Identification and quantification of multivariate interval uncertainty in Finite Element models

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Abstract

The objective of this work is to develop and validate a methodology for the identification and quantification of multivariate interval uncertainty in finite element models. The principal idea is to find a solution to an inverse problem, where the variability on the output side of the model is known from measurement data, but the multivariate uncertainty on the input parameters is unknown. For this purpose, the uncertain simulation results set created by propagating interval uncertainty through the model is represented by its convex hull. The same concept is used to model the uncertainty in the measurements. A metric to describe the discrepancy between these convex hulls is defined based on the difference between their volumes and their mutual intersection. By minimisation of this metric, the interval uncertainty on the input side of the model is identified. It is further shown how the procedure can be optimized with respect to output quantity selection. Validation of the methodology is done using simulated measurement data in two case studies. Numerically exact identification of multiple, coupled parameters having interval uncertainty is possible following the proposed methodology. Furthermore, the robustness of the method with respect to the analysts initial estimate of the input uncertainty is illustrated. The method presented in this work is generic, but for the examples in this paper, it is specifically applied to dynamic models, using eigenfrequencies as output quantities, as

commonly applied in modal updating procedures.¹²

Keywords:

Finite Element Modelling, Multivariate Uncertainty, Possibilistic, Uncertainty Quantification, Interval Field

1. INTRODUCTION

1.1. *Non-determinism in FE computations*

Numerical modelling techniques have gained increasing interest in engineering, due to the exponential growth in computing power. The deterministic Finite Element (FE) method, an efficient tool to accurately solve the Partial Differential Equations (PDE) that govern most real-world problems, has become an indispensable tool for an engineer in various design stages. Improvements in computing power allow for making finer meshes; hence more detailed models can be used in engineering practice. A more recent approach however is to use enhanced computing power to integrate model parameter uncertainty into these FE models [1]. By applying non-deterministic approaches for FE simulations throughout different stages of a product design, a more robust and reliable design can be achieved, while reducing the cost of the experimental test phase. This is because the design can account for variability in early design stages. Especially for these design purposes, the study of inter-variability (variability between different nominal identical products) as well as intra-variability (variability of model properties within a single component, also referred to as ‘spatial variability’) proves to be extremely valuable.

Two distinctive philosophies exist for propagating non-deterministic quantities through FE analyses, i.e. the probabilistic and the possibilistic approach. In a probabilistic approach, uncertainty is represented as a probability density function (PDF) and thus propagated through an FE model (probabilistic Finite Element, PFE) using sampling or perturbation methods, possibly through a meta-model [2]. This technique has reached wide acceptance among academia and industry and it has been validated in a wide variety of applications varying from machine design to biomedics [3, 4, 5, 6, 7].

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However, application of this theory in practice often proves to be inconvenient, due to the large amount of data necessary for the identification of a PDF or the corresponding statistical moments. In this context, a review of literature shows that especially in industrial applications, many authors assume the data to follow a Gaussian distribution. This however can be a severe misjudgement of the non-deterministic data structure, resulting in an unrealistic assessment of the non-deterministic model behaviour, and consequently, the design quality. Furthermore, these results often come at a very high computational cost [8, 9].

Alternatively, possibilistic approaches such as interval Finite Elements (IFE) or Fuzzy Finite Elements (FFE) were introduced, where the uncertainty is represented by an interval number or fuzzy set and thus propagated. In the IFE concept, the uncertainty is modelled as two crisp bounds on the variable. The FFE concept is an extension to this principle, where the membership of a variable to an interval is considered as a continuous function, ranging from 0 (the variable is certainly not within the interval) to 1 (the variable is for sure in the interval). These concepts relax the need for the identification of a full probabilistic data description, which may be very cumbersome. Moreover, less expensive numerical procedures are necessary for the description of the variability [10, 11, 12, 13].

1.2. Problem statement, literature review and general overview

Despite the availability of very advanced techniques for the uncertainty propagation, the use of representative measurement data to quantify the input uncertainty of FE analyses remains nowadays the major challenge. Especially for a designer who is facing a substantial (and often beforehand unknown) variability in the realisation of his design, this knowledge is indispensable for the creation of reliable products. Techniques for updating IFE and FFE models on experimental data were introduced only very recently in literature. **Haag et al.** introduced in this context an inverse methodology for the identification of the parameters of FFE models [14]. **Khodaparast et al.** employed a Kriging predictor as a meta-model for the updating of interval [15] and fuzzy [16] finite element models, based on the minimisation of the difference between simulation and measurement data. **Erdogan and Bakir** used a hybrid combination of a Genetic Algorithm and Particle Swarm as updating scheme for the minimisation of the difference between measurement and simulation data in the context of the updating of Fuzzy

Finite Element models [17]. **Fedele et al.** used an adjoint optimisation technique to estimate the bounds of a model parameter, based on uncertain raw measurements [18]. They handled overestimation of the interval width using the inclusion isotonicity property of interval arithmetic. Finally, **Fang et al.** employed interval response surface models for the inverse propagation of the interval uncertainty in the framework of incorporating measurements into IFE analyses [19]. An alternative route of identifying possibilistic parameter uncertainty was introduced by **Elishakoff and Sarlin**. They proposed the concept of enclosing uncertainty in the parameters of a numerical model by a convex geometric figure having a minimal area. They then inflate these figures following a Chebyshev inequality in order to take into account the forecasted data [20, 21].

This paper focuses on the identification of multivariate interval uncertainty on the input parameters of an FE model following an inverse approach. The variability in the measured responses (e.g., eigenfrequencies) is represented as the multidimensional convex hull of the complete set of measurement replica. Identification is performed based on the minimisation of the discrepancy between this convex hull and the variability that is predicted by the responses that are computed using the IFE method. Also the optimal responses for the identification are selected from the measurement data set following a novel approach. Simulated measurement data are used for demonstrating the developed principle and an accurate identification and quantification of the interval uncertainty is shown to be obtainable. Moreover, because of the fact that the method takes into consideration the dependency between output responses, an identification of spatial uncertainty (e.g., in the context of interval fields) is shown to be possible, which is novel with respect to the approaches that exist to date in literature.

The method is of particular interest for an analyst who has a numerical model containing unknown non-determinism in its input parameters. This non-determinism can be identified and quantified by comparison of the result of the non-deterministic numerical model with measurement data, obtained via an experimental procedure. The paper is structured as follows. Section 2 gives an overview of the interval Finite Element method. The identification procedure is introduced in section 3. Section 4 introduces a reduction scheme for selecting the optimal responses in the context of multidimensional interval identification. An illustration of the methodology using an academic discrete mass-spring system is shown in section 5. In section 6, a more in-depth validation of the method using a dynamic finite element model of a cantilever

beam is presented. Finally, the most important conclusions are listed in section 7.

2. The interval field Finite Element Method

This section describes the interval field FE method, which can be used to model spatially varying possibilistic uncertainty in a numerically efficient way. By definition, an interval parameter x is indicated using apex I: x^I . Vectors are expressed as lower-case boldface characters $\mathbf{x}$, whereas matrices are expressed as upper-case boldface characters $\mathbf{X}$. For the remainder of the text, interval parameters are either represented using the bounds of the interval $x^I = [\underline{x}; \bar{x}]$ or the centre point $\hat{x} = \frac{\underline{x} + \bar{x}}{2}$ and interval radius $r_x = \frac{\bar{x} - \underline{x}}{2}$.

2.1. Interval Finite Element Analysis

The goal of an interval FE calculation is to find the bounds on the uncertainty in a general d -dimensional result $\mathbf{y} \in \mathbb{R}^d$ of an FE calculation $f() : \mathbb{R}^k \mapsto \mathbb{R}^d$, based on an interval description of the uncertainty in k physical properties $\boldsymbol{\gamma}^I \in \mathbb{IR}^k$, which are defined as parameters on the input side of the model. $\mathbb{IR}^k$ denotes the space of k -dimensional real-valued interval vectors. In the specific case when $f()$ is solved for the first d eigenfrequencies, $\mathbf{y}$ is the result vector containing these eigenfrequencies; in the case when a quasi-static model $f()$ is considered, $\mathbf{y}$ may contain the displacements of the d degrees of freedom of the model or corresponding strains. $\boldsymbol{\gamma}^I$ spans a hypercubic set by definition, as the nature of interval vectors does not allow for any dependence between their components [11]. The interval FE method can thus be expressed as finding the solution set $\tilde{\mathbf{y}}$:

$$\tilde{\mathbf{y}} = \{\mathbf{y} | \mathbf{y} = f(\boldsymbol{\gamma}) \wedge \boldsymbol{\gamma} \in \boldsymbol{\gamma}^I\} \quad (1)$$

Generally, $\tilde{\mathbf{y}}$ spans an output set in the d -dimensional simulation space $\mathbb{R}^d$. This output set is not necessarily convex, as the output responses y_i are coupled through the PDE of the FE model under consideration. It therefore cannot be represented as an interval vector, and is very hard to compute numerically. Therefore, instead of calculating the *real* uncertain solution set spanned by $\tilde{\mathbf{y}}$, the uncertainty at the output of the FE model generally is approximated using an interval vector $\mathbf{y}^I$:

$$\mathbf{y}^I = [y_1^I \ y_2^I \ \dots \ y_d^I]^T \quad (2)$$

$\mathbf{y}^I$ is usually calculated following an optimisation approach, where the bounds $\underline{y}_i$ and $\bar{y}_i$ on each output quantity y_i of the solution interval vector $\mathbf{y}^I$ are determined by searching the domain, defined by $\boldsymbol{\gamma}^I$ [11]:

$$\begin{aligned}\underline{y}_i &= \min_{\boldsymbol{\gamma} \in \boldsymbol{\gamma}^I} f_i(\boldsymbol{\gamma}) & i = 1, \dots, n \\ \bar{y}_i &= \max_{\boldsymbol{\gamma} \in \boldsymbol{\gamma}^I} f_i(\boldsymbol{\gamma}) & i = 1, \dots, n\end{aligned}\tag{3}$$

where $y_i^I = [\underline{y}_i; \bar{y}_i]$ is the result interval scalar for the i^{th} component of the solution interval vector of the model.

Solution of equation (3) returns the smallest hypercubic approximation $\mathbf{y}^I$ of the uncertainty in the solution set $\tilde{\mathbf{y}}$. This concept is also visualised in figure 1, where the d -dimensional uncertain solution set $\tilde{\mathbf{y}}$ and its hypercubic approximation $\mathbf{y}^I$ are shown as a cross-section on two arbitrary output quantities y_i and y_j . The interval results can become extremely conservative, as all possible dependency between the d output quantities in $\tilde{\mathbf{y}}$ is neglected. Realisations of the output vector which are in the hatched area in figure 1 are unphysical, even if they are included within $\mathbf{y}^I$.

Figure 1: Illustration of the concept of hypercubic approximation $\mathbf{y}^I$ of the uncertain solution set $\tilde{\mathbf{y}}$, shown as a cross-section on two arbitrary output quantities y_i and y_j of the result vector of the model.

Due to the large computational cost usually associated with the solution of equation (3), it is common practice to replace the FE calculation $f()$ by a surrogate meta-model such as an adaptive response surface model [22], local expansion technique [23] or sparse grid models [24]. In case of strict monotonicity of $f()$, the transformation method [13] can also be applied,

where the minimisation is replaced by a linear interpolation between the output quantities, calculated through the propagation of the vertices of the hypercubic uncertain input set $\boldsymbol{\gamma}^I$.

2.2. Interval fields

Interval fields (IF) are a specific case of multivariate interval uncertainty, where the interval parameters are continuous functions of the spatial variable $\mathbf{r}$, which is defined over the model domain Ω [25, 26, 27, 28, 29]. The concept provides an extension to the classic interval paradigm for the representation of spatial uncertainty, where one independent interval per individual element in the FE model is considered, neglecting all possible dependence throughout Ω . By neglecting dependence between two adjacent locations in Ω , unphysical realisations of the uncertainty throughout the model are explicitly included in the uncertainty representation. Moreover, the numerical cost of evaluating the model increases drastically as the minimisation in eq. (3) needs to be performed in the full k -dimensional input space on which the input intervals are defined. This is particularly problematic for industrially sized FE models containing up to millions of nodes in its discretisation.

Following the method of **Verhaeghe et al.**, the description of an IF is based on the superposition of n_b base vectors $\boldsymbol{\psi}_i$, which represent a set of spatial uncertainty *patterns*, scaled by independent interval scalars α_i^I , which represent the uncertainty at the input side of the model. These base vectors provide a mapping from the full k -dimensional input space to a reduced n_b dimensional input space. Formally, an interval field for the description of spatial uncertainty in the input parameters $\boldsymbol{\gamma}_{\mathbf{F}}^I(\mathbf{r}) : \mathbb{R}^k \mapsto \mathbb{R}^{n_b}$ is expressed in the explicit form as:

$$\boldsymbol{\gamma}_{\mathbf{F}}^I(\mathbf{r}) = \sum_{i=1}^{n_b} \alpha_i^I \boldsymbol{\psi}_i(\mathbf{r}) \quad (4)$$

with $n_b \leq k$. Next to a more realistic representation of the actual realisations, which are physically possible, also the dimension of the input space is reduced drastically as compared to defining an independent local interval for each node of the model. This approach therefore reduces the numerical cost of an evaluation of the interval field FE model. Thanks to the specific nature of the interval field concept, shown in equation 4, commonly used techniques for the propagation of the uncertainty in the input space, such as response surfaces or the transformation method, still remain applicable. This may be attributed

to the fact that in the explicit interval field formulation (4), the uncertainty is contained only in the interval scalars α_i^I , which are decoupled from the spatial nature of the uncertainty, which is contained in the base functions $\psi_i(\mathbf{r})$. The uncertainty at the input side thus remains as yet hypercubic .

3. Identification of multivariate interval uncertainty

The application of the interval concept for the representation of multivariate uncertainty requires the identification of the interval vector $\gamma^I \in \mathbb{R}^k$ which contains this uncertainty for the input parameters of the model. The proposed methodology focusses on this identification, by comparing the result of an interval FE computation to measurement data. The uncertainty in the measurement data, usually acquired through repeated experimental testing of the modelled physical structure (such as e.g., experimental modal analysis), is compared to the uncertainty in the simulation results, as it originates from the propagation of the analyst's initial estimate on the spatial uncertainty in the input parameters of the FE model. These parameters may include material properties such as Young's modulus or geometrical properties such as a plate thickness. The uncertainty in both the measurement and simulation data is represented as the convex hull which contains respectively the measured and simulated uncertainty in both spaces. Identification of γ^I is performed by minimising a well-defined objective function δ , which describes the mismatch between the convex hull of the simulation data and the convex hull of the measurement data.

In order to demonstrate the applicability of the proposed method to the identification of multivariate interval uncertainty having a spatial component through the model domain Ω , the method is presented in the context of the identification of an interval field γ_F^I , in the case when the analyst has expert knowledge on n_b and $\psi_i(\mathbf{r})$. However, this does not limit the generic nature of the method, as interval fields are a specific case of multivariate interval uncertainty, as explained in section 2.2.

3.1. Convex hull in the simulation space

The propagation of the multivariate uncertainty in the physical input parameters of a dynamic FE model that is captured in an interval field $\gamma_F^I(\mathbf{r})$, typically yields a multidimensional non-convex uncertain output set $\tilde{\mathbf{y}}$, as defined in (1). $\gamma_F^I(\mathbf{r})$ is defined through the n_b base vectors $\psi_i(\mathbf{r})$ and their

corresponding interval scalars α_i^I . When dynamic models are considered, $\tilde{\mathbf{y}}$ contains all eigenfrequencies f as they are represented in an interval format.

The exact solution set of the interval FE model $\tilde{\mathbf{y}}$, is approximated by the construction of an uncertain realisation set $\tilde{\mathbf{y}}_s$, obtained by propagating deterministic realisations $\gamma_{F,j}(\mathbf{r})$ of the interval field $\gamma_{\mathbf{F}}^I(\mathbf{r})$:

$$\tilde{\mathbf{y}}_s = \{\mathbf{y}_{sj} | \mathbf{y}_{sj} = f(\gamma_{F,j}(\mathbf{r})) \wedge \gamma_{F,j}(\mathbf{r}) \in \gamma_{\mathbf{F}}^I(\mathbf{r})\} \quad (5)$$

with $\mathbf{y}_{sj} \in \mathbb{R}^d$ a vector containing the d output responses of the j^{th} deterministic model solution, with $j \in [1, q]$:

$$\mathbf{y}_{sj} = [y_{s1}, y_{s2}, \dots, y_{sd}]^T \quad (6)$$

The cardinality of $\tilde{\mathbf{y}}_s$ is q , the total number of propagated realisations of the interval field $\gamma_{\mathbf{F}}^I(\mathbf{r})$. These q realisations should represent the solution set $\tilde{\mathbf{y}}$ as closely as possible, and may stem from e.g., the optimisation approach as presented in (3), or sampling techniques such as e.g., the transformation method [13], Design Of Experiments, uniform Monte Carlo sampling or Latin Hypercube sampling. In the specific case of the identification and quantification of multivariate interval uncertainty in dynamical FE models, one vector $\mathbf{y}_{sj}$ in the uncertain realisation set $\tilde{\mathbf{y}}_s$ consists of the computed eigenfrequencies f for the j^{th} realisation of the input interval field $\gamma_{\mathbf{F}}^I(\mathbf{r})$:

$$\mathbf{y}_{sj} = [f_1, f_2, \dots, f_d]^T \quad (7)$$

In the context of expressing a measure of discrepancy between measurement data and the uncertain realisation set, the *boundaries* of $\tilde{\mathbf{y}}_s$ need to be expressed in a mathematical formulation. Therefore, the convex hull $\tilde{\mathbf{y}}_s^C$ of $\tilde{\mathbf{y}}_s$ is computed, as also proposed by [20, 21]. In general, the convex hull of a finite set of vectors in Euclidean space is defined as the smallest possible convex set, containing the vectors in $\tilde{\mathbf{y}}_s$. It is the intersection of all possible convex sets containing these vectors. Formally, it is expressed as:

$$\tilde{\mathbf{y}}_s^C = \left\{ \sum_{j=1}^q \beta_j \mathbf{y}_{sj} \mid (\forall j : \beta_j \geq 0) \wedge \sum_{j=1}^q \beta_j = 1 ; \mathbf{y}_{sj} \in \tilde{\mathbf{y}}_s \right\} \quad (8)$$

where β is a vector of weighting factors, such that all β_j are non-negative and sum to one [30]. In the specific case when $\tilde{\mathbf{y}}_s$ is computed following (3), $\mathbf{y}^I$, the hypercubic approximation of $\tilde{\mathbf{y}}$, and $\tilde{\mathbf{y}}_s^C$ span the same uncertain solution set.

This concept is illustrated in figure 2, where an interval field $\gamma_{\mathbf{F}}^{\mathbf{I}}(\mathbf{r})$ with $n_b = 3$ input interval scalars is propagated using the transformation method. Thus, $q = 2^3 = 8$ deterministic realisations of this field are propagated. The figure shows the cross-section of the convex hull $\tilde{\mathbf{y}}_s^C$ with the f_i, f_j plane of two arbitrary eigenfrequencies.

Figure 2: Illustration of the convex hull principle on an uncertain simulation set $\tilde{\mathbf{y}}_s$, obtained by propagating an interval field $\gamma_{\mathbf{F}}^{\mathbf{I}}(\mathbf{r})$ with $n_b = 3$ input dimensions using the transformation method, leading to 8 realisations of $\mathbf{y}_{sj}$. The convex hull $\tilde{\mathbf{y}}_s^C$ has d dimensions, but only the cross-section with the f_i, f_j plane of two arbitrary eigenfrequencies is shown.

Finally the d -dimensional volume V_s of $\tilde{\mathbf{y}}_s^C$ is computed for the comparison to the measurement data. Intuitively, the d -dimensional volume of $\tilde{\mathbf{y}}_s^C$ can be interpreted as a measure for the uncertainty that is present in the data, and can thus be regarded as a generalisation of the one-dimensional interval width to higher dimensions. In general, the d -dimensional volume V_s of $\tilde{\mathbf{y}}_s^C$ may be expressed as a d -dimensional integral over the region bounded by $\tilde{\mathbf{y}}_s^C$:

$$V_s = \int \cdots \int_{\tilde{\mathbf{y}}_s^C} dy_{s1} dy_{s2} \dots dy_{sd} \quad (9)$$

The numerical computation of the convex hull and the corresponding multi-dimensional volume is done using the QHULL library, which uses the "Quickhull" algorithm, as developed by **Barber et al.** This is a generalisation of the Quicksort algorithm to higher dimensions [31]. Care should be

taken when computing high dimensional convex hulls, as the time complexity of the *Quickhull* computation of a convex hull is in a worst case scenario $O(\lfloor v_c^{\frac{d}{2}} \rfloor / \lfloor \frac{d}{2} \rfloor!)$, with v_c the number of vertices of $\tilde{\mathbf{y}}_s^C$ [31]. This means that computational cost of the algorithm scales exponentially with the dimension d of the space $\mathbb{R}^d$ in which the uncertain realisation $\tilde{\mathbf{y}}_s$ set is defined.

3.2. Convex hull in the measurement space

The measurement data used for the identification are usually obtained through an experimental procedure where a real life model is physically tested repeatedly in close relation to the FE model under consideration. The model responses, measured on the j^{th} test replica are grouped in the vector $\mathbf{y}_{mj} \in \mathbb{R}^d$:

$$\mathbf{y}_{mj} = [y_{m1}, y_{m2}, \dots, y_{md}]^T \quad (10)$$

In the specific case of dynamical models, one measurement vector $\mathbf{y}_{mj}$ contains the measured eigenfrequencies f , corresponding to the eigenfrequencies in $\tilde{\mathbf{y}}_s$:

$$\mathbf{y}_{mj} = [f_{m1}, f_{m2}, \dots, f_{md}]^T \quad (11)$$

These responses are subsequently used to construct a measurement set $\tilde{\mathbf{y}}_m$, which is defined as:

$$\tilde{\mathbf{y}}_m = \{\mathbf{y}_{mj} \mid j \in 1, \dots, t\} \quad (12)$$

with the cardinality of $\tilde{\mathbf{y}}_m$ equal to t , the number of conducted physical experiments. These may include both tested realisations of the structure as replicated measurements on the same realisation. For the sake of simplicity but without any further limitation to the validity of the proposed approach, it is assumed here that the experimental procedure gives information on the same response quantities which are modelled in the simulation model. In general this may not be true and it would imply that a suitable data reduction or interpolation technique should be selected for an objective and correct comparison between measurement and simulation data. For the particular case with resonance frequencies as response quantities which is considered here, modes may not be ordered in the same sequence in both data sets.

When mode switching occurs in the simulation model during the identification procedure, appropriate mode tracking algorithms, such as MAC-based mode tracking (see e.g. [32]) may be applied to assure that the same eigenmodes in simulation and measurement data are compared. The convex hull $\tilde{\mathbf{y}}_m^C$ of $\tilde{\mathbf{y}}_m$ is calculated as:

$$\tilde{\mathbf{y}}_m^C = \left\{ \sum_{j=1}^t \beta_j \mathbf{y}_{mj} \mid (\forall j : \beta_j \geq 0) \wedge \sum_{j=1}^t \beta_j = 1 \right\} \quad (13)$$

With corresponding d -dimensional volume V_m :

$$V_m = \int \cdots \int_{\tilde{\mathbf{y}}_m^C} dy_{m1} dy_{m2} \dots dy_{md} \quad (14)$$

The convex hull principle is illustrated in figure 3, where it is applied to the measurement set $\tilde{\mathbf{y}}_m$, for $t = 50$ individual measurements. The figure shows the cross-section of the convex hull $\tilde{\mathbf{y}}_m^C$ with the f_i, f_j plane of two arbitrary eigenfrequencies. The multidimensional volume V_m of $\tilde{\mathbf{y}}_m^C$ is computed using the functionality that is provided by the QHULL library.

Figure 3: Illustration of the convex hull $\tilde{\mathbf{y}}_m^C$ of the measurement set $\tilde{\mathbf{y}}_m$. $\tilde{\mathbf{y}}_m$ is obtained by performing 50 repeated measurements on a physical realisation. The convex hull $\tilde{\mathbf{y}}_m^C$ has d dimensions, but only the cross-section with the f_i, f_j plane of two arbitrary eigenfrequencies is shown.

3.3. Identification of the interval field

Identification of $\boldsymbol{\alpha}^I \in \mathbb{IR}^{n_b}$, the interval vector that is used for the construction of an interval field $\boldsymbol{\gamma}_{\mathbf{F}}^{\mathbf{I}}(\mathbf{r})$, as shown in eq. (4), is done through the minimisation of a cost function $\delta(\boldsymbol{\alpha}^I)$, which expresses the discrepancy between the uncertain realisation set $\tilde{\mathbf{y}}_s$ of the IFE simulation and the measurement set $\tilde{\mathbf{y}}_m$. The construction of $\delta(\boldsymbol{\alpha}^I)$ is based on the geometrical properties of shape and size of both hulls $\tilde{\mathbf{y}}_m^C$ and $\tilde{\mathbf{y}}_s^C$ and it contains three terms.

First, the difference between the multidimensional volumes V_m and V_s of the respective hulls is computed, as this is a measure for the amount of uncertainty captured by the convex hull. Also the multidimensional volume V_o of the intersection $\tilde{\mathbf{y}}_o$ of $\tilde{\mathbf{y}}_m^C$ and $\tilde{\mathbf{y}}_s^C$ is taken into account. The intersection between $\tilde{\mathbf{y}}_m^C$ and $\tilde{\mathbf{y}}_s^C$ is formally defined as:

$$\tilde{\mathbf{y}}_o = \tilde{\mathbf{y}}_s^C \cap \tilde{\mathbf{y}}_m^C \quad (15)$$

The computation of $\tilde{\mathbf{y}}_o$ is based on Minkowski-Weyl's theorem [33], which states that for any polyhedron, there always exists a duality in its representation (i.e. as a set of vertices (a convex hull) or as a set of half-spaces (a set of linear inequalities)). For the computation of $\tilde{\mathbf{y}}_o$, $\tilde{\mathbf{y}}_m^C$ and $\tilde{\mathbf{y}}_s^C$ are rewritten as a set of linear inequalities:

$$\begin{aligned} \tilde{\mathbf{y}}_m^C &\iff \mathbf{A}_m \mathbf{y}^T - \mathbf{b}_m \geq 0 \\ \tilde{\mathbf{y}}_s^C &\iff \mathbf{A}_s \mathbf{y}^T - \mathbf{b}_s \geq 0 \end{aligned} \quad (16)$$

with $\mathbf{A}_m \in \mathbb{R}^{h_m \times d}$, $\mathbf{A}_s \in \mathbb{R}^{h_s \times d}$, $\mathbf{b}_s \in \mathbb{R}^{h_s}$ and $\mathbf{b}_m \in \mathbb{R}^{h_m}$. h_m and h_s are the number of hyperplanes defining the boundaries of respectively $\tilde{\mathbf{y}}_m^C$ and $\tilde{\mathbf{y}}_s^C$. Specifically, the vertices constituting $\tilde{\mathbf{y}}_m^C$ and $\tilde{\mathbf{y}}_s^C$ are used to construct these sets of hyperplanes. $\tilde{\mathbf{y}}_o$ is then defined as the set of all $\mathbf{y}_j$ complying with both sets of linear inequalities:

$$\tilde{\mathbf{y}}_o = \{\mathbf{y}_j \mid \mathbf{A}_o \mathbf{y}_j^T - \mathbf{b}_o^T \leq 0 \wedge \mathbf{y}_j \in \mathbb{R}^d\} \quad (17)$$

with $\mathbf{A}_o \in \mathbb{R}^{(h_m+h_s) \times d}$:

$$\mathbf{A}_o = [\mathbf{A}_m; \mathbf{A}_s]^T \quad (18)$$

and $\mathbf{b}_o \in \mathbb{R}^{(h_m+h_s)}$:

$$\mathbf{b}_o = [\mathbf{b}_m; \mathbf{b}_s]^T \quad (19)$$

The multidimensional volume V_o of $\tilde{\mathbf{y}}_o$ is computed using the functionality that is provided by the QHULL library. $\tilde{\mathbf{y}}_o$ is illustrated for the 2D case in figure 4, as the intersection between $\tilde{\mathbf{y}}_s^C$ (as shown in figure 2) and $\tilde{\mathbf{y}}_m^C$ (as shown in figure 3). Due to the convexity of $\tilde{\mathbf{y}}_m^C$ and $\tilde{\mathbf{y}}_s^C$, $\tilde{\mathbf{y}}_o$ is also convex [34].

Figure 4: Illustration of the convex hull $\tilde{\mathbf{y}}_o$ of the overlap between the convex hull of the measurement set $\tilde{\mathbf{y}}_m^C$ and uncertain realisation set $\tilde{\mathbf{y}}_s^C$, shown as their cross-section with the plane y_i, y_j of two arbitrary eigenfrequencies.

Finally, $\delta(\boldsymbol{\alpha}^I)$ should contain a desirable descent direction for the specific case when the intersection between $\tilde{\mathbf{y}}_s^C$ and $\tilde{\mathbf{y}}_m^C$ is empty (i.e., $\tilde{\mathbf{y}}_o = \emptyset$). This condition may occur when the analyst's initial knowledge on the input parameters of the FE model has a large bias with respect to the actual uncertainty in the model. In this specific case, there is no descent direction for the optimisation algorithm that ensures overlap between $\tilde{\mathbf{y}}_s^C$ and $\tilde{\mathbf{y}}_m^C$. Therefore, also the squared L_2 norm of the difference between $\mathbf{c}_m$ and $\mathbf{c}_s$ is incorporated in the discrepancy metric. $\mathbf{c}_m$ and $\mathbf{c}_s$ are defined as the geometrical centre of gravity of $\tilde{\mathbf{y}}_m$ and $\tilde{\mathbf{y}}_s$:

$$\begin{aligned}
\mathbf{c}_m &= \frac{1}{t} \sum_{j=1}^t \mathbf{y}_{mj} \\
\mathbf{c}_s &= \frac{1}{q} \sum_{j=1}^q \mathbf{y}_{sj}
\end{aligned} \tag{20}$$

Combining these three elements, $\delta(\boldsymbol{\alpha}^I)$ is constructed according to:

$$\delta(\boldsymbol{\alpha}^I) = (\Delta V_m^2 + w_o \Delta V_o^2 + \Delta c^2) \tag{21}$$

with:

$$\begin{aligned}
\Delta V_m &= 1 - \frac{V_s(\boldsymbol{\alpha}^I)}{V_m} \\
\Delta V_o &= 1 - \frac{V_o(\boldsymbol{\alpha}^I)}{V_m} \\
\Delta c &= \|\mathbf{c}_m - \mathbf{c}_s(\boldsymbol{\alpha}^I)\|_2
\end{aligned} \tag{22}$$

and w_o a weighting factor. This weighting factor is of specific importance when the measurement data set is very small. In that case, more weight in the optimisation is given to maximising the overlap between the uncertain realisation set and the measurement data set, in order to ensure that all measurement points are included in the identified interval field.

$\boldsymbol{\alpha}^{I,*}$, the interval vector containing the identified interval scalars is finally computed as:

$$\boldsymbol{\alpha}^{I,*} = \operatorname{argmin} \delta(\boldsymbol{\alpha}^I) \tag{23}$$

Due to the general non-linearity of $\delta(\boldsymbol{\alpha}^I)$, the minimisation is performed following an iterative optimisation procedure. Specifically, the sequential quadratic programming (SQP) algorithm is employed for this reason [35]. It is an iterative optimisation algorithm that optimises the nonlinear objective function by minimising a series of quadratic approximations of $\delta(\boldsymbol{\alpha}^I)$. During each iteration of the SQP algorithm, $\tilde{\mathbf{y}}_s^C$, V_s , V_o and $\mathbf{c}_s$ are computed using eq. (5) - eq. (20) in order to obtain $\delta(\boldsymbol{\alpha}^I)$. A general overview of the identification procedure is given in the pseudo code, shown in algorithm 1. $\boldsymbol{\alpha}_j^I$ denotes the interval vector, computed in the j^{th} iteration of the SQP algorithm.

Algorithm 1 Identification of an interval field, based on measurement data

- 1: Compute initial uncertain realisation set $\tilde{\mathbf{y}}_s(\boldsymbol{\alpha}_0^I)$ by propagating an initial guess on the interval field $\boldsymbol{\gamma}_F^I(\mathbf{r}, \boldsymbol{\alpha}_0^I)$
 - 2: Select set of measurement data $\tilde{\mathbf{y}}_m$ (eq. (10))
 - 3: Compute convex hull $\tilde{\mathbf{y}}_m^C$ (eq. (13))
 - 4: Compute multi-dimensional volume V_m (eq. (14))
 - 5: **procedure** SQP OPTIMISATION
 - 6: $j = 0$
 - 7: **while** $\epsilon_r \geq \textit{limit}$ **do**
 - 8: Compute optimisation step for $\boldsymbol{\alpha}_j^I$
 - 9: Compute convex hull $\tilde{\mathbf{y}}_s^C(\boldsymbol{\alpha}_j^I)$ (eq. (8))
 - 10: Compute multi-dimensional volume $V_s(\boldsymbol{\alpha}_j^I)$ (eq. (9))
 - 11: Compute overlap $\tilde{\mathbf{y}}_o(\boldsymbol{\alpha}_j^I)$ between $\tilde{\mathbf{y}}_s^C(\boldsymbol{\alpha}_j^I)$ and $\tilde{\mathbf{y}}_m^C$ (eq. (15))
 - 12: Compute multi-dimensional volume V_o of $\tilde{\mathbf{y}}_o(\boldsymbol{\alpha}_j^I)$
 - 13: Compute objective function $\delta(\boldsymbol{\alpha}_j^I)$ (eq. (21))
 - 14: Compute relative change ϵ_r in $\delta(\boldsymbol{\alpha}_j^I)$
 - 15: $j = j+1$
 - 16: $\boldsymbol{\alpha}^{I,*} = \boldsymbol{\alpha}_j^I$
-

4. Optimal response selection

As mentioned in section 3, the computational cost for the determination of a convex hull $\tilde{\mathbf{y}}_s^C$ scales exponentially with the dimension d of the space $\mathbb{R}^d$ in which the uncertain realisation set $\tilde{\mathbf{y}}_s$ is defined. Since during the optimisation, this computation has to be performed during each iteration of the optimisation algorithm, this may lead to long computation times. However, not all responses $y_{si} \in \mathbf{y}_{sj}$ contained in the result vector represent an equally significant contribution to the convex hull $\tilde{\mathbf{y}}_s^C$ in $\mathbb{R}^d$. Therefore, the dimension of the space in which the convex hull is defined, can be reduced by selecting those responses that contribute the most to the uncertainty that is modelled. Specifically, a reduced result vector consisting of selected output quantities $\mathbf{y}_{sj}^r \subset \mathbf{y}_{sj}$ with $\mathbf{y}_{sj}^r \in \mathbb{R}^{d_r}$, $\mathbf{y}_{sj} \in \mathbb{R}^d$ and $d_r \ll d$ is constructed as such that a maximum amount of information on the constituting uncertainty at the input side of the model is maintained. The reduced uncertain realisation set $\tilde{\mathbf{y}}_s^r$, based on these reduced result vectors is then defined as:

$$\tilde{\mathbf{y}}_s^r = \{\mathbf{y}_{sj}^r | \mathbf{y}_{sj}^r \subset \mathbf{y}_{sj} = f(\gamma_{F,j}(\mathbf{r})) \wedge \gamma_{F,j}(\mathbf{r}) \in \gamma_F^I(\mathbf{r})\} \quad (24)$$

with $\mathbf{y}_{sj}^r$ a vector containing the d_r selected output responses out of the full-dimensional output $\mathbf{y}_{sj}$ of the deterministic solution of the propagation of the j^{th} input uncertainty realisation, with $j \in [1, q]$:

$$\mathbf{y}_{sj}^r = [y_{s1}, y_{s2}, \dots, y_{sd_r}] \quad (25)$$

In a first step of the selection procedure, the number of responses d_r in $\mathbf{y}_{sj}$ that are needed to accurately represent the uncertainty in the uncertain realisation set $\tilde{\mathbf{y}}_s$ is computed based on an singular value decomposition of the covariance matrix of $\tilde{\mathbf{y}}_s$, which is constructed over all computed realisations. Selection is subsequently based on the computation of the multidimensional interval sensitivity of all possible candidate $\tilde{\mathbf{y}}_s^r$ and the aspect ratio of the convex hull of $\tilde{\mathbf{y}}_s^r$.

4.1. Output space dimensionality

A first step in the selection procedure is the determination of the dimension d_r of the vector space $\mathbb{R}^{d_r}$ in which $\tilde{\mathbf{y}}_s^r$ has to be defined. This selection has to be made such that the q realisations of $\mathbf{y}_{sj}^r \in \mathbb{R}^{d_r} \subset \mathbf{y}_{sj} \in \mathbb{R}^d$ lose maximally a predefined fraction ϵ of the original variance contained in $\tilde{\mathbf{y}}_s$.

To this purpose, the covariance matrix Ξ of $\tilde{\mathbf{y}}_s$ is computed over the q realisations that are computed for each response in $\mathbf{y}_{sj}$. These realisations are concatenated for each response j in a vector $\mathbf{y}_{sj}^t \in \mathbb{R}^q$. Ξ is then defined as:

$$\Xi = \begin{bmatrix} \text{var}(\mathbf{y}_{s1}^t) & \text{cov}(\mathbf{y}_{s1}^t, \mathbf{y}_{s2}^t) & \dots & \text{cov}(\mathbf{y}_{s1}^t, \mathbf{y}_{sd}^t) \\ \text{cov}(\mathbf{y}_{s2}^t, \mathbf{y}_{s1}^t) & \text{var}(\mathbf{y}_{s2}^t) & \dots & \text{cov}(\mathbf{y}_{s2}^t, \mathbf{y}_{sd}^t) \\ \vdots & & \ddots & \vdots \\ \text{cov}(\mathbf{y}_{sd}^t, \mathbf{y}_{s1}^t) & \text{cov}(\mathbf{y}_{sd}^t, \mathbf{y}_{s2}^t) & \dots & \text{var}(\mathbf{y}_{sd}^t) \end{bmatrix} \quad (26)$$

Subsequently, a singular value decomposition of Ξ is performed so that:

$$\Xi = \Phi \Lambda \Phi^T \quad (27)$$

with $\Lambda \in \mathbb{R}^{d \times d}$ the diagonal matrix of the ordered eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_d$ of Ξ , and $\Phi \in \mathbb{R}^{d \times d}$ a matrix containing the orthogonal eigenvectors. d_r is then chosen such that maximally a fraction ϵ of the uncertainty that is predicted by the *full* uncertain realisation set $\tilde{\mathbf{y}}_s$ is left unexplained by the first d_r responses that are used to construct the reduced uncertain realisation set $\tilde{\mathbf{y}}_s^r$:

$$\sum_{i=1}^{d_r} \frac{\lambda_i}{\text{tr}(\Xi)} \geq 1 - \epsilon \quad (28)$$

with $\text{tr}(\Xi)$ the trace of the covariance matrix. These computations can also be done using the measurement data set $\tilde{\mathbf{y}}_m$, which is of particular interest when the number of measured replica is larger than the number of propagated realisations of the interval uncertainty that has to be identified.

4.2. Construction of possible subsets

Based on the computation of the d_r in eq. (28), possible candidate reduced uncertain output sets $\tilde{\mathbf{y}}_{s,l}^r$, their convex hull $\tilde{\mathbf{y}}_{s,l}^{r,C}$ and multidimensional volume $V_{s,l}^r$ are computed. $\tilde{\mathbf{y}}_{s,l}^{r,C}$ is defined as:

$$\tilde{\mathbf{y}}_{s,l}^{r,C} = \left\{ \sum_{j=1}^q \beta_j \mathbf{y}_{sj,l}^r \mid (\forall j : \beta_j \geq 0) \wedge \sum_{j=1}^q \beta_j = 1 ; \mathbf{y}_{sj,l}^r \in \tilde{\mathbf{y}}_{s,l}^r \subset \tilde{\mathbf{y}}_s \right\} \quad (29)$$

Following a brute-force approach, the candidate reduced uncertain realisation set and their convex hull $\tilde{\mathbf{y}}_{s,l}^{r,C}$ with corresponding volume $V_{s,l}^r$ are

computed for each possible d_r -dimensional responses $\mathbf{y}_{sj,l}^r$ that can be deduced from the result vector $\mathbf{y}_{sj}$. The number of convex hull computations is in that case:

$$\frac{d!}{d_r!(d-d_r)!} \quad (30)$$

The number of convex hull computations can in this case become prohibitively large when numerical models containing a large number of degrees of freedom or eigenfrequencies are considered. In that case, a more intelligent selection of the optimal responses can be performed following a global optimisation approach of the optimal subset selection metric that will be presented in section 4.5.

4.3. Multidimensional interval sensitivity

The subset that is selected should be sensitive to changes at the input side of the numerical model. In order to take into account the coupling between the regarded output responses, a modified version of the interval sensitivity, as proposed by **Moens & Vandepitte** in [36], is employed. In its original form, the interval sensitivity is defined as :

$$s_{il} = \frac{\partial r_{y_l}}{\partial r_{\gamma_i}} \quad (31)$$

with r_{y_l} and r_{γ_i} the interval radius of respectively the l^{th} model response and the i^{th} interval scalar at the input of the model. This concept is further expanded to consider the sensitivity of multiple output responses and their dependence w.r.t. changes in the interval uncertainty at the input side of the model $f()$. Specifically, the sensitivity s_{il} of the volume $V_{s,l}^r$ of the d_r -dimensional convex hull $\tilde{\mathbf{y}}_{s,l}^{r,C}$ of the l^{th} candidate reduced uncertain realisation set $\tilde{\mathbf{y}}_{s,l}^r$ with respect to changes in the interval radius r_{α_i} of a locally defined interval scalar α_i^I of $\boldsymbol{\gamma}_F^I(\mathbf{r})$ is calculated as:

$$s_{il} = \frac{\partial V_{s,l}^r}{\partial r_{\alpha_i}} \quad (32)$$

The computation of s_{il} is performed for all possible reduced uncertain realisation sets $\tilde{\mathbf{y}}_{s,l}^r$, constructed with the d_r -dimensional responses $\mathbf{y}_{sj}^r$, with respect to the k locally defined intervals α_i^I . This calculation only needs $2k$ calls to the IFE solver (for the computation of the finite differences). All $\tilde{\mathbf{y}}_{s,l}^r$

are extracted from the propagated full dimensional uncertain realisation set $\tilde{\mathbf{y}}_s$. In order to facilitate the comparison between different candidate reduced uncertain realisation sets based on the sensitivity, normalised relative interval sensitivities $s_{il}^{n,r}$ are more relevant, as they give the relative width of the intervals and are dimensionless. The normalised relative interval sensitivity is defined as:

$$s_{il}^{n,r} = \frac{V_{s,l}^r}{\partial r_{\alpha_i}} \cdot \frac{r_{\alpha_i}}{V_s} \cdot \frac{1}{s_i^r} \quad (33)$$

with:

$$s_i^r = \sum_j s_{il}^r = \sum_j \frac{V_{s,l}^r}{\partial r_{\alpha_i}} \cdot \frac{r_{\alpha_i}}{V_s} \quad (34)$$

and s_{il}^r the non-normalized relative interval sensitivity.

4.4. Dependence

Also the dependence between the q realisations of a response within a candidate $\mathbf{y}_{sj,l}^r$ is computed in order to prevent the selection two highly interdependent responses out of $\tilde{\mathbf{y}}_s$, which would increase the computational cost without adding information. Moreover, this may lead to convex hulls with very small internal angles, making the computation of the intersection in the objective function, as presented in eq. (21), numerically very hard. In the context of selecting the optimal responses within $\mathbf{y}_{sj,l}^r$ for the construction of a candidate $\tilde{\mathbf{y}}_{s,l}^r$, the dependence between these responses is regarded as the aspect ratio ν of the convex hull $\tilde{\mathbf{y}}_{s,l}^{r,C}$ of that candidate reduced uncertain realisation set $\tilde{\mathbf{y}}_{s,l}^r$. When all responses are completely interdependent (i.e. show a high correlation), these responses describe a line in $\mathbb{R}^{d_r}$. The aspect ratio of $\tilde{\mathbf{y}}_s^C$ will therefore tend to zero. Conversely, when they are completely independent (i.e. the correlation is zero), the aspect ratio will approach unity.

In general, the multidimensional aspect ratio is calculated as the ratio between the smallest and largest eigenvalue of the covariance matrix $\mathbf{\Xi}_c^r$ of $\tilde{\mathbf{y}}_{s,l}^r$:

$$\nu_l = \frac{\min(\mathbf{\Lambda}_c^r)}{\max(\mathbf{\Lambda}_c^r)} \quad (35)$$

with $\mathbf{\Lambda}_c^r \in \mathbb{R}^{d_r \times d_r}$ the diagonal matrix containing the ordered d_r eigenvalues of $\mathbf{\Xi}_c^r \in \mathbb{R}^{d_r \times d_r}$, the covariance matrix of $\tilde{\mathbf{y}}_{s,l}^r$. $\mathbf{\Xi}_c^r$ is calculated similarly to $\mathbf{\Xi}$ of $\tilde{\mathbf{y}}_s$, as shown in equation (26), where instead of the complete result vector $\mathbf{y}_{sj}$, the candidate reduced result vectors $\mathbf{y}_{sj,l}^r$ are used for the computation of the covariance matrix $\mathbf{\Xi}_c^r$.

4.5. Selection of the optimal subset

In order to select the optimal subset, a metric μ_{il} combining the sensitivity of all possible subsets and dependence between the dimensions of these subsets is calculated for each separate locally defined interval $i = 1, \dots, k$:

$$\mu_{il} = |s_{il}^{n,r}| \nu_l \quad (36)$$

with $s_{il}^{n,r} \in [-1; 1]$, as the normalised relative interval sensitivity is used, and $\nu \in [0; 1]$. The absolute value of the normalised relative interval sensitivity is used as only the magnitude of the sensitivity is of interest for the selection. A candidate subset is generally desirable when μ_{il} is maximal among all candidate reduced uncertain realisation sets, as this means that the subset is sensitive to changes in the input space and that these changes in the input space affect all dimensions in the subset as evenly as possible. Finally, this subset is selected as:

$$l = \operatorname{argmax} \left(\sum_{i=1}^k \mu_{il} \right) \quad (37)$$

As mentioned in subsection 4.2, solving eq. (37) can be performed following a *brute-force* approach when the number of considered responses is relative low, or in a global optimisation approach for larger models.

5. Example 1: 3-DOF mass-spring system

Validation of the methodology is first performed on an academic example of a discrete mass-spring system consisting of six springs and three masses. This system is shown in figure 5. Analytical equations are derived following Lagrange's method and are solved for the three eigenfrequencies and -modes of the system by solving the eigenvalue problem.

Although the model is small-scale, it serves as an illustration of the identification procedure for discrete multivariate interval uncertainty (i.e. interval uncertainty which is completely decoupled). All masses are assumed to be

Figure 5: Three degree-of-freedom mass-spring system that is used for the validation. The interval description of the spring stiffness values is listed in table 1.

deterministic with $m_1 = 1.0 \text{ kg}$, $m_2 = 2.0 \text{ kg}$ and $m_3 = 3.0 \text{ kg}$, in order to make the system non-symmetric. The nominal stiffness of the springs is $k_1 = 5.0 \text{ N/m}$, $k_2 = 6.0 \text{ N/m}$, $k_3 = 1.5 \text{ N/m}$, $k_4 = 8.0 \text{ N/m}$, $k_5 = 3.0 \text{ N/m}$ and $k_6 = 8.0 \text{ N/m}$. Solution of the nominal model yields $f_1 = 1.32 \text{ Hz}$, $f_2 = 2.82 \text{ Hz}$ and $f_3 = 4.89 \text{ Hz}$.

Using this academic example, three case studies are performed:

1. Three of the six springs (k_1 , k_2 and k_4) have an uncertain stiffness bounded by an interval $[\underline{k}_m; \bar{k}_m]$, while the other springs are considered to be *certain*, with $k_3 = 1.5 \text{ N/m}$, $k_5 = 3.0 \text{ N/m}$ and $k_6 = 8.0 \text{ N/m}$. As an initial estimate of the uncertainty, all springs are considered to have interval uncertainty $[\underline{k}_i; \bar{k}_i]$ associated to their stiffness.
2. Five springs (k_1 , k_2 , k_4 , k_5 and k_6) have an uncertain stiffness bounded by an interval $[\underline{k}_m; \bar{k}_m]$, while $k_3 = 1.5 \text{ N/m}$. As an initial estimate, all intervals $[\underline{k}_i; \bar{k}_i]$ associated to the uncertain spring stiffness values are considered as infinitely thin. This therefore corresponds to an entirely deterministic initial estimate.
3. All six springs have an uncertain stiffness bounded by an interval $[\underline{k}_m; \bar{k}_m]$. As an initial estimate, also all springs are considered to have interval uncertainty $[\underline{k}_i; \bar{k}_i]$ associated to their stiffness.

Measurement data are constructed by sampling the vertices of the pre-defined intervals k_m^I (or scalars for cases 1 and 2). An overview of the intervals on the spring stiffness values that were used for the construction of the measurement data, is given in table 1. Also the initial estimates k_i^I on the uncertainty in these springs are given in this table. Case 1 and case 2 are

specifically aimed at showing the performance of the method in identifying which parameters at the input of the model contain interval uncertainty and quantifying the corresponding intervals. Note that this is a purely hypothetical example with no link to the physical world. Case 2 is moreover aimed at showing the robustness of the methodology w.r.t. the initial estimate of the analyst by making the initial estimate completely deterministic. This case might for example exist when the analyst has no objective information on the uncertainty in the spring stiffness and leaves the identification and quantification of the uncertainty entirely up to the algorithm. Finally, case 3 is aimed at illustrating the performance of the methodology in identifying a high-dimensional hypercubic input set. For demonstrative purposes, care was taken that the initial estimate of the uncertain realisation set is largely disjunct with the measurement data set to further show the performance of the algorithm.

Identification of the uncertainty in the spring stiffness parameters that were used for the construction of the measurement data set is performed following algorithm 1. Application of eq. (28) yields that all three eigenfrequencies are necessary in order to obtain an accurate identification.

Table 1: Uncertain spring stiffness values that were used for the construction of the measurement data and the corresponding initial estimates on the spring stiffness uncertainty. k_m denotes the interval that was sampled for the construction of the measurement data, while k_i denotes the initial guess on the uncertainty.

Spring	Case 1				Case 2				Case 3			
	k_m	$\overline{k_m}$	k_i	$\overline{k_i}$	k_m	$\overline{k_m}$	k_i	$\overline{k_i}$	k_m	$\overline{k_m}$	k_i	$\overline{k_i}$
$k_1 [N/m]$	4.0	9.0	7.0	9.0	4.0	6.0	1.0	1.0	1.0	2.0	5.0	7.0
$k_2 [N/m]$	4.0	12.0	7.0	9.0	4.0	12.0	1.0	1.0	4.0	7.0	6.0	9.0
$k_3 [N/m]$	1.5	1.5	7.0	9.0	1.5	1.5	1.0	1.0	1.0	2.0	0.5	3.0
$k_4 [N/m]$	7.0	9.0	7.0	9.0	4.0	9.0	1.0	1.0	7.0	9.0	1.0	5.0
$k_5 [N/m]$	3.0	3.0	7.0	9.0	3.0	8.0	1.0	1.0	1.0	5.0	2.0	5.0
$k_6 [N/m]$	8.0	8.0	7.0	9.0	4.0	9.0	1.0	1.0	6.0	9.0	1.0	4.0

The results of the identification in these three cases are listed in table 2. A highly accurate identification for all three cases is obtained following the proposed methodology. Note that in case 1, although initially all springs are considered to have an interval uncertainty associated with their stiffness, the identification algorithm manages to eliminate the uncertainty in those spring stiffness values that were considered to be deterministic for the construction

of the measurement data set by imposing (numerically) thin intervals. Also, even coming from a deterministic initial estimate of the spring stiffness values, as is in case 2, an accurate identification of the constituting uncertainty at the input side of the model is attainable.

Table 2: Identified uncertainty in the spring stiffness. k^* denotes the identified and quantified uncertainty in the corresponding spring stiffness.

Spring	Case 1		Case 2		Case 3	
	$\underline{k^*}$	$\overline{k^*}$	$\underline{k^*}$	$\overline{k^*}$	$\underline{k^*}$	$\overline{k^*}$
k_1 [N/m]	4.0000	9.0001	4.0006	6.0005	0.9997	1.9997
k_2 [N/m]	4.0000	12.000	4.0000	12.000	4.0002	7.0002
k_3 [N/m]	1.5000	1.5000	1.4999	1.4996	1.0001	2.0001
k_4 [N/m]	6.9999	8.9999	3.9998	8.9998	6.9997	8.9999
k_5 [N/m]	3.0000	3.0000	2.9999	8.0001	0.9998	5.0000
k_6 [N/m]	8.0001	8.0001	3.9997	8.9998	6.0001	9.0005

These two cases illustrate that the method is capable of identifying which parameters are uncertain and quantifying the amount of uncertainty in these parameters, based on a set of measurement data. Moreover, it is illustrated that the identification works independent of the initial estimate of the analyst, as it is even capable of performing an accurate identification and quantification starting from an entirely deterministic estimate. This will further be elaborated in the second example in section 5.

Figure 6 shows for case 3 the uncertain realisation set, stemming from the identified intervals, the set of measurement data and the initial uncertain realisation set for the three two-dimensional intersections of the three-dimensional convex hulls with the three eigenfrequencies of the discrete mass-spring model. As can be seen in this figure, an exact identification of the interval uncertainty is attained following the minimisation of eq. 21.

Figure 6: Case 3: convex hulls of the measurement set, together with the convex hulls of the uncertain realisation sets before and after identification, based on the propagation of the 5-dimensional hypercubic input. These hulls are shown for the three combinations of two eigenfrequencies, as well as in three dimensions

6. Example 2: Cantilever beam case

A second illustration is a one-dimensional cantilever beam (figure 7). For illustration purposes, this example is designed to represent spatial uncertainty in Young's modulus. The nominal value is taken to be 1.5 GPa. Young's modulus of the beam is subjected to multivariate interval uncertainty with a continuous spatial component over Ω , which is modelled using the interval field concept. The beam has a length of 300 mm and a constant rectangular cross section of 40 mm $\times$ 2 mm. The validation case concerns the inverse identification of spatial variability based on measured eigenfrequencies from experimental modal analysis.

Figure 7: Top: Illustration of the Finite-Element model of the cantilever beam on which the methodology is validated. $\mathbf{r}$ indicates the positive direction along the beam, the numbers in the finite elements indicate the element numbers. Bottom: Geometrical properties of the cantilever beam.

The governing second order ordinary differential equation is discretised and solved for the first 10 eigenfrequencies using a finite element (FE) model containing 10 linear shell elements with 4 nodes (figure 7). Both in-plane and out-of-plane bending modes, as well as torsional modes are considered. The result vector $\mathbf{y}_{sj}$ is defined as:

$$\mathbf{y}_{sj} = [f_1, f_2, \dots, f_{10}]^T \quad (38)$$

Table 3 shows the first 10 eigenfrequencies of the beam with the corresponding mode shapes, for the nominal case.

Table 3: The first 10 eigenfrequencies f and mode shapes of the cantilever beam model that is shown in figure 7

Number	f [Hz]	modeshape
1	2.7	1 st out-of-plane bending mode
2	16.6	2 nd out-of-plane bending mode
3	22.5	1 st torsion mode
4	46.1	3 rd out-of-plane bending mode
5	52.3	1 st in-plane bending mode
6	67.4	2 nd torsion mode
7	89.8	4 th out-of-plane bending mode
8	112.7	3 rd torsion mode
9	149.7	5 th out-of-plane bending mode
10	159.1	4 th torsion mode

Two versions of this case are considered. In the first, the interval field is modelled using two base vectors and corresponding interval scalars. In this example, the sensitivity of the methodology w.r.t. the selected subset and the initial guess is illustrated. Secondly, also a four-dimensional interval field on the beam’s Young’s modulus is considered to prove the applicability to interval fields with higher dimensions.

Specifically for this case study and without loss of generality, the base vectors $\boldsymbol{\psi}_i$ are constructed according to an inverse distance weighting (IDW) scheme, as proposed by the authors in [37], due to its intuitive nature. This scheme controls the complexity of the field realisations by choosing an appropriate number of control points on predefined locations $\mathbf{r}_i$ inside the topology Ω of the FE model. At each $\mathbf{r}_i$, an independent interval scalar α_i^I determines the local value of the uncertain parameter. The uncertain field realisation is then constructed by extrapolating these local values towards neighbouring elements according to a weighting scheme inversely proportional to the Euclidean distance. Based on this approach, the number of control points directly affects the spatial unevenness, which as such can be tuned by the analyst to represent spatial uncertainty as closely as possible to the expected profile of the uncertainty. The base vectors $\boldsymbol{\psi}_i$ that are used to construct the interval field $\boldsymbol{\gamma}_F^I(\mathbf{r})$ are then formally constructed according to:

$$\boldsymbol{\psi}_i(\mathbf{r}) = \frac{w_i(\mathbf{r})}{\sum_{j=1}^{n_b} w_j(\mathbf{r})} \quad (39)$$

with $w_i(\mathbf{r})$:

$$w_i(\mathbf{r}) = \frac{1}{[d(\mathbf{r}_i, \mathbf{r})]^p} \quad (40)$$

The weighting factors $w_i(\mathbf{r})$ are calculated as the inverse of $d(\mathbf{r}_i, \mathbf{r})$ to the power of a strictly positive scalar $p \in \mathbb{R}^+$. $d(\mathbf{r}_i, \mathbf{r})$ is a distance measure between the location $\mathbf{r}_i$ where the interval scalar is defined and any other location $\mathbf{r}$ in the model. The scalar p influences the weight that is given to the locally defined interval in $\mathbf{r}_i$; higher values of p give more weight to those $\mathbf{r}_i$ that are closest to the interpolation point under consideration [38]. It should be noted that other distance-based interpolation functions, such as radial base functions are equivalently applicable as base vectors, as long as they maintain the decoupling of the interval scalars. Also other principles for interval field decomposition can be equally applied in the framework of the identification method presented in this work, such as the Local Interval Field Decomposition technique as developed by Imholz et al. [39], as the presented procedure is independent of the parameters of $\psi_i(\mathbf{r})$.

6.1. Cantilever Beam example with a 2-dimensional input

In the first validation case, the analyst has interval information about Young's modulus in elements 2 and 7. Therefore locally defined intervals α_1^I and α_2^I , are defined at $\mathbf{r}_1 = 2$ and $\mathbf{r}_2 = 7$. Young's modulus in other elements is inter- or extrapolated following eq. (39). In this specific case, the interval field FE model is evaluated using the transformation method, leading to $q = 2^2 = 4$ realisations in the simulation space. These realisations are used as basis for the convex hull representations of the uncertain realisation set $\tilde{\mathbf{y}}_s$, which is defined as:

$$\tilde{\mathbf{y}}_s = \{\mathbf{y}_{s1}, \mathbf{y}_{s2}, \mathbf{y}_{s3}, \mathbf{y}_{s4}\} \quad (41)$$

6.1.1. Measurement data

For benchmarking purposes, measurement data are simulated using an *a priori* defined interval field, based on known local intervals $\check{\alpha}_1^I$ and $\check{\alpha}_2^I$, located at $\mathbf{r}_1$ and $\mathbf{r}_2$. Specifically, the goal local intervals are defined as $\check{\alpha}_1^I = [1.35; 1.65] \text{ GPa}$ and $\check{\alpha}_2^I = [1.55; 1.85] \text{ GPa}$. The 10-dimensional measurement set $\tilde{\mathbf{y}}_m$, which consists of the 10 *measured* eigenfrequencies f_m , is generated by conducting a Monte Carlo simulation which contains 500 samples, with a uniform distribution between the bounds of the local intervals $\check{\alpha}_1^I$

and $\check{\alpha}_2^I$. These samples are used to construct 500 deterministic input fields $\gamma_F(\mathbf{r})$ which are solved for $\mathbf{y}_m$. These $\mathbf{y}_m$ are finally used to construct the measurement data set $\tilde{\mathbf{y}}_m$.

6.1.2. Subset selection

A first step of the identification procedure consists of selecting a subset $\tilde{\mathbf{y}}_s^r \subset \tilde{\mathbf{y}}_s$ to speed up the computation of the convex hull $\tilde{\mathbf{y}}_s^C$. The selection of this subset is based on the methodology that is presented in section 4. Application of eq. (28) using the measurement data set $\tilde{\mathbf{y}}_m$ yields that with $d_r = 2$, an $\epsilon = 2.81 \cdot 10^{-06}$ is obtained. This implies that the 10-dimensional uncertain realisation set (i.e. containing the 10 computed eigenfrequencies) can be reduced to a 2-dimensional set (i.e. containing only 2 eigenfrequencies) without excessive loss of information. This is in fact rather intuitive, since it is a linear numerical model, consisting of a two-dimensional uncertain input space.

The selection metric for each possible reduced uncertain realisation set, as computed using eq. (31) to (36), is shown in figure 8. From this figure, it is evident that the reduced uncertain realisation set, spanned by the 5th and 6th eigenfrequency, is the optimal set for the identification procedure. The normalised interval sensitivities of the subspace spanned by f_5 and f_6 with respect to α_1^I and α_2^I are respectively $4.04 \cdot 10^{-02}$ and $6.43 \cdot 10^{-02}$, while the aspect ratio of this subspace measures 0.21. These eigenfrequencies respectively correspond to the 1st in-plane bending mode and the 2nd torsional mode. By the selection of these two eigenfrequencies, the result vector $\mathbf{y}_{sj}$ is reduced to a reduced result vector $\mathbf{y}_{sj}^r$:

$$\mathbf{y}_{sj}^r = [f_5, f_6]^T \quad (42)$$

The measurement set $\tilde{\mathbf{y}}_m$ is reduced to $\tilde{\mathbf{y}}_m^r$ in a similar way. Besides, it can also be noted that the combinations of the 5th eigenfrequency with the 7th, 8th, 9th and 10th also show to be close to optimal. In order to demonstrate the performance of the subset selection algorithm, also all other possible 2-dimensional combinations are used as a basis for the identification and the performance in terms of accuracy and convergence is compared. The weighting factor is simply taken to be $w_o = 1$. Exactly the same measurement data set $\tilde{\mathbf{y}}_m$ is used for all identifications to prevent bias.

Figure 8: Reduced uncertain realisation set selection metric, computed for each possible combination of the 10 eigenfrequencies of the model

6.1.3. Identification procedure

Identification of $\alpha^{I,*}$ is performed by applying the procedure, as presented in algorithm 1. In order to check the accuracy of the identification of the interval field, also a metric κ_i is defined for both the upper and lower bound, which describes the discrepancy between $\check{\alpha}_i^I$ and $\alpha_i^{I,*}$ of the interval field. Specifically, it is defined as:

$$\begin{aligned}\kappa_i(\underline{\alpha}_i)^* &= \left(1 - \frac{\underline{\alpha}_i^*}{\check{\alpha}_i}\right) \times 100 \\ \kappa_i(\overline{\alpha}_i)^* &= \left(1 - \frac{\overline{\alpha}_i^*}{\check{\alpha}_i}\right) \times 100\end{aligned}\tag{43}$$

$\kappa_{0,i}$ is defined in the same way for the discrepancy between $\check{\alpha}^I$ and α_0^I . The results of the identification procedure are shown in table 4. For illustrational purposes, this identification is based on a conservative initial estimate. This table shows the interval vector $\check{\alpha}^I$ that was used for the construction of the measurement data set, the interval vector containing the initial guess

α_0^I and the identified interval vector $\alpha^{I,*}$. Also the discrepancy metrics that have been defined in eq.(43) are listed. The proposed methodology is capable of determining the goal interval field, with a maximum error κ of 0.042%. Convergence of the discrepancy metric δ was obtained after 37 iterations.

Table 4: Results of the identification procedure on an interval field capturing the spatial variability in the cantilever beam example, based on an over-conservative initial guess. $\kappa_{0,i}$ is the error of the initial guess, the κ_i^* is the error after application of the identification procedure. $\check{\alpha}^I$ is the interval vector containing the interval scalars that were sampled for the construction of the measurement data. α_0^I contains the initial estimates on the interval scalars. Finally, $\alpha^{I,*}$ contains the identified interval scalars.

	$\underline{\alpha}_1 [MPa]$	$\bar{\alpha}_1 [MPa]$	$\underline{\alpha}_2 [MPa]$	$\bar{\alpha}_2 [MPa]$
$\check{\alpha}^I$	$1.35 \cdot 10^3$	$1.65 \cdot 10^3$	$1.55 \cdot 10^3$	$1.85 \cdot 10^3$
α_0^I	$1.0 \cdot 10^3$	$1.81 \cdot 10^3$	$1.10 \cdot 10^3$	$2.3 \cdot 10^3$
$\kappa_{0,i}$	15.5%	35.5%	9.7%	24.3%
$\alpha^{I,*}$	$1.356 \cdot 10^3$	$1.652 \cdot 10^3$	$1.554 \cdot 10^3$	$1.849 \cdot 10^3$
κ_i^*	0.042%	0.025%	0.001%	0.007%

Figure 9 shows the convergence of the cost function δ as well as its three main terms (ΔV_m , ΔV_o and Δc , as defined in (22)). The fact that the initial guess is highly conservative, but encompasses the measurement data completely, may be noted from the fact that at iteration 1, ΔV_o is zero and ΔV_m is the main contributing factor in δ . This means that the discrepancy between the multi-dimensional volumes V_m and V_s , due to the conservatism in $\tilde{\mathbf{y}}_s^C$, dominates the objective function at this part of the iterative procedure. During the next iterations, this discrepancy decreases gradually, at the cost that a slight part of the overlap between $\tilde{\mathbf{y}}_s^C$ and $\tilde{\mathbf{y}}_m^C$ is lost (as indicated by the fact that ΔV_o becomes non-zero). After 37 iterations, the main factor in δ is ΔV_o , which may be explained by the fact that the measurement data set, which was constructed through Monte Carlo sampling, does not necessarily contain all extreme realisations of the uncertain realisation set. This leads to an incomplete mapping of $\tilde{\mathbf{y}}_m^C$ and $\tilde{\mathbf{y}}_s^C$, and consequently their intersection. Convergence is obtained after 37 iterations of the optimisation process, whereas the separate factors do not change significantly after 22 iterations. The discrepancy metric δ , defined in eq. (21) decreases during the optimisation from 8.99 to 0.017.

Figure 9: Convergence of the cost function δ and each of its terms (ΔV_m , ΔV_o and Δc , as defined in (22)) as a function of the iteration number.

Figure 10 shows the resulting extreme realisations of the interval field before and after updating, as well as the extreme realisations of the interval field that was used to produce the measurement set. This graph also proves that the methodology is capable of accurately identifying the interval field corresponding to the model's local uncertainty.

Figure 11 shows the convex hulls in several arbitrary subspaces of the model response before and after identification, together with the simulated measurement points. The identification based on the subspace spanned by f_5 and f_6 also leads to an exact identification of the uncertainty in other subspaces for this specific model.

Figure 10: Illustration of the extreme realisations of the input field $\gamma_F^I(\mathbf{r})$ before and after identification, as well as the *goal* input field as a function of the element number. A numerically exact identification of the goal interval field is achieved.

Figure 11: Convex hulls of the measurement set, together with the convex hulls of the uncertain realisation sets before and after identification, based on an over-conservative initial guess. These hulls are shown for four arbitrary sets of two eigenfrequencies

6.1.4. Influence of selected subset

In order to demonstrate the performance of the optimal subset selection method, as presented in section 4, the identification procedure is performed using each possible 2-dimensional subset that can be constructed from $\mathbf{y}_{sj}$. The influence of the selected subset on the accuracy of the identification is indicated in figure 12. The accuracy is here computed as the sum of the absolute values of each individual discrepancy metric κ_i . Also the corresponding selection metric μ_l is shown. The same simulated measurement data set was used for all conducted identification procedures. All identifications were initiated from the same initial guess on the interval field uncertainty, with $\alpha_{1,0}^I = [1.00; 1.81]$ GPa and $\alpha_{2,0}^I = [1.10; 2.00]$ GPa.

A first conclusion is that, as predicted by the selection metric, an accurate identification is attained for the reduced uncertain realisation sets built on the 5th eigenfrequency in combination with the 4th, 6th, 7th, 8th, 9th and 10th eigenfrequency. The combined error of the identification using all these reduced uncertain realisations sets is only 0.2%. Moreover, also other combinations of eigenfrequencies (e.g. $\{3, 6\}$, $\{3, 8\}$, $\{3, 9\}$ and $\{3, 10\}$) yield accurate results, as predicted by the fact that their corresponding selection metric is rather good. Looking at μ_l , the combination of the 3rd and the 7th should as well be able to deliver an accurate identification of the interval field, as it is approximately equal to the selection metric of the other reduced uncertain realisation sets consisting of the 3rd eigenfrequency. This anomaly can be explained by the fact the optimisation solver converged on a local minimum during the iterative procedure. Note however, that μ_l for this candidate reduced uncertain realisation set is an order of magnitude lower than the optimal one (i.e., $\{5, 6\}$). Also, in some cases it can be seen that the procedure is capable of performing an accurate identification, even though the selection metric predicts otherwise (e.g., $\{1, 10\}$). This is more due to chance, since the high aspect ratio of the uncertain realisation set makes it very hard for the optimisation solver to fit it around the set of measurement data. As a final remark, it can be seen that all identifications that succeed seem to yield the same error after identification. This can be explained by the fact that the same measurement data set was used in combination with equal stopping criteria for the optimisation solver. Therefore, to conclude, an accurate identification is attainable when the optimal reduced uncertain realisation set is used for the procedure. However, accurate identification is also possible with other sets, but it is then not guaranteed that the optimisa-

tion solver will converge on the global optimum, and thus the correct interval scalars.

Figure 12: Accuracy of the identification as a function of the selected reduced realisation set that was used for the identification. Also the subspace selection metric μ_l , as defined in eq. 36 is given for each of these candidate reduced uncertain realisation sets.

6.1.5. Influence of initial guess

In order to demonstrate the robustness of the methodology with respect to the initial guess, the identification is performed starting from 39 different initial guesses on the uncertainty that is captured by $\boldsymbol{\alpha}^I$ of the $\boldsymbol{\gamma}_F^I(\mathbf{r})$. Figure 13 shows a schematic overview of all initial guesses from where the identification was started.

Specifically, a hypercubic *design space* (in blue in the figure) is defined by expanding the interval range of the local intervals $\check{\alpha}_1^I$ and $\check{\alpha}_2^I$ that were used for the construction of the measurement data (the shaded area in the figure). The bounds of this design space are taken as $\alpha_{0,1}^I = [0.35; 2.5] \text{ GPa}$ and $\alpha_{0,2}^I = [0.35; 2.5] \text{ GPa}$. This space is subsequently subdivided in 36 equivolumetric and evenly spaced hypercubes, and identification is started from each of these hypercubes. These hypercubes are indicated with the dashed lines in figure 13; one such hypercube is hatched as illustration. Additionally, identification is also performed starting from the bounds of the design space (blue) and two additional hypercubes (red and green). In this way, a systematic overview of the robustness of the methodology is obtained. The base assumption for this study is that the analyst performing the identification has sufficient knowledge on the order of magnitude of both the deterministic value of the parameters, as on the order of magnitude of the possible uncertainty that is to be identified. This knowledge is based on the engineering judgement of the analyst, or can also be deduced from qualitative analysis of the measurement data.

For this study, both the subset spanned by the 5th and 6th eigenfrequency (i.e. an optimal subset) and spanned by the 1st and 10th eigenfrequency (i.e. a sub-optimal subset) are used for the identification. The same set of measurement data as for the study on the influence of the selected subset is used for all performed identifications.

Starting from these 39 initial guesses, the average identification errors and the standard deviation hereon for all converged identification procedures, based on the optimal subset, are listed in table 5. A highly accurate identification is obtained following the proposed methodology, starting from nearly all initial guesses that were tested. Moreover, the standard deviation on this error is negligible, proving that the accuracy of the methodology is robust w.r.t. the initial guess on the interval field. This is further proven by the lack of correlation between the error of the initial guess and the error after identification, as shown in figure 14. On average, 40 ± 7 iterations were

Figure 13: Schematic overview of the tested initial guesses

necessary before convergence was obtained.

Table 5: Average identification error and the standard deviation hereon of all converged identification procedures for the subset spanned by the 5th and 6th eigenfrequency.

	$\underline{\alpha}_1$ [Pa]	$\overline{\alpha}_1$ [Pa]	$\underline{\alpha}_2$ [Pa]	$\overline{\alpha}_2$ [Pa]
$\check{\alpha}^I$	$1.35 \cdot 10^6$	$1.65 \cdot 10^6$	$1.55 \cdot 10^6$	$1.85 \cdot 10^6$
$mean(\kappa_i^*)$	0.0906 %	0.0361 %	0.0843 %	0.0233 %
$std(\kappa_i^*)$	0.0015 %	0.0009 %	0.0009 %	0.0001 %

It should however be noted that in three cases the identifications failed to converge to the exact value, because the optimisation solver converged on a local minimum in the objective function, instead of on the global minimum. This is remediable by using another, possibly global, optimisation algorithm for the minimisation of eq. (21).

For the subset spanned by 1st and 10th eigenfrequency, the optimisation converged for 22 of the 39 initial guesses, indicating that the procedure is less robust when a sub-optimal subset is selected. For these 22 converged identification procedures, the average error and standard deviation are shown in table 6. As can be noted, the identification *in se* is equally accurate as

Figure 14: Scatterplot of the sum of absolute errors before ($\kappa_{i,0}$) and after (κ_i^*) application of the identification procedure.

when a less-optimal subset is selected, but identification is far less robust because of the higher sensitivity to local minima. Moreover, on average 47 ± 21 iterations were needed, which is considerably more as compared to the identification with the optimal subset. Finally, when using the least-optimal subset (i.e. the subset spanned by the 1st and 5th eigenfrequency), not a single identification procedure managed to converge to the global optimum.

Table 6: Average identification error and the standard deviation hereon of all converged identification procedures for the subset spanned by the 1st and 10th eigenfrequency.

	$\underline{\alpha}_1 [Pa]$	$\overline{\alpha}_1 [Pa]$	$\underline{\alpha}_2 [Pa]$	$\overline{\alpha}_2 [Pa]$
$\tilde{\alpha}^I$	$1.35 \cdot 10^6$	$1.65 \cdot 10^6$	$1.55 \cdot 10^6$	$1.85 \cdot 10^6$
$mean(\kappa_i^*)$	0.0756 %	0.0478 %	0.0825 %	0.0232 %
$std(\kappa_i^*)$	0.0603 %	0.0402 %	0.0064 %	0.0002 %

The fact that even when an optimal subset is used for the identification, minimisation in some cases converged to a local minimum, should not be prohibitive for the application of the technique. In this case, the analyst has a clear indication of the failure to converge by plotting 2 or 3-dimensional cross-sections of $\tilde{\mathbf{y}}_s^C$ and $\tilde{\mathbf{y}}_m^C$. As such, the analyst can see that both sets

are completely, or partially disjunct, indicating that the uncertain realisation set does not describe the same uncertainty as the one captured by the measurement set. In that case, simply another initial guess can be tried for the identification, as it has been proven that, as long as the optimiser has converged, a robust and accurate identification can be obtained. Also, as an alternative solution to this, other (possibly global) optimisers can be used to minimise the objective function.

6.2. Cantilever Beam example with a 4-dimensional input

A second validation using an interval field in a higher dimensional input space is made using the same cantilever beam example that is shown in figure 7. In this case, the analyst has interval information about Young's modulus in elements 2, 4, 6 and 8. Therefore, locally defined intervals α_1^I , α_2^I , α_3^I and α_4^I are defined respectively at $\mathbf{r}_1 = 2$, $\mathbf{r}_2 = 4$, $\mathbf{r}_3 = 6$ and $\mathbf{r}_4 = 8$. Young's modulus in other elements is inter- or extrapolated following eq. (39). In this specific case, the interval field FE model is evaluated using the transformation method, leading to $q = 2^4 = 16$ realisations in the simulation space. These realisations are used as a basis for the convex hull representations of the uncertain realisation set $\tilde{\mathbf{y}}_s$, which is defined as:

$$\tilde{\mathbf{y}}_s = \{\mathbf{y}_{s1}, \mathbf{y}_{s2}, \dots, \mathbf{y}_{s16}\} \quad (44)$$

Measurement data are simulated using an *a priori* defined interval field, based on known local intervals $\check{\alpha}^I \in \mathbb{R}^4$, located at $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_4$. $\check{\alpha}^I$ is defined as:

$$\check{\alpha}^I = \begin{bmatrix} \check{\alpha}_1^I \\ \check{\alpha}_2^I \\ \check{\alpha}_3^I \\ \check{\alpha}_4^I \end{bmatrix} = \begin{bmatrix} 1.35 ; & 1.65 \\ 1.55 ; & 1.85 \\ 1.45 ; & 1.75 \\ 1.25 ; & 1.55 \end{bmatrix} GPa \quad (45)$$

A first step of the identification procedure consists of the selection a subset $\tilde{\mathbf{y}}_s^r \subset \tilde{\mathbf{y}}_s$ to speed up the computation of the convex hull $\tilde{\mathbf{y}}_m^C$. The selection of this subset is based on the methodology that is presented in section 4. In this specific case, the 2nd, 3rd, 4th and 6th eigenfrequencies of the model are selected as they show the optimal combination of multidimensional interval sensitivity with respect to changes in the uncertainty in the input parameters of the model and multidimensional aspect ratio for all input interval scalars. These eigenfrequencies correspond respectively to the 2nd out-of-plane bending mode, 1st torsion mode, 3rd out-of-plane bending mode and 2nd torsion mode. The result vector $\mathbf{y}_{sj}$ is thus reduced to a reduced result vector $\mathbf{y}_{sj}^r$:

$$\mathbf{y}_{sj}^r = [f_2, f_3, f_4, f_6]^T \quad (46)$$

The measurement set $\tilde{\mathbf{y}}_m$ is analogously reduced to $\tilde{\mathbf{y}}_m^r$. The results of the identification procedure, based on the 4-dimensional input interval field are shown in table 7. This table shows the interval vector $\check{\alpha}^I$ that was used for the construction of the measurement data set, the interval vector containing

the first initial guess α_0^I and the identified interval vector $\alpha^{I,*}$. Also the discrepancy metrics that have been defined in eq.(43) are listed. The table shows that the proposed methodology is capable of determining the goal interval field, with a local maximum error of 2.19%. This shows that also for problems having more than two dimensions at the input side of the model, a very accurate identification is feasible using the proposed technique.

Table 7: Results of the identification procedure on an interval field capturing the spatial variability in the cantilever beam example, based on the propagation of the 4-dimensional input interval field. $\kappa_{0,i}$ is the error of the initial guess, the κ_i^* is the error after application of the identification procedure. $\check{\alpha}^I$ is the interval vector containing the interval scalars that were sampled for the construction of the measurement data. α_0^I contains the initial estimates on the interval scalars. Finally, $\alpha^{I,*}$ contains the identified interval scalars.

	$\underline{\alpha}_1$ [GPa]	$\overline{\alpha}_1$ [GPa]	$\underline{\alpha}_2$ [GPa]	$\overline{\alpha}_2$ [GPa]
$\check{\alpha}^I$	1.35	1.65	1.55	1.85
α_0^I	1.00	1.81	1.10	2.30
$\kappa_{0,i}$	25.9%	-9.69%	29.0%	-24.3%
$\alpha^{I,*}$	1.36	1.64	1.58	1.85
κ_i^*	-0.85%	0.66%	-2.19%	0.27%

	$\underline{\alpha}_3$ [GPa]	$\overline{\alpha}_3$ [GPa]	$\underline{\alpha}_4$ [GPa]	$\overline{\alpha}_4$ [GPa]
$\check{\alpha}^I$	1.45	1.75	1.25	1.55
α_0^I	1.20	1.95	1.30	2.10
$\kappa_{0,i}$	17.2%	-11.4%	-4.00%	-35.5%
$\alpha^{I,*}$	1.46	1.74	1.24	1.55
κ_i^*	-1.04%	0.57%	0.56%	0.24%

7. Conclusions

In order to represent spatial uncertainty in the physical model parameters constituting the input of a Finite Element (FE) model, the interval field concept proves to deliver accurate results with limited data. The use of this concept however, requires the identification of the interval scalars and base vectors. A general methodology to identify the interval scalars, based on measurement data is introduced. The concept of the identification is based on the convex hull of the uncertain output set, which originates from propagating the input interval field through the interval FE solver on the one hand, and the calculation of the convex hull of the measurement data on the other hand. Identification is performed by minimising a cost function which contains of the discrepancy between the multidimensional volumes of both convex hulls and their overlap.

A first validation is performed on an academic example of a dynamic model of a mass-spring system containing three degrees of freedom and six spring with uncertain stiffness. Three case studies are performed and an accurate identification of the uncertainty in the spring stiffness is obtained in each case.

The methodology is also validated on a dynamic FE model of a cantilever beam, which was solved for the 10 first in-plane and out-of-plane eigenfrequencies. Measurement data are generated through Monte Carlo sampling of a predefined interval field for benchmarking purposes. The general conclusion is that the interval scalars can be identified with high accuracy using the proposed methodology within a limited number of iterations. When the optimal subset is selected, the method is also found to be robust against the initial guess on the constituting uncertainty in the model parameters of the analyst. The method is thus capable of identifying interval field uncertainty based on measurement data and some expert knowledge on the amount of control points that are needed for the representation of the spatial nature of the uncertainty. Because of this limited amount of necessary assumptions, the technique is widely applicable to any model containing interval uncertainty in its input parameters.

The method is validated on interval fields that are built on the Inverse Distance Weighting technique. It should however be noted that no assumptions with regard to the formal definition of the base vectors are made in the identification procedure, neither does the methodology make use of these parameters, as it purely looks at the discrepancy between the response of

the interval field model and measurement data to identify these parameters. The proposed method is therefore independent of the definition and forward propagation of the interval field, as long as the interval field is parametric.

In theory, the developed methodology is independent of the number of output responses that are considered. In practice however, the computation of the convex hulls suffers from the so-called curse of dimensionality; the computational cost increases exponentially with the number of responses that are considered, as each response corresponds to a dimension over which the convex hulls needs to be computed. Therefore, also an automated procedure to select the optimal subset of responses out of the full dimensional result of the interval field computation in the context of the identification procedure is introduced in this paper. It is shown that this method does select a subset out of the result that is capable of providing a robust and accurate identification of the interval field at the input side of the model.

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