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An isogeometric indirect boundary element method for solving acoustic problems in open-boundary domains

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Abstract

This work introduces an indirect Boundary Element Method (BEM) within a NURBS-based isogeometric framework for solving three-dimensional acoustic problems in the frequency domain. Developments of isogeometric boundary elements so far have focused on the direct BEM. Yet, a multitude of common problems in acoustics involves open-boundary surfaces, which require a more involved, indirect, boundary element formulation. The current work presents an indirect variational BEM which makes use of NURBS shape functions. Additionally, a novel technique for coupling (strongly) non-conforming patches is introduced to allow the analysis of more complex geometries. The proposed isogeometric indirect boundary element method is verified against analytical solutions and benchmarked against the conventional polynomial-based indirect BEM. Also two open-boundary problems are studied, including analyses over wider frequency ranges, and one industrial-type, complex geometry containing multiple non-conforming patches. The proposed method is found to be not only significantly more efficient than its polynomial-based counterpart, but also very robust against strong non-conformities in the NURBS descriptions.

Keywords: time-harmonic acoustics, isogeometric analysis, boundary element method, NURBS, non-conforming meshes

1. Introduction

The vibro-acoustic performance of a product has become a key design factor in recent years, in particular in the machine industry. This is partially due to ever tightening regulations on noise emission and exposure to vibration levels, but also commercial considerations play an important role: the acoustic character of an object has become a distinguishing feature that is often linked to a quality assessment of the entire product. In addition, acoustic engineers have increasingly less freedom in their designs due to the current trend toward lighter and more energy-efficient products. The lower weight of such designs inevitably leads to deteriorated noise and vibration properties. These strict legislative and commercial requirements together with the availability of ever stronger

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computer systems have made Computer Aided Engineering (CAE) techniques indispensable to design engineers. These techniques enable the accurate prediction of product performance, even in the early design stages.

Current state-of-the-use CAE tools in acoustic problems (in the lower frequency range) are the Finite Element Method (FEM) [1, 2] and the Boundary Element Method (BEM) [3, 4]. Although these tools have led to significantly shorter design cycles, their use still poses some problems. Geometry representations in Computer Aided Design (CAD) on the one hand, and in the FEM and BEM on the other hand, are very different and mostly incompatible with each other. Whereas CAD geometries are spline-based, FE and BE meshes are polynomial approximations of the exact CAD geometry. Because the accuracy of the numerical results relies heavily on the mesh regularity, this incompatibility requires a tedious and time-consuming meshing step to obtain a suitable (but still approximative) geometry. Moreover, if mesh refinement is required, the necessary communication with the CAD model is often difficult, if possible at all. Especially for problems of industrial complexity, where the analysis steps are typically embedded in an optimisation process, this gap between CAD and CAE can drastically slow down product development cycles. To remedy this, IsoGeometric Analysis (IGA) [5, 6] was developed as an alternative to the FEM. More specifically, IGA is a generalisation of the FEM that aims at bridging the gap between design and analysis by introducing CAD descriptions into a CAE environment. The conventional element-based discretisation and the associated (usually low-order polynomial) shape function expansions are replaced by CAD-based mappings and associated functions, typically spline-based ones. IGA has already been successfully applied in various fields of study, an overview of which can be found in [7].

Although the intrinsic surface representations in CAD geometries directly and strongly link the BEM and CAD, comparably little effort was originally put into isogeometric boundary element formulations. More recently, they have received more attention [8–13]. Some work has even been carried out in isogeometric boundary elements for acoustic problems. A direct BEM using an isogeometric approach employing T-splines is applied to time-harmonic acoustics in [14], while a boundary element formulation using Non-Uniform Rational B-Splines (NURBS) bases enriched with plane wave functions for solving acoustic scattering problems is presented in [15]. Although there are various BEM formulations available in literature, the most commonly used ones are the direct collocation BEM (dBEM) [16] and the indirect variational BEM (iBEM) [17] (which is sometimes referred to as symmetric Galerkin BEM [18]). Even though a direct formulation is not strictly tied to a collocation scheme, nor an indirect formulation to a variational scheme, these combinations are typically used. It should be noted that direct BEM approaches (in their standard form) require fully closed boundaries and are limited to either an interior or an exterior problem, while various industrial cases require the modelling of open boundaries and/or combined interior-exterior problems. An indirect formulation, on the other hand, does allow this. Moreover, in contrast to the direct collocation BEM, the indirect variational BEM yields symmetric system matrices and thus reduces the memory requirements by half and allows more efficient solving algorithms. Although the evaluation of double surface integrals (containing strong singularities) makes the matrix assembly for iBEM significantly more complex and costly, iBEM generally converges slightly faster than dBEM due to the variational scheme instead of the collocation one. This paper therefore introduces a NURBS-based isogeometric implementation of an indirect variational boundary integral formulation. Aside from some preliminary studies applying the iBEM in IGA in two dimensions [19, 20], most isogeometric BEM implementations so far have used a direct approach. To the authors’ knowledge, this work is the first application

of an indirect boundary integral formulation within an isogeometric framework in three dimensions.

The remainder of the text is structured as follows. First, the mathematical formulation of the acoustic problems under study is given in section 2. Section 3 then introduces the indirect variational boundary element approach. After that, section 4 discusses NURBS discretisations and their application in this work. The proposed method is then applied to several numerical examples in section 5, verifying it using analytical solutions, and comparing its performance to that of the conventional BEM. All studied problems are three-dimensional, with the main focus on exterior problems defined by open boundaries. The paper ends with some concluding remarks.

2. Time-harmonic acoustics in unbounded domains

Consider a general unbounded acoustic problem in the frequency domain as depicted in fig. 1. The acoustic domain V is filled with a fluid characterised by its speed of sound c and its mass density ρ_0 . The steady-state dynamic behaviour in this acoustic domain is described by the scalar acoustic pressure field $p_a(\mathbf{r})$, which is governed by the inhomogeneous Helmholtz equation [21]:

$$\nabla^2 p_a(\mathbf{r}) + k^2 p_a(\mathbf{r}) = -j\rho_0\omega q_a\delta(\mathbf{r}, \mathbf{r}_q), \quad \mathbf{r} \in V, \quad (1)$$

where $\mathbf{r}$ is the position vector, ω the angular frequency, $k = \frac{\omega}{c}$ the acoustic wavenumber, q_a the strength of an acoustic volume velocity source at position $\mathbf{r}_q$, and $\delta(\bullet, \star)$ the Dirac-delta function. The term $\nabla^2 \bullet = \frac{\partial^2 \bullet}{\partial x^2} + \frac{\partial^2 \bullet}{\partial y^2} + \frac{\partial^2 \bullet}{\partial z^2}$ is the Laplacian operator and j denotes the imaginary unit ($j^2 = -1$). Often the acoustic wavelength $\lambda = \frac{2\pi}{k} = \frac{c}{f}$ or the frequency $f = \frac{\omega}{2\pi} = \frac{kc}{2\pi}$ is used instead of the wavenumber k or angular frequency ω when defining the problem.

Figure 1: Description of a general unbounded acoustic problem in three dimensions.

In order to have a unique solution, the Helmholtz equation (1) requires one imposed boundary condition at each point on the problem boundary $\Omega = \partial V$. In the case of unbounded fluid domains, this boundary consists of two parts: the finite part of the boundary, Ω_f , and the boundary at infinity, Ω_∞ . The finite part Ω_f can be divided into three non-overlapping parts, assuming three types of common acoustic boundary

conditions: $\Omega_f = \Omega_p \cup \Omega_v \cup \Omega_Z$. These are the Dirichlet, Neumann and Robin boundaries, respectively. The boundary conditions on Ω_f can then be written as follows:

$$p_a(\mathbf{r}) = \bar{p}_a(\mathbf{r}) \quad \mathbf{r} \in \Omega_p, \quad (2a)$$

$$\frac{j}{\rho_0 \omega} \frac{\partial p_a(\mathbf{r})}{\partial n} = \bar{v}_n(\mathbf{r}) \quad \mathbf{r} \in \Omega_v, \quad (2b)$$

$$\frac{j}{\rho_0 \omega} \frac{\partial p_a(\mathbf{r})}{\partial n} = \frac{p_a(\mathbf{r})}{\bar{Z}_n(\mathbf{r})} \quad \mathbf{r} \in \Omega_Z. \quad (2c)$$

The quantities $\bar{p}_a$, $\bar{v}_n$ and $\bar{Z}_n$ are the imposed pressure, imposed normal velocity and imposed normal impedance, respectively. The vector $\mathbf{n}$ is the unit normal on the boundary. At the boundary at infinity Ω_∞ , non-reflecting boundary conditions are imposed. This ensures that no energy is reflected back into the problem domain and that the resulting acoustic waves can propagate freely toward infinity. This is known as the Sommerfeld radiation condition:

$$\lim_{\|\mathbf{r}\| \rightarrow \infty} \left(\frac{\partial p_a(\mathbf{r})}{\partial n} + jk p_a(\mathbf{r}) \right) = 0. \quad (3)$$

The Helmholtz equation (1) together with the boundary conditions (2) and (3) fully define the acoustic pressure field $p_a(\mathbf{r})$ in the entire problem domain V .

3. The indirect variational boundary element method in acoustics

The BEM, like the FEM, is an element-based simulation technique. But unlike the FEM, it uses a boundary integral formulation of the problem under study. This formulation relates the field variables in the continuum domain to the distribution of associated boundary variables on the boundary surface of the domain. Therefore the BEM requires only the boundary to be discretised into elements instead of the entire acoustic volume. While the resulting system matrices are fully populated, complex and frequency-dependent, this feature can significantly decrease the number of Degrees Of Freedom (DOFs) to be solved for. Moreover, in representing the field variables, the BEM uses Green's kernel functions, which inherently satisfy the Sommerfeld radiation condition (3). This makes the BEM especially well-suited for modelling unbounded acoustic problems.

This section briefly presents the indirect boundary integral formulation and its variational solution, as well as the employed technique for reducing the resulting hypersingular integral operators to weakly-singular ones.

3.1. The indirect boundary integral formulation for Helmholtz problems

The use of an indirect formulation of the Boundary Integral Equation (BIE) is crucial for the modelling of open-boundary domains. This indirect boundary integral formulation relates the acoustic pressure in any point of an acoustic field that satisfies the Helmholtz equation (1) (in its homogeneous¹ form) and the Sommerfeld radiation condition (3), to the distribution of potential functions on the boundary surface. What follows is only a

¹By regarding the total inhomogeneous pressure field as a superposition of a homogeneous pressure field and an inhomogeneous free-field pressure, a numerical solution procedure is required only for the homogeneous problem (after reformulating the boundary conditions for the total pressure in terms of the homogeneous pressure field). The remaining part of this section therefore assumes a homogeneous acoustic problem, without loss of generality.

brief overview, but a detailed and mathematically firm derivation of the indirect BIE can be found in [22].

An acoustic problem with a (thin) open boundary surface can be regarded as a special case of an exterior problem with a closed boundary by viewing both surface sides as two separate parts of a closed boundary surface. Figure 2 illustrates this, with the dashed line representing the open boundary surface Ω_f , and Ω_f^+ and Ω_f^- forming the separate sides of the corresponding closed boundary. The indirect boundary integral formulation uses the differences in pressure and in normal pressure gradient between these two sides as primary variables. These variables are called the double layer potential $\mu(\mathbf{r}_f)$ and the single layer potential $\sigma(\mathbf{r}_f)$, respectively²:

$$\mu(\mathbf{r}_f) = p_a(\mathbf{r}_f^+) - p_a(\mathbf{r}_f^-) \quad \mathbf{r}_f \in \Omega_f, \quad (4)$$

$$\sigma(\mathbf{r}_f) = \frac{\partial p_a(\mathbf{r}_f^+)}{\partial n} - \frac{\partial p_a(\mathbf{r}_f^-)}{\partial n} \quad \mathbf{r}_f \in \Omega_f, \quad (5)$$

with $\mathbf{r}_f$ the position vector on the boundary surface. The acoustic pressure is then given by the following representation formula:

$$p_a(\mathbf{r}) = \int_{\Omega_f} \left(\mu(\mathbf{r}_f) \frac{\partial G(\mathbf{r}, \mathbf{r}_f)}{\partial n} - \sigma(\mathbf{r}_f) G(\mathbf{r}, \mathbf{r}_f) \right) d\Omega_f(\mathbf{r}_f) \quad \forall \mathbf{r} \in V \setminus \Omega_f. \quad (6)$$

This formulation makes use of the Green's kernel function $G(\mathbf{r}, \mathbf{r}_f)$, which represents the free-field pressure in any point $\mathbf{r}$ due to an acoustic point source located at $\mathbf{r}_f$. It is defined as follows for three-dimensional domains:

$$G(\mathbf{r}, \mathbf{r}_f) = \frac{e^{-jk\|\mathbf{r}-\mathbf{r}_f\|}}{4\pi\|\mathbf{r}-\mathbf{r}_f\|}. \quad (7)$$

Assuming a thin boundary surface ($\Omega_f^+ \approx \Omega_f^- \approx \Omega_f$) and symmetric impedance boundary conditions ($\bar{Z}_n^+ = \bar{Z}_n^- = \bar{Z}_n$), the original boundary conditions (2) can be reformulated in terms of the single and the double layer potential as follows:

$$\mu(\mathbf{r}_f) = 0 \quad \mathbf{r}_f \in \Omega_p, \quad (8)$$

$$\sigma(\mathbf{r}_f) = 0 \quad \mathbf{r}_f \in \Omega_v, \quad (9)$$

$$\sigma(\mathbf{r}_f) = -jk\bar{\beta}(\mathbf{r}_f)\mu(\mathbf{r}_f) \quad \mathbf{r}_f \in \Omega_Z, \quad (10)$$

where $\bar{\beta} = \rho_0 c / \bar{Z}_n$. This means that in each point on the boundary, there is only one independent variable: either the single layer potential or the double layer potential. Equation (6) then becomes:

$$\begin{aligned} p_a(\mathbf{r}) = & - \int_{\Omega_p} \sigma(\mathbf{r}_f) G(\mathbf{r}, \mathbf{r}_f) d\Omega_p(\mathbf{r}_f) + \int_{\Omega_v} \mu(\mathbf{r}_f) \frac{\partial G(\mathbf{r}, \mathbf{r}_f)}{\partial n} d\Omega_v(\mathbf{r}_f) \\ & + \int_{\Omega_Z} \mu(\mathbf{r}_f) \left(\frac{\partial G(\mathbf{r}, \mathbf{r}_f)}{\partial n} + jk\bar{\beta}(\mathbf{r}_f) G(\mathbf{r}, \mathbf{r}_f) \right) d\Omega_Z(\mathbf{r}_f) \quad \forall \mathbf{r} \in V \setminus \Omega_f. \end{aligned} \quad (11)$$

²It should be noted that the term *layer potentials* is the name conventionally used for these variables in the field of numerical acoustics, which can be different from the nomenclature in more mathematically oriented BEM text books (e.g. [23]). The latter ones typically employ the term *layer potentials* to refer to the solutions $p_a(\mathbf{r})$ in V of the form $p_a(\mathbf{r}) = \int_{\Omega_f} \sigma(\mathbf{r}_f) G(\mathbf{r}, \mathbf{r}_f) d\Omega_f$ and $p_a(\mathbf{r}) = \int_{\Omega_f} \mu(\mathbf{r}_f) \frac{\partial G(\mathbf{r}, \mathbf{r}_f)}{\partial n} d\Omega_f$, cf. eq. (6). In this sense, σ and μ are actually layer potential *distributions*.

Figure 2: Boundary description for an open boundary surface in an indirect boundary integral formulation (for the sake of clarity presented in 2D). The surface is assumed to be thin (i.e. $\Omega_f^+ \approx \Omega_f^- \approx \Omega_f$).

Applying a weighted residual approach to the boundary residuals yields the variational formulation [4]:

$$\begin{aligned} \forall (\delta\sigma, \delta\mu) : \quad & \int_{\Omega_p} R_p(\sigma, \mu, \mathbf{r}) \delta\sigma \, d\Omega_p(\mathbf{r}) + \int_{\Omega_v} R_v(\sigma, \mu, \mathbf{r}) \delta\mu \, d\Omega_v(\mathbf{r}) \\ & + \int_{\Omega_Z} R_Z(\sigma, \mu, \mathbf{r}) \delta\mu \, d\Omega_Z(\mathbf{r}) = 0, \end{aligned} \quad (12)$$

where R_p , R_v and R_Z are the boundary residuals for the Dirichlet, Neumann and Robin boundaries, respectively. They are calculated by expressing the boundary conditions (2) using the boundary integral formulation, yielding

$$\begin{aligned} R_p = & - \int_{\Omega_p} \sigma(\mathbf{r}_f) G(\mathbf{r}, \mathbf{r}_f) d\Omega_p(\mathbf{r}_f) + \int_{\Omega_v} \mu(\mathbf{r}_f) \frac{\partial G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r}_f)} d\Omega_v(\mathbf{r}_f) \\ & + \int_{\Omega_Z} \mu(\mathbf{r}_f) \left(\frac{\partial G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r}_f)} + jk\bar{\beta}(\mathbf{r}_f) G(\mathbf{r}, \mathbf{r}_f) \right) d\Omega_Z(\mathbf{r}_f) - \bar{p}_a(\mathbf{r}), \end{aligned} \quad (13)$$

$$\begin{aligned} R_v = & - \int_{\Omega_p} \sigma(\mathbf{r}_f) \frac{\partial G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r})} d\Omega_p(\mathbf{r}_f) + \int_{\Omega_v} \mu(\mathbf{r}_f) \frac{\partial^2 G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r}) \partial n(\mathbf{r}_f)} d\Omega_v(\mathbf{r}_f) \\ & + \int_{\Omega_Z} \mu(\mathbf{r}_f) \left(\frac{\partial^2 G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r}) \partial n(\mathbf{r}_f)} + jk\bar{\beta}(\mathbf{r}_f) \frac{\partial G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r})} \right) d\Omega_Z(\mathbf{r}_f) \\ & + j\rho_0\omega\bar{v}_n(\mathbf{r}), \end{aligned} \quad (14)$$

$$\begin{aligned} R_Z = & - \int_{\Omega_p} \sigma(\mathbf{r}_f) \frac{\partial G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r})} d\Omega_p(\mathbf{r}_f) + \int_{\Omega_v} \mu(\mathbf{r}_f) \frac{\partial^2 G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r}) \partial n(\mathbf{r}_f)} d\Omega_v(\mathbf{r}_f) \\ & + \int_{\Omega_Z} \mu(\mathbf{r}_f) \left(\frac{\partial^2 G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r}) \partial n(\mathbf{r}_f)} + jk\bar{\beta}(\mathbf{r}_f) \frac{\partial G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r})} \right) d\Omega_Z(\mathbf{r}_f) \\ & - \int_{\Omega_p} jk\bar{\beta}(\mathbf{r}) \sigma(\mathbf{r}_f) G(\mathbf{r}, \mathbf{r}_f) d\Omega_p(\mathbf{r}_f) + \int_{\Omega_v} jk\bar{\beta}(\mathbf{r}) \mu(\mathbf{r}_f) \frac{\partial G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r}_f)} d\Omega_v(\mathbf{r}_f) \\ & + \int_{\Omega_Z} \mu(\mathbf{r}_f) \left(jk\bar{\beta}(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r}_f)} - k^2\bar{\beta}(\mathbf{r})\bar{\beta}(\mathbf{r}_f) G(\mathbf{r}, \mathbf{r}_f) \right) d\Omega_Z(\mathbf{r}_f). \end{aligned} \quad (15)$$

The terms $\delta\sigma$ and $\delta\mu$ are test functions. Following a Galerkin approach, they are chosen the same as the potentials. Solving eq. (12) for $\sigma(\mathbf{r}_f)$ on $\Omega_\sigma = \Omega_p$ and for $\mu(\mathbf{r}_f)$ on $\Omega_\mu = \Omega_v \cup \Omega_Z$ yields the distributions of the single and the double layer potential on the boundary surface. Numerical integration can be carried out using standard Gauss rules. However, eq. (12) still contains (strong) singularities that prevent an accurate solution. The section below discusses how this can be remedied.

3.2. Transformation of hypersingular operators to weakly-singular ones

The Green's kernel function (7) introduces singularities in the weighted residual formulation (12) for $\mathbf{r} = \mathbf{r}_f$. First, the highly singular parts (containing second-order derivatives of the Green's kernel) need to be rewritten to contain only removable singularities. This can be done by applying Stokes's theorem and by using properties of the Green's kernel and the continuity of the double layer potential [4, 24]:

$$\begin{aligned} & \int_{\Omega_\mu} \int_{\Omega_\mu} \mu(\mathbf{r}) \mu(\mathbf{r}_f) \frac{\partial^2 G(\mathbf{r}, \mathbf{r}_f)}{\partial n(\mathbf{r}) \partial n(\mathbf{r}_f)} d\Omega_\mu(\mathbf{r}) d\Omega_\mu(\mathbf{r}_f) \\ &= \int_{\Omega_\mu} \int_{\Omega_\mu} G(\mathbf{r}, \mathbf{r}_f) [k^2 \mu(\mathbf{r}) \mu(\mathbf{r}_f) \mathbf{n}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}_f) - \mathbf{n}(\mathbf{r}) \times \nabla \mu(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}_f) \times \nabla \mu(\mathbf{r}_f)] d\Omega_\mu(\mathbf{r}) d\Omega_\mu(\mathbf{r}_f). \end{aligned} \quad (16)$$

Now the weighted residual formulation only contains singularities of order $1/r$ and $1/r^2$. These can be removed by applying regularising coordinate transformations that render the singular integrals analytic, which then allows to numerically calculate them by means of standard quadrature methods. Sauter and Schwab [23] have derived such transformations by decomposing the integration domain (in this context the elements) into several subdomains and rewriting the integral in terms of relative coordinates. The price to pay for this regularisation is more expensive numerical integration: The number of quadrature points after applying the Sauter–Schwab transformations increases rapidly with increasing quadrature order. If an $n_{\text{GP}} \times n_{\text{GP}}$ Gauss rule is used for the standard quadrature in both the inner and the outer surface integrals, the number of quadrature points needed for computing a single regularised auto-element contribution is $8n_{\text{GP}}^4$. Although this is a strong increase (by a factor $4n_{\text{GP}}^2$) with respect to the standard quadrature scheme, the requirements for n_{GP} for these (regularised) auto-element contributions can be somewhat relaxed as compared to that standard quadrature rule. This makes the increase in quadrature cost less drastic.

The weighted residual formulation contains double surface integrals, which means that the numerical integration is defined on a pair of surface elements. This work applies regularising coordinate transformations, as presented in [23], to element pairs that are either identical elements (auto-singularities) or elements with a common edge (edge singularities³). The influence of the singularities for element pairs with a common vertex (corner singularities) or without any common point is considered negligible. Such pairs do not contribute notably to the singularity issue, since the strength of the singularity decreases with the distance between the quadrature points. Numerical tests carried out with the proposed method (but not shown here for the sake of brevity) have confirmed that ignoring edge singularities can prevent full convergence of the method, while corner singularities have not been found to pose a problem.

³Although there is in theory no singularity for non-identical element pairs, the small distance between quadrature points in neighbouring elements can introduce inaccuracies in the numerical solution.

4. NURBS-based isogeometric indirect boundary element analysis

This section introduces some basic terminology regarding NURBS surfaces and their corresponding basis functions in the context of IGA. First the construction of NURBS from B-splines is discussed. Then their use as test and trial functions in the indirect boundary element formulation is presented. The proposed method is subsequently introduced, including an explanation about the treatment of non-conforming NURBS patches. A more detailed explanation about IGA can be found in [5, 6] and about NURBS in particular in [25].

4.1. NURBS bases and surfaces

NURBS are built up from B-splines, which are piecewise polynomial functions. A basis of one-dimensional B-splines of polynomial order p is defined by a knot vector $\Xi = [\xi_1, \xi_2, \dots, \xi_{n+p+1}]$. This is a sequence of non-decreasing coordinates in the parameter space ξ . For a B-spline formed by n basis functions, $n + p + 1$ knots are required: $\xi_k \in \mathbb{R}$ ($k = 1, 2, \dots, n + p + 1$). Open knot vectors, where the first and the last knot each have a multiplicity of $p + 1$, are standard in CAD. Starting from a knot vector, B-spline basis functions $N_i^p(\xi)$ ($i = 1, \dots, n$) are defined recursively using the Cox–de Boor recursion formula [26, 27]:

$$\begin{aligned} p = 0 : \quad N_i^0(\xi) &= \begin{cases} 1 & \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}, \\ p > 0 : \quad N_i^p(\xi) &= \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_i^{p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1}^{p-1}(\xi). \end{aligned} \quad (17)$$

These functions form a partition of unity. To generate a NURBS basis from them, a weight w_i is assigned to every B-spline function $N_i^p(\xi)$. The NURBS basis functions $R_i^p(\xi)$ of polynomial order p are then given by

$$R_i^p(\xi) = \frac{N_i^p(\xi)w_i}{W(\xi)} = \frac{N_i^p(\xi)w_i}{\sum_{i_w=1}^n N_{i_w}^p(\xi)w_{i_w}} \quad \text{with } i = 1, \dots, n, \quad (18)$$

with $W(\xi)$ the weighting function. Each $R_i^p(\xi)$ is a piecewise rational function, since $N_i^p(\xi)$ and $W(\xi)$ are both piecewise polynomial. This definition implies that B-splines are in fact a subset of NURBS. Indeed, if all the weights are equal, then $R_i^p(\xi) = N_i^p(\xi)$; the basis functions are no longer rational and the NURBS degenerates into a B-spline.

Multivariate NURBS entities are generated by taking the tensor product of univariate NURBS bases. Given two polynomial orders p and q , and a pair of knot vectors $\Xi = [\xi_1, \xi_2, \dots, \xi_{n+p+1}]$ and $\mathbf{H} = [\eta_1, \eta_2, \dots, \eta_{m+q+1}]$, a bivariate NURBS basis is defined as

$$R_{i,j}^{p,q}(\xi, \eta) = \frac{N_i^p(\xi)M_j^q(\eta)w_{i,j}}{\sum_{i_w=1}^n \sum_{j_w=1}^m N_{i_w}^p(\xi)M_{j_w}^q(\eta)w_{i_w,j_w}} \quad \text{with } \begin{matrix} i = 1, \dots, n \\ j = 1, \dots, m \end{matrix}, \quad (19)$$

where $N_i^p(\xi)$ and $M_j^q(\eta)$ represent univariate B-spline basis functions of order p and q , associated with knot vectors Ξ and $\mathbf{H}$, respectively. To obtain a NURBS surface geometry $\mathbf{S}(\xi, \eta)$, a linear combination of these basis functions is generated using a net of vector-valued weighting coefficients $\mathbf{B}_{i,j} \in \mathbb{R}^3$ called control points:

$$\mathbf{S}(\xi, \eta) = \sum_{i=1}^n \sum_{j=1}^m R_{i,j}^{p,q}(\xi, \eta) \mathbf{B}_{i,j}. \quad (20)$$

It is worth noting that the control points are in general not interpolatory, i.e. they do not necessarily lie on the surface they define.

Equation (20) represents a single NURBS surface patch. A general NURBS surface can consist of a combination of several patches. Each one is a concatenation of knot spans. These are delimited by (non-identical) knots and represent the elements in the mesh, where the basis functions are smooth (i.e. C^∞ -continuous). Across a knot ξ_i , the basis functions have a C^{p-m_i} continuity, with m_i the multiplicity of knot ξ_i . NURBS bases are therefore quite flexible regarding inter-element continuity: virtually arbitrarily high continuities can be obtained.

4.2. NURBS in an IsoGeometric indirect Boundary Element Method (IGiBEM)

To numerically solve eq. (12), the boundary surfaces Ω_σ and Ω_μ are discretised into boundary elements. The potentials $\sigma(\mathbf{r}_f)$ and $\mu(\mathbf{r}_f)$ are then approximated by shape function expansions $\hat{\sigma}(\mathbf{r}_f)$ and $\hat{\mu}(\mathbf{r}_f)$:

$$\sigma(\mathbf{r}_f) \approx \hat{\sigma}(\mathbf{r}_f) = \sum_{i=1}^{n_\sigma} N_{\sigma,i}(\mathbf{r}_f) \cdot d_{\sigma,i} \quad \mathbf{r}_f \in \Omega_\sigma, \quad (21)$$

$$\mu(\mathbf{r}_f) \approx \hat{\mu}(\mathbf{r}_f) = \sum_{i=1}^{n_\mu} N_{\mu,i}(\mathbf{r}_f) \cdot d_{\mu,i} \quad \mathbf{r}_f \in \Omega_\mu, \quad (22)$$

where $n_\bullet$ is the number of (prescribed) shape functions $N_{\bullet,i}$ (with associated contribution factors $d_{\bullet,i}$) in the expansion. Using the isoparametric paradigm, these shape functions are identical to the ones used for the surface geometry mapping. Following a Galerkin approach, these same functions are then used as test functions. By introducing expansions (21) and (22) into the weighted residual formulation (12), a (symmetric) global system of equations can be assembled. Solving this for $d_{\sigma,i}$ and $d_{\mu,i}$ yields the discretised single layer potential $\hat{\sigma}(\mathbf{r}_f)$ and double layer potential $\hat{\mu}(\mathbf{r}_f)$. Post-processing this using eq. (11) then allows to calculate the pressure at any position inside the acoustic domain.

In the traditional Lagrangian BEM, the shape functions are Lagrangian polynomials (cf. fig. 3a) that are only locally defined within a single boundary element. They have unit value at one node and zero at all the other ones, making $d_{\bullet,i}$ nodal DOFs. The proposed method instead uses NURBS shape functions (cf. fig. 3b) as in eq. (18), which, although also highly locally supported, typically span several elements. A consequence of this wider support is that the shape functions — although they form a partition of unity — are in general not interpolatory. The contribution factors $d_{\bullet,i}$ (called the control variables in IGA terminology) are therefore not nodal DOFs anymore. Figure 3 illustrates the differences between conventional polynomials and NURBS when used for quadratic shape functions in the one-dimensional case. It is clear that the Lagrangian functions are interpolatory but only C^0 -continuous. Quadratic NURBS on the other hand are C^1 -continuous but clearly do not interpolate the nodes. They also span more nodes than the polynomials, although both the polynomials and NURBS in any given point share support with the same number of basis functions (namely $p+1$, or 3 in this quadratic case). It is worth noting that the quadratic Lagrange elements all require a middle node, meaning that the 7 nodes in fig. 3a represent 3 elements. In the NURBS case, each node (i.e. unique knot) represents an element boundary, resulting in 6 elements in fig. 3b. In general, for a given polynomial order and number of elements, a NURBS mesh contains fewer shape functions than a conventional polynomial mesh.

The non-interpolatory properties of the NURBS basis also have consequences for applying essential boundary conditions. Consider the essential boundary condition $\mu(\mathbf{r}_f) =$

(a) Quadratic Lagrangian shape functions for equally spaced nodes.

(b) Quadratic NURBS shape functions for a uniform open knot vector. These NURBS functions are B-splines $N_i^2(\xi)$ as defined in eq. (17).

Figure 3: One-dimensional quadratic shape functions for a mesh consisting of seven nodes (indicated by $\bullet$) using (a) conventional polynomials and (b) NURBS.

$g(\mathbf{r}_f)$ on $\mathbf{r}_f \in \Gamma_f$. Note that this is a condition on the boundary curve $\Gamma_f = \Gamma(\Omega_f)$ of the surface Ω_f (and not on the boundary surface Ω_f itself). It should therefore be regarded as a boundary condition for the reduced acoustic problem defined on the boundary surface Ω_f (and not for the original volumetric problem defined in V). The function g does not necessarily lie within the function space $\mathcal{S}^h$ spanned by the NURBS basis, i.e. a function $g^h \in \mathcal{S}^h$ such that $g^h|_{\Gamma_f} = g$ might not exist. The boundary condition is then only approximatively satisfied: $g^h|_{\Gamma_f} \approx g$. The traditional BEM satisfies such conditions pointwise by interpolating g at the boundary nodes. In IGA, however, this can lead to a smeared g^h . Alternatively, a weakly imposed essential condition may be used [6]. Luckily, homogeneous conditions do not pose a problem, since they can easily be built into the solution space by applying them directly to the control variables. This is done by setting the contribution factors of all functions N_{μ, Γ_f} that are non-zero on Γ_f to zero: $d_{\mu, \Gamma_f} = 0$. The partition of unity of the basis on the boundary curve then ensures that $\mu(\mathbf{r}_f) = 0$ on $\mathbf{r}_f \in \Gamma_f$ — the same reasoning holds for non-homogeneous but constant boundary conditions. This is important in the proposed method for applying a zero jump of pressure condition at free edges of an open-boundary domain, i.e. $\mu(\mathbf{r}_f|_{\Gamma(\Omega_f)}) = 0$, with $\Gamma(\Omega_f)$ the free boundary edges. As such, these boundary conditions can be exactly satisfied.

There are several advantages to an isogeometric implementation (based on NURBS), the most obvious one being the possibility of directly using CAD geometry descriptions as field variable approximations and thus eliminating or at least reducing costly meshing steps. But also the use of the exact geometric descriptions can be a crucial advantage. For instance when needing to refine the discretisation, but also simply for the sake of obtaining accurate solutions. This is especially true in a BEM formulation, where all elements are geometrically flawed in a conventional polynomial-based implementation (as opposed to only a fraction of the elements in the conventional FEM). Even more so toward higher frequencies, where geometric details can strongly influence the acoustic

behaviour. NURBS meshes (thanks to their higher degree of smoothness) have also been shown to exhibit better spectral approximation properties with respect to conventional polynomial elements [28–30]. Moreover, NURBS allow more efficient mesh refinement than conventional polynomial meshes [31]. One of the drawbacks of NURBS is that they are not always well-suited for analysis [32–34] and might therefore still require some additional pre-processing treatment. An important issue in this context is the coupling of non-conforming patches, as described in the next section.

4.3. Enforcing continuity across non-conforming NURBS patches

The weighted residual formulation (12) only contains first-order derivatives of the double layer potential $\mu(\mathbf{r}_f)$ (after application of Stokes’s theorem), and no derivatives of the single layer potential $\sigma(\mathbf{r}_f)$. Therefore, to ensure convergence, the compatibility condition for the shape functions requires that the single layer potential expansion (21) be non-singular within each element — although it may be discontinuous across element boundaries. But the double layer potential expansion (22) should be continuous across element boundaries, and its first-order derivatives non-singular within elements. The NURBS-based shape function expansions proposed in eqs. (21) and (22) always satisfy these requirements, except across patch boundaries, where the C^0 continuity of $\hat{\mu}(\mathbf{r}_f)$ needs to be enforced explicitly. This is straightforwardly done in the case of conforming⁴ patches by a one-to-one matching of the control variables on the interface. Coupling non-conforming patches, however, can require significant additional treatment. It is nonetheless an important issue, since complex multipatch geometries (like the one presented in section 5.4) are often non-conforming. The remainder of this section therefore briefly explains how such patches are coupled in this work, using a novel master-slave formulation. A more detailed overview of this issue in general can be found in e.g. [35].

The underlying idea behind the proposed coupling technique was inspired by a study by Cottrell et al. [31], where a special case of this technique is applied to hierarchical meshes. It is in fact similar to the coupling method presented by Lei et al. [36], but the implementation is quite different. The relationships between interface variables in neighbouring patches are derived in the same manner, but result in different coupling equations. Moreover, in [36] the restrictions are enforced by means of a penalty method, whereas the coupling technique presented here employs static condensation. The proposed procedure is explained below; a more detailed study of it can be found in [37].

The two non-conforming patches (numbered 1 and 2, and assumed to be of the same degree) are virtually refined until they form conforming patches, after which the regular one-to-one matching is enforced. This virtual refinement is done by inserting into both interface knot vectors the knots that are present in the other patch’s knot vector, but not in that of the patch itself (taking into account knot multiplicities). That is; given the two interface knot vectors Ξ_1 and Ξ_2 , refine them until they are both identical to $\Xi_{1\cup 2} = \Xi_1 \cup \Xi_2$. This refinement process is referred to here as virtual uncommon knot insertion. It can be represented by extraction operators A_1 and A_2 (similar to those for Bézier extraction presented in [38]) that link the original control variables $\mathbf{d}_1$ and $\mathbf{d}_2$ to the control variables $\mathbf{d}_1^v$ and $\mathbf{d}_2^v$ of the virtually refined versions:

$$\mathbf{d}_i^v = A_i \mathbf{d}_i \quad i = 1, 2. \quad (23)$$

⁴Conforming patches have a matching discretisation at their interface, with an equal number of interface variables in each patch. Non-conforming patches on the other hand have different basis functions and not even necessarily the same number of interface variables.

In other words, the extraction operators A_i express the linear combinations of the original control points that result in each refined control point. This refinement step is virtual in the sense that these finer discretisations are never used for shape function expansions; only the coupling equations resulting from a one-to-one matching of the refined control variables $\mathbf{d}_i^v$ are used to link the original control variables $\mathbf{d}_i$. Ensuring the continuity then consists of enforcing $\mathbf{d}_1^v = \mathbf{d}_2^v$, or equivalently, using the original control variables $\mathbf{d}_i$:

$$A_1 \mathbf{d}_1 = A_2 \mathbf{d}_2. \quad (24)$$

In a general (arbitrary) patch configuration, the severity of the non-conformity will lead to an overconstraining of the interface if this entire set of restrictions is enforced. Therefore, both patches are weakly coupled by expressing the variables of the more finely discretised patch (the slave) in function of those of the more coarse one (the master). This reduces the number of interface constraints by satisfying eq. (24) in a least-squares sense, allowing an accurate solution without overconstraining the model. Using subscripts ‘m’ and ‘s’ for the master and slave patch, respectively, this yields the following continuity relationship:

$$\mathbf{d}_s = A_s^+ A_m \mathbf{d}_m, \quad (25)$$

with A_s^+ the pseudo-inverse of A_s . By condensing out the DOFs $\mathbf{d}_s$ according to this equation, the global system of equations can then be solved while taking into account the continuity conditions. In the special case of hierarchical meshes, these coupling restrictions exactly enforce continuity across the entire interface (since in that case A_s is the identity matrix and no least-squares approximation is needed).

Figure 4 illustrates this coupling procedure. The top part of the figure represents the coupled patches (with $\mathbf{d}_s$ and $\mathbf{d}_m$ linked by eq. (25)), the bottom left part the uncoupled ones. The bottom right illustration shows the refined (virtual) configuration, where $\mathbf{d}_1^v = \mathbf{d}_2^v$ has to hold. As will be shown in the numerical examples in the next section, the proposed coupling approach leads to a good satisfaction of the continuity requirements, without overconstraining the original system of equations.

5. Numerical examples

This section presents some numerical examples to verify the accuracy of the proposed IsoGeometric indirect Boundary Element Method (IGiBEM) and evaluate its performance compared to the conventional iBEM. Also the proposed patch coupling procedure is validated. First, two problems with an analytical solution are studied, which allows an accurate assessment of the proposed method’s convergence. After that, the method is compared to the conventional iBEM for more general problems, containing open boundaries. All IGiBEM simulations in this work are performed with an in-house MATLAB code that makes use of some functionalities of the open source IGA framework IGAFEM [7]. All conventional iBEM simulations are carried out using the Siemens PLM Software LMS Virtual.Lab [39]. Unless otherwise stated, a 3×3 Gauss point rule is used for numerical quadrature. The authors have found that for quadratic, cubic and quartic shape functions, using more Gauss points than this did not improve the accuracy for most of the studied examples.

5.1. Pulsating sphere

Consider an infinite acoustic medium of density ρ_0 and speed of sound c , containing a sphere of radius R pulsating at an angular frequency $\omega = kc = 2\pi f$ with a constant

Figure 4: Conceptual illustration of the patch coupling method. Using restrictions resulting from the matching of (virtually) uncommon knot inserted meshes (bottom right) that are transformed back to the original control points (bottom left), C^0 continuity conditions can be (weakly) enforced within a master-slave relationship (top). The right patch is the less refined one and therefore the master.

radial velocity v_R (i.e. a velocity boundary condition as in eq. (2b) with $\bar{v}_n = v_R$ for the entire surface). The radiated acoustic pressure field $p_r(r)$ can be expressed in closed form as follows [40]:

$$p_r(r) = v_R \frac{j\omega\rho_0 R^2}{1 + jkR} \frac{e^{-jk(r-R)}}{r}, \quad (26)$$

with r the radial distance from the sphere's centre. This boundary value problem is numerically solved with the proposed IGiBEM and with the conventional iBEM, as a benchmarking case.

Figure 5 shows the employed NURBS model of the sphere, generated by rotating a semicircle line over 360° . The result is an exact sphere represented by a single patch of quadratic NURBS functions. At the poles, the quadrilateral elements degenerate into triangular ones. During analysis, all the basis functions meeting at a pole are added together

(a) The most coarse representation of a sphere, consisting of 8 elements. (b) A more refined mesh, consisting of 200 elements.

Figure 5: Mesh topology of the NURBS representations of the sphere, represented by a semicircle rotated over 360° , yielding a biquadratic, exactly spherical surface.

Figure 6: A Lagrangian mesh of a sphere, consisting of 92 quadratic elements. To avoid mesh distortion, triangular elements are used (CTRIA6).

to form a single basis function with a single control variable. The same is done for basis functions meeting at the original semi-circle line. The resulting mesh is C^1 -continuous across elements, except across the poles, the equator, and four meridians (including the original semi-circle line), where it is C^0 -continuous. Figure 6 shows a Lagrangian⁵ mesh of a sphere as used in Virtual.Lab. In order to minimise mesh distortion, the sphere is meshed using triangular elements (quadratic CTRIA6 elements).

Simulations are carried out using $R = 0.25$ m, $v_R = 1$ m/s, $c = 340$ m/s and $\rho_0 = 1.225$ kg/m³. Numerical solutions are evaluated at 50 different radiuses r , ranging between R and $5R$. An average relative error on the radiated pressure field $p_a(r)$ is then calculated according to the discrete l^2 relative error norm $\|\epsilon\|_2$:

$$\|\epsilon\|_2 = \frac{\|p_a(\mathbf{r}) - p_{a,\text{ref}}(\mathbf{r})\|_2}{\|p_{a,\text{ref}}(\mathbf{r})\|_2} = \frac{\sqrt{\sum_{i=1}^{n_p} |p_a(\mathbf{r}_i) - p_{a,\text{ref}}(\mathbf{r}_i)|^2}}{\sqrt{\sum_{i=1}^{n_p} |p_{a,\text{ref}}(\mathbf{r}_i)|^2}}, \quad (27)$$

with $n_p = 50$, $\mathbf{r} = r$ (due to spherical symmetry) and $p_{a,\text{ref}}(r)$ as given by eq. (26). This formula uses the complex pressure values and therefore not only takes into account the pressure amplitudes, but also the phase errors. Figure 7 shows this averaged relative error for various solutions calculated with both the proposed IGibEM and the conventional ibEM at a frequency of 100 Hz.

⁵In fact, the simulations make use of Serendipity elements because of their overall higher efficiency as compared to Lagrangian elements. But for the sake of clarity, the traditional polynomial meshes are referred to here as ‘Lagrangian’, adhering to the common use of the term Lagrangian when referring to the traditional polynomial-based FEM or BEM in contrast to isogeometric approaches.

(a) Convergence curves for the IGiBEM and conventional iBEM solutions as function of the number of DOFs n_{DOF} . A 2×2 Gauss quadrature is used in both cases.

(b) Convergence of the IGiBEM solution with increasing quadrature order $n_{\text{GP}} \times n_{\text{GP}}$, for quadratic and cubic models of different sizes (8 elements and 200 elements, cf. fig. 5).

Figure 7: Convergence curves for the IGiBEM and conventional iBEM solutions to the pulsating sphere problem. The curves are calculated for an analysis frequency of 100 Hz. The data points indicated by $\times$ and $\star$ indicate corresponding models between both subfigures.

Figure 7a shows the convergence as a function of the fineness of discretisation, calculated using a 2×2 quadrature rule for both methods. The number of elements in the IGiBEM models ranges from 8 to 8712, corresponding to a range in DOFs from 26 to 9114. The curves clearly indicate the superiority of the proposed method over conventional iBEM for this problem. IGiBEM is more than an order of magnitude more accurate for a given number of DOFs, and this difference can in fact even be increased by using higher order quadrature rules. Indeed, the only factor limiting the IGiBEM accuracy for this problem is the quadrature error: The exact double layer potential is uniform across the spherical surface (and can therefore be represented correctly even with constant shape functions), and the NURBS discretisation can also exactly represent the problem geometry. In contrast to the conventional BEM models (which still suffer from geometrical errors), the resulting IGiBEM solution error is therefore completely due to the numerical quadrature. Figure 7b confirms this, showing the continuing convergence with increasing quadrature order for the meshes from fig. 5. Even the most coarse model with only 8 elements can yield arbitrarily high accuracies by using a sufficiently high quadrature order. The curves also show that increasing the polynomial order p of the NURBS basis does not improve the accuracy: only the quadrature and the number of elements influence the result. The IGiBEM convergence rate slowing down in fig. 7a is also caused by the numerical quadrature. This can be realised by looking at the data point marked by a blue asterisk in both plots. This point indicates the error level resulting from using 2×2 quadrature for a 200 element NURBS mesh (cf. fig. 5b). This is the mesh where the stagnation starts in fig. 7a, but fig. 7b shows that this model's error can in fact be further reduced by increasing the quadrature order.

5.2. Rigid sphere scattering a plane wave

As a more complex but still analytical example, the second problem studied here is that of a plane pressure wave of unit amplitude propagating through an unbounded air volume and impinging on a rigid sphere (i.e. a velocity boundary condition $\bar{v}_n = 0$ m/s for the entire surface, cf. eq. (2b)). The resulting scattered acoustic pressure field $p_s(r, \theta)$ can be calculated analytically as an infinite sum [41]:

$$p_s(r, \theta) = \sum_{\nu=0}^{\infty} -(2\nu+1)j^{\nu} \frac{(\nu+1)j_{\nu+1}(kR) - \nu j_{\nu-1}(kR)}{(\nu+1)h_{\nu+1}^{(2)}(kR) - \nu h_{\nu-1}^{(2)}(kR)} P_{\nu}(-\cos \theta) h_{\nu}^{(2)}(kr), \quad (28)$$

with r the radial distance from the sphere's centre and θ the accompanying azimuthal angle (measured with respect to the plane wave propagation direction). The function P_{ν} is the Legendre polynomial, j_{ν} the spherical Bessel function of the first kind, and $h_{\nu}^{(2)}$ the spherical Hankel function of the second kind, all of order ν .

Simulations are carried out with the same problem parameters as used for the pulsating sphere ($R = 0.25$ m, $c = 340$ m/s and $\rho_0 = 1.225$ kg/m³). Again, the mesh topology shown in fig. 5 is used, but in addition calculations are also carried out using a multipatch NURBS discretisation, in order to evaluate the coupling technique described in section 4.3. For the single patch calculations, the poles of the mesh are placed at $\theta = \pm 90^\circ$ in order to not build the solution's axisymmetry into the NURBS function space.

5.2.1. Single patch NURBS

In contrast to the pulsating sphere, the solution is now no longer uniform across the entire surface. Given the axisymmetric character of the problem, directivity patterns showing the amplitude of the radiated pressure as a function of the azimuthal angle can be calculated. Figure 8a plots such a pattern for a radial distance of $2R = 0.5$ m at a frequency of 100 Hz. Both the IGiBEM and the conventional iBEM model consist of about 180 DOFs, and both solutions seem to coincide well with the analytical scattering pattern. Analysis of the relative error (fig. 9a), however, shows that the IGiBEM model yields a significantly better approximation: for the same number of DOFs, IGiBEM is more than two orders of magnitude more accurate than conventional iBEM. Even the most coarse NURBS model (fig. 5a) of only 26 DOFs performs equally well as a conventional iBEM model containing almost 400 times more DOFs. This is partly due to the rather low frequency under study, yielding a relatively simple scattered pressure field. The geometrically exact NURBS model therefore quite easily gives a very accurate approximation, while the conventional iBEM models are still limited by their geometrical error.

This is confirmed by fig. 8b, which plots the directivity patterns at a higher frequency of 1000 Hz. The solution is now more dynamic and the conventional iBEM model is clearly a lot less accurate than at 100 Hz. But although not really visible in the directivity pattern, the relative error plots in fig. 9b show that also the IGiBEM accuracy has decreased. The NURBS model of 182 DOFs is still more than an order of magnitude more accurate than the corresponding conventional iBEM one, but now 266 NURBS DOFs (the mesh in fig. 5b) are necessary to match the performance of the conventional model of 9770 DOFs. A higher frequency typically leads to larger discretisation errors, both for polynomial and for NURBS-based shape functions. However, for the conventional iBEM these errors can be masked by the geometrical errors, which is not the case for IGiBEM solutions. This explains why the growing discretisation error for higher frequencies is more visible in the IGiBEM results than in the conventional iBEM ones, but it does not mean that

(a) Directivity pattern at 100 Hz.

(b) Directivity pattern at 1000 Hz.

Figure 8: Directivity pattern of the acoustic pressure [Pa] scattered by the rigid sphere as function of the azimuthal angle θ [°] where $\theta = 0^\circ$ corresponds to the plane wave propagation direction (i.e. the plane wave travels from left to right). Patterns at a radius of 0.5 m ($r = 2R$).

the NURBS discretisation errors are larger. On the contrary, NURBS have already been shown to exhibit superior approximation properties, e.g. by Hughes et al. in [29].

(a) Relative error at 100 Hz.

(b) Relative error at 1000 Hz.

Figure 9: Relative error plot of the scattered acoustic pressure at a radius of 0.5 m as function of the azimuthal angle θ for various IGiBEM and iBEM models.

5.2.2. Multipatch NURBS

To directly assess the accuracy of the patch coupling technique proposed in section 4.3, the double layer potential distribution calculated using a multipatch discretisation of the sphere is studied. Figure 10 presents the topology of the employed NURBS representation, consisting of six identical biquartic patches.

Figure 11a shows the analytical reference solution for the double layer potential (de-

(a) The most coarse multipatch representation of a sphere, consisting of 6 biquartic single-element patches.

(b) A more refined multipatch mesh, consisting of 150 elements. The thicker lines indicate the patch boundaries.

Figure 10: Mesh topology of the (geometrically exact) multipatch NURBS representations of the sphere, built up from 6 identical biquartic patches.

(a) Analytical double layer potential magnitude [Pa], derived from eq. (28).

(b) Relative error [-] on the double layer potential calculated using the multipatch mesh from fig. 10b, containing 386 biquartic DOFs. The patch boundaries are indicated in black.

Figure 11: Assessment of the double layer potential across the scattering sphere, calculated at 100 Hz. Top view of the sphere, with the plane wave impinging from left to right.

rived from eq. (28)⁶, while fig. 11b shows the relative error $|\hat{\mu} - \mu_{\text{ref}}| / |\mu_{\text{ref}}|$ of the double layer potential calculated with the multipatch NURBS model from fig. 10b. This model, with 386 DOFs, consists of conforming patches, meaning that the coupling across patches can be done by simply adding up the matching basis functions to form a single basis function with a single control variable. This is a trivial coupling that yields good results, with the error not exceeding $2 \cdot 10^{-4}$ anywhere and with an average value of $1.6 \cdot 10^{-5}$.

In the case of non-conforming patches, the coupling cannot be done one-to-one or even exactly in an arbitrary situation. To test the proposed coupling technique, a strongly non-conforming mesh is generated by refining the most coarse model (fig. 10a) until each patch contains $(i_{\text{patch}} + 1)^2$ elements, with $i_{\text{patch}} = 1, \dots, 6$ being the patch number. Figure 12a shows the top view of this mesh. Most patch boundaries are non-hierarchical, further complicating the coupling. Without further treatment, the patches are fully decoupled

⁶Although the double layer potential in the indirect BIE does not in general represent any direct physical quantity of the pressure field, for closed problems it can be considered to equal the total acoustic pressure on the (outer) boundary surface $\mathbf{r}_f^+$ (by taking $p_a(\mathbf{r}_f^-) = 0$).

Figure 12: Assessment of the double layer potential across the scattering sphere for the non-conforming multipatch mesh, calculated at 100 Hz. Top view of the sphere, with the plane wave impinging from left to right. The patch boundaries are indicated in black.

and the resulting solution is completely erroneous, as illustrated by fig. 12b. Using the coupling technique from section 4.3, however, leads to an accurate double layer potential, as shown by the relative error plot in fig. 12c. Although the error level near the patch boundaries is somewhat higher than in the conforming case (fig. 11b), the accuracy is still very good, with the error not exceeding $5 \cdot 10^{-3}$ anywhere and an average value of $3.1 \cdot 10^{-4}$. As expected, the largest errors occur at non-hierarchical boundaries, where an exact coupling cannot be guaranteed.

It is worth noting that the coupling of the patches does not reduce the computational performance of the overall method. The master-slave relationships only depend on the mesh itself and can therefore be generated in a pre-processing step. In fact, they only depend on the interface knot vectors, making them inexpensive to compute. Condensation of the system matrix afterwards reduces the final size of the system matrix (which is fully populated), lowering the computational cost for solving the system of equations.

Figure 13 shows error convergence curves for the different models, generated using k-refinement⁷. The relative error is calculated at 100 Hz in 50 points (at 5 different radii and 10 different azimuthal angles), according to the discrete l^2 relative error norm. Both multipatch topologies converge, although the convergence of the non-conforming models is somewhat slower. This is partly due to the coupling error at non-hierarchical boundaries, but also to the non-uniform and less efficient distribution of the DOFs in the non-conforming case. Moreover, the singularities have a stronger influence, since the regularising transformations discussed in section 3.2 cannot be implemented for adjacent elements without full common boundaries. The conventional iBEM models are again less accurate as compared to IGiBEM, but this is now also partly because of their lower polynomial order (quadratic instead of quartic). This is also the reason why the (quadratic) single patch models converge more slowly than the (quartic) multipatch ones.

⁷This isogeometric mesh refinement technique first elevates the polynomial degree before inserting additional knots. It increases not only the number of shape functions but also their smoothness, and has no analogue in the conventional Lagrangian BEM.

Figure 13: Convergence curves for the scattering sphere problem at 100 Hz. The abbreviation *SP* denotes the single patch models (fig. 5), *MP* the conforming multipatch discretisations (fig. 10) and *MP NC* the non-conforming ones (fig. 12a).

5.3. Curved plate

The next numerical example is an acoustic scattering problem with a geometrically relatively simple open boundary surface, as shown in fig. 14. It concerns a zero thickness panel of dimensions of 1 m × 1 m that is curved in one direction with a radius of curvature of 0.5 m. It acts as a rigid boundary (boundary condition $\bar{v}_n = 0$ m/s) and is placed in an unbounded acoustic domain filled with air ($\rho_0 = 1.225$ kg/m³, $c = 340$ m/s). The domain is acoustically excited by a unit point source located at $\mathbf{r}_q$ about 2 metres above the plate.

This problem is studied in a frequency range from 100 to 1000 Hz, for both IGiBEM and conventional iBEM models of various sizes. All models use quadratic shape functions and consist of meshes of square elements, which in the case of Lagrangian models are CQUAD8 elements. Because no analytical solution is available, a very fine Lagrangian model (containing 7701 quadratic DOFs) is used as a reference. To compare the different models, the scattered pressure field is investigated by looking at the averaged response over the 75 response points indicated in fig. 14. The averaged pressure response is plotted in fig. 15a, using a frequency resolution of 10 Hz. All the models match the reference solution well. In order to better assess the performance of the different models, fig. 15b shows the relative error with respect to the reference model in the discrete l^2 norm $\|\epsilon\|_2$ (cf. eq. (27) with $n_p = 50$).

These error curves present several observations. First of all, the IGiBEM model of 484 DOFs is consistently more accurate than the conventional iBEM model of 481 DOFs, with an improvement up to a factor 4 (i.e. with the same number of DOFs, the IGiBEM error is up to 4 times smaller than the conventional iBEM one). An IGiBEM model of only 100 DOFs already matches the performance of this Lagrangian model, whereas about 1800 Lagrangian DOFs are necessary to match the performance of the 484 NURBS DOFs. Also, the accuracy of a given model decreases with frequency, which is a well-known effect in dynamic analysis. As the frequency increases, the interpolation and pollution errors increase. However, the higher continuity of a NURBS basis as compared to conventional polynomials makes isogeometric methods less sensitive to this effect, leading to

Figure 14: The curved plate problem setting. The symbol $\bigcirc$ indicates the acoustic point source location $\mathbf{r}_q$, $\times$ the response points. The points $\mathbf{P}_i$ ($i = 1, \dots, 4$) are the corner nodes of the curved panel.

(a) The average scattered field pressure amplitude over the 75 response points. All models approximate the reference solution well.

(b) Averaged relative error on the scattered pressure field, showing IGiBEM clearly outperforming conventional iBEM.

Figure 15: Assessment of the scattered acoustic pressure response (in the 75 response points) for the curved plate problem in a frequency range from 100 to 1000 Hz. All models use quadratic shape functions.

less performance deterioration with increasing frequency. This is studied in more detail for acoustic problems in [30] and for approximation of eigenmodes in general in [29]. Also the geometrical error plays a role here, since higher-frequency fields are more sensitive to

geometric details.

It is worth noting that even an increase in accuracy (for a given model size) of a factor 4 is already a very significant improvement, since in contrast to the FEM, the BEM yields fully populated matrices. Not only do both the assembly and storage of such matrices scale with n^2 (n being the number of DOFs), solving them typically scales with n^3 [42, 43]. This means that a factor 4 efficiency gain can quickly lead to a speed-up of factor 64 to obtain a given accuracy.

For a further assessment of the IGiBEM performance, fig. 16 plots the convergence expressing average l^2 error versus the number of DOFs for IGiBEM models of varying polynomial order p , calculated at 500 Hz. Also the curve for the conventional iBEM of second order is plotted. Figure 16a shows the convergence curves using the same

Figure 16: Convergence curves for the averaged relative error (eq. (27)) for the curved plate problem at 500 Hz, plotted for models of varying polynomial order p . The Lagrangian reference solution used in (a) is not accurate enough to allow full convergence of the IGiBEM models.

Lagrangian reference model (of 7701 biquadratic DOFs) as used in fig. 15. For quadratic IGiBEM models, this reference solution seems accurate enough. The higher order models, however, stagnate below an accuracy of about 1%. Given the previous findings, this is likely due to an insufficiently accurate (Lagrangian) reference solution. Figure 16b therefore plots the same curves but now using a very fine isogeometric reference model containing 8836 biquartic DOFs. This now does allow for full convergence of the IGiBEM models. Also note that higher-order models converge faster than the lower-order ones, as expected. The last data point on the conventional iBEM curve in fig. 16b shows that the accuracy of the very fine Lagrangian reference is indeed limited just below 1% (at this frequency). Changing the reference solution for the curves in fig. 15 does not affect those curves, as it is already sufficiently accurate for those models. It should however not be used for evaluating solutions of finer models.

5.4. Car bonnet

The last numerical example studies a complex multipatch object generated using the CAD programme Rhino3D [44]. The geometry is that of a sports car bonnet structure, as shown in fig. 17. The bonnet's outer box dimensions are roughly $1.8 \times 1.4 \times 0.4$ m and

Figure 17: The car bonnet structure, consisting of 14 NURBS patches of which some are non-conforming (indicated with red boundaries). The thicker lines are the patch boundaries, the thinner ones represent the elements.

it is modelled using 14 biquadratic NURBS patches, some of which are non-conforming (indicated by the red patch boundaries). The model contains 1053 independent DOFs (after taking into account the patch coupling dependencies). The structure is placed in an unbounded acoustic domain filled with air ($\rho_0 = 1.225 \text{ kg/m}^3$, $c = 340 \text{ m/s}$). It excites that domain by vibrating at a unit velocity (boundary condition $\bar{v}_n = 1 \text{ m/s}$ across the entire surface, cf. eq. (2b)). The purpose of this case study is to assess the performance and robustness of the proposed method for a more complex geometry containing several non-conforming patches.

A first analysis is performed at a frequency of 500 Hz. Figure 18 shows a top view of the structure, with a plot of the resulting double layer potential distribution. The colour plot clearly shows the good coupling between the 14 patches, including the non-conforming ones. The double layer potential is nicely continuous, although the non-conforming

Figure 18: Double layer potential amplitude distribution [Pa] for the vibrating car bonnet structure (top view) at 500 Hz, calculated using IGiBEM with the model shown in fig. 17.

boundaries are still slightly visible in the plot. This is due to the inevitable non-exact character of the coupling at those boundaries. Nonetheless, the error introduced by the non-conformities remains very limited, as shown by the relative error plot of the double layer potential in fig. 19. This relative error ϵ is calculated as $\epsilon = ||\hat{\mu}| - |\mu_{\text{ref}}|| / |\mu_{\text{ref}}|$,

Figure 19: Relative error [-] of the double layer potential distribution for the vibrating car bonnet structure in fig. 17 (top view) at 500 Hz, with the patch boundaries indicated in black (cf. the thicker lines in fig. 17).

where μ_{ref} is computed using a detailed conventional iBEM model using 11301 biquadratic DOFs. The high relative error near the bonnet edges is due to the very low absolute values of the reference values there (reaching exactly zero on the edge itself, as imposed by the free edge boundary conditions). Other than that, the error is mostly in the order of an engineering accuracy of 1 %, with an average value of 2.9 %. On the non-conforming boundaries it is slightly higher but still well within bounds. Refining the mesh can decrease the error further. The solution and also the error plot should be symmetric (left-right symmetric in the figures) due to the uniform boundary conditions and the symmetric geometry. Nonetheless there are some slight error asymmetries, which are mainly due to the reference model not using an exactly symmetrical mesh.

To demonstrate the robustness of the coupling technique, a discretisation containing strongly non-conforming (non-hierarchical) patches is studied. Figure 20 shows this model. It is obtained from the one in fig. 17 by strongly refining the two patches that are fully surrounded by non-conforming patch boundaries. The resulting model contains 1965 independent DOFs. As illustrated by the resulting relative error plot in fig. 21, these strong non-conformities do not really affect the solution accuracy. In fact, near the strong non-conformities, this solution is if anything more accurate than the one in fig. 19, because of the richer bases in the two refined patches. This observation illustrates that the method is inherently the least accurate when the bordering patches are non-conforming but have a nearly equal number of elements (as in fig. 17). Indeed, such configurations have the lowest degree of flexibility to ensure the correct continuity enforcement between the bordering but non-matching discretisations. If the non-conformities are very pronounced (i.e. the numbers of elements differ strongly, as in fig. 20), the finer mesh in the

Figure 20: The car bonnet structure (cf. fig. 17) with several strongly non-conforming patch boundaries.

Figure 21: Relative error [-] of the double layer potential distribution for the pulsating car bonnet structure (top view) at 500 Hz, obtained using the strongly non-conforming mesh from fig. 20.

slave patch introduces extra DOFs that facilitate the continuity enforcement.

For further benchmarking, a broader frequency range is studied by looking at the radiated pressure response. The response is evaluated in 75 points scattered in the volume surrounding the car bonnet structure, up to one metre away from it. As reference solution, the same refined conventional iBEM model as above (containing 11301 biquadratic DOFs) is used. For IGiBEM, the mesh from fig. 17 is studied, as well as a more refined version containing the same non-conformities (resulting in 1745 DOFs). The presented conventional iBEM models are built using quadrilateral (CQUAD8) and triangular (CTRIA6) elements (for minimal mesh distortion). All studied models employ quadratic basis functions. Figure 22a shows the averaged pressure response, between 100 and 500 Hz, visualised with a frequency resolution of 10 Hz. All models match the reference solution well. The averaged relative error in fig. 22b shows the performance of the different models in more detail. Again eq. (27) is used as error measure, taking into account both magnitude and phase errors. Although the errors are in the same order of magnitude,

(a) The average radiated field pressure amplitude over the 75 response points. All models approximate the reference solution well.

(b) Averaged relative error on the radiated pressure field, showing IGiBEM clearly outperforming conventional iBEM.

Figure 22: Assessment of the radiated acoustic pressure response (in the 75 response points) for the car bonnet in a frequency range from 100 to 500 Hz. All models use quadratic shape functions.

the IGiBEM models clearly perform better than the conventional iBEM solutions. The model from fig. 17 is up to two times more accurate than the iBEM model with slightly more DOFs, while the finer IGiBEM solution outperforms the iBEM model containing 30 % more DOFs by the same improvement factor.

In the spirit of directly using CAD models in an analysis framework, the analysis models are generated straight from the CAD data without considerations for mesh quality. However, it is worth noting that the IGiBEM performance could be improved by generating more uniformly sized meshes instead of simply refining the CAD discretisation. Indeed, some element sizes and aspect ratios in fig. 17 are not ideal from an analysis point of view. Moreover, the previous case studies (cf. fig. 16) have shown that conventional iBEM models can be inadequate as references for IGiBEM solutions. Using an IGiBEM reference would therefore most likely further increase the performance difference — particularly in the higher frequency region, where geometrical errors become more of an issue. But even so, the authors consider the straightforward translation from CAD to CAE to be the main strength of isogeometric frameworks, and by extension of the proposed IGiBEM (in particular with the presented non-conforming patch coupling method). An increase in modelling efficiency should in that sense mainly be regarded as a useful bonus, especially for industrial problems.

6. Conclusion

An isogeometric indirect BEM is presented and verified against analytical solutions. Benchmarking against the conventional, polynomial-based indirect BEM shows the proposed method's superior accuracy, both for radiation and for scattering problems. Accuracy improvements of a factor 4 up to 400 are observed on a per-DOF basis. Given the

fully populated character of BEM system matrices, this presents a significant efficiency improvement. Contributing to this is the fact that the IGiBEM does not suffer from the geometrical errors that conventional iBEM models typically experience. These errors are much more important in the BEM than in the FEM, since in the BEM all the elements are geometrically flawed when using an approximate geometry, while FE models only suffer from geometrical errors in a (typically small) fraction of their elements. An investigation of the frequency behaviour of the proposed method shows that also for higher frequencies, it outperforms the conventional iBEM. In addition, a method for coupling arbitrarily non-conforming patches is presented and verified for several configurations, including a complex car bonnet structure. Assessments of double layer potential distributions show that this method also works well for strongly non-conforming, non-hierarchical patches. In fact, it is exact in the case of hierarchical meshes, and is a general and straightforward coupling technique that can be extended to other NURBS-based isogeometric methods as well. In addition, the car bonnet structure shows that the IGiBEM can also outperform the conventional iBEM in more industrial-like cases. In contrast to existing isogeometric boundary element methods, which so far mainly used direct (collocation) approaches, the proposed indirect (variational) implementation can treat open boundaries as well as combined interior-exterior problems, and moreover yields symmetric system matrices.

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