The Category-theoretic Solution of Recursive Ultra-metric Space Equations

Amin Timany Bart Jacobs

iMinds-Distrinet KU Leuven

CoqPL – January 23rd, 2016
St. Petersburg, Florida

- In order to define Kripke-style semantics of a higher-order imperative programming language we need to solve equations

$$\mathcal{W} = \mathbb{N} \rightharpoonup_{\text{fin}} \mathcal{T}$$

and

$$\mathcal{T} = \mathcal{W} \to_{\text{mon}} 2^{V}$$

- In order to define Kripke-style semantics of a higher-order imperative programming language we need to solve equations

$$\mathcal{W} = \mathbb{N} \rightharpoonup_{\text{fin}} \mathcal{T}$$

and

$$\mathcal{T} = \mathcal{W} \rightarrow_{\text{mon}} 2^{V}$$

- $\mathcal{W}$ is the set of Kripke worlds (each assigns types to locations)

- In order to define Kripke-style semantics of a higher-order imperative programming language we need to solve equations

$$\mathcal{W} = \mathbb{N} \rightharpoonup_{\text{fin}} \mathcal{T}$$

and

$$\mathcal{T} = \mathcal{W} \rightarrow_{\text{mon}} 2^{V}$$

- $\mathcal{W}$ is the set of Kripke worlds (each assigns types to locations)
- $\mathcal{T}$ is the set of interpretations of types (depends on worlds to determine interpretation of refrences)

- In order to define Kripke-style semantics of a higher-order imperative programming language we need to solve equations

$$\mathcal{W} = \mathbb{N} \rightharpoonup_{\text{fin}} \mathcal{T}$$

and

$$\mathcal{T} = \mathcal{W} \rightarrow_{\text{mon}} 2^{V}$$

- $\mathcal{W}$ is the set of Kripke worlds (each assigns types to locations)
- $\mathcal{T}$ is the set of interpretations of types (depends on worlds to determine interpretation of refrences)
- Impossible due to cardinality issues

- We use step-indexing

$$\widehat{\mathcal{T}} \simeq \blacktriangleright((\mathbb{N} \rightharpoonup_{\text{fin}} \widehat{\mathcal{T}}) \to_{\text{mon}} \mathbb{P}(V))$$

$$\mathbb{P}(V) \triangleq \{p \in n \to 2^V \mid \forall n, v.\ v \in p(n) \to \forall m \leq n.\ v \in p(m)\}$$

- We use step-indexing

$$\widehat{\mathcal{T}} \simeq \blacktriangleright((\mathbb{N} \rightharpoonup_{\text{fin}} \widehat{\mathcal{T}}) \to_{\text{mon}} \mathbb{P}(V))$$

$$\mathbb{P}(V) \triangleq \{p \in n \to 2^{V} \mid \forall n, v.\ v \in p(n) \to \forall m \le n.\ v \in p(m)\}$$

- And define

$$\mathcal{W} \triangleq \mathbb{N} \rightharpoonup_{\text{fin}} \widehat{\mathcal{T}}$$

and

$$\mathcal{T} \triangleq \mathcal{W} \to_{\text{mon}} \mathbb{P}(V)$$

Outline

- Bounded Ultra-metric Space:

  A space A with a distance function $\delta : A \times A \to [0, b]$ such that:

- Bounded Ultra-metric Space:

  A space A with a distance function $\delta : A \times A \rightarrow [0, b]$ such that:
  1. $\forall x, y.\ \delta(x, y) = 0 \Leftrightarrow x = y$

- Bounded Ultra-metric Space:

  A space A with a distance function $\delta : A \times A \to [0, b]$ such that:

  1. $\forall x, y.\ \delta(x, y) = 0 \Leftrightarrow x = y$
  2. $\forall x, y.\ \delta(x, y) = \delta(y, x)$

- Bounded Ultra-metric Space:

A space A with a distance function $\delta : A \times A \to [0, b]$ such that:

1. $\forall x, y.\ \delta(x, y) = 0 \Leftrightarrow x = y$
2. $\forall x, y.\ \delta(x, y) = \delta(y, x)$
3. $\forall x, y, z.\ \delta(x, y) \leq max(\delta(x, z), \delta(y, z))$

- Bounded Ultra-metric Space:

  A space A with a distance function $\delta : A \times A \to [0, b]$ such that:

  1. $\forall x, y.\ \delta(x, y) = 0 \Leftrightarrow x = y$
  2. $\forall x, y.\ \delta(x, y) = \delta(y, x)$
  3. $\forall x, y, z.\ \delta(x, y) \leq max(\delta(x, z), \delta(y, z))$

- δ can be though of as *degree of similarity*

- Bounded Ultra-metric Space:

  A space A with a distance function $\delta : A \times A \to [0, b]$ such that:
  1. $\forall x, y.\ \delta(x, y) = 0 \Leftrightarrow x = y$
  2. $\forall x, y.\ \delta(x, y) = \delta(y, x)$
  3. $\forall x, y, z.\ \delta(x, y) \leq max(\delta(x, z), \delta(y, z))$
- δ can be though of as *degree of similarity*
- An Ultra-metric space is *complete* if every Cauchy sequence $\{a_n\}_{n \in \mathbb{N}}$ converges:

$$\forall \varepsilon > 0.\ \exists N.\ \forall m, k \geq N.\ \delta(a_m, a_k) < \varepsilon$$

- Bounded Ultra-metric Space:

  A space A with a distance function $\delta : A \times A \to [0, b]$ such that:
  1. $\forall x, y.\ \delta(x, y) = 0 \Leftrightarrow x = y$
  2. $\forall x, y.\ \delta(x, y) = \delta(y, x)$
  3. $\forall x, y, z.\ \delta(x, y) \leq max(\delta(x, z), \delta(y, z))$
- δ can be though of as *degree of similarity*
- An Ultra-metric space is *complete* if every Cauchy sequence $\{a_n\}_{n \in \mathbb{N}}$ converges:

$$\forall \varepsilon > 0.\ \exists N.\ \forall m, k \geq N.\ \delta(a_m, a_k) < \varepsilon$$

- Example (bisected distance):

$$\delta : S^{\mathbb{N}} \times S^{\mathbb{N}} \to [0, 1]$$

with

$$\delta(f, g) = \begin{cases} 0 & \text{if } f = g \\ 2^{-max\{n \mid \forall m \leq n.\ f(n) = g(n)\}} & \text{otherwise} \end{cases}$$

forms a complete bounded ultra-metric space

- for (A, δ) and (B, δ'), $f : A \to B$ is *non-expansive* if:

$$\forall x, y : A.\ \delta'(f(x), f(y)) \leq \ \delta(x, y)$$

- for (A, δ) and (B, δ'), $f : A \to B$ is *non-expansive* if:

$$\forall x, y : A.\ \delta'(f(x), f(y)) \leq \ \delta(x, y)$$

- Example:

$$f : S^{\mathbb{N}} \to S^{\mathbb{N}} \ \text{ where } \ f(x)(n) = h(x(n)) \ \text{ for some } \ h : S \to S$$

- for (A, δ) and (B, δ'), $f : A \to B$ is *non-expansive* if:

$$\forall x, y : A.\ \delta'(f(x), f(y)) \leq \ \delta(x, y)$$

- Example:

$$f : S^{\mathbb{N}} \to S^{\mathbb{N}} \ \text{ where } \ f(x)(n) = h(x(n)) \ \text{ for some } \ h : S \to S$$

- for (A, δ) and (B, δ'), $f : A \to B$ is *contractive* if:

$$\forall x, y : A.\ \delta'(f(x), f(y)) \leq c \cdot \delta(x, y) \ \text{ for some } \ 0 \leq c < 1$$

- for (A, δ) and (B, δ'), $f : A \to B$ is *non-expansive* if:

$$\forall x, y : A.\ \delta'(f(x), f(y)) \leq \ \delta(x, y)$$

- Example:

$$f : S^{\mathbb{N}} \to S^{\mathbb{N}} \ \text{ where } \ f(x)(n) = h(x(n)) \ \text{ for some } \ h : S \to S$$

- for (A, δ) and (B, δ'), $f : A \to B$ is *contractive* if:

$$\forall x, y : A.\ \delta'(f(x), f(y)) \leq c \cdot \delta(x, y) \ \text{ for some } \ 0 \leq c < 1$$

- Example:

$$f : S^{\mathbb{N}} \to S^{\mathbb{N}} \ \text{ where } \ f(x)(n) = \begin{cases} a & \text{if } n = 0 \\ x(n-1) & \text{otherwise} \end{cases} \ \text{ for some fixed } \ a \in S$$

- An *M-category* $\mathcal{C}$ is a category such that:

- An *M-category* $\mathcal{C}$ is a category such that:
  - For $A, B \in \mathcal{C}$, $\delta_{A,B} : hom(A, B) \times hom(A, B) \to [0, b]$ makes $hom_{A,B}$ a *complete bounded ultra-metric space*

- An *M-category* $\mathcal{C}$ is a category such that:
  - For $A, B \in \mathcal{C}$, $\delta_{A,B} : hom(A, B) \times hom(A, B) \to [0, b]$ makes $hom_{A,B}$ a *complete bounded ultra-metric space*
  - The composition functions are *non-expansive*

- An *M-category* $\mathcal{C}$ is a category such that:
  - For $A, B \in \mathcal{C}$, $\delta_{A,B} : hom(A, B) \times hom(A, B) \to [0, b]$ makes $hom_{A,B}$ a *complete bounded ultra-metric space*
  - The composition functions are *non-expansive*
- A functor $\mathcal{F}$ on M-categories is *locally non-expansive* (resp. *locally contractive*) if its morphism map is non-expansive (resp. contractive)

- An *M-category* $\mathcal{C}$ is a category such that:
  - For $A, B \in \mathcal{C}$, $\delta_{A,B} : hom(A,B) \times hom(A,B) \to [0,b]$ makes $hom_{A,B}$ a *complete bounded ultra-metric space*
  - The composition functions are *non-expansive*
- A functor $\mathcal{F}$ on M-categories is *locally non-expansive* (resp. *locally contractive*) if its morphism map is non-expansive (resp. contractive)
- An *increasing Cauchy tower* in $\mathcal{C}$ is a diagram

$$A_0 \underset{g_0}{\overset{f_0}{\rightleftarrows}} A_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} A_2 \underset{g_2}{\overset{f_2}{\rightleftarrows}} \cdots$$

such that:

- An *M-category* $\mathcal{C}$ is a category such that:
  - For $A, B \in \mathcal{C}$, $\delta_{A,B} : hom(A,B) \times hom(A,B) \to [0,b]$ makes $hom_{A,B}$ a *complete bounded ultra-metric space*
  - The composition functions are *non-expansive*
- A functor $\mathcal{F}$ on M-categories is *locally non-expansive* (resp. *locally contractive*) if its morphism map is non-expansive (resp. contractive)
- An *increasing Cauchy tower* in $\mathcal{C}$ is a diagram

$$A_0 \underset{g_0}{\overset{f_0}{\rightleftarrows}} A_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} A_2 \underset{g_2}{\overset{f_2}{\rightleftarrows}} \cdots$$

  such that:
  - $g_i \circ f_i = id_{A_i}$

- An *M-category* $\mathcal{C}$ is a category such that:
  - For $A, B \in \mathcal{C}$, $\delta_{A,B} : hom(A,B) \times hom(A,B) \to [0,b]$ makes $hom_{A,B}$ a *complete bounded ultra-metric space*
  - The composition functions are *non-expansive*
- A functor $\mathcal{F}$ on M-categories is *locally non-expansive* (resp. *locally contractive*) if its morphism map is non-expansive (resp. contractive)
- An *increasing Cauchy tower* in $\mathcal{C}$ is a diagram

$$A_0 \underset{g_0}{\overset{f_0}{\rightleftarrows}} A_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} A_2 \underset{g_2}{\overset{f_2}{\rightleftarrows}} \cdots$$

such that:

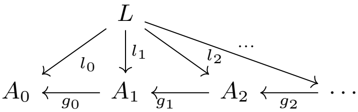
  - $g_i \circ f_i = id_{A_i}$
  - $\lim_{i \to \infty} \delta(f_i \circ g_i, id_{A_{i+1}}) = 0$

- An *M-category* $\mathcal{C}$ is a category such that:
  - For $A, B \in \mathcal{C}$, $\delta_{A,B} : hom(A, B) \times hom(A, B) \to [0, b]$ makes $hom_{A,B}$ a *complete bounded ultra-metric space*
  - The composition functions are *non-expansive*
- A functor $\mathcal{F}$ on M-categories is *locally non-expansive* (resp. *locally contractive*) if its morphism map is non-expansive (resp. contractive)
- An *increasing Cauchy tower* in $\mathcal{C}$ is a diagram

$$A_0 \underset{g_0}{\overset{f_0}{\rightleftarrows}} A_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} A_2 \underset{g_2}{\overset{f_2}{\rightleftarrows}} \cdots$$

such that:

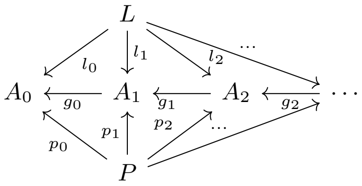
  - $g_i \circ f_i = id_{A_i}$
  - $\lim\limits_{i\to\infty} \delta(f_i \circ g_i, id_{A_{i+1}}) = 0$
- The inverse limit of an increasing Cauchy tower is the (category theoretical limit) of:



- An *M-category* $\mathcal{C}$ is a category such that:
  - For $A, B \in \mathcal{C}$, $\delta_{A,B} : hom(A,B) \times hom(A,B) \to [0,b]$ makes $hom_{A,B}$ a *complete bounded ultra-metric space*
  - The composition functions are *non-expansive*
- A functor $\mathcal{F}$ on M-categories is *locally non-expansive* (resp. *locally contractive*) if its morphism map is non-expansive (resp. contractive)
- An *increasing Cauchy tower* in $\mathcal{C}$ is a diagram

$$A_0 \underset{g_0}{\overset{f_0}{\rightleftarrows}} A_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} A_2 \underset{g_2}{\overset{f_2}{\rightleftarrows}} \cdots$$

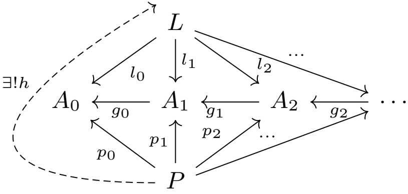
  such that:
  - $g_i \circ f_i = id_{A_i}$
  - $\lim\limits_{i \to \infty} \delta(f_i \circ g_i, id_{A_{i+1}}) = 0$
- The inverse limit of an increasing Cauchy tower is the (category theoretical limit) of:



- An *M-category* $\mathcal{C}$ is a category such that:
  - For $A, B \in \mathcal{C}$, $\delta_{A,B} : hom(A,B) \times hom(A,B) \to [0,b]$ makes $hom_{A,B}$ a *complete bounded ultra-metric space*
  - The composition functions are *non-expansive*
- A functor $\mathcal{F}$ on M-categories is *locally non-expansive* (resp. *locally contractive*) if its morphism map is non-expansive (resp. contractive)
- An *increasing Cauchy tower* in $\mathcal{C}$ is a diagram

$$A_0 \underset{g_0}{\overset{f_0}{\rightleftarrows}} A_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} A_2 \underset{g_2}{\overset{f_2}{\rightleftarrows}} \cdots$$

  such that:
  - $g_i \circ f_i = id_{A_i}$
  - $\lim_{i \to \infty} \delta(f_i \circ g_i, id_{A_{i+1}}) = 0$
- The inverse limit of an increasing Cauchy tower is the (category theoretical limit) of:



Theorem

- *Let*
  - $\mathcal{C}$ *be an M-category*

Theorem

- *Let*
  - $\mathcal{C}$ *be an M-category*
  - *with a terminal object 1*

Theorem

- *Let*
  - $\mathcal{C}$ *be an M-category*
  - *with a terminal object 1*
  - *such that* $\mathcal{C}$ *has inverse limit of all increasing Cauchy towers*

Theorem

- *Let*
  - $\mathcal{C}$ *be an M-category*
  - *with a terminal object 1*
  - *such that* $\mathcal{C}$ *has inverse limit of all increasing Cauchy towers*
  - $\mathcal{F} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ *be a mixed-variance locally contractive functor*

Theorem

- *Let*
  - $\mathcal{C}$ *be an M-category*
  - *with a terminal object 1*
  - *such that* $\mathcal{C}$ *has inverse limit of all increasing Cauchy towers*
  - $\mathcal{F} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ *be a mixed-variance locally contractive functor*
  - *such that* $m : 1 \to F(1, 1)$

Theorem

- *Let*
  - $\mathcal{C}$ *be an M-category*
  - *with a terminal object 1*
  - *such that* $\mathcal{C}$ *has inverse limit of all increasing Cauchy towers*
  - $\mathcal{F} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ *be a mixed-variance locally contractive functor*
  - *such that* $m : 1 \to F(1, 1)$
- *Then,* $\mathcal{F}$ *has a unique fixed point, i.e.,*

$$\exists! A.\ A \simeq F(A, A)$$

- We use the theory to solve

$$\widehat{\mathcal{T}} \simeq \blacktriangleright((\mathbb{N} \rightharpoonup_{\text{fin}} \widehat{\mathcal{T}}) \to_{\text{mon}} \mathbb{P}(V))$$

- We use the theory to solve

$$\widehat{\mathcal{T}} \simeq \blacktriangleright((\mathbb{N} \rightharpoonup_{\text{fin}} \widehat{\mathcal{T}}) \rightarrow_{\text{mon}} \mathbb{P}(V))$$

- In the M-category **CBULt** of

- We use the theory to solve

$$\widehat{\mathcal{T}} \simeq \blacktriangleright((\mathbb{N} \rightharpoonup_{\text{fin}} \widehat{\mathcal{T}}) \rightarrow_{\text{mon}} \mathbb{P}(V))$$

- In the M-category **CBULt** of
  - Objects: (bisected) Complete Bounded Ultra metric spaces

- We use the theory to solve

$$\widehat{\mathcal{T}} \simeq \blacktriangleright((\mathbb{N} \rightharpoonup_{\text{fin}} \widehat{\mathcal{T}}) \to_{\text{mon}} \mathbb{P}(V))$$

- In the M-category **CBULt** of
  - Objects: (bisected) Complete Bounded Ultra metric spaces
  - Morphisms: non-expansive maps

- We use the theory to solve

$$\widehat{\mathcal{T}} \simeq \blacktriangleright((\mathbb{N} \rightharpoonup_{\text{fin}} \widehat{\mathcal{T}}) \to_{\text{mon}} \mathbb{P}(V))$$

- In the M-category **CBULt** of
  - Objects: (bisected) Complete Bounded Ultra metric spaces
  - Morphisms: non-expansive maps
- By constructing a locally contractive functor

$$\mathcal{F}(X, Y) = \blacktriangleright((\mathbb{N} \rightharpoonup_{\text{fin}} X) \to_{\text{mon}} \mathbb{P}(V))$$

- We use the theory to solve

$$\widehat{\mathcal{T}} \simeq \blacktriangleright((\mathbb{N} \rightharpoonup_{\text{fin}} \widehat{\mathcal{T}}) \to_{\text{mon}} \mathbb{P}(V))$$

- In the M-category **CBULt** of
  - Objects: (bisected) Complete Bounded Ultra metric spaces
  - Morphisms: non-expansive maps
- By constructing a locally contractive functor

$$\mathcal{F}(X, Y) = \blacktriangleright((\mathbb{N} \rightharpoonup_{\text{fin}} X) \to_{\text{mon}} \mathbb{P}(V))$$

  - For any complete bounded ultra metric space X

$$\mathbb{N} \rightharpoonup_{\text{fin}} X$$

  is a complete bounded ultra metric space with a partial order relation:

$$f \sqsubseteq g \Leftrightarrow \forall x \in dom(f).\ f(x) = g(x)$$

$$\delta(f, g) = \begin{cases} b & \text{if } dom(f) \neq dom(g) \\ \bigsqcup\limits_{x \in dom(f)} \delta(f(x), g(x)) & \text{otherwise} \end{cases}$$

- $\mathbb{P}(V)$ is a complete bounded ultra metric space with a partial order

$$p \sqsubseteq q \Leftrightarrow \forall n \in \mathbb{N}.\ p(n) \subseteq q(n)$$

$$\delta(p,q) = \begin{cases} 0 & \text{if } p = q \\ 2^{-max\{n \mid \forall m \leq n.\ p(n)=q(n)\}} & \text{otherwise} \end{cases}$$

- $\mathbb{P}(V)$ is a complete bounded ultra metric space with a partial order

$$p \sqsubseteq q \Leftrightarrow \forall n \in \mathbb{N}.\ p(n) \subseteq q(n)$$

$$\delta(p,q) = \begin{cases} 0 & \text{if } p = q \\ 2^{-max\{n | \forall m \leq n.\ p(n) = q(n)\}} & \text{otherwise} \end{cases}$$

- $\mathcal{G}(X,Y) = (\mathbb{N} \rightharpoonup_{\text{fin}} X) \rightarrow_{\text{mon}} \mathbb{P}(V)$ is locally non-expansive

- $\mathbb{P}(V)$ is a complete bounded ultra metric space with a partial order

$$p \sqsubseteq q \Leftrightarrow \forall n \in \mathbb{N}.\ p(n) \subseteq q(n)$$

$$\delta(p,q) = \begin{cases} 0 & \text{if } p = q \\ 2^{-max\{n | \forall m \leq n.\ p(n)=q(n)\}} & \text{otherwise} \end{cases}$$

- $\mathcal{G}(X,Y) = (\mathbb{N} \rightharpoonup_{\text{fin}} X) \to_{\text{mon}} \mathbb{P}(V)$ is locally non-expansive
- $\blacktriangleright(X) : \mathbf{CBULt} \to \mathbf{CBULt}$ is space X with distances halved

- $\mathbb{P}(V)$ is a complete bounded ultra metric space with a partial order

$$p \sqsubseteq q \Leftrightarrow \forall n \in \mathbb{N}.\ p(n) \subseteq q(n)$$

$$\delta(p,q) = \begin{cases} 0 & \text{if } p = q \\ 2^{-max\{n | \forall m \leq n.\ p(n) = q(n)\}} & \text{otherwise} \end{cases}$$

- $\mathcal{G}(X,Y) = (\mathbb{N} \rightharpoonup_{\text{fin}} X) \rightarrow_{\text{mon}} \mathbb{P}(V)$ is locally non-expansive
- $\blacktriangleright(X) : \mathbf{CBULt} \rightarrow \mathbf{CBULt}$ is space X with distances halved
- $\blacktriangleright$ is locally contractive

- $\mathbb{P}(V)$ is a complete bounded ultra metric space with a partial order

$$p \sqsubseteq q \Leftrightarrow \forall n \in \mathbb{N}.\ p(n) \subseteq q(n)$$

$$\delta(p, q) = \begin{cases} 0 & \text{if } p = q \\ 2^{-max\{n \mid \forall m \leq n.\ p(n) = q(n)\}} & \text{otherwise} \end{cases}$$

- $\mathcal{G}(X, Y) = (\mathbb{N} \rightharpoonup_{\text{fin}} X) \rightarrow_{\text{mon}} \mathbb{P}(V)$ is locally non-expansive
- $\blacktriangleright(X) : \mathbf{CBULt} \rightarrow \mathbf{CBULt}$ is space X with distances halved
- $\blacktriangleright$ is locally contractive
- Thus, $\mathcal{F} = \blacktriangleright \circ \mathcal{G}$ is locally contractive

- $\mathbb{P}(V)$ is a complete bounded ultra metric space with a partial order

$$p \sqsubseteq q \Leftrightarrow \forall n \in \mathbb{N}.\ p(n) \subseteq q(n)$$

$$\delta(p,q) = \begin{cases} 0 & \text{if } p = q \\ 2^{-max\{n \mid \forall m \leq n.\ p(n)=q(n)\}} & \text{otherwise} \end{cases}$$

- $\mathcal{G}(X,Y) = (\mathbb{N} \rightharpoonup_{\text{fin}} X) \rightarrow_{\text{mon}} \mathbb{P}(V)$ is locally non-expansive
- $\blacktriangleright(X) : \mathbf{CBULt} \rightarrow \mathbf{CBULt}$ is space X with distances halved
- $\blacktriangleright$ is locally contractive
- Thus, $\mathcal{F} = \blacktriangleright \circ \mathcal{G}$ is locally contractive

- The fix point is $\widehat{\mathcal{T}}$ is uniquely determined:

$$\widehat{\mathcal{T}} \simeq F(\widehat{\mathcal{T}}, \widehat{\mathcal{T}}) = \blacktriangleright((\mathbb{N} \rightharpoonup_{\text{fin}} \widehat{\mathcal{T}}) \rightarrow_{\text{mon}} \mathbb{P}(V))$$

- Implementation:

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\perp$

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\perp$
  - Has a top element $\top$

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\perp$
  - Has a top element $\top$
  - Has meet ($\sqcup$) of arbitrary subsets of X

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\perp$
  - Has a top element $\top$
  - Has meet ($\sqcup$) of arbitrary subsets of X
  - $Appr(X)$ is a subset of X of approximation elements

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\bot$
  - Has a top element $\top$
  - Has meet ($\sqcup$) of arbitrary subsets of X
  - $Appr(X)$ is a subset of X of approximation elements
  - $\forall a.\ a \in Appr(X) \to \bot \sqsubset a$

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\bot$
  - Has a top element $\top$
  - Has meet ($\sqcup$) of arbitrary subsets of X
  - $Appr(X)$ is a subset of X of approximation elements
  - $\forall a.\ a \in Appr(X) \to \bot \sqsubset a$
  - $\forall a.\ \bot \sqsubset a \to \exists b \in Appr(X).\ b \sqsubseteq a$

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\bot$
  - Has a top element $\top$
  - Has meet ($\sqcup$) of arbitrary subsets of X
  - $Appr(X)$ is a subset of X of approximation elements
  - $\forall a.\ a \in Appr(X) \to \bot \sqsubset a$
  - $\forall a.\ \bot \sqsubset a \to \exists b \in Appr(X).\ b \sqsubseteq a$
  - $\forall a.\ (\forall b \in Appr(X).\ a \sqsubset b) \to a = \bot$

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\bot$
  - Has a top element $\top$
  - Has meet ($\sqcup$) of arbitrary subsets of X
  - $Appr(X)$ is a subset of X of approximation elements
  - $\forall a.\ a \in Appr(X) \rightarrow \bot \sqsubset a$
  - $\forall a.\ \bot \sqsubset a \rightarrow \exists b \in Appr(X).\ b \sqsubseteq a$
  - $\forall a.\ (\forall b \in Appr(X).\ a \sqsubset b) \rightarrow a = \bot$
  - $(\forall a \in Appr(X).\ \exists b \in Appr(X).\ b \sqsubset a) \vee (\exists c \in Appr(X).\ \forall a.\ a \sqsubset c \rightarrow a = \bot)$

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\bot$
  - Has a top element $\top$
  - Has meet ($\sqcup$) of arbitrary subsets of X
  - $Appr(X)$ is a subset of X of approximation elements
  - $\forall a.\ a \in Appr(X) \rightarrow \bot \sqsubset a$
  - $\forall a.\ \bot \sqsubset a \rightarrow \exists b \in Appr(X).\ b \sqsubseteq a$
  - $\forall a.\ (\forall b \in Appr(X).\ a \sqsubset b) \rightarrow a = \bot$
  - $(\forall a \in Appr(X).\ \exists b \in Appr(X).\ b \sqsubset a) \vee (\exists c \in Appr(X).\ \forall a.\ a \sqsubset c \rightarrow a = \bot)$
- Bisected spaces can be represented using M-lattice $(\mathbb{B}, \sqsubseteq_{\mathbb{B}})$:

$$\mathbb{B} = \{f : \texttt{nat} \rightarrow \texttt{Prop} \mid \forall m, n.\ m \leq n \rightarrow f(n) \rightarrow f(m)\}$$

$$x \sqsubseteq_{\mathbb{B}} y \ \text{ iff } \ \forall n.\ y(n) \rightarrow x(n)$$

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\bot$
  - Has a top element $\top$
  - Has meet $(\sqcup)$ of arbitrary subsets of X
  - $Appr(X)$ is a subset of X of approximation elements
  - $\forall a.\ a \in Appr(X) \to \bot \sqsubset a$
  - $\forall a.\ \bot \sqsubset a \to \exists b \in Appr(X).\ b \sqsubseteq a$
  - $\forall a.\ (\forall b \in Appr(X).\ a \sqsubset b) \to a = \bot$
  - $(\forall a \in Appr(X).\ \exists b \in Appr(X).\ b \sqsubset a) \vee (\exists c \in Appr(X).\ \forall a.\ a \sqsubset c \to a = \bot)$
- Bisected spaces can be represented using M-lattice $(\mathbb{B}, \sqsubseteq_{\mathbb{B}})$:

$$\mathbb{B} = \{f : \texttt{nat} \to \texttt{Prop} \mid \forall m, n.\ m \leq n \to f(n) \to f(m)\}$$

$$x \sqsubseteq_{\mathbb{B}} y \ \text{ iff } \ \forall n.\ y(n) \to x(n)$$

  - $\bot_{\mathbb{B}} = \texttt{fun \_} \Rightarrow \texttt{True}$

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\bot$
  - Has a top element $\top$
  - Has meet ($\sqcup$) of arbitrary subsets of X
  - $Appr(X)$ is a subset of X of approximation elements
  - $\forall a.\ a \in Appr(X) \rightarrow \bot \sqsubset a$
  - $\forall a.\ \bot \sqsubset a \rightarrow \exists b \in Appr(X).\ b \sqsubseteq a$
  - $\forall a.\ (\forall b \in Appr(X).\ a \sqsubset b) \rightarrow a = \bot$
  - $(\forall a \in Appr(X).\ \exists b \in Appr(X).\ b \sqsubset a) \vee (\exists c \in Appr(X).\ \forall a.\ a \sqsubset c \rightarrow a = \bot)$
- Bisected spaces can be represented using M-lattice $(\mathbb{B}, \sqsubseteq_{\mathbb{B}})$:

$$\mathbb{B} = \{f : \texttt{nat} \rightarrow \texttt{Prop} \mid \forall m, n.\ m \leq n \rightarrow f(n) \rightarrow f(m)\}$$

$$x \sqsubseteq_{\mathbb{B}} y \quad \text{iff} \quad \forall n.\ y(n) \rightarrow x(n)$$

  - $\bot_{\mathbb{B}} = \texttt{fun \_} \Rightarrow \texttt{True}$
  - $\top_{\mathbb{B}} = \texttt{fun \_} \Rightarrow \texttt{False}$

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\bot$
  - Has a top element $\top$
  - Has meet $(\sqcup)$ of arbitrary subsets of X
  - $Appr(X)$ is a subset of X of approximation elements
  - $\forall a.\ a \in Appr(X) \to \bot \sqsubset a$
  - $\forall a.\ \bot \sqsubset a \to \exists b \in Appr(X).\ b \sqsubseteq a$
  - $\forall a.\ (\forall b \in Appr(X).\ a \sqsubset b) \to a = \bot$
  - $(\forall a \in Appr(X).\ \exists b \in Appr(X).\ b \sqsubset a) \lor (\exists c \in Appr(X).\ \forall a.\ a \sqsubset c \to a = \bot)$
- Bisected spaces can be represented using M-lattice $(\mathbb{B}, \sqsubseteq_{\mathbb{B}})$:

$$\mathbb{B} = \{f : \texttt{nat} \to \texttt{Prop} \mid \forall m, n.\ m \leq n \to f(n) \to f(m)\}$$

$$x \sqsubseteq_{\mathbb{B}} y \text{ iff } \forall n.\ y(n) \to x(n)$$

  - $\bot_{\mathbb{B}} =$ `fun _ => True`
  - $\top_{\mathbb{B}} =$ `fun _ => False`
  - for $F : I \to A$, $(\sqcup_{\mathbb{B}} F)(n) = \forall x : I.\ f(x)(n)$

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\bot$
  - Has a top element $\top$
  - Has meet ($\sqcup$) of arbitrary subsets of X
  - $Appr(X)$ is a subset of X of approximation elements
  - $\forall a.\ a \in Appr(X) \to \bot \sqsubset a$
  - $\forall a.\ \bot \sqsubset a \to \exists b \in Appr(X).\ b \sqsubseteq a$
  - $\forall a.\ (\forall b \in Appr(X).\ a \sqsubset b) \to a = \bot$
  - $(\forall a \in Appr(X).\ \exists b \in Appr(X).\ b \sqsubset a) \vee (\exists c \in Appr(X).\ \forall a.\ a \sqsubset c \to a = \bot)$
- Bisected spaces can be represented using M-lattice $(\mathbb{B}, \sqsubseteq_{\mathbb{B}})$:

$$\mathbb{B} = \{f : \texttt{nat} \to \texttt{Prop} \mid \forall m, n.\ m \leq n \to f(n) \to f(m)\}$$

$$x \sqsubseteq_{\mathbb{B}} y \quad \text{iff} \quad \forall n.\ y(n) \to x(n)$$

  - $\bot_{\mathbb{B}} = \texttt{fun \_} \Rightarrow \texttt{True}$
  - $\top_{\mathbb{B}} = \texttt{fun \_} \Rightarrow \texttt{False}$
  - for $F : I \to A$, $(\sqcup_{\mathbb{B}} F)(n) = \forall x : I.\ f(x)(n)$
  - $Appr(\mathbb{B}) = \{(1/2)^n \top_{\mathbb{B}} \mid n \in \mathbb{N}\}$

$$((1/2)f)(n) = \begin{cases} \texttt{True} & \text{if } n = 0 \\ f(n+1) & \text{otherwise} \end{cases}$$

- Implementation:
- Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:
  - Has a bottom element $\bot$
  - Has a top element $\top$
  - Has meet $(\sqcup)$ of arbitrary subsets of X
  - $Appr(X)$ is a subset of X of approximation elements
  - $\forall a.\ a \in Appr(X) \to \bot \sqsubset a$
  - $\forall a.\ \bot \sqsubset a \to \exists b \in Appr(X).\ b \sqsubseteq a$
  - $\forall a.\ (\forall b \in Appr(X).\ a \sqsubset b) \to a = \bot$
  - $(\forall a \in Appr(X).\ \exists b \in Appr(X).\ b \sqsubset a) \vee (\exists c \in Appr(X).\ \forall a.\ a \sqsubset c \to a = \bot)$
- Bisected spaces can be represented using M-lattice $(\mathbb{B}, \sqsubseteq_{\mathbb{B}})$:

$$\mathbb{B} = \{f : \texttt{nat} \to \texttt{Prop} \mid \forall m, n.\ m \leq n \to f(n) \to f(m)\}$$

$$x \sqsubseteq_{\mathbb{B}} y \quad \text{iff} \quad \forall n.\ y(n) \to x(n)$$

  - $\bot_{\mathbb{B}} = \texttt{fun \_ } \Rightarrow \texttt{True}$
  - $\top_{\mathbb{B}} = \texttt{fun \_ } \Rightarrow \texttt{False}$
  - for $F : I \to A$, $(\sqcup_{\mathbb{B}} F)(n) = \forall x : I.\ f(x)(n)$
  - $Appr(\mathbb{B}) = \{(1/2)^n \top_{\mathbb{B}} \mid n \in \mathbb{N}\}$

$$((1/2)f)(n) = \begin{cases} \texttt{True} & \text{if } n = 0 \\ f(n+1) & \text{otherwise} \end{cases}$$

- Distance $\delta : S^{\mathbb{N}} \times S^{\mathbb{N}} \to \mathbb{B}$ in a bisected space (A, δ):

$$\delta(f, g)(n) = \bigwedge_{0 \leq i \leq n} f(n) = g(n)$$

- To represent contractiveness in L, we use a contraction rate $\rho : L \to L$

- To represent contractiveness in L, we use a contraction rate $\rho : L \to L$
  - $\forall x.\ \rho(x) \sqsubseteq x$

- To represent contractiveness in L, we use a contraction rate $\rho : L \to L$
  - $\forall x.\ \rho(x) \sqsubseteq x$
  - $\forall x.\ \bot \sqsubset x \to \rho(x) \sqsubset x$

- To represent contractiveness in L, we use a contraction rate $\rho : L \to L$
  - $\forall x.\ \rho(x) \sqsubseteq x$
  - $\forall x.\ \bot \sqsubset x \to \rho(x) \sqsubset x$
  - $\forall x, y \in Appr(A).\ \exists n \in \mathbb{N}.\ \rho^n(x) \sqsubseteq y$

- To represent contractiveness in L, we use a contraction rate $\rho : L \to L$
  - $\forall x.\ \rho(x) \sqsubseteq x$
  - $\forall x.\ \bot \sqsubset x \to \rho(x) \sqsubset x$
  - $\forall x, y \in Appr(A).\ \exists n \in \mathbb{N}.\ \rho^n(x) \sqsubseteq y$
- Example: $(1/2) : \mathbb{B} \to \mathbb{B}$ is a contraction rate:

$$((1/2)f)(n) = \begin{cases} \texttt{True} & \text{if } n = 0 \\ f(n+1) & \text{otherwise} \end{cases}$$

- To represent contractiveness in L, we use a contraction rate $\rho : L \to L$
  - $\forall x.\ \rho(x) \sqsubseteq x$
  - $\forall x.\ \bot \sqsubset x \to \rho(x) \sqsubset x$
  - $\forall x, y \in Appr(A).\ \exists n \in \mathbb{N}.\ \rho^n(x) \sqsubseteq y$
- Example: $(1/2) : \mathbb{B} \to \mathbb{B}$ is a contraction rate:

$$((1/2)f)(n) = \begin{cases} \texttt{True} & \text{if } n = 0 \\ f(n+1) & \text{otherwise} \end{cases}$$

- For (A, δ) and (B, δ'), $f : A \to B$ we change the contractiveness condition from:

$$\forall x, y : A.\ \delta'(f(x), f(y)) \leq c \cdot \delta(x, y) \quad \text{for some} \quad 0 \leq c < 1$$

to

$$\forall x, y : A.\ \delta'(f(x), f(y)) \sqsubseteq \rho(\delta(x, y)) \quad \text{for some contraction rate } \rho$$

- It is all implemented on top of a general purpose category theory library[1]

[1] `https://github.com/amintimany/Categories`

- It is all implemented on top of a general purpose category theory library[1]
- All category theoretical constructions and facts, e.g., (co)limits, their uniqueness, etc. are taken from there

[1] https://github.com/amintimany/Categories

- It is all implemented on top of a general purpose category theory library[1]
- All category theoretical constructions and facts, e.g., (co)limits, their uniqueness, etc. are taken from there
- M-categories are defined as:

```
Record MCat (L : MLattice) : Type :=
    { MC_Obj : Type;
      MC_Hom : MC_Obj → MC_Obj → (Complete_UltraMetric L);
      MC_compose : forall {a b c : MC_Obj}, NonExpansive
           (product_CUM (MC_Hom a b) (MC_Hom b c)) (MC_Hom a c);
       ...
      MC_Cat :> Category := {| Obj := MC_Obj; Hom := MC_Hom;
          compose := fun _ _ _ x y ⇒ MC_compose (x, y);
          ... |} }.
```

[1] https://github.com/amintimany/Categories

- It is all implemented on top of a general purpose category theory library[1]
- All category theoretical constructions and facts, e.g., (co)limits, their uniqueness, etc. are taken from there
- M-categories are defined as:

```
Record MCat (L : MLattice) : Type :=
    { MC_Obj : Type;
      MC_Hom : MC_Obj → MC_Obj → (Complete_UltraMetric L);
      MC_compose : forall {a b c : MC_Obj}, NonExpansive
            (product_CUM (MC_Hom a b) (MC_Hom b c)) (MC_Hom a c);
       ...
      MC_Cat :> Category := {| Obj := MC_Obj; Hom := MC_Hom;
          compose := fun _ _ _ x y ⇒ MC_compose (x, y);
          ... |} }.
```

- Primitive projections guarantee that `MC_Cat` projection of an M-category $\hat{\mathcal{C}}$ constructed out of a category $\mathcal{C}$ is definitionally equal to $\mathcal{C}$

[1] https://github.com/amintimany/Categories

- It is all implemented on top of a general purpose category theory library[1]
- All category theoretical constructions and facts, e.g., (co)limits, their uniqueness, etc. are taken from there
- M-categories are defined as:

```
Record MCat (L : MLattice) : Type :=
    { MC_Obj : Type;
      MC_Hom : MC_Obj → MC_Obj → (Complete_UltraMetric L);
      MC_compose : forall {a b c : MC_Obj}, NonExpansive
           (product_CUM (MC_Hom a b) (MC_Hom b c)) (MC_Hom a c);
       ...
      MC_Cat :> Category := {| Obj := MC_Obj; Hom := MC_Hom;
          compose := fun _ _ _ x y ⇒ MC_compose (x, y);
          ... |} }.
```

- Primitive projections guarantee that `MC_Cat` projection of an M-category $\hat{\mathcal{C}}$ constructed out of a category $\mathcal{C}$ is definitionally equal to $\mathcal{C}$
- We can use all facts about $\mathcal{C}$ on $\hat{\mathcal{C}}$

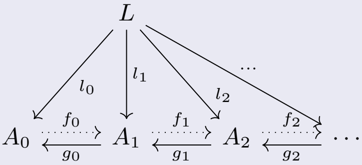
[1] https://github.com/amintimany/Categories

- Similarly for locally-contractive functors

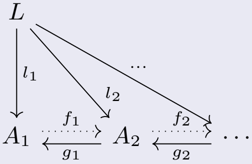
```
Record LocallyContractive {L : MLattice} (M M' : MCat L) : Type :=
    {
      LCN_FO : M → M';
      LCN_ContrRate : ContrRate L;
      LCN_FA : forall {a b}, Controlled_Contractive LCN_ContrRate
           (MC_Hom M a b) (MC_Hom M' (LCN_FO a) (LCN_FO b));
      ...
      LCN_Func :> Functor M M' :=
        {|
          FO := LCN_FO; FA := @LCN_FA;
          ...
        |}
    }.
```

Lemma (1)

If

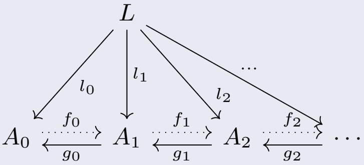


is a limit diagram, so is

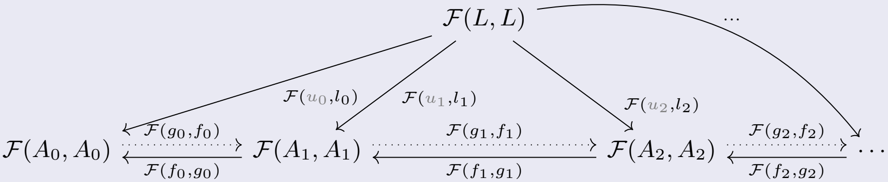


Lemma (2)

If $\mathcal{F} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ *be a mixed-variance locally contractive functor and*



is a limit diagram, so is



Proof of Theorem (existence).

Note that the following is an increasing Cauchy tower and has a limit in $\mathcal{C}$

$$1 \underset{g_0 = !_{F(1,1)}}{\overset{f_0 = m}{\rightleftarrows}} F(1,1)$$

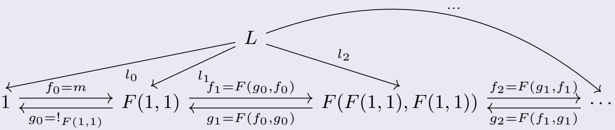
Proof of Theorem (existence).

Note that the following is an increasing Cauchy tower and has a limit in $\mathcal{C}$

$$1 \underset{g_0 = !_{F(1,1)}}{\overset{f_0 = m}{\rightleftarrows}} F(1,1) \underset{g_1 = F(f_0, g_0)}{\overset{f_1 = F(g_0, f_0)}{\rightleftarrows}} F(F(1,1), F(1,1)) \underset{g_2 = F(f_1, g_1)}{\overset{f_2 = F(g_1, f_1)}{\rightleftarrows}} \cdots$$

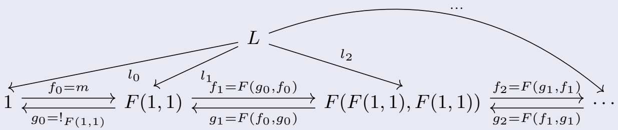
Proof of Theorem (existence).

Note that the following is an increasing Cauchy tower and has a limit in $\mathcal{C}$

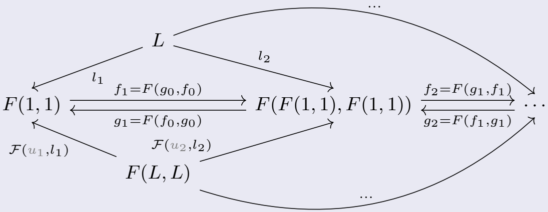


Proof of Theorem (existence).

Note that the following is an increasing Cauchy tower and has a limit in $\mathcal{C}$

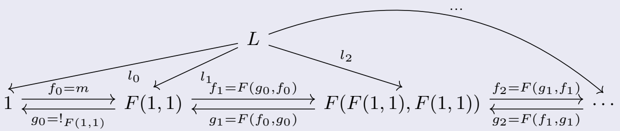


By Lemma 1 and Lemma 2

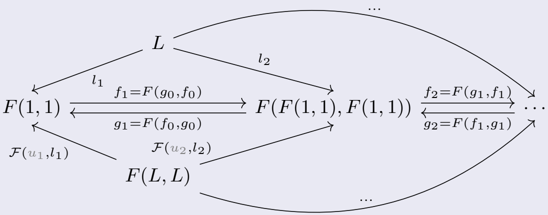


Proof of Theorem (existence).

Note that the following is an increasing Cauchy tower and has a limit in $\mathcal{C}$



By Lemma 1 and Lemma 2



By uniqueness of limits we have $L \simeq F(L, L)$ $\square$

- Available on: https://github.com/amintimany/CTDT

Thanks!