Analysis of the effect of uncertain clamping stiffness on the dynamical behaviour of structures using interval field methods

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Abstract. In uncertainty calculation, the inability of interval parameters to take into account mutual dependency is a major shortcoming. When parameters with a geometric perspective are involved, the construction of a model using intervals at discrete locations not only increases the problem dimensionality unnecessarily, but it also assumes no dependency whatsoever, including unrealistic parameter combinations leading to results that probably overestimate the true uncertainty. The concept of modelling uncertainty with a geometric aspect using interval fields eliminates this problem by defining basis functions and expressing the uncertain process as a weighted sum of these functions. The definition of the functions enables the model to take into account geometrically dependent parameters, whereas the coefficients in a non-interactive interval format represent the uncertainty. This paper introduces a new type of interval field specifically tailored for geometrically oriented uncertain parameters, based on a maximum gradient condition to model the dependency. This field definition is then applied to a model of a clamped plate with uncertain clamping stiffness with the purpose of identifying the effects of spatial variability and mean value separately.

Introduction

General concept of uncertainty quantification. Uncertainty Quantification (UQ) is an aspect of design that all engineers encounter at some point during their working life. It lies at the base of producing reliable products. Reliability analysis aims at coming to quantitative conclusions on the likelihood of a product to perform its intended action successfully. In practice, this comes down to identifying all possible configurations of a product and its environment that will lead to failure of the product and determining the likelihood of occurrence of such an event. For a designer, both the product and its working environment is subject to all kinds of uncertainties. Some of these are inherent to the non-deterministic character of nature, others are due to a lack of knowledge on the product or the environment. UQ attempts to find the uncertain response characteristics of any product given information on the uncertain nature of it.

In numerical modelling, UQ comes down to numerically determining uncertainty on the output quantities of a model, given the uncertainty on the model input or parameters. More specifically, in FE analysis, uncertainty can be present in the model parameters, e.g. the element mass and stiffness values, but also in the model input, i.e. load or excitation characteristics. For numerical UQ to be successful, the modelling of the uncertainty should accurately represent the uncertainty present in real life, and the numerical model needs to propagate it accurately to obtain reliable uncertainty results at the output.

Uncertainty can be quantified in many ways, which can be categorized in two groups. Stochastic methods use stochastic parameters such as variance and mean values to model non-deterministic variables. Although well-established in FE analysis (see e.g. [1],[2]), their application in industrial context remains limited because of the increased computational cost of a non-deterministic analysis, but more dominantly because of the high degree of knowledge required to perform it. Usually, stochastic parameters are expensive to obtain and assumptions on the probability distribution are necessary in most cases. Possibilistic methods model uncertainty on model input or model parameters using possibilistic parameters, of which intervals, fuzzy numbers and convex regions are the most common.

An interval models uncertainty on a variable as two extreme bounds which are not to be exceeded. A fuzzy number is an extension to this concept, by considering the membership of a parameter to a certain interval as a continuous function, given by a membership function μ that ranges from 0 to 1. A value of 1 indicates the value is certainly part of the interval, whereas a value of 0 indicates that it is definitely not. Convex regions define an uncertainty region in a multidimensional space and they attempt to capture uncertainty on parameter combinations by convex shapes such as ellipses. An overview of convex regions is given by [3].

In general, possibilistic modelling of uncertainty requires less extensive knowledge, but consequently, UQ using possibilistic methods provides less information. However, acquiring reliable knowledge on the bounds of the response of a system is of great interest to a designer. Possibilistic methods therefore provide a low-threshold analysis method for UQ compared to stochastic methods.

Non-deterministic fields. When uncertain parameters with a spatial character are involved (i.e. geometric shape uncertainties or material properties), care must be taken. In FE analysis, such a parameter is discretized to element level, leading to localized values in each element. Physically, some degree of dependence always exist within the discretized set. For this, a field representation $x(\mathbf{r})$ is suitable, which describes the field variable x as a function of a spatial co-ordinate $\mathbf{r}$. A single field realisation is then comprised of the value of the variable x in each point of the spatial domain considered. Non-deterministic field representations are possible as well. The theory of random fields (RF) [4] defines fields in a probabilistic way, with each field realisation to be a function of both the spatial co-ordinate and some well-specified random parameters. Next to RF, interval fields (IF) were introduced by Moens et al.[5] to model non-deterministic field variables in a possibilistic context. In its simplest form, interval fields are defined as a sum of weighted basis functions, in which the weights are interval parameters. The general equation of an IF is given in equation (1):

$$x^I(\mathbf{r}) = \sum_{i=0}^n \alpha_i^I \cdot \phi_i(\mathbf{r}). \quad (1)$$

The functions $\phi_i(\mathbf{r})$ are deterministic functions of the spatial co-ordinate $\mathbf{r}$, α_i^I are simple interval parameters. The objective of interval fields is to define an uncertain set of field realisations that are physically plausible, i.e. obey some dependency criterion. The use of IF can be exploited fully if the definition of α_i^I and $\phi_i(\mathbf{r})$ is done in such a way that the interval parameters are all non-interactive. In this case, the dependency in the field is captured exclusively by the basis functions $\phi_i(\mathbf{r})$, whereas the uncertainty is captured only by the interval parameters α_i^I . This separation enables efficient uncertainty propagation throughout all steps of the analysis to the final output quantities. [6] considers an analysis with a single interval parameter to model the interval field. However, when more interval parameters are necessary to model an uncertain field with some degree of correlation between parameters, a decomposition has to be applied which ensures the interval parameters are truly non-interactive.

This paper will introduce a general IF definition that enables an analyst to model knowledge which he/she possibly has on the uncertainty of the application as straightforwardly as possible, while at the same time ensures this strict division of uncertainty and dependency between the α_i^I and $\phi_i(\mathbf{r})$. Once constructed, the interval parameters α_i^I will span a non-interactive interval space, but the field $x^I(\mathbf{r})$ will nevertheless obey a dependency criterion, defined by a bound on the maximum gradient the field variable can show. As an illustration, this field will be applied to a clamped plate model with uncertain clamping conditions, which will be propagated to obtain intervals on the plate's natural frequencies. This will illustrate the possibility to introduce a maximum gradient and show the effect on the set of field realisations.

The paper exclusively discusses intervals to model uncertainty. Throughout this work, an interval parameter x^I with upper bound $\bar{x}$ and lower bound $\underline{x}$ is represented in two ways: using the bounds themselves: $x^I = \langle \underline{x} | \bar{x} \rangle$ or using the centre point $x_c = \frac{\bar{x} + \underline{x}}{2}$ and the interval radius $r_x = \frac{\bar{x} - \underline{x}}{2}$: $x^I = \langle x \rangle_{x_c}^{r_x}$

Uncertainty definition

The starting point of any uncertainty analysis is the knowledge of the analyst. Being an expert at a specific application or product, the analyst may have some knowledge on the uncertainty present. The first section introduces the global uncertainty parameters which is used further on to define an IF decomposition. A second section discusses in more detail the mathematical background of the presented field decomposition method.

Definition of global uncertainty parameters. Consider a field variable $x(\mathbf{r})$. The value in each point is expressed as the sum of the mean field value μ_x and the deviation from the mean value in that point:

$$x(\mathbf{r}) = \mu_x + x(\mathbf{r}) - \mu_x = \mu_x + s_x(\mathbf{r}) \quad (2)$$

where $\mu_x = \frac{1}{\Omega} \int_{\Omega} x(\mathbf{r}) d\Omega$ and Ω a measure for the range of the field domain (i.e. the length, surface, or volume). When the field variable is uncertain, both the mean value and the deviation are subject to uncertainty. Equation (2) is then left unchanged, with the superscript I added to the variables to indicate they are now interval parameters:

$$x^I(\mathbf{r}) = \mu_x^I + s_x^I(\mathbf{r}) \quad (3)$$

In this description, $\mu_x^I = \langle \mu_x | \overline{\mu_x} \rangle$ is a simple interval parameter, but $x^I(\mathbf{r})$ and $s_x^I(\mathbf{r})$ are both interval fields, i.e. a set of interactive intervals in each point of the field domain. For the deviation it is assumed that its absolute value is bounded by a specified value $s_{x,max}$. This indicates the field variable itself never exceeds the interval $\langle \mu_x - s_{x,max} | \overline{\mu_x} + s_{x,max} \rangle$ in any point. To introduce the spatial dependency within the field, it is assumed that the difference between two adjacent points of the field is bounded as well, as most physical parameters normally exhibit some smooth evolution. This obliges the field values in adjacent points to vary within reasonable limits. To be able to express this property in a continuous field, *the gradient of the field is constrained*. This gradient constraint is converted to an interval with its centre at 0. The uncertainty is assumed to be homogeneous with respect to the spatial dimension, leading to an interval on the gradient, the magnitude of which is constant over the entire domain. The bounds are given by a single value $\left. \frac{\partial x}{\partial \mathbf{r}} \right|_{max}$ leading to the interval $\langle -\left. \frac{\partial x}{\partial \mathbf{r}} \right|_{max} | \left. \frac{\partial x}{\partial \mathbf{r}} \right|_{max} \rangle$.

In summary, the interval field $x^I(\mathbf{r})$ in this section has four major parameters defining the uncertainty:

1. The *lower bound on the mean field value* $\underline{\mu_x}$.
2. The *upper bound on the mean field value* $\overline{\mu_x}$.
3. The maximum absolute value of the *deviation from the mean value* $s_{x,max}$.
4. The maximum absolute value of the *gradient of the field* $\left. \frac{\partial x}{\partial \mathbf{r}} \right|_{max}$.

Implications on local uncertainty in FE meshes. When a field is applied in FE models, the field is discretized to the nodes and elements of the FE model. Normally, the points of the field can be chosen to correspond to the nodes of the FE model. It is important to mention that the definition of a non-zero maximum first derivative will lead to an interval of non-zero size in each point of the field. *The dimensionality of the uncertainty is equal to the number of points in the field*, as the interval parameters in each point are not fully dependent of each other (this would mean that the value in one point completely determines the value in another point). However, they are not fully independent either. The field decomposition presented below respects the dimensionality of the uncertainty and only tackles the dependency within the field.

The Local Interval Field Decomposition

Summary of objectives. The starting point for this field decomposition is the explicit interval field description introduced by Moens et al. Equation (1) expresses the field $x^I(\mathbf{r})$ as a weighted sum of

basis functions, in which the weights are interval parameters. To adapt the field description to the definition of the global uncertainty parameters from the previous section, the IF definition of equation (1) is slightly altered to:

$$x^I(\mathbf{r}) = \mu_x^I + \sum_{i=0}^n \alpha_i^I \cdot \phi_i(\mathbf{r}). \quad (4)$$

Next to this, we define the set of field realisations X_s as:

$$X_s = \left\{ \tilde{x}(\mathbf{r}) \left| \tilde{x}(\mathbf{r}) = \tilde{\mu}_x + \sum_{i=0}^n \tilde{\alpha}_i \cdot \phi_i(\mathbf{r}), \forall \tilde{\alpha}_i \in \alpha_i^I, \forall \tilde{\mu}_x \in \mu_x^I \right. \right\}. \quad (5)$$

The analyst may specify the four previously defined global uncertainty parameters, which impose conditions on the field that have to be met. Each field realisation that is part of X_s has to obey the following conditions:

1. $\underline{\mu}_x \leq \int_{\Omega} \tilde{x}(\mathbf{r}) d\mathbf{r} = \tilde{\mu}_x \leq \overline{\mu}_x$
2. $\forall \mathbf{r} \in \Omega : \tilde{x}(\mathbf{r}) - \tilde{\mu}_x \leq s_{x,max}$
3. $\forall \mathbf{r} \in \Omega : \tilde{\mu}_x - \tilde{x}(\mathbf{r}) \geq -s_{x,max}$
4. $-\left. \frac{\partial x}{\partial \mathbf{r}} \right|_{max} \leq \frac{\partial \tilde{x}(\mathbf{r})}{\partial \mathbf{r}} \leq \left. \frac{\partial x}{\partial \mathbf{r}} \right|_{max}$

In addition, the interval parameters α_i^I need to be *non-interactive* without violating this conditions for any field realisation $\tilde{x}(\mathbf{r})$ withi the uncertain set. The remainder of this section discusses the definition of the field decomposition for a *1D spatial domain*, i.e. a variable which can vary along a line in space. This will suffice for the application on the clamped plate model. However, all properties of the field can be extended to 2D and 3D domains, at the cost of increased mathematical complexity. The spatial co-ordinate will be indicated by the variable y throughout the remainder of this paper.

Definition of basis functions. In equation (4), $\phi_i(\mathbf{r})$ are deterministic basis functions. For the purpose of propagating the interval field to obtain the output uncertainty, it is beneficial that the interval parameters α_i^I span a non-interactive interval space, since existing uncertainty propagation methods are based on non-interactive interval spaces. The basis functions possess the following properties to ensure this condition is met:

1. All ϕ_i are piecewise second order polynomial functions so their first spatial derivative is continuous (see figure (1)).
2. A single ϕ_i is positioned at the center point y_i of each element of the FE mesh, centered on this node (see figure (2)).
3. All ϕ_i are identically shaped
4. All ϕ_i satisfy $\int_0^{L_y} \phi_i(y) dy = 0$

This leads to basis functions of equation (6):

$$\phi_i(y, y_i) = f(|y - y_i|, a, R). \quad (6)$$

In this equation, a is referred to as the *influence strenght* of the field, which determinies the ability of an individual point to influence the value in nearby points. It directly determines the maximum gradient bound. R is the *influence radius* and determines the distance over which a point can influence the value in other points. The basis functions $\phi_i(y)$ are defined in each point along the field, as illustrated in figure (2). Figure (2) shows that the basis functions have a local character: beyond a radius R their

Figure 1: Definition of a basis function and its first derivative.

Figure 2: Definition of a basis function and its first derivative.

Figure 3: Superposition of 4 basis functions to produce a single field realisation.

ability to change the field value vanishes, hence the name *Local Interval Field Decomposition* (LIFD). Using this basis functions, the field is expressed as

$$x^I(y) = \langle \mu_x \rangle_{C_\mu}^{R_\mu} + \sum_{i=1}^{N_k} \langle \beta_i \rangle_0^1 \cdot \phi_i(y, a, R). \quad (7)$$

The field description of equation (7) puts an interval $\langle \beta_i \rangle_0^1$ in each node/element of the FE mesh and therefore it *does not reduce the dimensionality of the uncertainty*. However, *these intervals can now be considered as non-interactive while still obeying the maximum gradient condition*, which was the initial goal of this decomposition method. The $\langle \beta_i \rangle_0^1$ combined with $\langle \mu_x \rangle_{C_\mu}^{R_\mu}$ now span a $(n + 1)$ -dimensional non-interactive uncertainty region. Figure (3) illustrates how field realisations are created from random parameter selection within this hypercubic space. In figure (3), the dotted lines represent basis functions in four points with a random scaling factor within the interval $\langle -a|a \rangle$. Their sum is given by the solid line.

Non-deterministic analysis case: clamped plate

The purpose of this numerical example is to illustrate the use of LIFD on a clamped plate FE model with an uncertain clamping stiffness K_θ . For a few values of the maximum gradient, the uncertain clamping stiffness set will be visualised and the effect on the uncertainty on the natural frequencies is assessed. We examine a homogenous thin square plate clamped at one edge. The FE model consists of 10x10 2D-plate elements. A clamping stiffness value is defined in each element along the clamped edge, leading to a total of 10 clamping stiffness values. Since the LIFD targets spatial uncertainty, we assume the clamping stiffness in each element can be different, and the maximum gradient condition enables the introduction of dependency within the uncertain set. It is common to express the clamping stiffness on a logarithmic scale, so the LIFD is applied as expressed by equation (8). As the influence strenght a serves as a scaling factor, it is taken out of the summation.

$$K_\theta^I(y) = \langle \mu_K \rangle_{C_\mu}^{R_\mu} \cdot \left(1 + \sum_{i=1}^{N_k} \langle \beta_i \rangle_0^1 \cdot \phi_i(y, a, R) \right) = \langle \mu_K \rangle_{C_\mu}^{R_\mu} \cdot \left(1 + a \sum_{i=1}^{N_k} \langle \beta_i \rangle_0^1 \cdot \phi_i(y, R) \right) \quad (8)$$

Global uncertainty parameters	Reference case 0	analysis case 1	analysis case 2	analysis case 3	analysis case 4	analysis case 5
μ_K^I	$\langle 10^1 10^2 \rangle$	$\langle 10^1 10^2 \rangle$	$\langle 10^1 10^2 \rangle$	$\langle 10^1 10^2 \rangle$	$\langle 10^1 10^2 \rangle$	$\langle 10^1 10^2 \rangle$
$\frac{1}{\mu} \frac{\partial K_\theta}{\partial y}$	0	3.682	4.044	5.048	6.428	10
$\frac{K_\theta - \mu_K}{\mu_K} \max$	0.9	0.9	0.9	0.9	0.9	0.9

Table 1: global uncertainty parameters for each analysis case.

Corresponding LIFD parameters	Reference case 0	analysis case 1	analysis case 2	analysis case 3	analysis case 4	analysis case 5
a	0	0.1841	0.2022	0.2524	0.3214	0.5
R	-	0.5	0.4	0.3	0.2	0.1

Table 2: LIFD parameters for each analysis case.

Corresponding to this definition, the maximum gradient condition is normalized to obtain bounds on $\frac{1}{\mu} \frac{\partial K_\theta}{\partial y}$. It can be shown that $\left(\frac{1}{\mu} \frac{\partial K_\theta}{\partial y} \right)_{\max} = 20a$, so a choice of the influence strength immediate sets the maximum gradient condition. The influence radius determines the maximum deviation from the mean value. We aim at performing a study on the gradient alone, as a measure for the quality of the clamping stiffness. For this, we assume an interval on the mean value of $\mu_K^I = \langle 10^1 | 10^2 \rangle$ or $\mu_{\log K}^I = \langle 1 | 2 \rangle$. Starting from this, we examine five different values for the maximum gradient. In each case, the influence radius is chosen in such a way that the deviation from the mean value remains constant, so the only global uncertainty parameter that is variable in this analysis is the maximum gradient condition. The five analysis cases are listed in table (1) and (2), including a reference case where $\left(\frac{1}{\mu} \frac{\partial K_\theta}{\partial y} \right)_{\max} = 0$, corresponding to a constant clamping stiffness value.

Figure (6) illustrates the uncertain set of clamping stiffness configurations for each of the cases by plotting 50 randomly generated samples. Each of the cases define a set of clamping stiffness configurations $\{K_\theta(y)\}$ for which we will determine intervals on the first 10 natural frequencies of the plate. Since the maximum gradient is monotonically increasing, it is important to notice that every new case expands the uncertain set and that therefore all configurations part of a set corresponding to a certain maximum gradient are also part of all sets corresponding to a larger maximum gradient. It is therefore impossible that the intervals on the natural frequencies decrease in size when the cases 0 to 5 are analysed. Figure (5) illustrates this.

Initial simulations show that the natural frequencies are monotonically increasing with respect to increasing mean value. We therefore only consider the extreme values of the mean value. A varying clamping stiffness extends the intervals, so for each case different from the reference case, two optimisation problems are solved: at $\mu_K = 10^2$ to find the configurations that provide maximum frequency values, and at $\mu_K = 10^1$ to find configurations that provide minimum frequency values. Since these optimisation problems are not necessarily monotonous, global optimisation is used, based on a constant-step gradient based optimisation method. The starting configuration in each case is a constant clamping stiffness value equal to the mean value.

The results of the optimisation routines are given in figure (4), which shows the intervals on the 10 first natural frequencies for each of the 6 cases. Figure (5) shows the increase in interval size compared to the reference case for each of the other cases. The initial increase in maximum gradient means a significant increase in interval size for all frequencies. After that; further increase is of much smaller influence. An exception to this seems to be the 10th frequency, for which the interval size continues to grow with increasing maximum gradient.

intervals on each of the natural frequencies for the 6 analysis cases from table 1

Figure 4: interval sizes for the 10 lowest natural frequencies for all analysis cases.

Figure 5: increase in interval sizes for the 10 lowest natural frequencies compared to the reference case.

some randomly generated samples for each analysis case

Figure 6: randomly generate randomly generated samples for each analysis case.

Figure 7: increase in interval sizes for the 10 lowest natural frequencies compared to the reference case.

Figure 8: clamping stiffness configurations that produce the extreme values for the interval on the 10th frequency.

To further analyse this observation, we examine the clamping stiffness configurations that produce the interval bounds on the 10th natural frequency and compare them to the corresponding mode shape. Figure (7) shows this mode shape, determined for a clamping stiffness equal to zero. At the supported edge, there are points of maximal angular displacement and points of no angular displacement. Introducing a clamping stiffness partially prohibits this angular displacement. From figure (8), we see that the maximum frequency bound is given by the clamping stiffness configuration that, given the bounds on the maximum gradient, *prohibits the displacement given by the mode shape the most*. This configuration exhibits large values at points of large angular displacement and smaller values at points of small angular displacement. Inversely, the minimum frequency bound is given by the configuration that provides the smallest influence on the displacement of the mode shape. This configuration exhibits smaller values at points of large angular displacement and large values at points with smaller angular displacement. This explains why the effect of increasing maximum gradient is mostly felt at high frequencies, as these frequencies correspond to high-order modes, with possibly multiple wavelengths present in the mode shape. The extreme configurations can only be part of the uncertain set for higher allowable gradients, whereas for lower-order modes, the increase in frequency is of no influence as the extreme configurations are already part of the uncertain set for lower maximum gradients.

Conclusion

This paper quantifies effects of uncertainty on the clamping stiffness of a clamped plate with possible spatially varying clamping stiffness value. Using the LIFD, which is based on the explicit interval definition, a non-interactive uncertainty region is formed which still obeys a certain maximum gradient condition. By this definition, the effect of the maximal gradient could be separated from influence of the mean value and the deviation of the mean value. For different values of the maximum gradient, the uncertainty was quantified by defining intervals on the natural frequencies of the plate using a basic gradient-based optimization routine. It was shown that the clamping stiffness configurations that produce these bounds are related to the corresponding mode shape. The lower and upper frequency bound for each mode shape are determined by the configurations that have the smallest and the greatest effect respectively on the mode shape.

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