 & \text{else.} \end{cases}$$

For $q = 1$, we have

$$(96) \quad \begin{aligned} \theta^{(1)} &= t_0 \pi_0^* \theta_L + t_1 \pi_1^* \theta_L, \\ F_{\theta^{(1)}} &= -dt_1 \wedge (\pi_0^* \theta_L - \pi_1^* \theta_L) - \frac{t_0 t_1}{2} [\pi_0^* \theta_L - \pi_1^* \theta_L, \pi_0^* \theta_L - \pi_1^* \theta_L]. \end{aligned}$$

Let $I: G \rightarrow G$ denote the inversion on G , L_g the diffeomorphism of M corresponding to g and let $\theta_R \in \Omega^1(G, \mathfrak{g})$ be the right-invariant Maurer-Cartan form. The differential of the section $s: G \times M \rightarrow G^2 \times M$ from (80) is

$$(97) \quad s_*|_{(g,p)}(\tilde{x}, w) = (0, I_*(\tilde{x}), (L_g)_*(w) - v_{\theta_R(\tilde{x})}|_{gp}) \quad \text{for } \tilde{x} \in T_g G, w \in T_p M.$$

Therefore,

$$(98) \quad \begin{aligned} s^* F_{\theta^{(1)}} &= -dt_1 \wedge \pi_G^* \theta_R - \frac{t_0 t_1}{2} \pi_G^* [\theta_R, \theta_R], \\ s^* F_{\theta^{(1)}}^i &= (-1)^i i \frac{(t_0 t_1)^{i-1}}{2^{i-1}} dt_1 \wedge \pi_G^* (\theta_R \wedge [\theta_R, \theta_R]^{i-1}) + (-\frac{t_0 t_1}{2} \pi_G^* [\theta_R, \theta_R])^i. \end{aligned}$$

The horizontal projection for the connection $\theta^{(1)}$ on $\Delta^1 \times E_1 G \times M = \Delta^1 \times G^2 \times M$ is given by

$$\begin{aligned} T_{(t, g_0, g_1, p)}(\Delta^1 \times G^2 \times M) &\rightarrow T_{(t, g_0, g_1, p)}(\Delta^1 \times G^2 \times M), \\ (a, \tilde{x}_0, \tilde{x}_1, w') &\mapsto (a, \tilde{x}_0, \tilde{x}_1, w') - v_{t_0 \theta_L(\tilde{x}_0) + t_1 \theta_L(\tilde{x}_1)}, \end{aligned}$$

where $v^{G^2 \times M}$ is the fundamental vector field for the action $G \curvearrowright G^2 \times M$ (79). In particular, its component in $T_p M$ is

$$(99) \quad (\pi_M)_*(\text{hor}_{\theta^{(1)}}(a, \tilde{x}_0, \tilde{x}_1, w')) = w' - t_0 v_{\theta_L(\tilde{x}_0)}|_p - t_1 v_{\theta_L(\tilde{x}_1)}|_p.$$

Remark B.13. Note that, since $F_{\theta^{(q)}}$ is of homogeneous total degree 2, but not of homogeneous bi-degree, the chain map

$$(100) \quad \text{Car}^\theta: C_{G, \text{spl}}^*(E_\bullet G \times M) \rightarrow \Omega_{\text{spl}}^*(G \times M)$$

only preserves the grading on the total complexes, but maps elements of homogeneous tri-degree to elements of inhomogeneous bi-degree.

Remark B.14. Note that $\theta = \mathcal{C}(\pi_0^*\theta_L)$, with $\pi_0^*\theta_L \in \Omega^1(E_0G \times M, \mathfrak{g})^G = \Omega^1(G \times M, \mathfrak{g})^G$ the pullback of the left-invariant Maurer-Cartan form.

B.5. Cartan complex and Bott-Shulman-Stasheff complex. We can now compose the chain maps defined above and, if G is compact, obtain a quasi-isomorphism between the Cartan complex and the Bott-Shulman-Stasheff complex:

Proposition B.15. *Let G be a Lie group and M a G -manifold. There is a natural chain map from the Cartan model to the Bott-Shulman-Stasheff model*

$$(101) \quad \Phi: C_G^*(M) \xrightarrow{j} C_{G,\text{spl}}^*(E_\bullet G \times M) \xrightarrow{\text{Car}^\theta} \Omega_{\text{spl}}^*(G \times M_\bullet) \xrightarrow{J} \Omega^*(G \times M_\bullet),$$

where

- j is the chain map from Prop. B.10,
- Car^θ is the simplicial Cartan map (92) for the simplicial connection θ (93) on $E_\bullet G \times M \rightarrow E_\bullet G \times_G M \cong G \times M_\bullet$,
- J is the quasi-isomorphism (82) defined by Dupont in Thm. B.5.

If G is compact, then all of the above are quasi-isomorphisms and hence $\Phi: C_G^*(M) \rightarrow \Omega^*(G \times M_\bullet)$ is a quasi-isomorphism.

Remark B.16. Note that if G is not compact, j and Car^θ can fail to be quasi-isomorphisms.

In the case of j , taking invariants will in general not commute with taking cohomology, [22, Thm. 6.5.1] and hence Lemma B.6 fails in general.

If G is non-compact, then, in general, the Cartan model does not compute the equivariant cohomology. Since the equivariant cohomology of a principal G -bundle equals the cohomology of the base, the Cartan map Car^θ is not an isomorphism in general.

B.6. Homotopy moment maps from Cartan cocycles. We are now in the position to prove Thm. 6.8 by applying Thm. 6.6 to the image of a cocycle in the Cartan complex under the chain map (101) and to obtain an explicit homotopy moment map:

Proof of Thm. 6.8. Let $\omega + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} P_i \in C_G^{n+1}(M)$ be a cocycle. If G is compact, then using the chain map Φ (101) and the second part of Thm. 6.6, we immediately obtain a homotopy moment map.

If G is an arbitrary Lie group, we will show that the $\Omega^{0,n+1}(G \times M_\bullet) = \Omega^{n+1}(M)$ and $\Omega^{1,n}(G \times M_\bullet) = \Omega^n(G \times M)$ -components of the image of $\omega + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} P_i$ in $\Omega^{n+1}(G \times M_\bullet)$ are, in fact, G -invariant. This will then allow us to use the first part of Thm. 6.6 to construct a homotopy moment map.

The elements

$$\begin{aligned} j(\omega) &\in C_{G,\text{spl}}^{0,0,n+1}(E_\bullet G \times M) \subset \prod_{q=0}^{\infty} \Omega^{0,n+1}(\Delta^q \times E_q G \times M)^G, \\ j(P_i) &\in C_{G,\text{spl}}^{i,0,n-i+1}(E_\bullet G \times M) \subset \prod_{q=0}^{\infty} \left(S^i(\mathfrak{g}^\vee) \otimes \Omega^{0,n-2i+1}(\Delta^q \times E_q G \times M) \right)^G, \end{aligned}$$

are given by the sequences $j(\omega)^{(q)} = \pi_M^* \omega$ and $j(P_i)^{(q)} = \pi_M^* P_i$, respectively.

The images of $j(\omega)$ and $j(P_i)$ under the simplicial Cartan map Car^θ are given by the sequences

$$(102) \quad \begin{aligned} \text{Car}^{\theta^{(q)}}(\pi_M^* \omega) &= s^* \text{hor}_{\theta^{(q)}} \pi_M^* \omega, \\ \text{Car}^{\theta^{(q)}}(\pi_M^* P_i) &= \langle s^* F_{\theta^{(q)}}^i, s^* \text{hor}_{\theta^{(q)}} \pi_M^* P_i \rangle, \end{aligned}$$

respectively. Here s is the trivializing section (80).

Keeping in mind that Thm. 6.6 only uses the components in $\Omega^{0,n+1}(G \times M_\bullet)$ and $\Omega^{1,n}(G \times M_\bullet)$, we only need to compute the $(0, n+1)$ - and $(1, n)$ -components of $\mathcal{J}(\text{Car}^\theta(j(\omega)))$ and $\mathcal{J}(\text{Car}^\theta(j(P_i)))$. Since \mathcal{J} is a map of bicomplexes, we only need the Cartan maps for $\theta^{(0)}$ and $\theta^{(1)}$.

Equation (95) implies that the $\Omega^{0,n+1}(G \times M_\bullet)$ -component of $\mathcal{J}(\text{Car}^\theta(j(\omega + \sum_i P_i)))$ is indeed

$$\int_{\Delta^0} \text{Car}^{\theta^{(0)}} \left(\pi_M^* \omega + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} P_i \right) = s^* \pi_M^* \omega = \omega,$$

and, in particular, G -invariant.

We now turn to the $\Omega^{1,n}(G \times M_\bullet)$ -component. Since $\text{Car}^{\theta^{(1)}}(\pi_M^* \omega) \in \Omega^{0,n+1}(\Delta^1 \times E_1 G \times M)^G$, we have

$$\int_{\Delta^1} \text{Car}^{\theta^{(1)}}(\pi_M^* \omega) = 0.$$

Thus, the $\Omega^{1,n}(G \times M_\bullet)$ -component of $\mathcal{J}(\text{Car}^\theta(j(\omega + \sum_i P_i)))$, from which the homotopy moment map is constructed, is

$$\sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \int_{\Delta^1} \text{Car}^{\theta^{(1)}}(\pi_M^* P_i) = \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \int_{\Delta^1} \langle s^* F_{\theta^{(1)}}^i, s^* \text{hor}_{\theta^{(1)}} \pi_M^* P_i \rangle.$$

We will now compute this explicitly, and also show that it defines a G -invariant n -form on $G \times M$, so that we can apply the first part of Thm. 6.6.

Combining (97) and (99), and using $I^* \theta_L = -\theta_R$ as well as $t_0 = 1 - t_1$ and $(L_{g^{-1}})_*(v_{\theta_R(\tilde{x})}) = v_{\theta_L(\tilde{x})}$, we have

$$\begin{aligned} (\pi_M)_*(\text{hor}_{\theta^{(1)}} s_*(\tilde{x}, w)) &= (\pi_M)_*(\text{hor}_{\theta^{(1)}}(0, I_*(\tilde{x}), (L_g)_*(w) - v_{\theta_R(\tilde{x})}|_{gp})) \\ &= (L_g)_*(w) - v_{\theta_R(\tilde{x})}|_{gp} - t_1 v_{I^* \theta_L(\tilde{x})}|_{gp} = (L_g)_*(w) - t_0 v_{\theta_R(\tilde{x})}|_{gp} \\ &= (L_g)_*(w - t_0 v_{\theta_L(\tilde{x})}|_p) \end{aligned}$$

for all $(\tilde{x}, w) \in T_{(g,p)}(G \times M)$. Using the G -invariance of P_i , i.e. $L_g^* P_i = \text{Ad}_g^\vee P_i$, we have

$$(103) \quad s^* \text{hor}_{\theta^{(1)}} \pi_M^* P_i = (\text{Ad}_g^\vee)^{\otimes i} P_i \circ \phi_{t_0}^{\otimes (n-2i+1)},$$

where ϕ_{t_0} denotes the map $T_g G \oplus T_p M \ni (\tilde{x}, w) \mapsto w - t_0 v_{\theta_L(\tilde{x})}|_p \in T_p M$. Combining (94), (98), (103) and $\text{Ad}_{g^{-1}} \theta_R = \theta_L$, we have

$$\text{Car}^{\theta^{(1)}}(\pi_M^* P_i) = \langle (-1)^i i \frac{(t_0 t_1)^{i-1}}{2^{i-1}} dt_1 \wedge \pi_G^*(\theta_L \wedge [\theta_L, \theta_L]^{i-1}) + (-\frac{t_0 t_1}{2} \pi_G^*[\theta_L, \theta_L])^i, P_i \circ \phi_{t_0}^{\otimes (n-2i+1)} \rangle.$$

Since ϕ_{t_0} is invariant under the left action $(g, (t, h, p)) \mapsto (t, gh, p)$, this also proves that

$$\text{Car}^{\theta^{(1)}}(\pi_M^* P_i) \in \Omega^n(\Delta^1 \times G \times M)^G,$$

where G acts by the same action. Hence also

$$(104) \quad \int_{\Delta^1} \text{Car}^{\theta^{(1)}}(\pi_M^* P_i) = \int_{\Delta^1} (-1)^i i \frac{(t_0 t_1)^{i-1}}{2^{i-1}} dt_1 \wedge \langle \pi_G^*(\theta_L \wedge [\theta_L, \theta_L]^{i-1}), P_i \circ \phi_{t_0}^{\otimes (n-2i+1)} \rangle \in \Omega^{1,n}(G \times M_\bullet)^G.$$

However, recall that ϕ_{t_0} depends on $t_0 = 1 - t_1$. For $x_1, \dots, x_k \in \mathfrak{g}$ and $w_1, \dots, w_{n-k} \in T_p M$ we have

$$\begin{aligned} &\langle \pi_G^*(\theta_L \wedge [\theta_L, \theta_L]^{i-1}), P_i \circ \phi_{t_0}^{\otimes (n-2i+1)} \rangle((x_1, 0), \dots, (x_k, 0), (0, w_1), \dots, (0, w_{n-k})) \\ &= \sum_{\sigma \in \text{Sh}(2i-1, k-2i+1)} (-1)^\sigma (-t_0)^{k-2i+1} \langle \theta_L \wedge [\theta_L, \theta_L]^{i-1}(x_{\sigma(1)}, \dots, x_{\sigma(2i-1)}), P_i(v_{x_{\sigma(2i)}}, \dots, v_{x_{\sigma(k)}}, w_1, \dots, w_{n-k}) \rangle \\ &= \frac{k! t_0^{k-2i+1}}{(k-2i+1)!} \text{Alt}_k \left(\underbrace{(\iota_{\mathfrak{g}}^{k-2i+1} P_i(\cdot, [\cdot, \cdot], \dots, [\cdot, \cdot]))}_{i-1} (x_1, \dots, x_k)(w_1, \dots, w_{n-k}) \right). \end{aligned}$$

Combining this with (104), and since $\int_0^1 t_0^{k-i} t_1^{i-1} dt_1 = \frac{(i-1)!(k-i)!}{k!}$, we see that the image of $\omega + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} P_i$ in $C_{\mathfrak{g}}^*(M)$ is

$$f^{\zeta} := \sum_{k=1}^n \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} r \left(\int_{\Delta^1} \text{Car}^{\theta^{(1)}}(\pi_M^* P_i) \right)_k = \underbrace{\sum_{k=1}^n \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{(-1)^i i!(k-i)!}{2^{i-1}(k-2i+1)!} \text{Alt}_k \left(t_{\mathfrak{g}}^{k-2i+1} P_i(\cdot, \underbrace{[\cdot, \cdot], \dots, [\cdot, \cdot]}_{i-1}) \right)}_{f_k^{\zeta}}.$$

With $f_k = \zeta(k) f_k^{\zeta}$, this completes the proof. □

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MATHEMATISCHES INSTITUT, GEORG-AUGUST UNIVERSITÄT GÖTTINGEN, BUNSENSTRASSE 3-5, D-37073 GÖTTINGEN, GERMANY. CURRENT ADDRESS: FAKULTÄT FÜR MATHEMATIK, SFB 701, UNIVERSITÄT BIELEFELD, POSTFACH 100131, 33501 BIELEFELD, GERMANY

E-mail address: mcallies@math.uni-bielefeld.de

MIT/UNIVERSITY OF ZURICH/UNIVERSITÉ D’ARTOIS

E-mail address: yael.fregier@math.mit.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEVADA, RENO, 1664 N. VIRGINIA STREET RENO, NV 89557-0084 USA

E-mail address: chrisrogers@unr.edu, chris.rogers.math@gmail.com

KU LEUVEN, DEPARTMENT OF MATHEMATICS, CELESTIJENLAAN 200B BOX 2400, BE-3001 LEUVEN, BELGIUM

E-mail address: marco.zambon@wis.kuleuven.be