COMPUTING JUMPING NUMBERS IN HIGHER DIMENSIONS

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Abstract. We design an algorithm that computes a small set containing the jumping numbers of an ideal in a regular local ring of arbitrary dimension. We also provide some criteria to decide whether these numbers are jumping numbers.

Introduction

The jumping numbers of an ideal sheaf on a smooth algebraic variety are very interesting geometric invariants, that were studied in [5], but also appeared earlier in [10], [12], and [15]. They indicate in some sense how bad a singularity is, and are determined by the exceptional divisors in a resolution of the ideal.

Several algorithms have been developed to compute jumping numbers in specific settings. Tucker [14] designed an algorithm to compute jumping numbers on surfaces with rational singularities. Alberich-Carramiñana, Àlvarez-Montaner and the second author [1] introduced another algorithm in that setting. Berkesch and Leykin [3] and Shibuta [13] developed algorithms using Bernstein-Sato polynomials.

We present a generalization of the algorithm in [1], that can be used for computing jumping numbers on higher-dimensional varieties. The idea is to compute a small subset of the candidate jumping numbers, containing all the jumping numbers, and then, in many cases, one can check whether these numbers are jumping numbers.

1. Multiplier ideals and jumping numbers

In this first section, we introduce some basic notions, such as multiplier ideals, jumping numbers and contribution. Let R be a regular local ring over $\mathbb{C}$ such that $X = \operatorname{Spec} R$ is the germ of a smooth algebraic variety. We will be particularly interested in the case where $d := \dim X \geqslant 3$, since the two-dimensional case has been worked out completely in [1].

Let $\mathfrak{a} \subset \mathcal{O}_X$ be a sheaf of ideals on X, and take a log resolution $\pi : Y \to X$ of $\mathfrak{a}$. Denote by K_π the relative canonical divisor of π, and by F the normal crossings divisor on Y satisfying $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. The *multiplier ideal* of $(X, \mathfrak{a})$ with coefficient $c \in \mathbb{Q}_{>0}$ is defined as

$$\mathcal{J}(X, \mathfrak{a}^c) = \pi_* \mathcal{O}_Y(K_\pi - \lfloor cF \rfloor).$$

There exists a sequence of numbers $0 < \lambda_1 < \lambda_2 < \dots$ such that, for all i, we have that $\mathcal{J}(X, \mathfrak{a}^{\lambda_i}) \supsetneq \mathcal{J}(X, \mathfrak{a}^{\lambda_{i+1}})$, and $\mathcal{J}(X, \mathfrak{a}^c)$ is constant for $c \in [\lambda_i, \lambda_{i+1})$. These numbers are called the *jumping numbers* of the pair $(X, \mathfrak{a})$. The multiplier ideals and jumping numbers do not depend on the chosen resolution.

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Note that if we write $K_\pi = \sum_{i\in I} k_i E_i$ and $F = \sum_{i\in I} e_i E_i$, where the E_i are the irreducible components of F, then the set $\left\{ \frac{k_i+n}{e_i} \middle| i \in I, n \in \mathbb{Z}_{>0} \right\}$ contains the jumping numbers. The numbers in this set are called *candidate jumping numbers*. The smallest candidate is always a jumping number, and is called the *log canonical threshold*. We say λ is a *candidate for* $G = E_1 + \cdots + E_r$ if λ can be expressed as $\frac{k_i+n_i}{e_i}$ for $i = 1, \ldots, r$, where $n_i \in \mathbb{Z}_{>0}$.

If λ is a jumping number, and $G = E_1 + \cdots + E_r$ is a divisor such that λ is a candidate for G, we say G *contributes* λ if $\mathcal{J}(X, \mathfrak{a}^\lambda) \subsetneq \pi_* \mathcal{O}_Y(K_\pi - \lfloor \lambda F \rfloor + G)$. This happens if and only if $H^0(G, \mathcal{O}_Y(K_\pi - \lfloor \lambda F \rfloor + G)|_G) \neq 0$.

2. π-antieffective divisors

Definition 2.1. Generalizing the notion of antinef divisors, we say that a divisor D on Y is π-*antieffective* if $H^0(E, \mathcal{O}_Y(-D)|_E) \neq 0$ for every π-exceptional prime divisor E. This is equivalent with saying that $-D|_E$ defines a class in $\operatorname{Pic} E$ that contains an effective divisor.

Given a divisor D, one can compute its π-*antieffective closure* by the *unloading procedure*, i.e., if $H^0(E, \mathcal{O}_Y(-D)|_E) = 0$ for some E, replace D by $D + E$, and continue until the obtained divisor $\tilde{D}$ is π-antieffective. This is a generalization of the unloading procedure for divisors on surfaces, described in [4], [6] or [8]. The π-antieffective closure satisfies $\pi_* \mathcal{O}_Y(-\tilde{D}) = \pi_* \mathcal{O}_Y(-D)$.

3. An algorithm to compute jumping numbers

The set of *supercandidates* is constructed as follows.

Algorithm 3.1 (Computing supercandidates)**.** `Input:` An ideal $\mathfrak{a}$ and a resolution of $\mathfrak{a}$.
`Output:` The set of supercandidates with their minimal jumping divisors.

- The first supercandidate is the log canonical threshold.
- If λ is a supercandidate, then the next supercandidate is $\lambda' = \min \left\{ \frac{k_i+1+e_i^\lambda}{e_i} \middle| i \in I \right\}$, where $D_\lambda := \sum_{i\in I} e_i^\lambda E_i$ is the π-antieffective closure of $\lfloor \lambda F \rfloor - K_\pi$. The *minimal jumping divisor* of λ' is the reduced divisor $G_{\lambda'}$ supported on those E_i where this minimum is achieved.

Theorem 3.2. *The set of supercandidates contains all the jumping numbers.*

Proof. This follows from the fact that

$$\mathcal{J}(X, \mathfrak{a}^\lambda) = \pi_* \mathcal{O}_Y(K_\pi - \lfloor \lambda F \rfloor) = \pi_* \mathcal{O}_Y(-D_\lambda),$$

so there can be no jumping numbers between two consecutive supercandidates. □

If $\dim X = 2$, the converse also holds. This is a consequence of Lipman's result in [11, Section 18], stating that there is a one-on-one relation between integrally closed ideals and π-antieffective divisors. In higher dimensions, different π-antieffective divisors might determine the same ideal. Therefore, we might have supercandidates that are not jumping numbers. However, in several cases, we can check that a supercandidate is a jumping number.

Proposition 3.3. *If λ is a supercandidate such that G_λ has an irreducible connected component, then λ is a jumping number.*

This is a very important case, since in many situations, a significant number of supercandidates seem to have an irreducible jumping divisor.

Proposition 3.4. *If λ is a jumping number, it is contributed by G_λ, and hence there is a minimal contributing divisor $G \leqslant G_\lambda$.*

So if we want to check whether a supercandidate is a jumping number, we only need to check contribution by divisors $G \leqslant G_\lambda$. This seems to be hard in general when G_λ is reducible, but the following result can help.

Proposition 3.5. *If λ is a candidate for $G = E_1 + E_2$, and if $\mathcal{O}_Y(K_\pi - \lfloor \lambda F \rfloor + G)|_{E_i} \cong \mathcal{O}_{E_i}$ for $i \in \{1, 2\}$, then λ is a jumping number contributed by $E_1 + E_2$.*

All together, we get the following algorithm.

Algorithm 3.6 (Computing jumping numbers)**.** `Input:` An ideal $\mathfrak{a}$ and a resolution of $\mathfrak{a}$. `Output:` The set of jumping numbers of $\mathfrak{a}$.

- Compute the supercandidates λ, along with their minimal jumping divisors G_λ.
- If G_λ has an irreducible connected component, λ is a jumping number.
- Otherwise, check whether λ is a jumping number.

By Skoda's theorem [9, Theorem 9.6.21], it suffices to compute the supercandidates in $(0, d]$, or even in $(0, n]$, where n is the number of generators of $\mathfrak{a}$.

Remark 3.7. It can be hard in general to determine whether a linear equivalence class contains an effective divisor, or to decide about the existence of a global section on reducible divisors. This might complicate the unloading procedure and make it hard to check whether a supercandidate is a jumping number when Proposition 3.5 does not apply. However, in many examples, the provided results suffice to determine all the jumping numbers.

Remark 3.8. Apart from the obstructions mentioned in Remark 3.7, the algorithm can be implemented as follows. For the computation of a log resolution, one could use the algorithm of [7], implemented in the packages "resolve.lib" and "reszeta.lib" in Singular. If we are able to describe the effective cones of the exceptional divisors in the resolution, the computation of supercandidates and their minimal jumping divisors is easy to implement.

Example 3.9. Let X be the germ of affine threespace around the origin, and $\mathfrak{a}$ the ideal generated by $f = x(yz - x^4)(x^4 + y^2 - 2yz) + yz^4 - y^5$. After six point blow-ups, we obtain a resolution $\pi : Y \to X$. We have $K_\pi = 2E_1 + 4E_2 + 8E_3 + 14E_4 + 6E_5 + 6E_6$ and $F = F_{aff} + 5E_1 + 9E_2 + 16E_3 + 27E_4 + 11E_5 + 11E_6$, where the E_i are the exceptional divisors, numbered in order of creation, and F_{aff} is the strict transform of $\{f = 0\}$. The supercandidates in $(0, 1]$ are $\frac{5}{9}, \frac{2}{3}, \frac{20}{27}, \frac{7}{9}, \frac{23}{27}, \frac{8}{9}, \frac{25}{27}, \frac{26}{27}$ and 1, and G_λ equals E_4 for $\frac{20}{27}, \frac{23}{27}, \frac{25}{27}$ and $\frac{26}{27}$, and $E_2 + E_4$ for $\frac{5}{9}, \frac{2}{3}, \frac{7}{9}$ and $\frac{8}{9}$. Since the ideal is principal, 1 is a jumping number. By Proposition 3.3, $\frac{20}{27}, \frac{23}{27}, \frac{25}{27}$ and $\frac{26}{27}$ are jumping numbers contributed by E_4. The log canonical threshold $\frac{5}{9}$ is a jumping number, and using Proposition 3.5, one can see that $\frac{2}{3}$ is a jumping number contributed by $E_2 + E_4$. Finally, one can check that $\frac{7}{9}$ and $\frac{8}{9}$ are jumping numbers contributed by E_2. By Skoda's theorem, we know all the jumping numbers.

In this example, our method is clearly faster than naively checking for all candidate jumping numbers whether they are jumping numbers. The algorithm of [3], that is implemented in Macaulay2, did not give a result after several days of computing.

Example 3.10. Let X be as in the previous example, and let $\mathfrak{a}$ be the ideal generated by $f = (x^d + y^d + z^d)^2 + g(x, y, z)$, with $d \geqslant 3$ and $g(x, y, z)$ a homogeneous polynomial of degree $2d + 1$. To compute a resolution, we first blow up at the origin, and then at the $k = d(2d + 1)$ singular points on the strict transform of D. The exceptional divisors are denoted E_1 and E_i^p for $i = 1, \dots, k$, respectively. After two more blow-ups, centered at a curve of genus $g = \frac{1}{2}(d-1)(d-2)$, with exceptional divisors E_2 and E_3, we have a resolution. We have $F = F_{aff} + 2dE_1 + (2d+2)\sum_{i=1}^k E_i^p + (2d+1)E_2 + (4d+2)E_3$ and $K_\pi = 2E_1 + 4\sum_{i=1}^k E_i^p + 3E_2 + 6E_3$. Since E_2 and E_3 are ruled surfaces over a curve of higher genus, it is not obvious to determine the classes in their Picard groups containing effective divisors. However, we can deduce enough information to run our algorithm. We find that the set of supercandidates in $(0, 1]$ is

$$\left\{ \frac{n}{2d} \,\middle|\, 3 \leqslant n < d \right\} \cup \left\{ \frac{2n+1}{4d+2} \,\middle|\, d \leqslant n \leqslant 2d \right\} \cup \left\{ \frac{n}{2d} \,\middle|\, d+3 \leqslant n \leqslant 2d \right\},$$

and all of them are jumping numbers contributed by E_1 or E_3.

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