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Maximum-entropy methods for Time-Harmonic Acoustics

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Abstract

This paper explores the application of *maximum-entropy* methods (max-ent) to time harmonic acoustic problems. Max-ent basis functions are mesh-free approximants that are constructed observing an equivalence between basis functions and discrete probability distributions and applying Jaynes's *maximum entropy* principle. They are C^∞ -continuous and therefore they are particularly suited for the resolution of Helmholtz problems, where classical finite element methods show a poor accuracy in the high frequency region. In addition, it was recently shown that max-ent approximants can be blended with isogeometric basis functions on the boundary of the domain. This preserves the correct representation of the boundary like in Isogeometric Analysis, with the advantage that the discretization of the interior of the domain is straightforward. In this paper the max-ent mathematical formulation is reviewed and then some numerical applications are studied, including a 2D car cavity geometry defined by B-spline curves. In all cases, if the same nodal discretization is used, finite elements results are significantly improved.

Keywords: maximum-entropy; meshless; high geometric fidelity; Helmholtz problems

1. Introduction

Over the last few years the acoustic properties of a product have become an important criterion in many design problems. This increasing demand on the acoustic performance raised a strong need for numerical prediction tools which allow a reliable evaluation of different design alternatives without expensive experimental studies.

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Several techniques are nowadays available for the simulation of the acoustic wave propagation, addressed by the Helmholtz equation. Among these, the Finite Element Method (FEM) [1] and the Boundary Element Method (BEM) [2] have been widely studied in the literature. The most important advantage of BEM is the dimension reduction of the physical problem. However, it requires significant computational times because it works with full, complex and frequency-dependent matrices. For this reason, the FEM is the most employed numerical technique in commercial simulation tools for interior problems. Nevertheless, some drawbacks are still present. It is well known in the FEM literature that the accuracy of the numerical results heavily depends on the regularity of the mesh used for the discretization [3]. Although many powerful mesh generation algorithms are available nowadays [4], this operation may take a significant part of the total analysis time, especially in three-dimensional applications. In addition, the mesh generation process can rarely be automated and therefore important human resources are required in the preprocessing stage. Another well known problem that affects the FEM is the poor accuracy in the high frequency region, due to pollution errors, [1, 5, 6] which makes the short wave acoustic problem one of the still unsolved problems in the FEM environment [7].

Numerical methods with higher continuity of the basis functions, such as Isogeometric Analysis (IGA) [8], are expected to better handle the latter drawback. In [9] IGA is applied to the simulation of structural vibration problems and it is shown to outperform the standard FEM in the high frequency region. The main purpose of IGA is to integrate the Computer Aided Design (CAD) and the analysis stages by using the same basis functions for the geometric representation and for the numerical analysis. By doing so, the geometrical errors that are introduced by the FEM discretization on the boundary of the domain are avoided. Unfortunately IGA does not possess the same flexibility as the standard FEM in discretizing complexly shaped domains. Some modified formulations based on T-splines [10], hierarchical B-splines [11] and trimming techniques [12] have been proposed for two-dimensional applications but the problem is still open in 3D. In [13] an isogeometric boundary element method based on T-splines is proposed; with this approach the interior parametrization of the domain is no longer required, but as in the classical BEM, fully populated matrixes are obtained. In contrast to this method, which is a direct BEM, the NURBS-based technique proposed in [14] is an indirect BEM, which also allows the modeling of open boundary domains. Such boundaries are very common in vibroacoustic problems.

Another family of higher order continuity schemes, whose application to

acoustic problems has been recently studied, are the mesh-free (or meshless) approximation schemes [15]. The earliest meshless methods are the somehow equivalent Element Free Galerkin Method (EFGM) [16] and Reproducing Kernel Particle Method (RKPM) [17]. These methods use a Moving Least Squares approach [18] to construct the basis functions that, as a consequent drawback, are not strictly positive and do not possess the Kronecker-Delta property on the boundary of the domain, which requires additional efforts to impose essential boundary conditions [19]. The latter problem is solved in the Point Interpolation Method (PIM) [20] where polynomial interpolants that pass through each node are obtained. By using Radial Basis Functions, the RPIM, which is better suited for arbitrarily scattered sets of points, was subsequently developed [21].

An interpolatory approximation is also obtained in the Natural Element Method (NEM) [22], where the basis functions are constructed using the Delaunay triangulation and the Voronoi diagram of the nodes. NEM basis functions are non-negative and possess the Kronecker-Delta property, but their evaluation requires a significant computational effort [23].

In more recent studies, the Jaynes's *maximum entropy* principle [24] was used for the construction of polygonal interpolants [25] and then generalized for the definition of the *local maximum-entropy* (LME) mesh-free approximants [26]. In contrast to the previous mesh-free schemes these approximants are C^∞ everywhere within the convex-hull of the computational grid, they are strictly non-negative and they possess the weak Kronecker-Delta property on the faces of the convex-hull. Additionally, it was recently shown that the max-ent formalism can be used to blend the LME approximants with isogeometric basis functions [27]. In particular, the boundary of the domain is described with a NURBS curve and isogeometric functions are associated to the control points that define the curve. Then the LME approximants are calculated in the interior of the domain and they are blended with the NURBS functions on a thin region close to the boundary. Thanks to this approach, the IGA difficulties in the parametrization of complexly shaped surfaces are avoided and, at the same time, the geometrical error on the boundary, which like for FEM is present also for Galerkin based meshless formulations, can be avoided as well.

Despite their recent introduction, many applications of max-ent methods have emerged over the last years, including thin shell analysis [28], reduced order modeling of mechanical systems [29], convection-diffusion problems [30], non-linear structural analysis [31], fracture mechanics [32], incompressible media problems [33], metal forming and cutting simulation [34], biasing of molecular simulations [35] and phase-field models applied to biomem-

branes [36, 37]. In more theoretical studies the continuity of max-ent basis functions has been proven by the theory of epi-convergence [38], quadratically complete convex approximations schemes have been proposed [39, 40] and it has been shown that by removing the strictly positivity constraint arbitrary order consistency conditions can be satisfied [41].

The good performance of max-ent methods with respect to other numerical techniques in the simulation of structural vibration problems was firstly noted in [39] and then also in [40] and [41]. This paper further investigates on the application of LME approximants to acoustic problems and attempts to take advantage of the blending method proposed in [27] to study arbitrarily shaped domains, whose boundary is given by a NURBS curve.

Due to the aforementioned drawbacks of the FEM, alternative methods for analyzing higher-frequency problems have been the subject of extensive research. Statistical techniques such as Statistical Energy Analysis (SEA) [42] are often used for vibroacoustic analysis. SEA divides the problem domain into a small number of subsystems in which a spatially averaged energy estimate is obtained. It is computationally efficient, but relies on assumptions that are only met above a certain frequency limit. This restricts the use of SEA to high-frequency problems, leaving a gap in the mid-frequency range. This gap can be filled by extending the frequency range of deterministic approaches, which is what the Wave Based Method (WBM) [43] aims to do. The WBM is a Trefftz-based approach that approximates the field variables by a weighted sum of wave functions, each of which is an exact solution of the governing differential equations. It has a higher convergence rate as compared to the FEM and therefore allows for a significant extension of the frequency range of analysis upwards. Also meshless schemes can increase the upper frequency limit and thus have been a subject of study in the last years literature. We refer to [44, 45, 46] for the EFGM, [47, 48] for the RKPM and [49] for the RPIM. These studies prove that mesh-free methods are able to reduce the dispersion effect significantly and, by appropriately choosing certain parameters, classical FEM results can be significantly improved [50]. More recently an improved meshless weighted least-square (IMWLS) method was applied to the simulation of Helmholtz problems [51] and it was shown to reduce the EFGM computational times while retaining equivalent accuracy.

Because the application of max-ent methods to acoustic problems has not been studied in detail yet, this paper focuses on it. It is organized as follows: in Section 2 the LME approximants are briefly introduced and the blending approach proposed in [27] is described. In Section 3 the time-harmonic acoustic problem, governed by the Helmholtz equation, is formulated and

then some numerical examples are considered in Section 4. Finally, in Section 5 some concluding remarks are given and possible future developments are outlined.

2. The *maximum entropy* approximants

In information theory the notion of entropy as a measure of uncertainty or incomplete knowledge was firstly introduced by Shannon [52]. Then Jaynes proposed the principle of *maximum entropy* [24] in which it was shown that maximizing entropy provides the least-biased statistical inference when insufficient information is available.

Consider now a set of distinct nodes in $\mathbb{R}^d$ that are located at $\mathbf{x}_a$ ($a = 1, 2, \dots, n$), with $\Omega = \text{con}(\mathbf{x}_1, \dots, \mathbf{x}_n) \subset \mathbb{R}^d$ denoting the convex hull of the nodal set. For a real-valued function $u(\mathbf{x}) : \Omega \rightarrow \mathbb{R}$, the numerical approximation for $u(\mathbf{x})$ is:

$$u^h(\mathbf{x}) = \sum_{a=1}^n \phi_a(\mathbf{x}) u_a, \quad (1)$$

where $\phi_a(\mathbf{x})$ is the basis function associated with node a , and u_a are coefficients. In the max-ent approach an entropy functional that depends on a discrete probability measure $\{p_a\}_{a=1}^n$ is maximized, subject to linear constraints on p_a . On noting the correspondence between basis functions $\{\phi_a\}_{a=1}^n$ and discrete probability measures $\{p_a\}_{a=1}^n$, the max-ent formalism was recently applied to construct basis functions. The idea was firstly introduced by Sukumar for the construction of polygonal interpolants [25]. The associated max-ent variational formulation was: find $\mathbf{x} \mapsto \phi(\mathbf{x}) : \Omega \rightarrow \mathbb{R}_+^n$ as the solution of the following constrained (concave) optimization problem:

$$\max_{\phi \in \mathbb{R}_+^n} H(\phi) = - \sum_{a=1}^n \phi_a(\mathbf{x}) \ln \phi_a(\mathbf{x}), \quad (2)$$

subject to the linear reproducing conditions:

$$\sum_{a=1}^n \phi_a(\mathbf{x}) = 1, \quad (3a)$$

$$\sum_{a=1}^n \phi_a(\mathbf{x})(\mathbf{x}_a - \mathbf{x}) = \mathbf{0}, \quad (3b)$$

where $\mathbb{R}_+^n$ is the non-negative orthant. It is easy to prove that the aforementioned problem has a solution if, and only if, $\mathbf{x}$ belongs to the convex hull of

the nodal set Ω . Furthermore this solution is unique. In fact, it comes from the definition of convex hull that, if the reproducing conditions given by Eq. (3) are verified, $\mathbf{x} \in \Omega$. On the other hand one can use Carathéodory's theorem to prove that, for any point $\mathbf{x}$ in Ω , at least $n - d - 1$ nodes are not necessary to verify Eq. (3). In addition, since Ω is a compact subset of $\mathbb{R}^d$, by the Weierstrass extreme value theorem, the entropy H attains a maximum in Ω . Observing that H is a strictly concave function of the basis ϕ [53], the maximum is unique [26].

The aforementioned approach is suitable for polygonal meshes, where the max-ent interpolants are calculated in each polygon, but it can not be directly applied to scattered clouds of nodes, because too many points would enter in the calculations. In order to extend the max-ent approach to the application in meshfree implementations, in [26] the character of the basis functions was modified by introducing a Pareto optimum between the maximization of the entropy and their locality. In fact, it was noted that, according to [54], the piecewise affine shape functions defined on the Delaunay triangulation of a set of nodes can be obtained through convex optimization by solving the following minimization problem

$$\min_{\phi \in \mathbb{R}_+^n} \sum_{a=1}^n \phi_a(\mathbf{x}) |\mathbf{x} - \mathbf{x}_a|^2, \quad (4)$$

subject to the linear reproducing constraints given by Eq. (3). Since the total width is minimized, these shape functions are the most local possible functions for a given cloud of nodes. Therefore Sukumar's formulation was modified by introducing a Pareto optimum between (4) and (2). In particular the following optimization problem is solved:

$$\min_{\phi \in \mathbb{R}_+^n} \sum_{a=1}^n \beta_a \phi_a(\mathbf{x}) |\mathbf{x} - \mathbf{x}_a|^2 + \sum_{a=1}^n \phi_a(\mathbf{x}) \ln(\phi_a(\mathbf{x})) \quad (5)$$

again subject to the constraints given by Eq. (3). The basis functions obtained with this scheme are known in the literature as *local maximum entropy* (LME) approximants. The Pareto parameter β_a controls the locality of the basis function of each node. In particular the functions become more local when β increases until the case of $\beta \rightarrow \infty$ where the Delaunay shape functions are recovered. On the contrary for $\beta = 0$ purely max-ent basis functions are obtained. Since the influence of β_a depends on the mesh size, normally a non-dimensional parameter $\{\gamma_a = \beta_a h_a^2\}_{a=1,\dots,N}$ is preferred. Typical values of γ employed in the literature are in the range from 0.5 to

2 (Figure 1). Like in the previous case, the objective function minimized in (5) is continuous and strictly convex and, therefore, the minimization problem has a unique solution if and only if $\mathbf{x} \in \Omega$ [26]. In particular, using the method of Lagrange multipliers, the solution of the variational problem stated in (5) is:

$$\phi_a(\mathbf{x}) = \frac{Z_a(\mathbf{x}; \boldsymbol{\lambda})}{Z(\mathbf{x}; \boldsymbol{\lambda})}, \quad Z_a(\mathbf{x}; \boldsymbol{\lambda}) = \exp -(\beta_a |\tilde{\mathbf{x}}_a|^2 + \boldsymbol{\lambda} \cdot \tilde{\mathbf{x}}_a) \quad (6)$$

where $\tilde{\mathbf{x}}_a = \mathbf{x}_a - \mathbf{x}$ ($\mathbf{x}, \mathbf{x}_a \in \mathbb{R}^d$) are shifted nodal coordinates, $\boldsymbol{\lambda}(\mathbf{x}) \in \mathbb{R}^d$ are the d Lagrange multipliers associated with the constraints given by Eq. (3b), and $Z(\mathbf{x}; \boldsymbol{\lambda}) = \sum_b Z_b(\mathbf{x}; \boldsymbol{\lambda})$ is known as the *partition function* in statistical mechanics. On considering the dual formulation, the solution for the Lagrange multipliers can be written as

$$\boldsymbol{\lambda}^* = \operatorname{argmin} F(\boldsymbol{\lambda}), \quad F(\boldsymbol{\lambda}) := \ln Z(\boldsymbol{\lambda}). \quad (7)$$

In [26] a proof of the uniqueness of the minimizer $\boldsymbol{\lambda}^*$ is also given. Since F is strictly convex in the interior of Ω and the minimizer is unique, the basis functions can be computed efficiently and robustly using numerical techniques such as the Newton-Raphson scheme that, for the aforementioned standard values of γ , converges to machine precision within a few iterations. From a theoretical point of view, the LME basis functions have global support in the domain Ω . However, according to Eq. (6), they exponential decay in function of the distance to the corresponding node. This has the important consequence that only a small number of nodes has a significant contribution to the partition function. Therefore, in the numerical practice, a tolerance Tol_0 is set and for each node $\mathbf{x}_a$ the corresponding basis function $\phi_a(\mathbf{x})$ is truncated when the term $\exp -(\beta_a |\tilde{\mathbf{x}}_a|^2)$ in Eq. (6) becomes smaller than Tol_0 . Since $|\tilde{\mathbf{x}}_a|$ is the distance between the evaluation point $\mathbf{x}$ and $\mathbf{x}_a$, this is equivalent to consider a circular support whose radius R_a is given by

$$R_a = \sqrt{-\log(\text{Tol}_0)/\beta_a} = h \sqrt{-\log(\text{Tol}_0)/\gamma_a}.$$

We refer to [26] for a more detailed discussion about the LME basis functions and the solution of the optimization problem and to [55, 56] for the computation of the first and second derivatives.

As it can be observed from Figure 1, LME approximants are endowed with features such as monotonicity and C^∞ smoothness. A rigorous proof of the continuity of LME and other max-ent basis functions can be found in [38]. They also possess the variation diminishing property and satisfy the

Figure 1: LME basis functions on a square domain discretized with a regular grid of nodes, computed for different values of γ . The nodes are depicted with a point. The shape and the support of the basis functions change with the locality parameter, which in this example takes the values: $\gamma = 0.4$ (blue), $\gamma = 0.8$ (green), $\gamma = 1.2$ (red), $\gamma = 1.6$ (yellow). For $\gamma \rightarrow \infty$ the Delaunay shape functions would be recovered. For illustrations purposes, the basis functions are truncated by choosing $\text{Tol}_0 = 10^{-6}$.

weak Kronecker-delta property on the faces of the convex hull of the nodes, which makes the imposition of essential boundary conditions in Galerkin methods straightforward. Moreover, the approximants are multidimensional and lead to well-behaved mass matrices.

2.1. Blending LME and isogeometric approximants

In [27] it was noted that the drawbacks of IGA and LME schemes are somehow complementary and that they can be avoided by blending the two methodologies. In this section we will refer for the sake of simplicity only to a two dimensional case, where the boundary of the domain $\partial\Omega$ is split into NURBS patches $\partial\Omega = \cup_{\alpha}\Gamma_{\alpha}$. Each patch is described by a parametric curve $\mathbf{C}_{\alpha} : [0, 1] \mapsto \Gamma_{\alpha}$ such that

$$\mathbf{C}_{\alpha}(\xi) = \sum_i M_i(\xi) \mathbf{P}_i, \quad (8)$$

where $M_i(\xi)$ are 1D NURBS basis functions calculated on a knot span in the interval $[0, 1]$, and $\mathbf{P}_i \in \mathbb{R}^2$ are the control points. In order to blend isogeometric and LME basis functions a ring of 2D multivariate NURBS functions associated with the boundary nodes is defined in a subset $\Omega_{\star}$ of

Figure 2: Two examples of surface parametrization employed for the blending. In the left a circumference is modeled with rational quadratic Bezier curves. Then the sub-domain $\Omega_\star$ is obtained by associating in the radial directions 3 quadratic B-spline basis functions to the 3 rings of control points (the internal ones in green are obtained by projecting the external ones in red). In the right the boundary of the domain is given by a closed cubic B-spline curve and then 4 rings of control points are employed to generate $\Omega_\star$.

the computational domain Ω (Figure 2). For this purpose, in each patch, a NURBS mapping $\mathbf{S} : [0, 1] \times [0, 1] \mapsto \Omega_\star$ is defined as

$$\mathbf{S}(\xi, \eta) = \sum_i \sum_j M_i(\xi) P_j(\eta) \mathbf{P}_{i,j}. \quad (9)$$

where $P_j(\eta)$ are 1D NURBS functions calculated on a semi-open knot span so that $P_1(0) = 1$. Associating to each point $\mathbf{x} = \mathbf{S}(\xi, \eta)$ the value $N_b(\mathbf{x}) = M_i(\xi) P_1(\eta)$, a set of 2D basis functions is obtained in $\Omega_\star$.

At this point the blending with the LME approximants is realized by including the NURBS functions in the constraints of the *maximum entropy* optimization problem :

$$\begin{aligned} \text{For fixed } \mathbf{x} \text{ minimize } & \sum_{a \in \mathcal{I}_{ME}} \phi_a(\mathbf{x}) \ln \phi_a(\mathbf{x}) + \sum_{a \in \mathcal{I}_{ME}} \beta_a \phi_a(\mathbf{x}) |\mathbf{x} - \mathbf{x}_a|^2 \\ \text{subject to } & \phi_a(\mathbf{x}) \geq 0, \quad a \in \mathcal{I}_{ME} \\ & \sum_{a \in \mathcal{I}_{ME}} \phi_a(\mathbf{x}) + \sum_{b \in \mathcal{I}_{NU}} N_b(\mathbf{x}) = 1 \\ & \sum_{a \in \mathcal{I}_{ME}} \phi_a(\mathbf{x}) \mathbf{x}_a + \sum_{b \in \mathcal{I}_{NU}} N_b(\mathbf{x}) \mathbf{x}_b = \mathbf{x}, \end{aligned} \quad (10)$$

where $\phi_a(\mathbf{x})$ are the basis functions of the nodes that lies in $\Omega_\star$, indicated with the indexes in $\mathcal{I}_{ME}$, so that the global set of indices $\{1, 2, \dots, N\}$ is given by $\mathcal{I}_{NU} \cup \mathcal{I}_{ME}$. While the above optimization problem is solved for all the nodes in $\Omega_\star$, nothing changes for the internal nodes, where standard LME functions are employed. Also in this case, the optimization problem stated in (10) has a unique solution in $\Omega_\star$ and, therefore, the calculation of the basis functions is straightforward applying duality methods. We refer to [27] for the expression of the Lagrangian and for the computation of the derivatives.

3. The time-harmonic acoustic problem

In this section we formulate the time-harmonic acoustic problem on a two-dimensional (bounded) domain Ω , as depicted in Figure 3. The domain is filled with an acoustic fluid with speed of sound c and fluid mass density ρ_0 . The steady-state dynamic behavior in this system is described by the acoustic pressure field $p_a(\mathbf{x})$, which is governed by the inhomogeneous Helmholtz equation [57]:

$$\nabla^2 p_a(\mathbf{x}) + k^2 p_a(\mathbf{x}) = -j\rho_0\omega q\delta(\mathbf{x}, \mathbf{x}_q), \quad \mathbf{x} \in V, \quad (11)$$

where $\mathbf{x}$ is the position vector, k the acoustic wavenumber, ω the angular frequency, q the strength of an acoustic volume velocity source at position $\mathbf{x}_q$, and $\delta(\bullet, \star)$ the Dirac-delta function. The wavenumber k is related to the angular frequency ω by the speed of sound $c = \frac{\omega}{k}$. The term $\nabla^2 \bullet = \frac{\partial^2 \bullet}{\partial x^2} + \frac{\partial^2 \bullet}{\partial y^2} + \frac{\partial^2 \bullet}{\partial z^2}$ is the Laplacian operator and j denotes the imaginary unit ($j^2 = -1$).

In order to have a unique solution, the Helmholtz equation (11) requires one imposed boundary condition at each point on the problem boundary $\Gamma = \partial\Omega$. This boundary can be divided into three non-overlapping parts, assuming three types of common acoustic boundary conditions: $\Gamma = \Gamma_p \cup \Gamma_v \cup \Gamma_Z$. These parts correspond to a Dirichlet, a Neumann and a Robin boundary condition, respectively. The boundary condition on Γ can then be written as follows:

$$p_a(\mathbf{x}) = \bar{p}_a \quad \mathbf{x} \in \Gamma_p, \quad (12)$$

$$\frac{j}{\rho_0\omega} \frac{\partial p_a(\mathbf{x})}{\partial n} = \bar{v}_n \quad \mathbf{x} \in \Gamma_v, \quad (13)$$

$$\frac{j}{\rho_0\omega} \frac{\partial p_a(\mathbf{x})}{\partial n} = \frac{p_a(\mathbf{x})}{\bar{Z}_n} \quad \mathbf{x} \in \Gamma_Z. \quad (14)$$

Figure 3: Description of a two-dimensional acoustic problem domain.

The quantities $\bar{p}_a$, $\bar{v}_n$ and $\bar{Z}_n$ are the imposed pressure, imposed normal velocity and imposed normal impedance, respectively, where $\mathbf{n}$ is the normal on the boundary. In the case of a (coupled) vibroacoustic problem, where a (typically thin-shell type) elastic structure is part of the boundary domain, a normal velocity continuity boundary condition can be applied on this part of the boundary. This expresses the vibroacoustic coupling condition, imposing that the normal fluid velocity must equal the normal structural velocity along the elastic structure. This paper however only considers the first three boundary conditions mentioned above.

The Helmholtz equation (11) together with the associated boundary conditions (12) – (14) fully define the acoustic pressure field $p_a(\mathbf{x})$ in the entire problem domain.

4. Numerical examples

The accuracy of max-ent methods is firstly analyzed and compared to other numerical techniques in the prediction of the eigenfrequencies and eigenmodes of simply shaped geometries. Then some practical applications on complexly shaped domains with general boundary conditions are presented and finally the performance of max-ent methods is studied in the simulation of high wavenumber problems, where pollution errors are a huge challenge for the classical FEM formulation.

4.1. Eigenvalue problem

4.1.1. Formulation

The good performance of LME approximants in the calculation of the eigenfrequency spectrum of a vibrating membrane (which is also addressed by the Helmholtz equation) was firstly noted in [39]. In order to maintain the analogy with vibrations of membranes and other eigenvalues problems, here we consider the homogeneous problem with Dirichlet boundary conditions:

$$\nabla^2 p_a(\mathbf{x}) + k^2 p_a(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega, \quad (15a)$$

$$p_a(\mathbf{x}) = 0 \quad \mathbf{x} \in \partial\Omega. \quad (15b)$$

The aforementioned problem can be either solved for the wavenumber k or for the frequency $\omega = kc$. In the examples that follow we will assume a speed of sound $c = 1$, in which case the eigenfrequencies correspond numerically to the related wavenumbers.

4.1.2. Square domain

If a unit square domain $\Omega = [0, 1]^2$ is considered, the exact eigenmodes and eigenfrequencies of problem (15) are given by

$$\Phi_{mn} = \sin(m\pi x) \sin(n\pi y), \quad \omega_{mn} = \pi \sqrt{m^2 + n^2}, \quad m, n = 1, 2, 3 \dots \quad (16)$$

In Figure 4 we can observe how the ratio between the numerical and the exact eigenfrequencies changes across the spectrum when LME approximants are used to solve the problem on a 100×100 tensor product regular grid. It is well known that these curves asymptotically converge to a given shape that can be used to determinate the accuracy of the numerical method concerning the pollution error. In particular, in the case of LME approximants, the error remains below 5% even in the high frequency region. It can also be observed that the quality of the approximation changes with the locality parameter γ . Low values of γ give, according to Figure 1, basis functions that have a larger support and therefore the approximated pressure field in each point will be influenced by a higher number of nodes. This improves the quality of the approximation but it also increases the bandwidth of the mass and stiffness matrices and, as a consequence, the computational times. The good behavior of LME approximants in the high frequency region can be observed also considering a comparison with IGA that, in this application, in the case of $p = 1$ recovers quadrilateral bilinear finite elements.

In addition to better managing pollution errors, LME approximants also improve the accuracy of the FEM in the estimation of the low frequency

Figure 4: Square domain - Normalized discrete eigenfrequency spectrum obtained with the LME approximants for different values of γ (a) and a comparison with IGA(b).

modes. In Figure 5 we can in fact observe the convergence of the LME solution for the first eigenmode and the first eigenfrequency, compared with quadrilateral and triangular finite elements defined on the Delaunay triangulation of the grid. The triangulation is also used for the numerical integration in the LME implementation, where a 6th order symmetric rule from [58] with 12 points per triangle is employed. According to [26] this rule is enough to ensure that approximation errors are predominant in the convergence study. Looking at Figure 5 we can see how, for $\gamma = 1$, the LME solution outperforms the FEM by more than two orders of magnitude in the L_2 norm of the error on the eigenmode and four in the relative error on the eigenfrequency.

Because of the bigger bandwidth of the mass and stiffness matrices, the computational times are higher in the LME case. However, this drawback is compensated by the significant improvement of accuracy. For this particular application, we considered the computational times required in our implementation to compute the first eigenmode and eigenfrequency, for the grids employed in the convergence analysis in Figure 5. A MATLAB implementation with sparse matrices and the function *eigs* were used. In Figure 6 we can observe how the LME results are still more accurate, if compared to the FEM.

As far as the computational times are concerned, it is worth mentioning that also the evaluation of the LME basis functions is more expensive with respect to the FEM. However, this is not expected to be a bottle-neck. In fact, the computational complexity of this operation is linear with respect to the number of Gauss points. But the number of Gauss points is also linear with respect to the total number of nodes. Therefore, the problem

Figure 5: Square domain - Convergence of the L_2 norm of the error on the first eigenmode (a) and of the relative error on the first eigenfrequency (b).

Figure 6: Square domain - L_2 norm of the error on the first eigenmode (a) and relative error on the first eigenfrequency (b) in function of the computational time required by the function *eigs* in MATLAB.

is still linear with respect to the number of nodes. On the other hand, the computation of the eigenmodes has a complexity that is higher than linear. In addition, in many acoustic applications, like the problems considered in Sections 4.2 and 4.3, the mass and stiffness matrices have to be computed only once and then a linear system of equations has to be solved multiple times. In that case the matrix assembly time is negligible anyway.

4.1.3. Triangular domain

An equilateral triangular domain (Figure 7a) is also considered in order to study the accuracy of the LME approximants and the pollution error on a different geometry and a different nodes configuration. In this case, if the nodes are distributed in a regular way, a hexagonal lattice is obtained in the

Figure 7: Triangular domain - Geometry and distribution of a 55 nodes grid (a) and normalized discrete eigenfrequency spectrum obtained with the LME approximants for different values of γ (b).

interior of the triangle. The exact eigenfrequencies for the problem stated in (15) are given by [59]:

$$\omega_{mn} = \frac{2\pi}{3\sqrt{3}} \sqrt{m^2 + mn + n^2} \quad m, n = 1, 2, 3 \dots \quad (17)$$

The ratio between the numerical and the analytical frequencies is plotted in Figure 7b. For this configuration a slightly higher peak is obtained in the high frequency region while, like for the square domain, the error remains in the order of a few percent digits in the mid-frequency region. On the other hand it is interesting to note how, according to Figure 8, the predictions of the first eigenmode and eigenfrequency show a super-optimal rate of convergence. It is worth remarking, however, that this is only a particular result that does not hold in the general case. For instance, the super-convergent behavior is lost if a quasi-regular lattice, created with a mesh generator, is employed. From the theoretical point of view, it can only be proved that the convergence rate is at least 2:1 [60], like in linear FEM. This comes from the fact that standard LME schemes are only linear complete or, in FEM terms, the order of the approximation is $p = 1$. Determining how the rate is influenced by the nodal distribution can be a topic of future research. In the general case, to achieve higher convergence rates, high order max-ent schemes [41] should be employed. The analytical value of the eigenmode is given by [59]:

$$\Phi_1 = 2 \cos \frac{\pi y}{\sqrt{3}} \sin \frac{\pi(2+x)}{3} - \sin \frac{2\pi(2+x)}{3}. \quad (18)$$

Figure 8: Triangular domain - Convergence of the L_2 norm of the error on the first eigenmode (a) and of the relative error on the first eigenfrequency (b).

On comparing with finite elements, the definition of a triangular mesh is straightforward on the grid while different algorithms may be used to generate a quadrilateral mesh and therefore the results may depend on the particular method employed. For this reason only triangular element were considered in the comparison. Like for the square domain, the LME prediction is significantly more accurate.

4.1.4. Circular domain

Although the basis functions can be directly calculated from the cloud of points, max-ent and other meshless methods require anyway a mesh or some background cells to perform the integration. Therefore, like in the FEM, a geometrical error is introduced when domains with curved boundaries are studied. The simplest shape to analyze this error is a circle. If a circular domain with unit radius is approximated with an n -sided regular polygon, its area is given by

$$A_p = n \frac{1}{2} \sin \frac{2\pi}{n} \quad (19)$$

which gives a relative error

$$e_\pi = \frac{A_c - A_p}{A_c} = \frac{\pi - n \frac{1}{2} \sin \frac{2\pi}{n}}{\pi} = 1 - \frac{n}{2\pi} \sin \frac{2\pi}{n}. \quad (20)$$

As we will see in the following, this error often represents a limit to the accuracy of the numerical solution of a problem defined on the polygonal domain. On the other hand, if the circular domain is correctly represented, like in the case of IGA, such problem is avoided. According to section 2,

Figure 9: Circular domain - Convergence of the L_2 norm of the error on the first eigenmode (a) and of the relative error on the first eigenfrequency (b).

one of the most appealing features of max-ent methods is the possibility to blend LME and NURBS basis functions to preserve the correct geometric representation. Therefore the circular domain is a good benchmark to study the performance of the IGA-LME approach. Also in this case, the eigenvalue problem with Dirichlet conditions stated in (15) is considered. For a unit radius circle the radially symmetrical eigenmodes and eigenfrequencies are given by

$$\Phi_{0n} = J_0(\alpha_{0n}r), \quad \omega_{0n} = \alpha_{0n}, \quad n = 1, 2, 3 \dots \quad (21)$$

where J_0 is the Bessel function of the first kind of order 0 [61] and α_{0n} are the n -th positive roots of J_0 .

In Figure 9 we can observe a comparison of the convergence in the predictions of the first eigenmode and eigenfrequency between the FEM, purely LME approximants and blended IGA-LME approximants. For the FEM solution a quadrilateral mesh was generated with the GEN4U tool [62] starting from a regular polygon. The nodes were used also for the triangular elements analysis (where the Delaunay triangulation was generated as a mesh) and for the LME one.

In the IGA-LME implementation different strategies can be used to construct the initial NURBS curve that represent the circumference and then to generate the grid. In particular, in this work, the approach presented in [63] to construct circular arcs with rational quadratic Bézier curves was employed. In this way, according to (Figure 2), a uniform distribution of control points is obtained on the boundary. Then, the interior grid can be generated with a standard FEM meshing tool like GEN4U. As far as the numerical integration is concerned, triangular cells defined on the Delaunay triangulation are still considered, but special quadrature rules are

implemented for the curved elements on the boundary [64], as described in [27].

Observing the convergence curves it is evident how in this application both the FEM and the LME results are limited by the geometrical error e_π . At the same time, introducing the correct NURBS representation of the circumference, the IGA-LME approach gives results that are consistent with the previous applications.

4.2. Acoustic field inside a star shaped polygonal domain

A star shaped polygonal domain is considered in order to compare the accuracy of max-ent and finite element methods in the resolution of the boundary value problem outlined in Figure 10. According to Eqs. (11) – (14) both Neumann and Robin condition are considered. The domain is discretized with a cloud of 374 nodes and the Frequency Response Function (FRF) is computed in the range from 0 to 2000 Hz for LME approximants and triangular finite elements. A reference solution is calculated on a 30000 nodes FE model. In Figure 11 we can observe a comparison between the two predictions of the FRF in the center of the star. In the FEM solution, a resonance frequency shift starts around 500 Hz and then the solution accuracy rapidly deteriorates further with increasing frequency. On the other hand, the LME solution remains significantly more accurate towards higher frequencies. Although the accuracy of the peak values starts to decrease around 900 Hz, the resonance frequencies and behavior of the curve remain well preserved up to 1700 Hz. It is interesting to note that the wavelength at this frequency is 20cm while the average mesh size h of the cloud of nodes is 6 cm. Therefore, the loss of accuracy can be attributed to approximation errors rather than dispersion problems.

4.3. Acoustic field inside a car cavity geometry defined by B-spline curves

In this section the potentialities of the blending between LME and iso-geometric approximants are shown in a typical acoustic problem where the wave propagation in the interior of a car is studied. The problem is outlined in Figure 12. A 2D section of a car cavity is defined by two closed B-spline curves. On using the IGA-LME approach, a cloud of nodes is generated in the interior of the cavity assuming as grid size the average spacing between the control points $\bar{h}$. If the problem is studied with finite elements or a purely meshless approach, the mesh is generated from a polygonal shape that approximates the B-spline curves and therefore a geometrical error is introduced. In section 4.1.4 this error was calculated analytically for the circular domain. Even if an analytical study is not possible for this application

Figure 10: Star shaped polygonal domain - Definition of the problem (a) and cloud of nodes used for the discretization (b).

Figure 11: FRF for the problem in Figure 10 - Comparison between the FEM solution(a) and the LME one (b).

(a)

(b)

Figure 12: Car cavity - Definition of the problem (a) and IGA-LME discretization (b). The green squares represent the control points which define the B-spline curve, while the black circle are the interior nodes.

Figure 13: Error on the first ten non-zero eigenfrequencies of the car cavity with different numerical techniques.

the importance of the correct geometric representation can be observed considering the error on the first ten non-zero eigenfrequencies, which is plotted in Figure 13. In particular, the error on the frequencies computed with the IGA-LME approach is compared with the FEM error and the purely LME error. To this end, the FEM meshes are generated from the image of the knots on the boundary curves, which result in a projection of the control points on the curves. The triangular mesh is generated with the DistMesh tool [65] and the quadrilateral one is generated with GEN4U. The former is also used for the purely LME simulation. In both cases a mesh size of $\bar{h}$ is considered. A reference solution is calculated on a FEM mesh with roughly one million nodes. Figure 13 shows how also in this case the FEM and the purely meshless solutions are limited by the geometrical error while the IGA-LME approach improves significantly the results.

To confirm the validity of the IGA-LME prediction an FRF for the problem outlined in Figure 12a is also considered. A normal velocity is imposed on the front panel in order to represent the vibrations generated by the engine. The surfaces of the seat and the roof are considered to be absorbent and therefore a Robin condition is implemented. The FRF is computed for the point P (Figure 12) in the spectrum from 1 to 4000 Hz. The FEM model with one million nodes is again assumed for reference. Like in the previous example, the finite elements prediction presents a resonance frequency shift that starts around 2kHz. On the other hand, the IGA-LME solution does not show any shift until 4kHz. At this frequency, the corresponding wavelength is 8.5 cm while the average node spacing is $\bar{h} = 1.81cm$, which confirms

Figure 14: FRF for the car cavity problem - Comparison between the FEM solution(a) and the IGA-LME one (b).

again the good behavior of max-ent methods concerning both dispersion and approximation errors.

4.4. High wavenumber problems and pollution error

The numerical examples presented in the previous sections showed that the resonance frequency shift is much more evident for FEM than for max-ent methods and therefore suggest that the latter can better handle pollution errors. In order to investigate this behavior more in detail, we consider a boundary value problem defined on a circular domain, with a radially symmetrical solution. In this case, the Helmholtz equation can be expressed in polar coordinates [57] as

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p(r)}{\partial r} \right) + k^2 p(r) = 0 \quad r \in [0 \quad R]. \quad (22)$$

The general solution of the aforementioned problem is given by

$$p(r) = AJ_0(kr) + BY_0(kr), \quad (23)$$

where Y_0 is the Bessel function of the second kind of order 0 [61], which would be unbounded for $r \rightarrow 0$. Thus, the constant B must be zero. A natural condition is imposed to the velocity on the boundary

$$\frac{j}{\rho_0 \omega} \frac{\partial p(r)}{\partial r} = \bar{v}_n \quad r = R. \quad (24)$$

On combining this condition with Eq. (23) the value of the constant A is obtained and the analytical expression of the pressure is given by:

$$p(r) = -\bar{v}_n \frac{J_0(kr)}{\frac{j}{\rho \omega} k J_1(kR)}, \quad (25)$$

where J_1 is the Bessel function of the first kind of order 1.

The problem is solved considering the following parameters: $R = 1m$, $\bar{v}_n = 1mm/s$, $c = 343m/s$ and $\rho = 1.225Kg/m^3$. In order to study the behavior of max-ent methods for different wavenumbers, 4 wavelengths $\lambda = [5m, 1m, 0.5m, 0.1m]$, corresponding to a spectrum going from low to mid-high frequencies, are considered and the related wavenumber $k = \frac{2\pi}{\lambda}$ is employed in Eq. (22). The LME and the IGA-LME methodologies are compared to triangular elements and the distmesh generator [65] is used for all the simulations.

In Figure 15 the convergence of the L_2 relative error norm is plotted for the different wavelengths. We can observe how, at low frequencies ($\lambda = 5m$), the LME and the FEM solutions give a very similar prediction, that is close to the geometrical error defined in Section 4.1.4. On the other hand, the accuracy is improved with the IGA-LME approach that takes into account the exact definition of the circle.

The situation changes when mid frequencies (corresponding to $\lambda = 1m$ and $\lambda = 0.5m$) are considered. In this case, both the LME and the IGA-LME solutions are more accurate than the FEM one and the geometric error is not an issue anymore. This fact is even more evident at higher frequencies ($\lambda = 0.1m$), where the FEM solution is significantly affected by pollution errors. In fact, for linear finite elements, the general estimate for the relative error e_h in the H^1 semi-norm is given by [5]

$$e_h < C_1 kh + C_2 k^3 h^2, \quad (26)$$

where C_1 and C_2 are two constants independent of k and h . In this equation, the first term corresponds to the approximation error and the second one to the pollution error. The former can be controlled by keeping kh constant,

Figure 15: Convergence of the L_2 relative error norm for problem (11), solved on a circular domain for different wave-numbers.

which gives the well known 'rule of thumb' of taking a certain number of elements per wavelength. However, this is not sufficient for the pollution effect, that requires finer meshes for higher frequencies. In fact, we can observe how using 10 elements per wavelength gives a relative error in the order of 10^{-1} for the mid frequencies (the ratio λ/h is plotted on the top of the figures), and around 60% for the highest frequency. On the other hand, both the IGA and the IGA-LME solutions are quite accurate if 10 nodes per wavelength are used (we prefer to say nodes rather than elements since the mesh is used only for the integration).

This behavior can be better observed in Figure 16a, where the problem is solved considering different values of k (chosen such that the corresponding frequency $w = kc$ is sufficiently far from the resonance frequencies) and keeping kh constant by fixing $h = \frac{\lambda}{10}$. In the case of FEM, we can see how the error increases with k , according to Eq. (26). In fact, the term $k^3 h^2$ gives a preasymptotic range in the convergence of the FE-solution [6] and, on keeping kh constant, its influence is progressively increased for higher k

Figure 16: Evolution of the L_2 relative error norm for kh constant. In (a) the LME and the IGA-LME methods are compared to FEM for $\gamma = 1$, while in (b) the evolution of the error for different values of γ is considered.

values. Therefore, in order to maintain the solution in the asymptotic range of convergence, where the FE-error is close to the approximation error, it is necessary to consider a mesh size such that the term $k^3 h^2$ is kept constant and the problem quickly becomes computationally challenging, even in 2 dimensions. This is not the case of max-ent methods, where the error remains in the order of 10^{-2} for wavenumbers in the region from 1 to 100 m^{-1} . It is a common practice to define a non dimensional wavenumber $\kappa = kL$, where L is a characteristic length of the domain. In this case, if we consider $\kappa = 2Rk$, the max-ent error is not affected by pollution for values of κ up to 200.

In both the LME and the IGA-LME simulations a value of $\gamma = 1$, which is often used in the literature, has been employed. However, if higher values of γ are considered (Figure 16b), the dependence of the error on the wavenumber becomes evident also for max-ent methods. This is not surprising because in the limit case, when $\gamma \rightarrow \infty$, the FEM results would be recovered.

Although from the theoretical point of view an error estimate like the one in Eq. (26) has not yet been found for max-ent methods, the numerical results in Figure 16 suggest that also in this case the performance deteriorates towards higher frequencies, but with the important advantage that the asymptotic range of convergence is maintained in a much larger zone, especially for lower values of γ . In particular, for the reference value $\gamma = 1$, the rule of thumb $h = \frac{\lambda}{10}$ is still sufficient to limit the error for much higher values of κ respect to the FEM.

5. Conclusions

This work investigated the application of max-ent methods to the simulation of acoustic problems. The numerical examples showed that the max-ent basis functions can handle pollution errors better than finite elements. This good behavior was observed both in the computation of the discrete eigen-frequency spectrum of simply-shaped domains and in the computation of the FRF for boundary value problems, where it was found that the resonance frequency shift is introduced at a lot smaller wavelengths with respect to the FEM. On studying the Helmholtz equation for high wavenumbers, it was also found that the rule of thumb of choosing a fixed number of nodes per wavelength can be used in a much larger zone, especially for lower values of the shape parameter γ .

In addition it was shown how to take advantage of the blending between max-ent and isogeometric functions defined on the boundary of the domain. On doing so, the correct geometrical description of the problem can be preserved like in IGA and, thanks to the blending, the discretization of the interior of the domain is straightforward. The extension of this approach to three-dimensional problems, where the volume parametrization of complexly shaped geometries is still a challenge for IGA, would be an interesting topic of future research.

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