Deterministic fractal models for transport properties, inspired by $d = 2$ random walks

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(Received 29 November 1988; revised manuscript received 27 June 1989)

We introduce a class of deterministic ultrametric fractal models in $d = 2$, which are expected to mimic some dynamic properties of random walks. The relative diffusion and dc conduction problems are solved exactly, showing both universal and nonuniversal regimes, as already found in simpler $d = 1$ hierarchical structures. For a natural choice of parameters, the model's spectral dimension takes the Alexander-Orbach value $\frac{4}{3}$, which was also conjectured for random walks in $d = 2$. The problem of self-avoiding walks on these structures is also briefly discussed.

I. INTRODUCTION

During the last few years, interest in the dynamics of fractal structures was to a large extent motivated by the observation that the spectral dimension of the infinite incipient cluster (IIC) of percolation is numerically very close to $\frac{4}{3}$ in all dimensions d .¹ Even if numerical evidence seems now to disfavor an exact satisfaction of the above conjecture,² this issue stimulated a lot of speculation about what would be the basic prerequisite of a random or deterministic fractal structure, in order to possess the spectral dimension $\bar{d} = \frac{4}{3}$, predicted in Ref. 1.

Arguments have been given that $\bar{d} = \frac{4}{3}$ could hold for percolation on Cayley trees or in high-dimensional regular lattices, and that, in general, fractal structures possessing this property should be "homogeneous."³ On the other hand, a picture has also been proposed in which the above criterion should fail as soon as the geometrical fractal dimension $\bar{d}$ of the structure is less than 2. For percolation in $d = 2$, e.g., a different relation, $\bar{d} = 2\bar{d}/(\bar{d} + 1)$, should hold.⁴

A common feature of all the above arguments is their rather speculative nature. All present ideas about possible mechanisms determining $\bar{d}$, and leading to $\bar{d} = \frac{4}{3}$ are very inadequate. Even in the context of deterministic fractal models, to our knowledge, structures possessing $\bar{d} = \frac{4}{3}$ have not yet been proposed.

The main purpose of this paper is to present and to solve examples of such deterministic fractal structures. We have indeed constructed a whole class of models, depending on a free parameter, the spectral dimension $\bar{d}$ being nonuniversal. $\bar{d}$ furthermore attains the value $\frac{4}{3}$ for a rather natural choice of the free parameter. We think that these examples are instructive in several respects and constitute a sort of counterexample to some of the previous speculations mentioned above.^{3,4} We hope that our study will represent an interesting step for future at-

tempts towards a better understanding of fractals obeying $\bar{d} = \frac{4}{3}$.

A further motivation of our study in this paper comes from recent results obtained by the present authors for the dynamics of lattice random walks in $d = 2$.⁵ In spite of the fact that the fractal structures generated by random walks are space filling for $d = 2$ ($\bar{d} = d = 2$), one finds⁵ evidence for a nontrivial dynamics with $\bar{d} = \frac{4}{3} \neq 2$. This somehow contrasts a naive picture which would imply $\bar{d} = d$ for a multiply connected space-filling structure. Of course, the claim that random walks in $d = 2$ have an anomalous $\bar{d} = \frac{4}{3}$ implies a specific choice of length and time scales on which one decides to test the dynamics of their structure. By studying, e.g., diffusion on a very long random walk for a relatively short time, one should recover normal diffusion, because an extremely long walk visits with probability one each site of a $d = 2$ lattice. Similarly, the asymptotic behavior for large times on a finite random walk will be trivial. The point is that between these two limits there is a whole range of intermediate time and length scales where the fractal structure of the random walk determines its dynamical properties. These intermediate scales are expected to control the global properties of the structure such as, e.g., the end-to-end resistance. Indeed, if the intermediate fractal region is large enough and spreads to the scale of the whole walk in the limit of very long walks, it will dominate the global properties. The end-to-end resistance will then be described by an exponent related to the anomalous diffusion through the Einstein relation.

The class of deterministic models presented is indirectly inspired by the above findings for random walks in $d = 2$ and it has the same feature of possessing a space-filling fractal structure. In some sense, we are here trying to repeat for the random-walk fractal the same steps that were done for explaining some properties of the IIC (Ref. 6) by constructing a simple deterministic structure

embedding some salient features of the original problem. Furthermore, the model studied here has the interesting feature of representing a nontrivial realization of ultrametricity in two dimensions, in the same way as other, more familiar problems realize ultrametricity within a one-dimensional context.⁷⁻⁹ Up to now, exact renormalization-group approaches for ultrametric models in $d=2$ were possible only for those which trivially reduce to the one-dimensional case.⁸ Our example provides thus a new result in this field.

After introducing our model in Sec. II and presenting an asymptotic solution of the diffusion problem by a renormalization technique in Sec. III, we also calculate in Sec. IV a set of recursive equations for the resistivity problem on our fractals. Besides constituting a relatively more simple approach, this also allows us to explicitly verify that the Einstein relation,¹⁰ linking these two problems, is exactly verified over the whole range of nonuniversal exponents in our class of models. Section V contains a discussion of the results and a comparison between them and those obtained in Ref. 5 by the same authors about clusters generated by random walks.

II. CONSTRUCTION OF THE MODELS

Our aim was to construct a deterministic model which, one could hope, would possess the essential properties that show the same anomalous dynamical behavior as was observed in two-dimensional random walks.⁵ We therefore looked at the patterns that were formed by some large, but finite, random walks on a regular lattice (see Fig. 1). These patterns seem to be composed of many densely visited regions of various sizes, interconnected by narrow links, with holes and missing links in between. This scheme, moreover, seems to repeat itself over many length scales. Led by these considerations, we looked for deterministic fractals incorporating the same succession

FIG. 1. Typical structure generated by a random walk (here of 40 000 steps) on the square lattice.

of dense regions and narrow links, hoping to be able to calculate exactly some dynamical properties, like diffusion rate or resistivity, on these fractals.

The iterative construction of the first such model is shown in Fig. 2. This structure is certainly space filling as far as the visited sites are concerned, but it contains many missing links, making the diffusion much more difficult than on a regular lattice. The resistivity analysis of this model, however, is somewhat trivial. Indeed, the resistance between two points that lie, e.g., diagonally on the structure can easily be seen to grow linearly with their distance, leading immediately to the result $\zeta=1$ for the scaling exponent ζ defined by

$$\Omega(r) \sim r^\zeta, \quad (2.1)$$

where $\Omega(r)$ is the resistance between two points at a distance r . The triviality arises from the fact that this fractal has the structure of a Cayley tree, when we forget about the small loops which are all of the same minimal size. Nevertheless, the diffusion on this fractal is not so trivial, and it may be calculated by a decimation renormalization procedure similar to the one described in Sec. III. Let it be sufficient to mention that the result is indeed $\bar{d} = \frac{4}{3}$, in full agreement with the Einstein relation¹⁰

$$\zeta = d_w - \bar{d}, \quad (2.2)$$

where¹

$$d_w = 2\bar{d}/\bar{d} \quad (2.3)$$

represents the diffusion dimensionality through

$$\langle r^2(t) \rangle \simeq t^{2/d_w}. \quad (2.4)$$

In Fig. 3 we present the iterative construction of a new model, in which at each step four blocks of the previous step are connected with one another to make a larger block. Since there is still some variety in the way in which one performs these connections, we actually have a class of models.

The links on the structure represent either resistances, in the resistivity problem, or paths along which a particle may diffuse, in the diffusion problem. Obviously the resistances of the links should be inversely proportional to the transition rates for the diffusion. The elementary units of the structure are simple squares with a given resistance R_a along the sides and infinite resistance along the diagonal. This means that we have some transition

FIG. 2. Iterative construction of a space-filling structure with $\bar{d} = \frac{4}{3}$.

FIG. 3. Iterative construction of a class of models, depending on a family of parameters $\{b^{(n)}\}$.

rate a along the sides and zero transition rate along the diagonal. The links connecting these elementary units may be subdivided according to the order in which they appear in the iterative construction of the fractal. We may thus introduce resistances $R_b^{(n)}$ or transition rates $b^{(n)}$ with $n=1,2,3,\dots$ for the properties along these links. The most simple choice might seem to take $b^{(n)}=b=a$ for all n , which would correspond to the problem of a regular lattice with missing links. Keeping in mind the pictures of our random-walk samples, however, we want to leave open the possibility of having longer links between the blocks at higher steps of the construction. The essential property of these links should be that they have a larger resistance or a smaller transition rate than the bonds of the elementary units.

III. THE DIFFUSION EQUATION

We will now solve the diffusion problem on the models presented in Fig. 3 and characterized by the transition rates a and $b^{(n)}$, through a decimation procedure. By this procedure we will map a block, constructed from four square units linked together, onto a single square (see Fig. 4), thus decimating out 12 from the original 16 sites of the block. If we repeat this decimation n times, we will reduce the linear size of our blocks in the n th step from 2^n-1 units to $2^{n-1}-1$ units, which means that asymptotically the lengths are rescaled by a factor 2.

We have numbered the sites of our original block from 1 to 16 in such a way that sites numbered from 1 to 4 are saved by the decimation procedure (see Fig. 4). We suppose that the transition rates for the site occupation probabilities inside this block are given by a , b , and c (see Fig. 5), where a and b are regular nearest-neighbor transition

FIG. 4. Scheme of the renormalization procedure for the diffusion equation, by decimation of sites 5 through 16.

FIG. 5. Transition rates for the diffusion ($b_1=0$ if this link is absent).

rates (the b being equal to $b^{(1)}$), and c is a diagonal transition rate. On the initial fractal we suppose $c=0$, but since it acquires a nonzero value through the renormalization procedure, we will include it in our calculation. Furthermore, sites 1 through 4 may or may not have an outgoing link with a transition rate $b^{(n)}$ (with $n > 1$), depending on their position in the fractal. We will indicate this transition rate by b_1 , which should be zero if the link is missing.

Let $P_k(t)$ indicate the probability for finding the particle at site k at time t . These quantities obey the master equation

$$\frac{dP_k}{dt} = \sum_j W_{kj}(P_j - P_k), \quad (3.1)$$

where $W_{ij}=W_{ji}$ denote the already mentioned transition rates. We define $X_k(\lambda)$ as the Laplace transform of the $P_k(t)$. From (3.1) these $X_k(\lambda)$ obey a set of 16 equations, which may easily be constructed from Fig. 4. Using the definition

$$p = 2a + c \quad (3.2)$$

and denoting with $X_k^0(\lambda)$ the Laplace transform of the occupation probability of an external neighbor to site k ($k=1,\dots,4$), we have, e.g., for site 1

$$(\lambda + p + b_1)X_1 = a(X_5 + X_{12}) + cX_{13} + b_1X_1^0 \quad (3.3)$$

and similar equations for the other sites. A straightforward elimination procedure of the functions X_5 through X_{16} can always be normalized to end up with the original value of a in equations such as

$$(\lambda' + p' + b'_1)X_1 = a(X_2 + X_4) + c'X_3 + b'_1X_1^0, \quad (3.4)$$

where

$$p' = 2a + c'. \quad (3.5)$$

Since a remains constant, we may arbitrarily choose it to be equal to unity from now on: $a=1$. The renormalized constants λ' , b' , and c' are determined by the following equations:

$$b^{(n)'} = \gamma b^{(n+1)}, \quad (3.6)$$

$$c' = \frac{b(1+c)}{2(1+c)+b(3+c)} + O(\lambda), \quad (3.7)$$

$$\lambda' = 4\gamma\lambda, \quad (3.8)$$

$$\gamma = \frac{(1+c)+b(2+c)}{b(1+c)} + O(\lambda). \quad (3.9)$$

One should remark that, together with their transformation, the b 's shift down one step in their eventual hierarchy of the $\{b^{(n)}\}$ values. Since we are interested in the asymptotic ($t \rightarrow \infty$) behavior of the $X_k(t)$, we expanded Eqs. (3.6)–(3.9) up to the lowest order in λ .

It is immediately clear from Eq. (3.6) that, if initially all $b^{(n)}$ are equal, they will finally be renormalized to infinity. In this case, we obtain a fixed point

$$b^* = \infty, \quad (3.10)$$

$$c^* = \sqrt{2} - 1, \quad (3.11)$$

and, from Eq. (3.8), follows in this case

$$\left[\frac{\lambda'}{\lambda} \right]^* = 4 + 2\sqrt{2}. \quad (3.12)$$

Since each step in our decimation procedure corresponds to a length rescaling by a factor 2, we expect from (2.4) that this implies a rescaling of the time by a factor 2^{d_w} , and thus a rescaling of its conjugate variable λ by 2^{-d_w} or

$$d_w = \frac{\ln(\lambda'/\lambda)}{\ln 2}. \quad (3.13)$$

In this case $d_w = 2.77$ and, from (2.3), $\bar{d} = 1.44$.

Another possibility is to start from a model, in which the $b^{(n)}$ are not equal to one another, but become smaller with growing n , indicating in this way some lengthening of the effective diffusion path along these links. If the hierarchy of $b^{(n)}$ values is chosen properly, one might obtain any desired finite fixed point value for b^* . In this case we have

$$c^* = \frac{[2(b^*)^2 + 4b^* + 1]^{1/2} - b^* - 1}{b^* + 2}, \quad (3.14)$$

$$\left[\frac{\lambda'}{\lambda} \right]^* = \frac{2(c^* + 1)}{c^*}. \quad (3.15)$$

In order to obtain these values we should, at least asymptotically for large n , have started with a situation where

$$b^{(n)} \simeq b f^{-n} \quad (3.16)$$

with

$$f = (c^* + 1)/2c^*. \quad (3.17)$$

From these formulas, it is clear that, when b^* goes from 0 to ∞ , c^* increases monotonically from 0 to $\sqrt{2} - 1$ and f from ∞ to $1 + \sqrt{2}/2$.

If in Eq. (3.16) f is smaller than $f_c = 1 + \sqrt{2}/2$, then b always iterates to $b^* = \infty$ and c to $c^* = \sqrt{2} - 1$.

Since our renormalization steps correspond to a rescaling of the lengths by a factor 2, we have, in general,

$$d_w = \begin{cases} 2 + \ln f / \ln 2 & (f \geq f_c) \\ 2 + \ln f_c / \ln 2 & (f < f_c) \end{cases} \quad (3.18)$$

$$d_w = \begin{cases} 2 + \ln f / \ln 2 & (f \geq f_c) \\ 2 + \ln f_c / \ln 2 & (f < f_c) \end{cases} \quad (3.19)$$

The two dynamical regimes described by (3.18) and (3.19) are qualitatively similar to those found in more simple one-dimensional realizations of ultrametric dynamics.^{8,9} It means that we have a whole class of models with nonuniversal diffusion dimensions $2.77 \leq d_w \leq \infty$. One special case immediately draws our attention. The $b^{(n)}$ are diffusion strengths on links between clusters with a linear dimension 2^n . In realistic fractal structures grown by some random procedures, a good guess would be that links between such clusters would themselves have lengths of the order 2^n . This can be modeled in our scheme by making the diffusion strengths $b^{(n)}$ shrink by factors $f = 2$. This special and obvious choice leads to the values $b^* = 4$, $c^* = \frac{1}{3}$, $(\lambda'/\lambda)^* = 8$, $d_w = 3$, and it gives us the Alexander-Orbach value $\bar{d} = \frac{4}{3}$.

IV. RESISTIVITY

An alternative approach to the transport properties of fractal structures is the direct determination of the exponent ζ , defined in Eq. (2.1). In this section we will show how the class of models proposed in Sec. II lend themselves to an exact calculation of this exponent.

After the n th step of the iterative construction of our models, we obtain a square structure with a linear dimension of $2^n - 1$ unit lengths. Let us denote by Ω_n and ω_n the diagonal, respectively, the lateral resistance on such a square. In the first stage, e.g., we have for $n = 1$ the simple square in Fig. 6. With unit resistors on the sides of the square, $\Omega_1 = 1$ is the resistance between sites A and C (or between B and D), whereas $\omega_1 = \frac{3}{4}$ is the resistance between sites A and B . We will now derive two recursive formulas between the Ω_n and ω_n , from which ζ may be determined.

In Fig. 7 we have drawn the structure obtained with $n = 4$. According to the diffusion rates chosen in Sec. III, we suppose that we have unit resistors everywhere, except on the b links, where we allow a hierarchy of values R_n . Eventually, we will assume them to behave asymptotically like

$$R_n \simeq f R_{n-1}, \quad (4.1)$$

in agreement with the supposition (3.16) in the diffusion problem.

If we apply a voltage between two diagonal points, e.g.,

FIG. 6. Elementary square with resistances $\Omega_1 = 1$ between A and C and $\omega_1 = \frac{3}{4}$ between A and B .

FIG. 7. Structure corresponding to $n=4$.

A and C , no current will flow through the sections that link equipotential points. Such points may easily be determined from symmetry considerations. If we then eliminate all such sections, we are left with the network represented in Fig. 8, from which we may conclude that the resistance between A and C is given by $\frac{1}{2}\Omega_3 + 1 + \omega_1 + \omega_2 + R_1 + R_2 + R_3$. This argument can immediately be generalized and yields the result

$$\Omega_{n+1} = \frac{1}{2}\Omega_n + 1 + \sum_{j=1}^{n-1} \omega_j + \sum_{j=1}^n R_j. \quad (4.2)$$

On the other hand, if we apply in Fig. 7 a voltage between two lateral points like A and B and like D and C , the intermediate points indicated by G and G' will be at equal potential and may thus be connected to one another. By symmetry, the resistance ω_4 between A and B will then be twice the resistance between A and G . This resistance can be represented by the Wheatstone scheme in

FIG. 8. Same as Fig. 7, but with all sections deleted, which connected equipotential points, given an applied voltage between A and C .

Fig. 9, where the XXY triangle is the equivalent of the n th block held at its three corners A , E , and F . In order to have the correct values Ω_n and ω_n between these corners, the resistances X and Y must have the values

$$X = (4\omega_n - \Omega_n)/2, \quad (4.3)$$

$$Y = X\Omega_n / (2\omega_n - \Omega_n), \quad (4.4)$$

and directly from Fig. 7 we can see that

$$U = R_n/2, \quad (4.5)$$

$$V = 3R_n/2 + \Omega_n. \quad (4.6)$$

A standard calculation of the Wheatstone bridge of Fig. 9 then tells us that

$$\omega_{n+1} = 2 \frac{(U^{-1} + X^{-1})(V^{-1} + X^{-1}) + Y^{-1}(U^{-1} + V^{-1} + 2X^{-1})}{\frac{U^{-1} + X^{-1}}{VX} + \frac{V^{-1} + X^{-1}}{UX} + \frac{2X^{-1}(U^{-1} + V^{-1})}{Y}}. \quad (4.7)$$

FIG. 9. Equivalent scheme for half the resistance between points A and B in Fig. 7.

Equations (4.2)–(4.7) allow to completely determine all ω_n and Ω_n in an iterative way. Furthermore, from the asymptotic behavior (4.1) of R_n , one can see that for all f both Ω_n and ω_n scale like $2^{n\zeta}$ with

$$\zeta = \begin{cases} \ln f / \ln 2 & (f \geq f_c) \\ \ln f_c / \ln 2 & (f < f_c) \end{cases} \quad (4.8)$$

$$\zeta = \begin{cases} \ln f / \ln 2 & (f \geq f_c) \\ \ln f_c / \ln 2 & (f < f_c) \end{cases}. \quad (4.9)$$

Comparing this with our result for d_w from Sec. III, this confirms that the Einstein relation Eq. (2.2) holds for the whole class of models.

V. DISCUSSION

We have presented a class of fractal models that are space filling ($\bar{d}=2$) but that present some lacunarity in the links between the occupied sites. They all have the property that they consist of a hierarchy of clusters of increasing sizes which are multiply connected. The different members of this class of models differ only in the hierarchy of strengths of the links connecting these clusters. Two dynamical properties, namely the time diffusion and the electrical resistivity, have been studied in an exact way and their scaling properties have been determined. These properties are described by exponents d_w and ζ , respectively, which show a nonuniversal behavior. Their difference, however, equals $\bar{d}$ in a universal way, as predicted by the so-called Einstein relation.¹⁰

An immediate consequence of the nonuniversality of d_w is that the Alexander-Orbach conjecture¹ does not hold for our whole class of models. Nevertheless, this conjecture is fulfilled for the special case, where the strength of the links between the clusters (the b links) scales in exactly the same way as the size of the clusters. It should be remarked also that in the simplest possible model, corresponding to the choice $f=1$ (which means that, before starting the decimation, all the occupied links are equivalent to one another), the diffusion dimensionality $d_w=2.77$ differs by less than 10% from the Alexander-Orbach value $d_w=3$. This model has some similarity with an anisotropic model studied by Dhar,¹¹ where one obtained $\bar{d}=\frac{3}{2}$ or $d_w=2.67$.

In order to facilitate the decimation calculation, we have interpreted the b links as single links between neighboring sites, with resistances $R_n \propto f^n$ or diffusion rates $b^{(n)} \propto f^{-n}$. When comparing this model with the pictures obtained for the random walk (as in Fig. 1), one sees that these links are actually composed from longer, almost one-dimensional chains between clusters. In the resistance calculation it is obvious that a chain of N unit resistances is equivalent to a single resistance of value N , just like N parallel unit links between two sites are equivalent to a single resistance of value $1/N$. This last case in the diffusion problem gives rise naturally to an N -fold diffusion rate, so that it should be also natural to assign a diffusion rate $1/N$ to a chain of length N . If, however, one performs the decimation procedure on such a chain, it turns out that the situation is not so clear. This procedure introduces on the corners of the blocks waiting

times that depend on the nature of the sites [this means that in Eq. (3.4) λ is to be replaced by $\lambda\tau_j$ with τ_j depending on the site number j]. These waiting times prohibit an easy analysis of the equations, and one might only hope that they will not change the essential part of the conclusions drawn from them. This should also be guaranteed if the Einstein relation remains valid, as we expect. On the other hand, introducing chains consisting of f^n links does not change the fractal dimension $\bar{d}=2$, as long as the Euclidian distances spanned by these chains do not grow faster than 2^n . For random walks, where we suppose that $f=2$ is the most natural choice, the maximum length of the chains is 2^n , but most likely these lengths will only be of the order of the square root of this value.

In order to obtain a more complete understanding of the conditions for the validity of the generalized Alexander-Orbach conjecture, a more comprehensive study will be necessary. We hope that our class of models at least shed some new light on this subject.

Let us end with an interesting side remark. In a previous paper⁵ we proposed an expression of the Flory-type for d_{SAW} , the fractal dimension of a self-avoiding walk (SAW) on a fractal,

$$d_{\text{SAW}} = \frac{2 + \bar{d}}{2(1 + d_w^{-1})}. \quad (5.1)$$

In order to test the validity of this equation, we have applied it to our models, and at the same time we calculated this dimensionality d_{SAW} by an iterative procedure.¹² This shows that the Flory approximation indeed gives a fairly good estimate to the true value. As an example, we mention the values for $f=1$, which is the most unambiguous with respect to the interpretation of the chains or b links. In this case one can prove that exactly $d_{\text{SAW}}=1.43$, whereas the Flory approximation gives 1.47, which is an accuracy of a few percent. For $f=2$ (random walk) the agreement is somewhat less satisfactory: d_{SAW} is 1.50 on the basis of (5.1), while the actual value should remain 1.43.

ACKNOWLEDGMENTS

One of us (A.S.) acknowledges support of Unità di Padova, Gruppo Nazionale di Struttura della Materia del Consiglio Nazionale delle Ricerche e Centro Interuniversitario di Struttura della Materia, Padova, Italy.

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