Children's Understanding of the Addition/Subtraction Complement Principle

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Abstract

Background. In the last decades, children's understanding of mathematical principles has become an important research topic. Different from the commutativity and inversion principles, only few studies have focused on children's understanding of the addition/subtraction complement principle (if $a - b = c$, then $c + b = a$), mainly relying on verbal techniques.

Aim. This contribution aims at deepening our understanding of children's knowledge of the addition/subtraction complement principle, combining verbal and non-verbal techniques.

Sample. Participants were 67 third- and fourth-graders (9-10-year-olds).

Methods. Children solved two tasks in which verbal reports as well as accuracy and speed data were collected. These two tasks differed only in the order of the problems and the instructions. In the looking back task children were told that sometimes the preceding problem might help to answer the next problem. In the baseline task no helpful preceding items were offered. The looking back task included 10 trigger-target problem pairs on the complement relation.

Results. Children verbally reported looking back on about 40% of all target problems in the looking back task; the target problems were also solved faster and more accurately than in the baseline task. These results suggest that children used their understanding of the complement principle. The verbal and non-verbal data were highly correlated.

Discussion. This study complements previous work on children's understanding of mathematical principles by highlighting inter-individual differences in 9-10-year-olds' understanding of the complement principle and indicating the potential of combining verbal and non-verbal techniques to investigate (the acquisition of) this understanding.

Children's Understanding of the Addition/Subtraction Complement Principle

In goals of elementary mathematics education all over the world, the importance of children learning to solve mathematical tasks in varied and flexible ways has been increasingly stressed over the last decades. To be flexible, children should use various meaningfully acquired strategies, based on their understanding of numbers, operations and mathematical principles and relationships (Anghileri, 1999; Baroody & Dowker, 2003; Kilpatrick, Swafford, & Findell, 2001; Verschaffel, Greer, & De Corte, 2007). As a consequence, children's knowledge of mathematical principles and the use of strategies based on the understanding of these principles have become important research topics (e.g., Baroody, Torbeyns, & Verschaffel, 2009; Verschaffel, Bryant, & Torbeyns, 2012). We extended this research by investigating children's understanding of the addition/subtraction complement principle combining verbal and non-verbal techniques.

Addition/subtraction related principles

In the domain of addition and subtraction, six important arithmetic principles are distinguished (Kilpatrick et al., 2001), namely (a) the *additive commutativity principle*, or the insight that changing the order of the addends does not change the result ($a + b = b + a$); (b) the *subtractive negation principle*, or the understanding that any collection (any number n) reduced by itself leaves nothing ($n - n = 0$); (c) the *subtractive identity principle*, prescribing that if nothing is removed from a collection (for any number n minus zero) its cardinal value remains unchanged ($n - 0 = n$); (d) the *addition/subtraction inversion principle*, or recognizing that adding a number (b) to a collection (a) can be undone by subtracting the same number (b) and vice versa ($a + b - b = a$ or $a - b + b = a$); (e) the *addition/subtraction complement principle*, referring to the insight that if $a + b = c$, then $c - b = a$ and $c - a = b$ or that the difference $c - b = ?$ can be efficiently determined by considering what can be added to b to make c ; (f) the *addition/subtraction associativity principle*, or the understanding that it does

not matter in what order the addition and subtraction operations are performed as long as the sequence of the operands is not changed ($a + b - c = [b - c] + a$).

Although children's understanding of the commutativity and inversion principle has received a lot of research attention (e.g., Bisanz, Watchorn, Piatt, & Sherman, 2009; Canobi, Reeve, & Pattison, 1998; Gaschler, Vaterrodt, Frensch, Eichler, & Haider, 2013; Gilmore & Bryant, 2006, Gilmore & Papadatou-Pastou, 2009; Gilmore & Spelke, 2008; Godau, Haider, Hansen, Schubert, & Gaschler, 2014; Piatt, Matejko, Watchorn, & Bisanz, 2011; Prather & Alibali, 2009; Rasmussen, Ho, & Bisanz, 2003; Robinson & Dubé, 2009; Siegler & Stern, 1998), only few studies investigated children's understanding of the other principles, particularly the addition/subtraction complement principle (e.g., Baroody, 1999; Canobi, 2004, 2005, 2009; Dowker, 2014). This is surprising because a good understanding of the addition/subtraction complement principle is assumed to enhance mathematical proficiency, specifically the application of the *subtraction by addition* strategy (Baroody et al., 2009).

The subtraction by addition strategy involves determining how much needs to be added to the subtrahend to get to the minuend (e.g., solving $81 - 79 = .$ by counting 79,..., 80, 81, so the answer is 2; solving $75 - 43$ by decomposing it into smaller problems such as $43 + 30 = 73$ and $73 + 2 = 75$, so the answer is $30 + 2 = 32$; or solving by $81 - 79 = .$ by retrieving the complementary addition fact, such as responding 2 because one immediately knows that $79 + 2 = 81$). Adults use this strategy frequently, efficiently and flexibly (e.g., Campbell, 2008; Geary, Frensch, & Wiley, 1993; LeFevre, DeStefano, Penner-Wilger, & Daley, 2006), but children hardly report using it (e.g., Heinze, Marschick, & Lipowsky, 2009; Selter, 2001; Verschaffel, Torbeyns, De Smedt, Peters, & Ghesquière, 2010). This limited use in children has been hypothetically explained by a lack of understanding of the addition/subtraction complement principle: If children do not understand that one part-part-whole combination refers both to the components of a subtraction problem $a - b = c$ and its complementary

addition problem $c + b = a$, they may not be able to discover or use this subtraction by addition strategy (e.g., Baroody, 2003; Baroody et al., 2009; De Smedt, Torbeyns, Stassens, Ghesquière, & Verschaffel, 2010; Verschaffel et al., 2010).

Investigating children's understanding of addition/subtraction related principles

The limited number of studies on the addition/subtraction complement principle have analysed children's understanding of this principle mainly via assessments primarily based on verbal techniques, further constraining our insights into children's acquisition of the addition/subtraction complement principle. These techniques typically include three types of tasks (see Prather and Alibali [2009] or Bisanz et al. [2009], for a discussion). The first type of tasks involves the *application* of a shortcut procedure that is based on the mathematical principle (application of procedures tasks). Two frequently used ways to measure this are:

- (a) “looking back” tasks, in which children are confronted with conceptually related trigger-target problem pairs and can “look back” at the previous problem (i.e., the trigger) to solve the actual problem (i.e., the target), as in $5 + 3 = 8$ (trigger) followed by $3 + 5 = ?$ (target); the explicit verbal report of the possibility to look back or of having looked back at the previous problem is taken as evidence for a child's understanding of the mathematical principle;
- (b) tasks consisting of a set of purposefully constructed problems that can be solved quickly and accurately relying on a principle-based shortcut procedure, such as problems of the type $a + b - b = ?$, that can be quickly and accurately answered without computation by using one's understanding of the inversion principle; the fast and correct solution of these problems is considered as evidence for children's understanding of the underlying mathematical principle. The second type of tasks requires children to *evaluate* the validity of a principle-based shortcut procedure applied by another person (evaluation of procedures task); if a child judges the use of the principle-based shortcut procedure as valid to solve the problem, the child is assumed to understand the underlying principle. The third type of tasks involves the

justification or explanation of the appropriateness of a principle-based shortcut procedure to solve a problem (justification or explanation of procedures tasks). Here, children have to argue why a specific principle-based procedure is valid to solve the problem; children's arguments are interpreted in terms of evidencing understanding of the principle or not. Although Bisanz et al. (2009) and Prather and Alibali (2009) plea for multi-faceted assessments wherein these three types of tasks are combined, previous studies mainly relied on the application of procedures tasks, using either verbal data (i.e., verbal reports of the use or usability of principle-based shortcut procedures) or non-verbal data (i.e., accuracy and speed of solving principle-based shortcut problems) (but see Robinson & Dubé [2009] and Siegler & Stern [1998] for exceptions, combining verbal and non-verbal data).

Previous work on the addition/subtraction complement principle

Children's understanding of the addition/subtraction complement principle mainly involved application of procedures tasks, specifically looking back tasks, with children's verbal reports of the usability or actual use of the principle-based shortcut as major source of evidence. Baroody (1999) studied this principle in 40 4- to 7-year-olds. In a complement task, children were presented with conceptually related trigger-target problem pairs in which the trigger was presented together with its correct answer (e.g., $3 + 3 = 6$), and children were then asked whether this trigger could help them in solving the target (e.g., $6 - 3$). If the children verbally reported that this trigger was helpful, they were asked to state their answer and explain how the trigger helped them. Of the 29 participants with at least some formal background in subtraction, only 17% showed understanding of the complement principle on at least 4 out of 6 problems; 45% on at least one problem. Baroody concluded that young children have difficulties in recognising the complement relation between addition and subtraction.

Dowker (2014) presented 6-7-year-olds an arithmetical reasoning test that targeted the use of mathematical principles including the complement principle, with similar trigger-target problem pairs and verbal reports on the usability of the trigger to solve the target. Dowker's results indicated that only about 1/10 children verbally reported the usability of the trigger to solve the addition/subtraction complement target and/or were able to verbally explain the underlying complement principle on at least 2 out of 3 such problems.

Finally, Canobi (2009) presented 8-year-old children a looking back task in which children were asked to solve addition and subtraction problems with numbers up to 20 and verbally explain how they solved each problem. For all problems, the most recently solved problem appeared on top of the screen together with its correct answer. Twelve out of 30 problems were presented as a conceptually sequenced group, in which the initial problem (e.g., $3 + 2$) was followed by one problem based on commutativity (e.g., $2 + 3$), a second problem based on the complement principle (e.g., $5 - 2$), and a final problem based on the part-whole relation within subtraction (e.g., $5 - 3$). Participants thus had three opportunities to verbally report the use of the complement principle by looking back to the previous problem. Only in about 30% of these problems children reported to do so, and only for about 16% of all complement problems children could correctly explain the use of this principle. In 2004 and 2005, Canobi published studies in which a similar looking back task was presented to, respectively, 6- to 8- and 5- to 7-year-olds. Again, hardly any of the children verbally reported the use of the preceding addition problem to solve a complementary subtraction problem. In these latter two studies, Canobi also tried to identify profiles of understanding, and found large individual differences in children's understanding of the complement principle.

Taken together, the limited number of studies on the development of the addition/subtraction complement principle did not only reveal large inter-individual differences in children's understanding of this principle up to the first years of elementary

school. They also demonstrated serious difficulties in the acquisition of this principle compared to other addition/subtraction related principles.

The present study

This study aims at complementing current insights into the acquisition of the addition/subtraction complement principle, by investigating 9-10-year-olds' understanding of this important principle relying (also) on application of procedures tasks, but, contrary to previous studies, combining verbal and non-verbal techniques to unravel children's understanding of the principle. As such, we aim to contribute both to current theoretical insights into the acquisition of the addition/subtraction complement principle and to the methodological discussion on the merits of combining multiple methods for investigating children's understanding of mathematical principles in general and the addition/subtraction complement principle in particular.

In line with previous studies (Baroody, 1999; Canobi, 2004, 2005, 2009; Dowker, 2014), we offered children a looking back task with complement-related trigger-target problem pairs, and registered whether they verbally reported having looked back to the trigger to solve the target problem. Additionally, in line with previous studies on the commutativity and inversion principle (Bisanz et al., 2009; Prather & Alibali, 2009), we gathered data on the accuracy and speed of answering the target problems as an additional non-verbal source of evidence on children's understanding of the complement principle. Although these two types of data are not new to empirical research on children's understanding of mathematical principles, only few studies on the inversion principle (see Robinson & Dubé [2009] and Siegler & Stern [1998]) combined verbal and non-verbal data in the analyses, but such combination of data has not been reported in the context of studies on the complement principle. Finally, compared to previous studies on the complement principle, we included a larger number of items to address children's understanding in a reliable way.

We formulated three research questions. First, we wondered whether we would find similar results as in previous studies on primary school children's understanding of the complement principle when we looked at their verbal reports in a looking back task including 10 trigger-target complement pairs (research question 1). To answer this question, we analysed verbal strategy reports for the problems on which looking back as complement-based shortcut strategy could be used. Second, we compared both accuracy and speed on a task involving problems in which the complement-based shortcut strategy could be used (the looking back task) and a task consisting of exactly the same problems but without looking back possibilities (the baseline task). We wondered if we could find indications in these two kinds of non-verbal data for the use of the complement-based shortcut strategy to solve subtraction problems (research question 2). If our target problems were solved significantly better in the looking back task than in the baseline task, this would indicate that the children used the previously presented problem and thus relied on their understanding of the complement principle (Hypothesis 2a). Similarly, if the target problems were solved significantly faster in the looking back task than in the baseline task, this would indicate that the children used the previously presented problem and thus applied the complement-based shortcut strategy (Hypothesis 2b). We conducted the same analyses for the trigger problems to exclude that the performance differences were explained by test-retest effects. Finally, we wondered whether the verbal and non-verbal data would lead to similar conclusions (research question 3) and investigated the association between these two measures.

Method

Participants. Participants were 67 third- and fourth-grade children from two primary schools in Flanders (Belgium). All were typically developing children without developmental disorders (dyslexia, dyscalculia, ADHD, autism) and scoring above the 20th percentile of a

standardised mathematics test (Dudal & Van Dooren, 2002)¹. Their mean age was 9 years 3 months ($SD = 7$ months).

Procedure. All children first completed Raven's Standard Progressive Matrices (Raven, Court, & Raven, 1992) to measure their intellectual abilities. Afterwards, they solved two computer tasks, which were administered individually in a quiet room. These tasks were presented in a fixed order on two separate days: On Day 1, children completed the baseline task, and on Day 2, they completed the looking back task. In both tasks, children had to solve the same 33 two-digit problems: 16 addition and 17 subtraction problems which all required crossing the tens. The crucial difference between the two tasks was that in the baseline task no helpful preceding item was offered, whereas in the looking back task the preceding item remained on the screen with its correct answer, being sometimes helpful to solve the next problem. Also the order of the problems differed in the two tasks, as did the instructions.

The tasks started with two or three practice problems to familiarise participants with the task administration. Afterwards, the test items were offered in 3 series of 11 trials. Each trial started with an asterisk that appeared for 1000 ms in the centre of the computer screen. Next, the item was presented horizontally in the middle of the screen. The characters were 2×2 cm large, black, separated by adjacent spaces and presented against a white background. The item remained on the screen until the child answered. Time started when the item appeared on the screen, and ended when the experimenter pressed the spacebar of an external keyboard. The experimenter entered the answer via the keyboard, and then the child was asked to verbally explain how (s)he solved the problem. Then the experimenter initiated the next trial by means of a key press. Children were asked to solve each problem as fast and accurate as possible. No feedback was given. Both tasks were audio-taped.

Materials. Because the order of the problems in the looking back task is most crucial for the design of the present study, we will start with explaining this task.

Looking back task. For each of the 33 problems in this task (except the first one) the preceding problem appeared in blue colour on top of the screen together with its correct answer (see Figure 1). In the instructions, the child was explicitly told that sometimes the preceding problem might be helpful to find an answer to the next one. The problems (see Table 1) were arranged such that for 10 of the 17 subtractions (*target problems*) the previously presented addition (*trigger problem*) was helpful in finding the correct answer by using the complement principle (e.g., $48 + 45 = 93$ presented on top of the screen when solving $93 - 48 = ?$). For all 10 target problems, the value of the subtrahend and the difference differed by less than 10. For half of these problems the subtrahend was smaller than the difference (e.g., $71 - 33 = 38$), for the other half the subtrahend was larger than the difference (e.g., $93 - 48 = 45$).

--Insert Figure 1 about here--

The 13 other problems were *buffer problems*; they were placed in between the 10 trigger-target sequences to avoid that children would recognize the addition/subtraction complement pattern as our main research interest. For 3 of these 13 buffer problems the previous problem was helpful as well: Two problems focused on the part-part-whole relation (e.g., $82 - 37 = 45$ presented on top when solving $82 - 45 = ?$), while one problem dealt with the commutativity principle ($38 + 35 = 73$ presented on top when solving $35 + 38 = ?$). For the 10 other buffer problems the previous problem was not helpful. To control for looking back on all problems, at least one of the numbers from the previous problem was repeated. Five of these 10 problems were *regular* two-digit subtraction problems (e.g., $61 - 26 = 35$ followed by $73 - 35 = ?$), three were *regular* addition problems (e.g., $71 - 33 = 38$ followed by $48 + 38 = ?$), whereas two more addition problems were *tricky* buffer problems. In these tricky buffers, 2 out of the 3 numbers from the preceding subtraction problem were repeated, but the problem was turned into a non-complement based addition problem (e.g., $56 - 38 = 18$ presented on

top when solving $38 + 56 = ?$). By including these tricky problems, we could check whether children did not just give the third number from the previous problem as their answer to all problems in which two numbers were repeated.

--Insert Table 1 about here--

Baseline task. We used exactly the same problems in the baseline task as in the looking back task. By offering the same problems twice, a comparison of accuracy and RT was possible between target problems with and without a helpful preceding item. All target items were presented in the same order and on the exact same place in the sequences of the looking back task and the baseline task (Table 1). Furthermore, all target items were always preceded by an addition problem, to rule out the possible role of a strategy switch cost (Luwel, Schillemans, Onghena, & Verschaffel, 2009).

Results

All children solved 66 test items. A minority of trials (14/4422 or 0.32%) were discarded due to technical problems during data collection. For the RT analyses, incorrect trials and trials in which children immediately corrected their answer immediately after initial response were excluded (612/4422 or 14%).

Verbal reports on target problems. We first distinguished between verbal reports in which the child reported having looked back to the previous problem (*looking back*) and verbal reports in which (s)he reported how (s)he had calculated the answer without making any reference to the previous problem (*calculation*). These latter reports were further categorised into sub-categories, based on the operation underlying the calculation process: (a) If a child determined the answer by asking how much was needed to be added to the subtrahend to get to the minuend (e.g., $75 - 43$ by “ $43 + 30 = 73$ and $73 + 2 = 75$, so the answer is $30 + 2 = 32$ ”), this calculation strategy was labelled as *subtraction by addition*; (b) when a child determined the answer to the problem by subtracting the subtrahend from the minuend (e.g., $75 - 43$ by

“ $75 - 40 = 35$, $35 - 3 = 32$ ”), the calculation strategy was labelled as *direct subtraction*; (c) when a child misinterpreted the subtraction problem as an addition problem (e.g., $75 - 43$ as $75 + 43$), the strategy was labelled as *addition*; (d) explanations that could not be categorised as one of these three categories were labelled as *other*. For both tasks, the verbal strategy reports on the target problems of 10 children (5 third- and 5 fourth-graders; 15% of the target problem strategy reports) were independently coded by two researchers. They agreed on 98% of the trials, resulting in a Cohen’s Kappa κ of .93, $p < .001$.

In the baseline task, children could not look back and all verbal reports were labelled as one of the four calculation strategies. As expected (Table 2), most target problems were solved with direct subtraction (648/667 or 97%). Furthermore, 15 problems were solved with subtraction by addition (2%), used by only 3/67 children: One child used this strategy on 9/10 target problems, and two children used it on 3/10 target problems. A very small number of problems was misinterpreted as regular addition problems (4/667 or 0.60%).

--Insert Table 2 about here--

In the looking back task, 382/669 target problems (57%) were solved with direct subtraction, 17/669 (3%) with subtraction by addition, 2/669 as regular addition (0.30%), and 1/669 as other (0.15%). Two children used the subtraction by addition strategy - both also did so in the baseline task - on 10/10 and 6/10 of the target problems. The remaining 267/669 problems (40%) were solved by looking back (research question 1), with more frequent looking back occurrences on the second half (46%) than on the first half of target items (33%), $t(66) = -4.32$, $p < .01$. There were large individual differences: 28/67 children never looked back to the trigger to solve a target (42%), 12/67 children looked back on 1-5 targets (18%), 19/67 children looked back on 6-9 targets (28%), and 8/67 children looked back at all 10 targets (12%). While 60% the children reportedly looked back to the trigger to solve the

target problem, the frequency of looking back across individual children largely varied from only once (i.e., looking back on only 1 target) up to 10 (i.e., looking back on all targets).

Comparison of accuracies between the two tasks. Overall accuracy was 89% (3944/4408) with 88% in the baseline task and 91% in the looking back task. Buffer problems elicited most incorrect answers, followed by the (subtraction) target problems and the (addition) trigger problems (see Table 3).

--Insert Table 3 about here--

A more detailed descriptive comparison of the two tasks suggested that the higher accuracy in the looking back task was due to fewer errors on the target problems. They dropped from 105 to 54, whereas the number of incorrect buffers only dropped from 115 to 90, and the number of incorrect triggers remained 50. This association between problem type and task (research question 2) was significant, $\chi(2) = 7.15, p = .03$. Additional z -tests for proportions showed that, indeed, only the target problems were answered significantly more correctly in the looking back task than in the baseline task, $z = 4.33, p < .01$ (Hypothesis 2a), and there was no difference in the count of incorrect answers for both buffers, $z = 1.84, p = .06$, and triggers, $z = 0.00, p > .99$. This suggests that the children indeed used the previously presented trigger problem to solve the target problems in the looking back task, and thus showed evidence of using their understanding of the complement principle.

Comparison of RTs between the two tasks. The comparison of RTs between the looking back task and the baseline task provided a non-verbal indication that in the looking back task children used the previously presented problem to answer a subsequent problem (also aiming at answering research question 2). The mean RTs for all problem types in both tasks can be found in Table 4.

--Insert Table 4 about here--

Paired *t*-tests compared the children's mean RTs on each problem type between the two tasks. In line with Hypothesis 2b, the target problems were solved significantly faster in the looking back task than in the baseline task, $t(63) = 5.91, p < .01$. The trigger problems were solved significantly faster in the baseline task than in the looking back task, $t(66) = -3.82, p < .01$. This suggests that the children searched for looking back possibilities on both target and trigger problems.

Association between verbal and non-verbal methods. To investigate the association between verbal and non-verbal measures of understanding the complement principle (research question 3), we calculated Spearman correlations. For the verbal measure, we used each child's frequency of looking back on target problems in the looking back task. For the non-verbal measure, we calculated an Improvement Score based on the comparison of a participant's mean RT on the target problems in the looking back task and in the baseline task, $Impr-Score = (MeanRT_{BT} - MeanRT_{LBT})/MeanRT_{BT}$. Both measures were highly correlated, even when controlling for intellectual ability, $\rho(64) = .80, p < .01$: Children who reported having looked back more often also had a larger speed difference in favour of the looking back task.

Discussion

The present study aimed at complementing previous work on children's acquisition of the addition/subtraction complement principle, using both verbal and non-verbal techniques. Whereas other mathematical principles such as commutativity and inversion were extensively studied in the past, only few researchers directly addressed children's acquisition of the complement principle (e.g., Baroody, 1999; Canobi, 2009; Dowker, 2014). Moreover, these few studies mainly relied on verbal reports of the use or usability of a complement-based shortcut strategy to determine children's understanding of the principle. We extended this research by using both verbal data and two kinds of non-verbal data that have been

successfully used to study other mathematical principles. As a result, this data increase our understanding of children's acquisition of the addition/subtraction complement principle and highlight the potential of combining verbal and non-verbal techniques in future studies on children's understanding of mathematical principles.

Adding to our theoretical understanding of children's acquisition of mathematical principles, and consistent with previous work, we observed large individual differences in children's understanding of the addition/subtraction complement principle and confirmed children's difficulties in acquiring this principle even at the ages of 9-10-years. While most 9-10-year-olds in our study had at least some understanding of the addition/subtraction complement principle, as evidenced by verbal reports and non-verbal measures (i.e., accuracy and speed differences between the looking back versus baseline task), these children's understanding covered the whole continuum of understanding, ranging from an initial understanding up to a good understanding of the addition/subtraction complement principle. Moreover, a substantial number of children hardly showed verbal and/or non-verbal evidence for understanding the addition/subtraction complement principle, pointing again to the difficulties in acquiring a good understanding of it. Such difficulties are in contrast with children's acquisition of other addition/subtraction related principles such as commutativity and inversion (cf. Cowan & Renton, 1996; Gilmore & Spelke, 2008; Prather & Alibali, 2009; Rasmussen et al., 2003; Sophian, Harley, & Manos Martin, 1995). It is important to note that we included children who had practice in elementary subtraction for at least 2.5 years and that we excluded the 20% lowest achieving children in mathematics. Future studies should therefore investigate this understanding of the addition/subtraction complement principle in children of other ages and ability levels.

From a methodological perspective, the present study adds to the methodological challenge of assessing children's understanding of mathematical principles (Bisanz et al.,

2009; Prather & Alibali, 2009). Although we still relied on only one type of assessment, namely application of procedures tasks, we combined verbal (reports) and non-verbal techniques (using accuracy and reaction time data). The non-verbal measure, namely the baseline to looking back task targets improvement score, significantly correlated with children's verbal complement-based strategy reports, even after controlling for intellectual ability. This suggests that both measures are valid measures of children's complement-principle understanding in future studies. However, given the selective sample in terms of age and mathematical achievement level, the combination of these two types of assessment for younger and less mathematically able children requires further attention.

Future studies should also consider other measures to address children's understanding of this principle via application of procedures tasks. First, it would be interesting to look for unreported occurrences of looking back in the looking back task, by using a cut-off time (e.g., Siegler & Stern, 1998). However, to determine the proper cut-off time to classify problems into looking back or not based on non-verbal data only, one needs more than 10 target and trigger problem pairs. Moreover, in this cut-off time individual differences in mental subtraction performance should be taken into account as well. Second, eye-movement data provide a promising avenue to determine looking back behaviour. For example, Gaschler et al. (2013) and Godau et al. (2014) investigated children's understanding of the commutativity principle and found that fixation patterns matched the use of shortcut strategies based on this commutativity principle. To the best of our knowledge, such studies have not been performed to investigate other mathematical principles in children.

Finally, future studies should also include other types of tasks to understand mathematical principles, in particular evaluation of procedures and justification of procedures tasks (cf. Bisanz et al., 2009; Prather & Alibali, 2009). The use of multi-faceted assessments can help to enhance our understanding of variety in developmental trajectories as well as the potential

constraints within the different developmental routes towards acquiring a good understanding of mathematical principles. However, justifying the use of the complement principle and/or evaluating its legitimate application is not easy for young and mathematically weaker children because one can only state the mathematical relation in its most abstract form as “*if $a + b = c$, then $c - b$ has to equal a* ”. This was evidenced in our pilot study (Footnote 1) and by previous researchers’ analyses, not (clearly) including children’s justification and/or evaluation of complement-principle applications in their analyses of understanding, and their results, pointing to only minor successes in adequately justifying and/or evaluating complement-principle applications, on justification and evaluation tasks (cf. Baroody, 1999; Canobi, 2004; 2005, 2009; Dowker, 2014; see also Peters, Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2013). It is therefore a great challenge to develop age-appropriate versions of these types of tasks, using both verbal and non-verbal techniques.

From an educational viewpoint, the present study also offers building blocks for optimizing current mathematics instruction on addition/subtraction related principles, and especially the addition/subtraction complement principle. Given the large difficulties in acquiring a good understanding of the complement principle, even in moderate to higher achieving 9-10-year-olds, this important mathematical principle does not only require more, but also qualitatively stronger, instructional attention. The looking back task might be a promising tool to enhance children’s understanding of the complement principle. More concretely, the task might be used for classroom discussions on “the how, the when, and the why” of the use of a certain strategy (Selter, 1998). As indicated by children’s verbal reports of using the looking back strategy in the first versus second half of the looking back task - with more frequent looking back reports in the second half of the task - providing children with thoughtfully designed and patterned series of arithmetic problems with looking back possibilities (see for example the German textbook *Das Zahlenbuch* [Wittmann & Müller,

2007]) can push them to reflect upon the suitability of clever shortcut strategies and their underlying mathematical principles. By listening to a peer's explanation that (s)he did not had to compute the answer to a problem because (s)he could derive it from the previous one and why it was allowed to do so, children may start to develop an approach whereby they deliberately look for patterns in their arithmetic exercises and for possibilities of solving them in a clever way by applying mathematical principles. More generally, such classroom discussions can also result in more positive affects towards the insightful and flexible use of strategies in mathematics education (see also Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009).

Footnotes

¹ We included third- and fourth-graders scoring above Pc20 on a standardized mathematics achievement test into this study on the basis of both the Flemish educational context and the results of a pilot study. First, we decided to only select children who had received basic instruction in the addition/subtraction complement principle (taking into account the results of Baroody, 1999; Canobi, 2004; 2005, 2009; Dowker, 2014). This implies that only children from at least the third grade could participate. Second, our pilot study revealed that children with low mathematics achievement had serious difficulties in verbally explaining their understanding and use of the addition/subtraction complement principle. While this sampling method improved the quality of our data collection and analysis, it limits the generalizability of our findings to achievement and age groups.

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Table 1

Overview of the Order of the Looking Back Task and the Baseline Task.

	Problem Type in Looking Back Task	Looking Back Task	Baseline Task
Practice problem 1		$22 + 22 = 44$	$73 - 57 = 16$
Practice problem 2		$44 - 22 = 22$	$66 + 18 = 82$
Practice problem 3		$22 + 44 = 66$	/
1	Subtraction Buffer	$56 - 38 = 18$	$82 - 45 = 37$
2	Tricky Buffer	$38 + 56 = 94$	$64 + 28 = 92$
3	Trigger	$33 + 38 = 71$	$27 + 36 = 63$
4	Target	$71 - 33 = 38$	$71 - 33 = 38$
5	Addition Buffer	$48 + 38 = 86$	$37 + 45 = 82$
6	Trigger	$48 + 45 = 93$	$36 + 28 = 64$
7	Target	$93 - 48 = 45$	$93 - 48 = 45$
8	Trigger	$37 + 45 = 82$	$33 + 38 = 71$
9	Target	$82 - 37 = 45$	$82 - 37 = 45$
10	Part-Whole Buffer	$82 - 45 = 37$	$61 - 26 = 35$
11	Subtraction Buffer	$82 - 28 = 54$	$56 - 38 = 18$
Pause			
12	Trigger	$36 + 28 = 64$	$44 + 47 = 91$
13	Target	$64 - 36 = 28$	$64 - 36 = 28$
14	Tricky Buffer	$64 + 28 = 92$	$47 + 38 = 85$
15	Trigger	$28 + 24 = 52$	$36 + 57 = 93$

16	Target	$52 - 28 = 24$	$52 - 28 = 24$
17	Subtraction Buffer	$75 - 28 = 47$	$81 - 27 = 54$
18	Trigger	$44 + 47 = 91$	$36 + 48 = 84$
19	Target	$91 - 44 = 47$	$91 - 44 = 47$
20	Trigger	$47 + 38 = 85$	$35 + 38 = 73$
21	Target	$85 - 47 = 38$	$85 - 47 = 38$
22	Addition Buffer	$38 + 35 = 73$	$35 + 26 = 61$
Pause			
23	Commutativity Buffer	$35 + 38 = 73$	$48 + 38 = 86$
24	Trigger	$35 + 26 = 61$	$38 + 35 = 73$
25	Target	$61 - 35 = 26$	$61 - 35 = 26$
26	Part-Whole Buffer	$61 - 26 = 35$	$82 - 28 = 54$
27	Subtraction Buffer	$73 - 35 = 38$	$73 - 35 = 38$
28	Trigger	$38 + 43 = 81$	$28 + 24 = 52$
29	Target	$81 - 38 = 43$	$81 - 38 = 43$
30	Subtraction Buffer	$81 - 27 = 54$	$75 - 28 = 47$
31	Trigger	$27 + 36 = 63$	$38 + 56 = 94$
32	Target	$63 - 27 = 36$	$63 - 27 = 36$
33	Addition Buffer	$36 + 57 = 93$	$48 + 45 = 93$

Table 2

Frequencies of Strategy Reports on the Target Problems (and Row Percentages), Comparing the Looking Back Task and the Baseline Task.

Task	Strategy					Total
	Addition	Direct subtraction	Subtraction by addition	Looking Back	Other	
Baseline task	4 (0.60)	648 (97.15)	15 (2.25)	0	0	667
Looking back task	2 (0.30)	382 (57.10)	17 (2.54)	267 (39.91)	1 (0.15)	669

Table 3

Mean Accuracy Rates (over 67 Subjects and Types of Problems) for both the Looking Back Task and the Baseline Task, with SD in Parentheses.

Problem Type	Baseline Task			Looking Back Task		
	Total number	Frequency of errors	Mean accuracy (SD)	Total number	Frequency of errors	Mean accuracy (SD)
Target	667	105	84.21 (21.70)	669	54	91.94 (17.94)
Trigger	668	50	92.50 (8.99)	668	50	92.46 (10.01)
Buffer	869	115	86.79 (15.60)	867	90	89.58 (11.90)

Table 4

Mean RTs in ms (over Subjects and Types of Problems) for both the Looking Back Task and the Baseline Task, with SD in Parentheses.

Problem Type	Baseline Task			Looking Back Task		
	Total number	N subjects	Mean RT (SD)	Total number	N subjects	Mean RT (SD)
Target	534	65	14385 (6406)	596	65	10101 (5744)
Trigger	595	67	10494 (3928)	597	67	12124 (5172)
Buffer	731	67	13471 (5313)	757	67	13202 (5061)

$$48 + 45 = 93$$

$$93 - 48 = .$$

Figure 1. Screenshot of a trigger-target sequence presented in the looking back task.