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Control of Linear Parameter-Varying Systems using B-Splines

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Abstract—This paper presents a novel approach to efficiently solve parameter-dependent (PD) linear matrix inequality (LMI) problems for, amongst others, linear parameter-varying (LPV) control design. Typically, stability and performance is guaranteed by finding a PD Lyapunov function such that a PD LMI is feasible on a parameter domain. To solve the resulting semi-infinite problems, we propose a novel LMI relaxation technique relying on B-spline basis functions. This technique provides less conservative solutions and/or a reduced numerical burden compared to existing approaches. Moreover, an elegant generalization of worst-case optimization to the optimization of any signal norm is obtained by expressing performance bounds as a function of the system parameters. This generalization yields better performance bounds in a large part of the parameter domain. Numerical comparisons with the current state-of-the-art demonstrate the generality and effectiveness of our approach.

I. INTRODUCTION

Bridging the gap between the restricted class of linear time-invariant (LTI) systems and the general class of nonlinear systems, the framework of LPV systems has been widely considered for decades, and proved successful in various analysis and control design applications [1], [2].

In the modern LPV controller design approaches, a controller is directly designed by solving a PD optimization problem [3], [4], [5], [6]. The main benefit of these approaches is that, in contrast to the classical interpolation based approaches (see, for instance, [7], [8]), they provide a certificate of stability and performance, since they rely on Lyapunov stability theory [9].

Typically, LPV synthesis problems are translated into the search for a PD Lyapunov function which guarantees feasibility of a PD LMI on a parameter domain. However, the corresponding PD LMI problem is numerically intractable, due to infinite-dimensional optimization variables and infinitely many LMI constraints. The former is relieved by imposing a parameterization on the Lyapunov function, such that a finite number of optimization variables results. Subsequently, so-called LMI relaxations provide an elegant and effective solution to derive a numerically tractable (i.e., finite) set of LMIs which guarantee feasibility of the parameterized PD LMI on the parameter domain. The relaxation step

amounts to replacing the infinite set of constraints by a more restrictive finite (i.e., conservative) set of constraints.

Considering polynomial parameterizations, Pólya's theorem and sum-of-squares decompositions have been widely applied to derive LMI relaxations [10], [11]. Both of these approaches allow systematic reductions of conservatism at the expense of an increased numerical burden.

In this paper, we present a novel LMI relaxation approach, which can be seen as an extension of Pólya relaxations. This approach relies on expressing polynomially PD LMIs in terms of so-called B-splines [12], [13], see also [14]. Since B-splines are positive, positivity (negativity) of the corresponding coefficients implies positivity (negativity) of the associated polynomially PD LMI. In addition to degree elevation, which is exploited to systematically reduce conservatism using Pólya relaxations, B-splines allow knot insertion as an attractive alternative. As a matter of fact, provided that a polynomially PD LMI is positive definite, its B-spline coefficients converge to positive definite matrices *linearly* with the number of degree elevations, and *quadratically* with the number of inserted knots, while the respective increases in computational cost are similar.

Furthermore, the B-spline framework can cope with *piece-wise* polynomial parameterizations of both the LPV system and the optimization variables, allowing for more accurate modeling and less conservative LMI relaxations compared to polynomial parameterizations.

Although mainly worst-case optimization is considered in the literature (see, amongst others, [15], [16], [5], [6]), an elegant and versatile generalization results when parameterizing performance bounds as a function of the parameter, and considering optimization of an arbitrary signal norm. Namely, optimizing a performance bound as a function of the parameter results in significantly better performance bounds on a large subset of the parameter space.

The paper is organized as follows. First, the problem is formulated and B-splines are defined in Section II. Then, our main results are discussed in Section III, followed by numerical validations in Section IV. Finally, the conclusions are provided in Section V.

Notation

The set of nonnegative integers is denoted by $\mathbb{N}$, while $\mathbb{R}^n$ ($\mathbb{R}^{m \times n}$) is the set of real vectors (matrices) of dimension n ($m \times n$). I_n denotes the identity matrix of dimension $n \times n$ and $0_{m \times n}$ denotes a zero matrix of dimension $m \times n$. The subscripts are omitted when the dimensions can be inferred from the context. The transpose of a matrix X is written as X' . The sets of real positive definite matrices of dimension

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n are denoted by $\mathbb{S}_+^n$. A star ($\star$) indicates a symmetric term in matrix inequalities.

II. PROBLEM FORMULATION

We consider the continuous-time LPV system

$$H: \begin{cases} \dot{x} = A(\alpha)x + B_u(\alpha)u + B_w(\alpha)w, & x(0) = 0, \\ z = C(\alpha)x + D_u(\alpha)u + D_w(\alpha)w, \end{cases} \quad (1)$$

with state $x: \mathbb{R}_+ \rightarrow \mathbb{R}^{n_x}$, control input $u: \mathbb{R}_+ \rightarrow \mathbb{R}^{n_u}$, exogenous input $w: \mathbb{R}_+ \rightarrow \mathbb{R}^{n_w}$ and regulated output $z: \mathbb{R}_+ \rightarrow \mathbb{R}^{n_z}$. All system matrices are real-valued and have a piecewise polynomial dependency on the exogenous multivariate parameter $\alpha = (\alpha_1, \dots, \alpha_N): \mathbb{R}_+ \rightarrow \mathbb{R}^N$ taking values in a Cartesian product of closed and bounded intervals:

$$\Lambda := [\underline{\alpha}_1, \bar{\alpha}_1] \times \dots \times [\underline{\alpha}_N, \bar{\alpha}_N] \subset \mathbb{R}^N. \quad (2)$$

For technical reasons, we require α to be continuously differentiable. Moreover, a priori known bounds on the rate of parameter variation are taken into account, yielding the set of possible parameter trajectories

$$\mathcal{T} := \left\{ \alpha: \mathbb{R}_+ \rightarrow \mathbb{R}^N \mid \begin{array}{l} (\alpha(t), \dot{\alpha}(t)) \in \Omega, \forall t \geq 0, \\ \alpha \text{ cont. differentiable} \end{array} \right\}. \quad (3)$$

Note that, by definition, $\Omega \in \mathbb{R}^{2N}$ is a Cartesian product of bounded intervals if the rate of parameter variation is bounded, and equals $\Lambda \times \mathbb{R}^N$ in the unrestricted case.

Remark 1: While $(\alpha, \dot{\alpha})$ lie in a hyperrectangle (i.e., a Cartesian product of intervals), in the discrete-time case the parameter domain is a convex polytope [15], [17]. Consequently, in the latter case, the derivation of LMI relaxations is more complicated.

The main objective is to derive tractable conditions for analysis and control of the LPV system (1), which can be efficiently solved with limited or no conservatism. To achieve this, we consider polynomial spline parameterizations, which are defined next.

A. Polynomial splines

Specifically, we assume that all system matrices in (1) have a tensor product polynomial spline dependency on α . To facilitate the introduction of tensor product polynomial splines, univariate polynomial splines are defined first. Subsequently, the extension to tensor product polynomial splines is briefly discussed.

1) *Univariate polynomial splines:* Consider a scalar parameter α on a closed and bounded interval $[\underline{\alpha}, \bar{\alpha}] \subset \mathbb{R}$, and let $\xi = (\xi_0, \dots, \xi_{l+1})$ be a sequence of points satisfying

$$\underline{\alpha} = \xi_0 < \xi_1 < \dots < \xi_l < \xi_{l+1} = \bar{\alpha}.$$

Then, a matrix $S(\alpha)$ is a polynomial spline (i.e., piecewise polynomial) of degree g with internal break points $\xi_1, \dots, \xi_l$ and continuity conditions $v_1, \dots, v_l$ if there exist polynomial matrices $P_0(\alpha), \dots, P_l(\alpha)$ of degree g such that

$$\begin{aligned} S(\alpha) &= P_i(\alpha), \quad \text{for } \alpha \in [\xi_i, \xi_{i+1}), \quad i = 0, \dots, l-1, \\ S(\alpha) &= P_l(\alpha), \quad \text{for } \alpha \in [\xi_l, \xi_{l+1}], \end{aligned}$$

Fig. 1. A polynomial spline is a piecewise polynomial with continuity requirements. A scalar univariate polynomial spline $s: [0, 5] \rightarrow \mathbb{R}$ of degree 3 with internal break points $(\xi_1, \xi_2, \xi_3) = (1, 2, 4)$ and continuity requirements $(v_1, v_2, v_3) = (2, 1, 2)$ is shown.

and

$$\left. \frac{d^{j-1} P_{i-1}}{d\alpha^{j-1}} \right|_{\alpha=\xi_i} = \left. \frac{d^{j-1} P_i}{d\alpha^{j-1}} \right|_{\alpha=\xi_i}, \quad \text{for } j = 1, \dots, v_i, \quad i = 1, \dots, l.$$

The latter conditions imply that, in a breakpoint ξ_i , $i \in \{1, \dots, l\}$, $S(\alpha)$ and its derivatives up to order $(v_i - 1)$ are continuous. Fig. 1 illustrates this concept for a scalar-valued function.

By virtue of the Curry-Schoenberg theorem [12], $S(\alpha)$ can always be expressed in terms of particular normalized (scalar) B-spline basis functions, by considering the knot sequence

$$\lambda = (\underbrace{\xi_0, \dots, \xi_0}_{g+1}, \underbrace{\xi_1, \dots, \xi_1}_{g+1-v_1}, \dots, \underbrace{\xi_l, \dots, \xi_l}_{g+1-v_l}, \underbrace{\xi_{l+1}, \dots, \xi_{l+1}}_{g+1}).$$

Denoting the i^{th} normalized B-spline basis function of degree g for the knot sequence $\lambda \in \mathbb{R}^{n_\lambda}$ by $B_{i,g,\lambda}(\alpha)$, the PD matrix $S(\alpha)$ is expressed as

$$S(\alpha) = \sum_{i=1}^{n_\lambda - g - 1} C_i B_{i,g,\lambda}(\alpha), \quad (4)$$

where C_i , $i = 1, \dots, n_\lambda - g - 1$ are matrix-valued coefficients. Specifically, for a given knot sequence λ , the B-splines $B_{i,g,\lambda}(\alpha)$, $\alpha \in [\underline{\alpha}, \bar{\alpha}]$, are computed using the Cox-de Boor recursive formula (see [12], p. 90):

$$\begin{aligned} B_{i,g,\lambda}(\alpha) &= \frac{\alpha - \lambda_i}{\lambda_{i+g} - \lambda_i} B_{i,g-1,\lambda}(\alpha) \\ &\quad + \frac{\lambda_{i+g+1} - \alpha}{\lambda_{i+g+1} - \lambda_{i+1}} B_{i+1,g-1,\lambda}(\alpha), \end{aligned}$$

starting with

$$B_{i,0,\lambda}(\alpha) = \begin{cases} 1 & \text{if } \alpha \in [\lambda_i, \lambda_{i+1}), \\ 0 & \text{else.} \end{cases}$$

B-splines are commonly used as basis functions for polynomial splines, since they possess various useful properties [12], [13]:

- Positivity:

$$B_{i,g,\lambda}(\alpha) \geq 0, \quad \forall \alpha \in [\underline{\alpha}, \bar{\alpha}], \quad i = 1, \dots, n_\lambda - g - 1.$$

- Partition of unity:

$$\sum_{i=1}^{n_{\lambda}-g-1} B_{i,g,\lambda}(\alpha) = 1, \quad \forall \alpha \in [\underline{\alpha}, \bar{\alpha}].$$

- Local support:

$$\begin{aligned} B_{i,g,\lambda}(\alpha) &> 0, \quad \forall \alpha \in (\lambda_i, \lambda_{i+g+1}), \\ B_{i,g,\lambda}(\alpha) &= 0, \quad \forall \alpha \notin [\lambda_i, \lambda_{i+g+1}]. \end{aligned}$$

2) *Tensor product polynomial splines*: Tensor product polynomial splines constitute a particular multivariate extension of univariate polynomial splines. By specifying a degree g_k and a knot sequence λ_k for every coordinate α_k , $k = 1, \dots, N$, a tensor product polynomial spline $S(\alpha)$ is defined on Λ in (2) as

$$S(\alpha) = \sum_{i_1=1}^{n_{\lambda_1}-g_1-1} \cdots \sum_{i_N=1}^{n_{\lambda_N}-g_N-1} C_{i_1, \dots, i_N} \underbrace{\left(\prod_{k=1}^N B_{i_k, g_k, \lambda_k}(\alpha_k) \right)}_{B_{i,g,\lambda}(\alpha)}, \quad (5)$$

where B_{i_k, g_k, λ_k} and $B_{i,g,\lambda}$ are univariate, respectively, tensor product B-splines. It should be emphasized that the properties of univariate B-splines transfer to tensor product B-splines: they are linearly independent, nonnegative, sum up to one (partition of unity), and have local (minimal) support. Also, note that the restriction of a multivariate polynomial to Λ is a tensor product polynomial spline. To simplify terminology, tensor product polynomial splines are named polynomial splines in the remainder of this paper.

III. MAIN RESULTS

In this section, we present a novel B-spline based approach to efficiently solve PD LMI problems. While the approach is applicable to a wide range of PD LMI problems, its presentation is facilitated by considering the design of an unstructured $\mathcal{H}_\infty$ state feedback controller for the LPV system (1). PD LMIs for, amongst others, robust stability analysis, $\mathcal{H}_\infty$ analysis, $\mathcal{H}_2$ analysis, and $\mathcal{H}_2$ state feedback synthesis are closely related and therefore omitted.

A. Parameter-dependent LMIs

To illustrate the B-spline based relaxation approach, we apply it to the PD LMI for unstructured $\mathcal{H}_\infty$ state feedback controller design presented in Theorem 1, see [18].

Theorem 1 (Unstructured $\mathcal{H}_\infty$ state feedback): If there exists a bounded matrix $Q(\alpha) : \Lambda \rightarrow \mathbb{S}_+^{n_x}$, for $\alpha \in \mathcal{T}$, such that the LMI

$$\underbrace{\begin{bmatrix} T(\alpha, \dot{\alpha}) & \star & \star \\ B_w(\alpha)' & -\gamma I_{n_w} & \star \\ C(\alpha)Q(\alpha) + D_u(\alpha)Z(\alpha) & D_w(\alpha) & -\gamma I_{n_z} \end{bmatrix}}_{\Upsilon(\alpha, \dot{\alpha})} \prec 0 \quad (6)$$

with

$$\begin{aligned} T(\alpha, \dot{\alpha}) &:= A(\alpha)Q(\alpha) + Q(\alpha)A(\alpha)' - \sum_{i=1}^N \frac{\partial Q(\alpha)}{\partial \alpha_i} \dot{\alpha}_i \\ &\quad + B_u(\alpha)Z(\alpha) + Z(\alpha)'B_u(\alpha)' \end{aligned}$$

is feasible for all $(\alpha, \dot{\alpha}) \in \Omega$, then the unstructured state feedback controller $u = Z(\alpha)Q(\alpha)^{-1}x$ stabilizes the LPV system (1) with a guaranteed bound on the closed-loop $\mathcal{H}_\infty$ performance $\|H\|_\infty < \gamma$, for all $\alpha \in \mathcal{T}$.

Although mainly worst-case optimization is considered in the literature (see, amongst others, [15], [16], [5], [6]), an elegant generalization results when parameterizing γ as a function of $(\alpha, \dot{\alpha})$, and considering the optimization problem

$$\begin{aligned} &\underset{Q(\cdot), Z(\cdot), \gamma(\cdot)}{\text{minimize}} : \|\gamma(\alpha, \dot{\alpha})\|_p \\ &\text{subject to} : \quad Q(\alpha) \succ 0, \\ &\quad \quad \quad \Upsilon(\alpha, \dot{\alpha}) \prec 0, \quad \forall (\alpha, \dot{\alpha}) \in \Omega, \end{aligned} \quad (7)$$

where $\|\cdot\|_p$, $1 \leq p \leq \infty$, denotes the $\mathcal{L}_p$ norm:

$$\|\gamma(\beta)\|_p := \left(\int_{\Omega} |\gamma(\beta)|^p d\beta \right)^{1/p}.$$

This generalization has the interesting implication that, for $p < \infty$, γ is optimized as a function of $(\alpha, \dot{\alpha})$, and thus directly provides useful information about the worst-case $\mathcal{H}_\infty$ bound on subdomains of the parameter space Ω . Although selecting $p < \infty$ might have a negative effect on the worst-case performance, it may constitute a significant performance improvement on a large part of the parameter domain.

As a matter of fact, in case of a time-invariant parameter $\alpha \in \Lambda$, the optimal solution of (7) for $p < \infty$ (e.g., $p = 1$) corresponds to the optimal $\mathcal{H}_\infty$ performance for each $\alpha \in \Lambda$.

Since the PD optimization problem (7) features infinite-dimensional optimization variables and infinitely many constraints, it is numerically intractable. To tackle this problem, an efficient solution exploiting polynomial spline parameterizations is explained next.

B. LMI relaxations with B-splines

In this subsection, we present a novel approach to derive a numerically tractable (i.e., finite) set of LMIs which, when feasible, guarantees feasibility of (7) for all $\alpha \in \mathcal{T}$. This approach consists of the following two steps:

- 1) We impose a polynomial spline parameterization on $Q(\alpha)$, $Z(\alpha)$ and $\gamma(\alpha, \dot{\alpha})$, resulting in a finite number of optimization variables.
- 2) An LMI relaxation is applied on $Q(\alpha)$ and $\Upsilon(\alpha, \dot{\alpha})$ by exploiting the structure of the parameter domain Ω in conjunction with the positivity of B-splines.

Applying step 1 yields a polynomial spline dependency of the PD LMI $\Upsilon(\alpha, \dot{\alpha})$ on $(\alpha, \dot{\alpha}) \in \Omega$, which is thus conveniently expressed in terms of B-splines, as in (5). Subsequently, the positivity property of B-splines reveals that

$$\begin{aligned} C_{i_1, \dots, i_{2N}} &\prec 0, \quad i_k = 1, \dots, n_{\lambda_k} - g_k - 1, \quad k = 1, \dots, 2N, \\ &\Rightarrow \Upsilon(\alpha, \dot{\alpha}) \prec 0, \quad \forall (\alpha, \dot{\alpha}) \in \Omega, \end{aligned}$$

and, in turn, $\Upsilon(\alpha, \dot{\alpha}) \prec 0$ for all $\alpha \in \mathcal{T}$. Hence, step 2 can be addressed by imposing positive, respectively, negative definiteness on all the B-spline coefficients of $Q(\alpha)$ and $\Upsilon(\alpha, \dot{\alpha})$. Generally speaking, imposing positive (negative) definiteness on all the B-spline coefficients of a polynomial

spline is sufficient for positive (negative) definiteness of the polynomial spline itself.

Although a finite set of sufficient conditions for the PD LMIs in (7) is readily derived from the associated B-spline coefficients, a less conservative but larger set of sufficient LMIs can be obtained by extending the B-spline bases of the PD LMIs $Q(\alpha)$ and $Y(\alpha, \dot{\alpha})$. This is explained next for univariate polynomial splines.

Consider the univariate polynomial spline $S(\alpha)$, defined in (4). It is always possible to express $S(\alpha)$ in terms of a higher dimensional B-spline basis, by applying either degree elevation or knot insertion. Then, the spline $S(\alpha)$ can be explicitly written as

$$S(\alpha) = \sum_{i=1}^{n_{\tilde{\lambda}}-(g+d)-1} \tilde{C}_i B_{i,g+d,\tilde{\lambda}}(\alpha),$$

where $\tilde{\lambda} \supset \lambda$ and $d \in \mathbb{N}$. Requiring $\tilde{C}_i \succ 0$, $i = 1, \dots, n_{\tilde{\lambda}} - (g + d) - 1$ is less conservative than $C_i \succ 0$, $i = 1, \dots, n_{\lambda} - g - 1$, at the expense of more coefficients.

1) *Degree elevation*: Increasing the degree of $S(\alpha)$ by $d \in \mathbb{N}$ on each subinterval $[\xi_i, \xi_{i+1})$ corresponds to the extended knot sequence

$$\tilde{\lambda} = (\underbrace{\xi_0, \dots, \xi_0}_{g+d+1}, \underbrace{\xi_1, \dots, \xi_1}_{g+d+1-v_1}, \dots, \underbrace{\xi_l, \dots, \xi_l}_{g+d+1-v_l}, \underbrace{\xi_{l+1}, \dots, \xi_{l+1}}_{g+d+1}),$$

with $n_{\tilde{\lambda}} = n_{\lambda} + d(l + 2)$, resulting in $d(l + 1)$ more coefficients.

2) *Knot insertion*: Let $\tilde{\lambda}$ be constructed from λ by adding a single knot λ_{add} between two breakpoints. Then, $n_{\tilde{\lambda}} = n_{\lambda} + 1$, and the coefficients $\tilde{C}_i$, $i = 1, \dots, n_{\tilde{\lambda}} - g - 1$ are related to the original coefficients of (4) as follows [12]:

$$\tilde{C}_i = (1 - \beta_{i,g}(\lambda_{\text{add}}))C_{i-1} + \beta_{i,g}(\lambda_{\text{add}})C_i,$$

where the function $\beta_{i,g}$ is defined by

$$\beta_{i,g}(\alpha) = \begin{cases} 0 & \text{if } \alpha \leq \lambda_i, \\ \frac{\alpha - \lambda_i}{\lambda_{i+g} - \lambda_i} & \text{if } \lambda_i < \alpha < \lambda_{i+g}, \\ 1 & \text{if } \lambda_{i+g} \leq \alpha. \end{cases}$$

The extension to tensor product polynomial splines follows by applying degree elevation and/or knot insertion for each coordinate separately.

Remark 2: Pólya relaxations for tensor product polynomials, i.e., with parameters in the multi-simplex (see [19]), constitute a particular case of the above degree elevation procedure. Namely, selecting $l = 0$ internal breakpoints for each coordinate, a tensor product polynomial results with the same basis as the homogeneous polynomial basis up to a binomial factor.

Additional reductions of conservatism can be achieved by increasing the polynomial degrees of $Q(\alpha)$, $Z(\alpha)$ and $\gamma(\alpha, \dot{\alpha})$, or by a proper extension of their knot sequences (e.g., equidistant spacing in each coordinate).

It is worth remarking that the above techniques can be applied to solve a wide variety of relevant PD LMI problems, ranging from stability analysis to involved output feedback control problems.

IV. NUMERICAL VALIDATIONS

This section demonstrates the merits of B-spline parameterizations for analysis and control of LPV systems, by means of numerical comparisons with existing approaches. The LMIs are implemented and solved in MATLAB using the software packages Yalmip [20] and MOSEK [21].

A. Example 1

In this first example, we investigate Schur stability of the multivariate matrix

$$A(\alpha) = A_0 + \alpha_1 A_1 + \alpha_2 A_2,$$

depending on the uncertain constant parameter $(\alpha_1, \alpha_2) \in [-\rho, \rho] \times [-\rho, \rho]$, where

$$A_0 = \begin{bmatrix} -0.7 & 0.7 & 0 \\ -0.1 & -0.3 & -0.3 \\ -0.1 & 0.3 & 0.3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.7 & -0.3 & 0.4 \\ 0.7 & 0.7 & -0.5 \\ -1.5 & 0.1 & 0.7 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & -1 & 0.6 \\ 0.4 & 0.9 & 0.1 \\ -2.7 & -1.2 & -0.6 \end{bmatrix}.$$

Consider $\rho = 0.38409$, which is the maximum value for which $A(\alpha)$ is Schur stable. It is well-known that $A(\alpha)$ is Schur stable if, and only if, the PD LMI

$$\begin{bmatrix} Q(\alpha) & Q(\alpha)A(\alpha) \\ A(\alpha)'Q(\alpha) & Q(\alpha) \end{bmatrix} \succ 0 \quad (8)$$

is feasible for all $(\alpha_1, \alpha_2) \in [-\rho, \rho] \times [-\rho, \rho]$.

Table I compares different parameterizations of $Q(\alpha)$, and different techniques to derive LMI relaxations for (8).

Firstly, the approach in [11] relies on so-called polytopic modeling. That is, the PD LMI (8) is expressed as a function of the four vertices of the parameter domain. The resulting LMI condition depends on four parameters taking values in a unit simplex. A polynomial parameter dependency of order $g = 3$ is required to certify robust stability, corresponding to $K = 120$ scalar optimization variables and $L = 45$ coefficients of dimension 6×6 .

Secondly, in [19], the multi-simplex is exploited to express polynomials depending on multiple mutually independent parameters, corresponding to tensor product polynomial splines without internal knots. The lowest polynomial degree certifying robust stability is $g = (g_1, g_2) = (2, 2)$, resulting in $K = 54$ and $L = 16$.

Lastly, a polynomial spline parameterization is imposed on $Q(\alpha)$, allowing the addition of internal knots as extra freedom compared to the approach in [19]. As a result, selecting $g = (1, 1)$ and inserting a single knot at 0.5 in the second coordinate basis proves robust stability, outperforming the approaches [11] and [19] with $K = 36$ and $L = 15$.

B. Example 2

Next, we consider the design of an unstructured $\mathcal{H}_{\infty}$ state feedback controller $u = K(\alpha)x$ for the continuous-time

TABLE I

EXAMPLE 1: COMPARISON BETWEEN THE APPROACHES [11], [19] AND THE B-SPLINE APPROACH OF SECTION III-B.

Approach	g	λ_{int}	K	L
[11]	3	-	120	45
[19]	(2,2)	-	54	16
B-splines	(1,1)	(-,0.5)	36	15

LPV system (1), where the state-space matrices, which are borrowed from [22], are explicitly given by

$$A(\alpha) = \begin{bmatrix} -4.1 - 3\alpha & 1 \\ -2\alpha & 2 - 3.2\alpha \end{bmatrix}, \quad B_u(\alpha) = \begin{bmatrix} 3 \\ 2 - \alpha \end{bmatrix},$$

$$B_w(\alpha) = \begin{bmatrix} -0.03 - 0.3\alpha \\ -0.47 + 0.9\alpha \end{bmatrix}, \quad C(\alpha) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_u(\alpha) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and $D_w(\alpha) = 0_{2 \times 1}$. The time-varying parameter α assumes values in the interval $[0, 6]$ and its derivative is bounded by $|\Delta\alpha| < 10$, corresponding to the parameter domain $\Omega = [0, 6] \times [-10, 10]$. To optimize the closed-loop $\mathcal{H}_\infty$ performance as a function of $(\alpha, \dot{\alpha})$, we consider the PD LMI problem (7) with $p = 1$.

1) *Comparing different LMI relaxations:* First, we compare the effect of degree elevation and knot insertion on the conservatism of a finite set of sufficient LMIs for (7). To this end, all the optimization variables are parameterized as polynomials of degree $g = 2$, resulting in polynomially PD LMIs with $K = 24$ optimization variables.

An initial set of sufficient LMIs is derived by imposing positivity (negativity) on the B-spline coefficients of $Q(\alpha)$ ($\Upsilon(\alpha, \dot{\alpha})$), resulting in $C_Q = 3$ coefficients of dimension 2×2 and $C_\Upsilon = 12$ coefficients of dimension 5×5 , and a bound $\|\gamma(\alpha, \dot{\alpha})\|_1 = 126$, as shown in the first row of Table II.

Subsequently, the B-spline bases corresponding to $Q(\alpha)$ and $\Upsilon(\alpha, \dot{\alpha})$ are systematically extended to obtain less conservative LMI relaxations, in two different ways:

- Degree elevation: for each coordinate, the degree is increased with $d = 2^m - 1$, for $m = 1, \dots, 5$.
- Knot insertion: for each coordinate, m midpoint refinements are applied, corresponding to $2^m - 1$ inserted knots. Specifically, a single knot is added in the middle of each breakpoint interval when m is increased by one.

In fact, d is chosen such that the numerical burdens associated with degree elevation and knot insertion coincide for fixed values of m . Both approaches guarantee a monotonic decrease of conservatism when increasing m . The results are summarized in Table II, showing the number of LMI coefficients C_Q and C_Υ , the performance norms $\|\gamma(\alpha, \dot{\alpha})\|_1$ and the computation times t_c (in seconds) corresponding to degree elevation/knot insertion, for $m = 1, \dots, 5$. Although the numerical burdens are similar, knot insertion provides significantly lower performance norms compared to degree elevation for all cases.

2) *Comparing different parameterizations:* Now, we investigate the effect of selecting different parameterizations for $Q(\alpha)$, $Z(\alpha)$ and $\gamma(\alpha, \dot{\alpha})$ on the optimized performance

TABLE II

EXAMPLE 2: EFFECT OF DEGREE ELEVATION/KNOT INSERTION, I.E. $(d, n_{\lambda_{\text{ADD}}}) = (2^m - 1, 0) / (0, 2^m - 1)$, ON PERFORMANCE OBJECTIVE AND NUMERICAL BURDEN

m	C_Q	C_Υ	$\ \gamma(\alpha, \dot{\alpha})\ _1$	t_c [s]
0	3	12	126	0.61
1	4	20	110 / 101	0.75 / 0.87
2	6	42	93.6 / 78.3	0.98 / 0.87
3	10	110	81.2 / 66.0	1.31 / 1.26
4	19	342	72.3 / 61.5	2.98 / 2.99
5	34	1190	66.6 / 59.9	10.3 / 9.73

TABLE III

EXAMPLE 2: EFFECT OF INCREASING DEGREE OF LMI VARIABLES ON PERFORMANCE OBJECTIVE AND NUMERICAL BURDEN ($n_{\lambda_{\text{INT}}} = 0$)

g	K	C_Q	C_Υ	$\ \gamma(\alpha, \dot{\alpha})\ _1$	t_c [s]
1	14	2	6	163	0.55
2	24	3	12	126	1.08
3	36	4	20	104	0.97
4	50	5	30	90.1	1.19
5	66	6	42	81.8	1.17
6	84	7	56	75.9	1.58
7	104	8	72	71.7	1.68
8	126	9	90	68.4	1.70
9	150	10	110	65.8	1.95
10	176	11	132	63.7	1.98

norm. It is remarked that no additional degree elevation or knot insertion is applied to the resulting PD LMIs $Q(\alpha)$ and $\Upsilon(\alpha, \dot{\alpha})$.

Initially, all the optimization variables are parameterized as polynomials of degree $g = 1$, resulting in $K = 14$ scalar variables, $C_Q = 2$, $C_\Upsilon = 6$, a performance objective $\|\gamma(\alpha, \dot{\alpha})\|_1 = 163$, and a computation time of $t_c = 0.55$ s.

The effect of an increased polynomial degree g (number of equidistantly spaced internal knots $n_{\lambda_{\text{int}}}$) on the performance objective and the numerical burden is shown in Table III (Table IV). Note that, when adding knots instead of increasing the degree of the optimization variables, significantly lower performance objectives are obtained with a similar numerical burden. Specifically, selecting $n_{\lambda_{\text{int}}} = 3$ internal knots already outperforms a degree $g = 10$ polynomial parameterization, both in terms of conservatism and numerical complexity.

TABLE IV

EXAMPLE 2: EFFECT OF ADDING INTERNAL KNOTS TO LMI VARIABLES ON PERFORMANCE OBJECTIVE AND NUMERICAL BURDEN ($g = 1$)

$n_{\lambda_{\text{int}}}$	K	C_Q	C_Υ	$\ \gamma(\alpha, \dot{\alpha})\ _1$	t_c [s]
0	14	2	6	163	0.55
1	24	3	18	94.4	0.62
2	36	4	36	73.1	0.72
3	50	5	60	62.6	0.84
4	66	6	90	57.2	1.01
5	84	7	126	54.1	1.25
6	104	8	168	52.1	1.59
7	126	9	216	50.7	1.70
8	150	10	270	49.5	2.23
9	176	11	330	48.7	2.71

Fig. 2. Optimization of $\|\gamma(\alpha, \dot{\alpha})\|_1$ versus a worst-case bound γ .

Lastly, optimization of the $\mathcal{L}_1$ norm $\|\gamma(\alpha, \dot{\alpha})\|_1$ and the worst-case norm γ_{wc} (i.e., $p = \infty$ in (7)) are compared. We select a polynomial degree $g = 2$ and $n_{\text{int}} = 20$ uniformly spaced knots for each coordinate of all LMI variables, while no degree elevation or knot insertion is applied. The resulting function $\gamma(\alpha, \dot{\alpha})$ is compared with γ_{wc} in Fig. 2. Although $\mathcal{L}_1$ norm optimization yields a worst-case bound of 0.91, which is 56% higher than the optimized worst-case bound 0.58, better performance is guaranteed on 86% of the parameter domain.

V. CONCLUSIONS

A novel B-spline based approach has been presented to derive LMI relaxations for PD LMIs related to, amongst others, LPV control design. Initial numerical results have demonstrated the merits of exploiting B-splines for LMI relaxations, providing less conservative solutions and/or a reduced numerical burden compared to existing approaches. In addition, by optimizing a performance bound as a function of system parameters, an elegant generalization of worst-case optimization was obtained, resulting in better performance bounds in a large part of the parameter domain.

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