Visualising the Boolean Algebra $\mathbb{B}_4$ in 3D

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Abstract. This paper compares two 3D logical diagrams for the Boolean algebra $\mathbb{B}_4$, viz. the rhombic dodecahedron and the nested tetrahedron. Geometric properties such as collinearity and central symmetry are examined from a cognitive perspective, focussing on diagram design principles such as congruence/isomorphism and apprehension.

Keywords: logical geometry, rhombic dodecahedron, nested tetrahedron, congruence, apprehension, central symmetry, Boolean algebra.

Introduction. Logical geometry systematically studies Aristotelian and related logical diagrams, focussing on both abstract-logical and visual-geometrical topics (cf. www.logicalgeometry.org). A major visual-geometrical issue is the fact that a single logical structure often gives rise to different visualisations. These diagrams are *informationally* equivalent, but they need not be *computationally* equivalent [3]: visual differences may significantly influence user comprehension. This paper presents a case study on this issue: we take one logical structure (viz. the Boolean algebra $\mathbb{B}_4$) and compare two 3D visualisations (viz. rhombic dodecahedron and nested tetrahedron) in the light of general principles of diagram design. The outcome of this comparison is a nuanced perspective: both diagrams are useful visualisations of $\mathbb{B}_4$; whichever one is ultimately adopted will depend on which logical properties of $\mathbb{B}_4$ the diagram author wants to highlight.

The Boolean algebra $\mathbb{B}_4$ will be represented by means of bitstrings of length 4, i.e. we take $\mathbb{B}_4 = \{0,1\}^4$. A bitstring's *level (L)* is defined as the number of bit positions with value 1; e.g. 1100 and 1101 are of L2 and L3, respectively. The Aristotelian relations in $\mathbb{B}_4$ are defined as follows: b_1 and b_2 are *contradictory (CD)* iff $b_1 \wedge b_2 = 0000$ and $b_1 \vee b_2 = 1111$; they are *contrary (C)* iff $b_1 \wedge b_2 = 0000$ and $b_1 \vee b_2 \neq 1111$, they are *subcontrary (SC)* iff $b_1 \wedge b_2 \neq 0000$ and $b_1 \vee b_2 = 1111$, and they are in subalternation iff $b_1 \wedge b_2 = b_1$ and $b_1 \vee b_2 \neq b_1$.

This Boolean algebra can be visualised using a *rhombic dodecahedron (RDH)* [5]. RDH has been used both as a Hasse diagram and as an Aristotelian diagram for $\mathbb{B}_4$; cf. Fig. 1(a–b) [1]. The second visualisation of $\mathbb{B}_4$ is the *nested tetrahedron (NTH)* in Fig. 1(c–d) [2,4]. Because a tetrahedron is *self-dual*, connecting the centres of its 4 faces yields a small, 'nested' tetrahedron.

Representing Levels. In a Hasse diagram of a Boolean algebra, the levels are visualised as (horizontal) hyperplanes that are orthogonal to the general (vertical) implication direction. This is a clear instance of the *Congruence Principle*, according to which the structure of the visualisation should correspond to the represented logical structure [7]. To what extent does this principle apply to the visualisation of levels in RDH/NTH? Fig. 1(e–j) shows the L1/L2/L3 distribution in RDH and NTH. It is clear that neither RDH nor NTH explicitly visualises the levels as (horizontal) hyperplanes orthogonal to one (vertical) implication direction. Still, NTH observes the Congruence Principle much better than RDH, albeit in a different way: levels no longer correspond to parallel hyperplanes ordered along one (vertical) dimension, but rather to the geometrical dimensions themselves. The natural geometrical ordering of 0-, 1- and 2-dimensionality for vertices, edges and faces thus corresponds to the logical ordering of the levels L1, L2 and L3.[1] By contrast, RDH is not level-preserving at all: levels do not correspond to parallel hyperplanes, and not to dimensionality either.

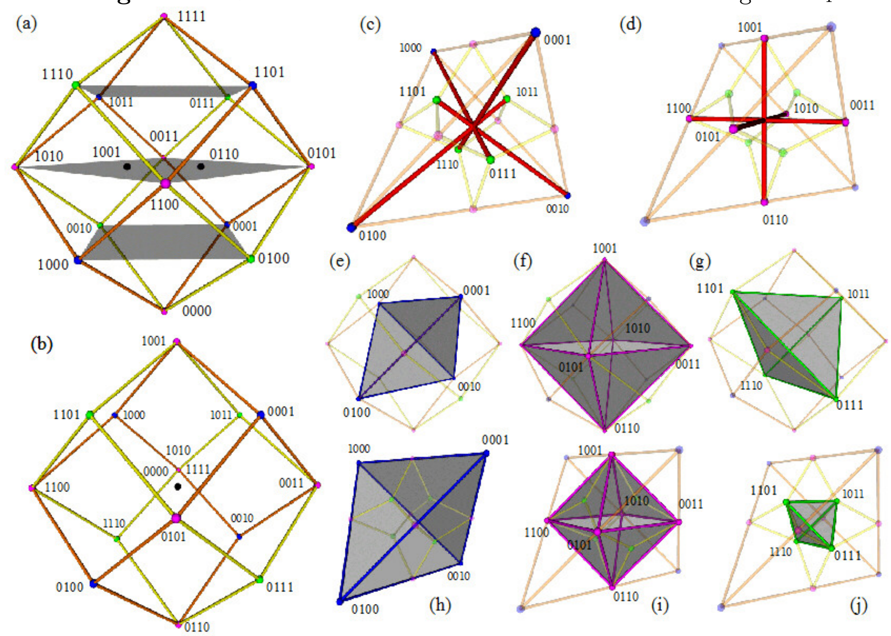
Representing Contradiction. Since the contradiction relation is symmetric and functional, Aristotelian diagrams often have the property of *central symmetry*: contradictory bitstrings are located at diametrically opposed vertices of the diagram and at the same distance from its centre. This is another clear instance of the Congruence Principle. To what extent does this principle also apply to the visualisation of contradictions in RDH/NTH? As to contradictory pairs of L2/L2 bitstrings, central symmetry holds in both RDH and NTH. However, as to contradictory pairs of L1/L3 bitstrings, central symmetry holds in RDH, but not in NTH: Fig. 1(c) shows that the L3 vertices are located at a much shorter distance from the centre than their contradictory L1 counterparts. Hence, NTH exhibits a lower degree of overall logico-geometrical congruence.

Contradiction is often argued to be the *strongest* opposition relation, in the sense that turning a bitstring into its contradictory involves switching the values in *all* of its bit positions. Because of the Congruence Principle, the idea of 'maximal logical distance' is sometimes visualised by means of 'maximal geometrical distance': the vertex that is farthest removed from the vertex representing a bitstring b is the vertex representing $\neg b$. Fig. 1(b) shows that this property is perfectly satisfied in RDH: vertices corresponding to contradictory bitstrings are systematically located at a maximal distance removed from one another. By contrast, Fig. 1(c–d) shows that in NTH, vertices corresponding to contradictory bitstrings are systematically *not* located at a maximal distance removed from one another. Hence, NTH exhibits a much lower degree of congruence.

Collinear Vertices. NTH contains several triples of collinear vertices. For example, the bitstrings 1000, 1001 and 0001 all lie on the top edge of NTH in Fig. 1(h–i). As to the Aristotelian relations between these bitstrings, we have a contrariety between 1000 and 0001, and subalternations from 1000 and 0001 to 1001. Since the vertices of these three bitstrings are collinear, the visualisation of the contrariety overlaps/coincides with the visualisations of the two subalternations. As a result of this overlap, the user looking at NTH might have difficulties in properly distinguishing the three distinct Aristotelian relations holding between 1000, 1001 and 1001. This is a serious violation of the *Apprehension Principle*, according to which "the structure and content of the visualization should

[1] Each L2/L3 bitstring is situated at the *midpoint* of its corresponding edge/face.

Fig. 1. The RDH and NTH visualisations of the Boolean Algebra $\mathbb{B}_4$.



be readily and accurately perceived and comprehended" [7, p. 37]. By contrast, RDH does not have any triples of collinear vertices. Consequently, all Aristotelian relations holding between the elements of $\mathbb{B}_4$ are visualised by means of distinct (non-coinciding) lines. This systematic avoidance of overlapping visual elements means that RDH is much more in accordance with the Apprehension Principle.

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