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Wave propagation in locally resonant cylindrically curved metamaterial panels

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Abstract

Combining lightweight characteristics with favourable noise and vibration behaviour, locally resonant metamaterials with vibro-acoustic stopband behaviour are gaining increasingly more attention. In view of applicability of metamaterials, this paper investigates the effects of curvature on stopband behaviour. Through derivation of curved unit cell models, dispersion curves for cylindrically curved infinite periodic structures are derived. The effect of curvature on metamaterials is investigated through both an academic example of an infinite cylindrically curved panel with added spring-mass systems and a realizable curved panel with resonant structures. It is shown that, whenever pure flexural motion exists in the host structure, the stopband mechanism can be applied to achieve flexural stopbands.

Keywords: metamaterial; Bloch-Floquet; wave propagation; cylindrically curved panels; complex wavenumber; periodic structures.

1. Introduction

In recent years, increasingly restrictive emission regulations forced the transportation industry to look at lightweight design to increase fuel efficiency of their vehicles. However, since generally structures with low mass and high stiffness suffer from unfavourable noise, vibration and harshness (NVH) behaviour, this leads to vehicles with possibly poor NVH behaviour. Therefore, considering the increasing awareness regarding the negative impact of unwanted noise on human health, post-treatment of lightweight vehicles seems to be a necessity. Albeit, only the example of automotive is given, these two often contradictory criteria of favourable NVH behaviour and lightweight design, hold for various sectors of industry and ask for new and innovative solutions which should be kept as simple, cheap and easy to apply as possible.

As one of these innovative and efficient lightweight NVH solutions, vibro-acoustic metamaterials are becoming of more interest. Metamaterials have shown a promising potential in efficiently insulating noise and vibration in a host structure, at least in targeted frequency ranges. These targeted frequency ranges are commonly referred to as stopbands and they are defined as zones in frequency in which no free wave propagation can occur. It has been shown that metamaterials can be designed to have multiple stopbands. The stopband phenomenon in metamaterials is the result of resonances of resonant cells distributed on a subwavelength scale (i.e. on a scale smaller than the governing wavelength in the host structure at the frequency

of interest) and can be explained based on Fano-type interference. Fano-type interference was first theoretically described in analysis of inelastic autoionizing resonances in atoms. This phenomenon is described as the interference between discrete resonances with the smooth continuum background in which they are embedded. Goffaux et. al. showed that Fano-type interference occurs in resonant metamaterials due to elastic waves travelling through a host structure and interacting with the resonant cells, leading to asymmetric transmission spectra which can be linked to stopbands . The major benefit of such metamaterials is that the stopband behaviour is independent of periodicity or spacing between resonant cells, as long as the subwavelength criterion is met, allowing the designer more freedom in design .

Examples of grids of mass-spring systems that can act as resonant cells are invaluable in order to better understand metamaterials , but in engineering applications, lots of different criteria from lightweight design and low production cost to ease of implementation and durability should be met. More importantly, the resonant cells should not influence the structural integrity of the structure or interfere with the primary function of the system in any way. In a recent work, Claeys et al. have proposed an innovative and practical metamaterial design, realised in a demonstrator validating the stopband phenomenon numerically and experimentally . Given the high amount of applications of curved panels in industry, the effect of curvature on stopband behaviour is discussed in this paper and cylindrical curvature is introduced to the demonstrators of Claeys et al. as a next step to obtain metamaterials solution for various applications.

Although periodicity is not a requirement, assuming periodicity simplifies the study of metamaterials. Considering a periodic structure and using the Bloch-Floquet theorem, it is possible to analyse the motion of the infinite periodic structure by considering a unit cell, i.e. a representative domain of the structure in space. To this end, an eigenvalue problem in terms of frequency and wavenumber vector is obtained, yielding dispersion curves which give information about how waves propagate in the considered periodic structure. This eigenvalue problem can be solved at given propagating wavenumbers to obtain the frequencies of free wave propagation, called the indirect approach or alternatively, it can be solved at given frequencies for complex wavenumbers, called the direct approach, providing information on both propagating and attenuated waves .

Wave propagation in cylindrical shells has been widely investigated by researchers using analytical or numerical methods. In simpler cases, such as circular cylindrical thin shells, analytical solutions can be obtained for the wave dispersion relation in the structure . Further investigation of wave propagation in thin cylindrical shells using three-dimensional elasticity has been done by Gazis . However, due to the complexity of wave motion in such structures, many researchers have used certain simplifying assumptions or approximate theories .

Pany and Parthan have investigated axial wave propagation in multi-supported curved panels using a finite element approach. Recently, Manconi and Mace have utilized the wave Finite Element Method (FEM) to analyze wave dispersion in isotropic and orthotropic cylindrical panels and closed cylindrical shells.

Due to the highly coupled nature of waves propagating in cylindrical structures, the wave dispersion is considerably different from those of flat structures, at least at lower frequencies. To reduce the computational effort needed to analyse wave motion in curved metamaterials, it is assumed that the resonant cells are distributed periodically and infinite cylindrically curved panels are investigated, allowing unit cell analysis based on the Bloch-Floquet theorem. Using the direct approach, eigen-wavenumbers are obtained in complex form, in which the imaginary and real parts represent propagative and attenuated parts of waves, respectively. These results can then be used to detect stopbands which show up as discontinuities in the dispersion curves of the propagating waves.

The paper is structured as follows. Section 2 first reviews the Bloch-Floquet theorem and its application in a cylindrical periodic elastic medium. The case of a flat plate is also addressed as a special case of the cylindrical case when the radius tends to infinity. Then the wave dispersion relation for such a structure is derived in the weak form and the application of the finite element method to solve this equation is explained. Section 3, first validates the numerical models by comparing the results of circular cylindrical shells and flat plates with those obtained from analytical solutions. Next, an academic representation of the cylindrically curved metamaterial panel as spring-mass Tuned Vibration Absorbers (TVA) is adopted to investigate the effect of curvature on the stopband phenomenon. To this end, panels of various radii are considered and the dispersion curves are compared with those of a flat panel. Afterwards, a practical, curved metamaterial design is introduced and following the unit cell approach, the wave propagation in this demonstrator is studied to further investigate the effects of the radius and wave coupling in cylindrical panels on the stopband zone. Last section summarizes the main conclusions of the paper. It is shown that radius plays a major effect on stopband behaviour of cylindrical metamaterials by determining the cut-on frequency of the flexural waves.

2. Unit cell modelling of cylindrically curved structures using Bloch-Floquet theorem

In this section, first the Bloch-Floquet theorem is briefly reviewed and next the application of this theory in the analysis of wave propagation in cylindrical panels is introduced. Finally, the weak form of the dispersion relation of a cylindrical unit cell is obtained and its solution using FEM is explained.

2.1. Bloch-Floquet Theorem

The Bloch-Floquet theorem states that in a three-dimensional infinite continuous medium composed of periodically repeating unit cells, the response of any point in the structure can be expressed in terms of the response of its corresponding point in a reference unit cell together with a complex phase shift which is related to the distance of the point to the unit cell as shown in Fig. 1.

Figure 1: Schematics of the unit cell in the periodic structure.

As can be seen in Fig. (1), the location of point “p” can be described in terms of the location of the corresponding point “u” in the reference unit cell as follows:

$$\mathbf{r}_p = \mathbf{r}_u + n_x \mathbf{d}_x + n_y \mathbf{d}_y + n_z \mathbf{d}_z, \quad (1)$$

in which $\mathbf{d}_x$, $\mathbf{d}_y$ and $\mathbf{d}_z$ are the basis vectors of the unit cell and n_x , n_y and n_z are the number of unit cells between the considered cell and the reference unit cell along the three basis vectors. According to Bloch-Floquet theorem, the displacement field $\mathbf{u}$ at point “p” can be expressed as:

$$\mathbf{u}(\mathbf{r}_p, f) = \mathbf{u}(\mathbf{r}_u, f) e^{k \cdot (n_x \mathbf{d}_x + n_y \mathbf{d}_y + n_z \mathbf{d}_z)} = \mathbf{u}(\mathbf{r}_u, f) e^{\boldsymbol{\mu} \cdot \mathbf{n}}, \quad (2)$$

where “dot” denotes the scalar product of two vectors, f represents the frequency, $\mathbf{k}$ is the wavenumber vector and $\boldsymbol{\mu} = \mathbf{k} \cdot \mathbf{d}$ is defined as the wave propagation vector which expresses the complex phase shift across one unit cell. Furthermore, considering that the propagation vector can be complex, the exponential term determines both the propagative and attenuated parts of the waves influencing both the changes in the phase and amplitude of the displacement field, respectively. Note that for the sake of simplicity, the coordinates of a point in the media is indicated by $\mathbf{r}$ instead of $\mathbf{r}_p$ from here onwards.

2.2. Application to cylindrical elasto-dynamic problems

The equations of motion of an infinite homogeneous structure can be represented with the following partial differential equations:

$$\nabla \cdot \boldsymbol{\sigma} = \rho \ddot{\mathbf{u}}(\mathbf{r}, t) \quad \forall \mathbf{r} \in \Omega_R \quad (3)$$

where Ω_R is the domain occupied by the continuous elastic medium, ρ is the density, $\ddot{\mathbf{u}}(\mathbf{r}, t)$ is the second order time derivative of the displacement field and $\nabla \cdot \boldsymbol{\sigma}$ represents the divergence of the stress tensor $\boldsymbol{\sigma}$. Moreover, $\boldsymbol{\sigma}$ can be defined in terms of strain tensor $\boldsymbol{\varepsilon}$ as follows:

$$\sigma_{ij} = \lambda \text{tr}(\boldsymbol{\varepsilon}) \delta_{ij} + 2\mu \varepsilon_{ij} \quad i, j = 1, 2, 3, \quad (4)$$

where $\text{tr}(\boldsymbol{\varepsilon})$ is the trace of strain tensor and δ_{ij} is the Kronecker delta. Moreover, λ and μ are Lamé constants which for a homogeneous isotropic material can be related to the Young's modulus E and Poisson ratio ν as follows:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}, \quad (5)$$

Finally, the strain tensor can be expressed in terms of the displacement field as follows:

$$\varepsilon_{ij} = \frac{1}{2} (\nabla u_i + \nabla u_j) = \frac{1}{2} (u_{i,j} + u_{j,i}) = \varepsilon_{ji} \quad i, j = 1, 2, 3, \quad (6)$$

Considering a time-harmonic dependency of the displacement field and assuming periodicity in the infinite structure, based on Eq. (2) the displacement field can be separated with respect to time and space as follows:

$$\begin{aligned} \mathbf{u}(\mathbf{r}, t) &= \mathbf{u}(\mathbf{r}, f) e^{i\omega t} \\ &= \mathbf{U}(\mathbf{r}_u, \mathbf{k}, f) e^{i\mathbf{k} \cdot \mathbf{r}_u} e^{i\boldsymbol{\mu} \cdot \mathbf{n}} e^{i\omega t} \\ &= \mathbf{U}(\mathbf{r}_u, \mathbf{k}, f) e^{i\mathbf{k} \cdot \mathbf{r}} e^{i\omega t} \end{aligned} \quad (7)$$

where $e^{i\mathbf{k} \cdot \mathbf{r}_u}$ defines the complex propagation phase shift within a unit cell while $e^{i\omega t}$ indicates the time dependency of the displacement field with ω being the angular frequency. From the second to the third line of Eq. (7) the displacement field in the unit cell is separated into a term accounting for the spatial harmonic displacement, dictated by $\mathbf{k}$, and the non-harmonic spatial displacement $\mathbf{U}(\mathbf{r}_u, \mathbf{k}, f)$. The use of this formulation allows further on deriving an eigenvalue problem in terms of wavenumber $\mathbf{k}$ and frequency f . Within a unit cell the displacement field thus reads:

$$\mathbf{u}(\mathbf{r}_u, f) = \mathbf{U}(\mathbf{r}_u, \mathbf{k}, f) e^{i\mathbf{k} \cdot \mathbf{r}_u}. \quad (8)$$

Once again, note that $\mathbf{k}$ is a vector with complex values, in which the imaginary and real parts express propagating and attenuated parts of waves, respectively.

The representations of the wavenumber vectors of two dimensional wave propagation in Cartesian and cylindrical coordinate systems are shown in Fig. (2). In the following pages, three kinds of waves will be investigated:

- Circumferential waves in which only the circumferential component of the wavenumber vector is non-zero;
- Axial waves in which only the axial component of the wavenumber vector is non-zero;
- Helical waves in which the wavenumber vector composes of a combination of the axial and circumferential wavenumber. Based on Fig. (2), the wavenumber vector $\mathbf{k}$ of helical waves, can be expressed as:

$$\mathbf{k} = \begin{bmatrix} 0 \\ k_\varphi \\ k_z \end{bmatrix} = k_0 \begin{bmatrix} 0 \\ \cos(\theta) \\ \sin(\theta) \end{bmatrix} = k_0 \Phi, \quad (9)$$

in which Φ defines the direction of wave propagation and k_0 is the amplitude of the wavenumber vector.

Figure 2: Representation of the wavenumber vector in 2D wave propagation in planar and cylindrical media.

Substituting Eqs. (8) and (9) into Eq. (6) gives the strain tensor within the unit cell in the cylindrical coordinate system as follows:

$$\varepsilon_{ij} = (\mathbf{E}_{ij} + \mathbf{I} k_0 \kappa_{ij}) e^{k_0 (a \cos(\theta) \varphi + \sin(\theta) z)} e^{i \omega t}, \quad (10)$$

where a is the radius corresponding to the central line of the cylinder and $\mathbf{E}$ and κ are symmetric tensors with components given as:

$$\begin{aligned} E_{rr} &= U_{,r}, \quad E_{\varphi\varphi} = \frac{(V_{,\varphi} + U)}{r}, \quad E_{zz} = W_{,z} \\ E_{\varphi z} &= \frac{1}{2} \left(\frac{W_{,\varphi}}{r} + V_{,z} \right), \quad E_{zr} = \frac{1}{2} (W_{,r} + U_{,z}), \quad E_{r\varphi} = \frac{1}{2} \left(\frac{U_{,\varphi}}{r} - \frac{V}{r} + V_{,r} \right) \\ \kappa_{rr} &= 0, \quad \kappa_{\varphi\varphi} = \frac{V \cos(\theta)}{r}, \quad \kappa_{zz} = W \sin(\theta) \\ \kappa_{\varphi z} &= \frac{1}{2} \left(\frac{W \cos(\theta)}{r} + V \sin(\theta) \right), \quad \kappa_{zr} = \frac{1}{2} (U \sin(\theta)), \quad \kappa_{r\varphi} = \frac{1}{2} \left(\frac{U \cos(\theta)}{r} \right), \end{aligned} \quad (11)$$

where U , V and W are the periodic amplitude of the displacement field in r , φ and z directions, respectively.

Substituting Eq. (10) into Eq. (4) provides components of the stress tensor as:

$$\begin{aligned} \sigma_{ij} &= (\Sigma_{ij} + \mathbf{K}_{ij}) e^{k_0 (a \cos(\theta) \varphi + \sin(\theta) z)} e^{i \omega t} \\ \Sigma_{ij} &= \lambda \text{tr}(\mathbf{E}) \delta_{ij} + 2\mu E_{ij} \\ \mathbf{K}_{ij} &= \lambda \text{tr}(\kappa) \delta_{ij} + 2\mu \kappa_{ij}. \end{aligned} \quad i, j = r, \varphi, z \quad (12)$$

Finally, substituting Eqs. (8) and (12) into the equations of motion, Eq. (3), leads to the following form of the equations of motion:

$$\begin{aligned} (\rho\omega^2 U(\mathbf{r}_u, \mathbf{k}, f) + \nabla \cdot (\boldsymbol{\Sigma}) + k_0 (\boldsymbol{\Sigma}) \cdot \boldsymbol{\Phi} + k_0 \nabla \cdot (\mathbf{K}) \\ + k_0^2 (\mathbf{K}) \cdot \boldsymbol{\Phi}) e^{k_0(a \cos(\theta)\varphi + \sin(\theta)z)} e^{i\omega t} = 0, \end{aligned} \quad \forall \mathbf{r} \in \Omega_{UC} \quad (13)$$

which then leads to the following quadratic eigenvalue problem in terms of the complex wavenumber amplitude k_0 within the unit cell domain Ω_{UC} :

$$\rho\omega^2 U(\mathbf{r}_u, \mathbf{k}, f) + \nabla \cdot (\boldsymbol{\Sigma}) + k_0 (\boldsymbol{\Sigma}) \cdot \boldsymbol{\Phi} + k_0 \nabla \cdot (\mathbf{K}) + k_0^2 (\mathbf{K}) \cdot \boldsymbol{\Phi} = 0 \quad \forall \mathbf{r} \in \Omega_{UC}, \quad (14)$$

with the following boundary condition on the boundaries of the unit cell shown in Fig. (1), enforcing the periodicity (i.e. representing the infinite dimensions of the structure):

$$U_{(\mathbf{r}_i - L_i, \mathbf{d}_i, \mathbf{k}, f)} = U_{(\mathbf{r}_i, \mathbf{k}, f)} \quad \text{for } i = \varphi, z \quad \forall \mathbf{r} \in \Gamma_{UC}. \quad (15)$$

Eq. (15) is known as the periodic boundary condition and ensures that the periodicity is met on the boundaries of the unit cell.

The two-dimensional wave propagation in a flat panel, as also shown in Fig. (2), can be considered as a special case of the wave propagation in a cylindrical panel. To this end, if one considers that the radius of the cylinder is tending towards infinity and considering the transformation between cylindrical and Cartesian coordinates, the components of the tensors $\mathbf{E}$ and $\boldsymbol{\kappa}$ can be found as follows:

$$\begin{aligned} E_{xx} = U_{,x}, E_{yy} = V_{,y}, E_{zz} = W_{,z} \\ E_{yz} = \frac{1}{2}(W_{,y} + V_{,z}), E_{zx} = \frac{1}{2}(W_{,x} + U_{,z}), E_{xy} = \frac{1}{2}(U_{,y} + V_{,x}) \\ \kappa_{xx} = 0, \kappa_{yy} = V \cos(\theta), \kappa_{zz} = W \sin(\theta) \\ \kappa_{yz} = \frac{1}{2}(W \cos(\theta) + V \sin(\theta)), \kappa_{zx} = \frac{1}{2}(U \sin(\theta)), \kappa_{xy} = \frac{1}{2}(U \cos(\theta)), \end{aligned} \quad (16)$$

and following the same procedure as shown in Eqs. (11) - (15), the quadratic eigenvalue problem in terms of wavenumber and frequency can be obtained as also presented by.

2.3. Weak formulation and numerical computation of the dispersion equation

In order to study wave propagation in an infinite periodic structure using the Bloch-Floquet theorem, the quadratic eigenvalue problem of Eq. (14) should be solved together with the periodic boundary condition given by Eq. (15). While many suitable numerical methods can be utilized to this end, in this paper, FEM is adopted as the numerical tool. In order to use the formulations discussed in the previous section in FEM, one needs to

obtain the weak form of the stated equations. If $\mathbf{U}(\mathbf{r}_u, \mathbf{k}, f)$ is a solution of Eq. (14) which satisfies the periodic boundary condition, the weak form of Eq. (14) can be obtained as:

$$\begin{aligned} \delta \Pi = \int_{\Omega_{UC}} & \left(\rho \omega^2 \delta(\mathbf{U}(\mathbf{r}_u, \mathbf{k}, f)) \mathbf{u}(\mathbf{r}_u, \mathbf{k}, f) - \delta(\mathbf{E}) : (\boldsymbol{\Sigma}) - k_0 \delta(\mathbf{E}) : (\mathbf{K}) \right. \\ & \left. + k_0 \delta(\boldsymbol{\kappa}) : (\boldsymbol{\Sigma}) + k_0^2 \delta(\boldsymbol{\kappa}) : (\mathbf{K}) \right) dv = 0 \quad \forall \mathbf{r} \in \Omega_{UC} \\ \forall \delta \left(U_{(r_i - L_i, d_i, \mathbf{k}, f)} \right) &= \delta \left(U_{(r_i, \mathbf{k}, f)} \right) \quad \text{for } i = \varphi, z \quad \forall \mathbf{r} \in \Gamma_{UC}, \end{aligned} \quad (17)$$

in which $\delta(\cdot)$ represents the test function. Finally, the weak form given by Eq. (17) can be used together with any standard FEM discretization tool leading to the following quadratic eigenvalue problem in terms of the wavenumber amplitude k_0 as:

$$k_0^2 \mathbf{A}_{(\Phi)} + k_0 \mathbf{B}_{(\Phi)} + (\mathbf{C} - \omega^2 \mathbf{D}) = 0, \quad (18)$$

where $\mathbf{A}_{(\Phi)}$, $\mathbf{B}_{(\Phi)}$, $\mathbf{C}$ and $\mathbf{D}$ are matrices defined based on Eq. (17), from which $\mathbf{C}$ and $\mathbf{D}$ are the standard stiffness and mass matrices, respectively; while $\mathbf{A}_{(\Phi)}$ and $\mathbf{B}_{(\Phi)}$ are symmetric and skew-symmetric matrices, respectively.

Note that Eq. (18) can be also considered as a linear eigenvalue problem in terms of angular frequency if the wavenumber vector is fixed; while at a given frequency, it is a quadratic eigenvalue problem in terms of wavenumber which can be solved by fixing the wave propagation direction and solving the problem for the wavenumber amplitude. The latter approach is adopted in this paper. The current approach is also of interest for being more flexible in case of dealing with frequency dependent material properties, which could be the case especially when dealing with damped structures, composed of viscoelastic or poroelastic materials.

Considering that the matrices $\mathbf{A}_{(\Phi)}$, $\mathbf{C}$ and $\mathbf{D}$ are symmetric and matrix $\mathbf{B}_{(\Phi)}$ is skew-symmetric, the solution of Eq. (18) is in the form of complex quadruple roots of $(k, -k, k^*, -k^*)$ where “*” indicates the complex conjugate. The complex wave solution provided by this approach is especially of importance when dealing with damped mechanical structures; since in this case all of the waves are attenuated, all of the solutions would be complex, where the imaginary parts of the eigen-wavenumbers define the propagating part of the wavenumber, and the real part is an indicator of the amount of wave attenuation.

Eventually, solving Eq. (18) provides the so-called dispersion curves which show the time and space dependency of existing waves through frequency and wavenumber definitions. Note that because of the periodic boundary condition given by Eq. (15), the solution of Eq. (18) is also periodic in the wavenumber domain with

the interval of $\left[-\frac{\pi}{L_i}, \frac{\pi}{L_i}\right]$ where $i = \varphi, z$. This periodic zone in the wave number domain is referred to as the Brillouin zone. It is common to use the wave propagation constant μ instead of the wavenumber k when plotting the dispersion curves as it is a non-dimensional parameter, changing the borders of the periodic zone to $[-\pi, \pi]$ in φ and z directions, making the interpretation of the dispersion curves easier and independent of the unit cell size. This approach has been adopted in this paper as well.

As stated, any standard FEM discretization packages can be utilized to solve the quadratic eigenvalue problem of Eq. (18) together with the periodic boundary condition of Eq. (15). In this paper, COMSOL Multiphysics® as a flexible FE tool is used to obtain the solutions of this eigenvalue problem.

3. Numerical results

In this section, first, the application of the unit cell model presented in the previous section is validated through the comparison of the results with those obtained from analytical solutions. Next, to better understand the effect of curvature on stopbands, cylindrical resonant metamaterial panels are considered as simple shell host structures with spring-mass TVA as local resonators. These academic panels allow studying the presence of stopbands and the effect of the radius on them. Next, more realistic designs of cylindrical locally resonant metamaterial panels are considered and the stopbands are investigated again using dispersion curves. For both cases the resonant cells are spaced on a sub-wavelength scale on the host structure according to the rules of metamaterial design; whether the design results in stopband behaviour and truly leads to a metamaterial structure, will become apparent through the analysis of the dispersion curves.

3.1. Validation of the model

In order to verify the accuracy of the model, dispersion curves of shear and flexural waves propagating in a flat panel are compared with those given by the analytical solution for the waves propagating in the x-direction corresponding to $\theta = 0$ as shown in Fig. (2). The plate is assumed to have a thickness of 1 mm with the material properties given in Table 1. A unit cell of dimensions 12.25 mm by 12.25 mm is chosen. As it can be seen in Fig. (3), the shear and flexural wave propagation constants found using the current model have good agreement with those obtained from the analytical solution.

Figure 3: Shear (left) and Flexural (right) waves propagating in a flat infinite plate in the direction of $\theta=0$.

Table 1: Material properties of the polyamide

Property	Value
Young modulus E	1.65 GPa
Poisson ratio ν	0.4
Density ρ	950 kg/m ³

To further validate the method, the results obtained by the model are compared to those given by the Flügge shell theory for the case of a thin cylindrical shell. The standard theories of thin shells define the motion in the structure based on an eighth-order system of differential equations which can take different forms depending upon their assumptions. Specifically, Flügge shell theory neglects shear deformation in the structure and it follows the assumptions made in Love's first approximation theory :

1. The thickness of the shell is small in comparison with other dimensions;
2. Deformations and strains are small enough to justify the second and higher order terms in the strain-displacement relations to be neglected;
3. The transverse normal stress is small enough to be neglected;
4. Straight lines which are normal to the mid-surface of the shell remain straight and normal to the mid-surface after deformation.

This theory is applied to closed cylinders, so it is assumed that the circumferential wavenumber is of the form

$k_\varphi = 0, \frac{1}{a}, \frac{2}{a}, \dots$ as the phase shift μ of a wave propagating around the circumference must be a multiple of 2π .

Therefore, the adopted approach to solve the eigenvalue problem is slightly different from what is explained in the previous section, as in this case the wavenumber vector is assumed as follows:

$$\mathbf{k} = \begin{bmatrix} 0 \\ k_\phi \\ k_z \end{bmatrix}, \quad (19)$$

where k_ϕ is given at $0, \frac{1}{a}, \frac{2}{a}$, etc. and k_z is obtained following the formulations provided in section 2.

Fig. (4) shows the comparison between numerical and analytical results for a closed cylindrical shell of thickness 0.005 m and central radius 1 m with the material properties given in table 1 for $k_\phi = 0, 1$ and 2 m^{-1} which correspond to breathing, bending and ovaling modes, respectively. Considering that such a configuration has a homogeneous geometry, it can be assumed that it is composed of unit cells of arbitrary length. In this case, the unit cell has an axial length of 5 mm and a circumferential length of 10 mm.

Figure 4: Dispersion curves of a closed cylindrical shell at the breathing, bending and ovaling modes

As it can be seen, there is a good agreement between numerical and analytical results. A major property of these dispersion curves is a transition frequency (commonly known as the ring frequency) after which the behaviour of the waves propagating in the cylindrical shell tend to those propagating in a flat panel, and shear, longitudinal and flexural waves can be clearly distinguished .

3.2. Cylindrically curved panels with TVAs

In this section, the potential for stopband creation using cylindrically curved resonant metamaterial panels is studied by considering a simple panel as the host structure with a spring-mass TVA as a resonant structure. To this end, a 1 mm thick 12.5x12.5 mm unit cell is used to calculate the dispersion relations, with a TVA attached to its centre as shown in Fig. (5). The host structure is composed of polyamide with material properties given in Table 1. The mass and stiffness of the TVA are tuned to create a stopband in line with the metamaterial design of Section 3.3, leading to a 29% mass addition and a resonance frequency at 1022 Hz. Each TVA has a mass of 0.043047 grams and a stiffness of 1775.0197 Pa.

In what follows, the stopband behaviour of a curved panel is analysed and compared to the flat plate case for various propagation directions. Next, the influence of the radius of curvature on the stopband behaviour is investigated.

Figure 5: Flat (left) and curved (right) unit cell with TVA

3.2.1. Stopband behaviour in infinite curved panels

To serve as a comparison, the dispersion relations for the host structure with resonators are calculated for the case of an infinite flat plate. Wave propagation is considered for various propagation directions $\theta = 0, \pi/4, \pi/2$; while the angle θ is defined in Fig. (2). For the curved panels, these angles correspond to circumferential, helical and axial propagation directions, respectively. The dispersion curves are calculated for frequencies up to 1500 Hz with a step of 1 Hz.

Figure 6: Dispersion curves for the flat host structure with TVAs for the unit cell shown in Fig. (5) for propagation directions $\theta = 0, \pi/4, \pi/2$ (only propagating parts of waves are presented)

Fig. (6) shows the propagating part of the waves for an infinite flat panel with the addition of TVAs. From here on, the dispersion curves corresponding to longitudinal, shear and flexural waves are numbered as 1, 2 and 3 in the figures, respectively. As shown in Fig. (6), the addition of the TVA leads to a stopband for the flexural vibration mode between 1011 Hz and 1157 Hz as no freely propagating solutions for the flexural mode are found in this frequency range. The other dispersion curves are continuous representing fully propagative longitudinal and shear waves as the TVAs are only tuned to create a stopband in the dispersion curves of the flexural mode. For the propagation directions $\theta = 0$ and $\pi/2$, which lead to the same solutions due to the symmetry of the unit

cell, no free wave propagation is found between 1001 Hz and 1157 Hz. For the propagation direction $\theta = \pi/4$, this frequency zone goes from 1011 Hz to 1157 Hz.

In order to get a better understanding of the effect of curvature on the stopband behaviour for cylindrically curved resonant metamaterials, first the effect of curvature on wave propagation through the host structure is discussed. A radius of 0.5 m is considered. The top three figures in Fig. (7) show the propagative dispersion curves for propagation directions $\theta = 0, \pi/4, \pi/2$ of a unit cell without TVAs. As it can be seen, for each propagation direction, three modes of vibration are found in the frequency range of 0-1500 Hz. Longitudinal and shear waves, i.e. dispersion curves numbered as 1 and 2, are not of direct interest to the current study, since they do not contribute to the acoustic radiation of the cylindrical shell.

The three bottom figures in Fig. (7) show the complex solution for wave propagation in the directions of $\theta = 0, \pi/4, \pi/2$. In these figures, real and imaginary wavenumbers represent attenuated and propagative parts of waves, respectively. Due to symmetry of these dispersion curves with respect to the real axis, only curves with imaginary wavenumbers equal to or bigger than zero are given in the figures. As it can be seen, the flexural wave appears as pairs of purely propagative and purely evanescence waves. It is observed that below a cut-on frequency at $f_c = 458$ Hz the behaviour is highly different from flat panels. This frequency can be calculated analytically using the expression for the ring frequency f_r for a cylindrical shell:

$$f_r = \frac{1}{2\pi a} \sqrt{\frac{E}{\rho(1-\nu^2)}}, \quad (20)$$

with a being the radius of the shell. For the considered radius of 0.5 m and the material parameters of Table 1, this frequency is $f_r = 457.7$ Hz, comparing well to the result obtained from the dispersion relations. Above this frequency, all three wave types are present for the considered propagation direction in the curved panel and the behaviour of the curved shell host structure can be compared to that of an infinite flat plate. While for the circumferential wave propagation direction, flexural wave propagation is possible from 0 Hz onwards, this is not the case for helically and axially propagating waves as the flexural waves cut on at higher frequencies. For the axial propagation direction, the cut-on frequency of the bending wave lies close to this ring frequency. Therefore, this cut-on phenomenon can significantly influence possible stopband behaviour in case of curved structures. However, above cut-on, stopband behaviour can be expected when resonant additions are made on a subwavelength scale.

Figure 7: Dispersion curves of propagating part (top) and complex waves (bottom) for the curved host structure with radius $a=0.5$ m without TVAs for the unit cell shown in Fig. (5) for propagation directions $\theta = 0, \pi/4, \pi/2$

In Fig. (8), the dispersion curves are shown for the curved infinite panel of radius 0.5 m with TVAs. Similar to the results shown for the flat plate with TVAs, a frequency zone is obtained in which no free flexural wave propagation is possible for all considered propagation directions. For the considered panel of radius 0.5 m, the stopband ranges from 1010 Hz to 1155 Hz. Outside this frequency zone, the dispersion curves show good agreement with those of the infinite curved panel without TVAs as shown in Fig. (7). However, each propagation direction should be inspected separately. For the circumferential wave propagation $\theta=0$, no free flexural wave propagation is found between 1000 Hz and 1055 Hz, comparing well to the solution of the flat unit cell with TVA. The helically propagating wave $\theta=\pi/4$ shows no free wave propagation between 1010 Hz and 1161 Hz, a slightly increased frequency range compared to the flat solution. For the axial wave propagation $\theta=\pi/2$, flexural wave propagation is hindered between 1000 Hz and 1175 Hz, with almost 20 Hz increase as compared to the axial solution found for the flat unit cell. This can signify a possible beneficial effect of the radius on the stopband width.

Figure 8: Dispersion curves of propagating part (top) and complex waves (bottom) for the curved host structure with radius $a = 0.5$ m with TVAs for the unit cell shown in Fig. (5) for propagation directions $\theta = 0, \pi/4, \pi/2$

For all of the considered propagation directions, the cut-on frequency is lowered compared to the host structure from 458 Hz to 396 Hz. Moreover, another cut-on frequency appears after the stopband at 1179 Hz. The lowering in cut-on frequency can be partially understood by inspecting Eq. (20) for the ring frequency. Since the addition of TVAs amounts to a mass addition of 29%, this frequency can be re-calculated considering a host structure of equivalent mass. Using an equivalent density $\rho_{eq} = 1225.5$ kg/m³, a ring frequency of $f_r = 403$ Hz is found, comparing relatively well to the numerically obtained cut-on frequency. Furthermore, it was shown that before the stopband, TVAs and the host structure move in-phase, while after the stopband, they move out-of-phase. Note that in-phase and out-of-phase motion can lead to different dynamic masses; while the static mass density relates to the mass per unit volume, dynamic mass density is defined as the inertial mass density, so unlike static mass, the dynamic mass is both dependent on mass distribution and the phase of the motion. When TVAs pass through their resonance, the in-phase motion between the TVAs and the host structure becomes out-of-phase. Below the resonance frequency, due to the in-phase motion, the effective dynamic mass is higher than that of the host structure while above this frequency the effective dynamic mass is lowered as a result of the out-of-phase motion of the host structure and the TVAs.

As stated, the addition of TVAs changes the dynamic mass of the host structure which results also in a change in the ring frequency which determines the cut-on frequency of the flexural mode in the axial direction of

wave propagation. Since the motion below stopband is in-phase, the effective dynamic mass will be higher than that of the host structure which as a result reduces the ring frequency. This effect can be seen in Fig. (7) where the onset of the flexural mode below the stopband is lowered as compared to that of the host structure given in Fig. (6). For the helically and axially propagating waves, the onset of the flexural mode is lowered, from 212 Hz to 186 Hz and from 421 Hz to 364 Hz, respectively. For these non-circumferential propagation directions, the frequency range of the flexural wave is consequently increased before the stopband zone.

3.2.2. Influence of radius on stopband behaviour

In section 3.2.1, it was shown that stopband behaviour can occur in infinite cylindrically curved panels with resonant additions. However, curvature leads to a cut-on phenomenon, above which the structure behaves as a flat plate, while below this frequency the behaviour is more complex. As it is shown in the complex dispersion curves in Fig. (7) this cut-on frequency defines the onset of the flexural wave propagation in the axial direction while in other directions, the flexural waves start to propagate at frequencies between 0 and the cut-on frequency of the axial direction. Below this cut-on frequency, the dispersion curves corresponding to the flexural waves appear as highly attenuated complex solutions. In addition, below the cut-on frequency, different waves (longitudinal, shear and flexural) are coupled to each other. Besides this, especially for waves propagating in axial direction, a possibly beneficial effect of the curvature on the width of the stopband was observed. Therefore, in this section, the effect of the radius is further analysed.

The considered values for the radius are 1, 0.5, 0.2 and 0.1 m. As in the previous section, for each radius, the dispersion relations are calculated for propagation along the directions $\theta = 0, \pi/4, \pi/2$. For the flat plate and for the cylindrically curved panel of radius $a=0.5$ m, these solutions have been presented in Figs. (6) and (8), respectively. The results for the remaining radii are presented in the following figures.

In Fig. (9), the dispersion curves of the propagating waves in a cylindrically curved panel of radius 1 m with TVAs are given. The cut-on frequency of the cylindrical host structure occurs at 229 Hz, far below the targeted stopband frequency zone. Consequently, as the panel behaves as an infinite flat plate in the targeted frequency zone, the stopband behaviour is expected to be similar to that of the flat plate; therefore, Fig. (9) indicates a stopband, between 1010 Hz and 1156 Hz, very similar to the stopband of the flat unit cell. In addition, these results show more resemblance with the flat panel case, as the infinite flat panel is a special case of the infinite curved panel with $a \rightarrow \infty$. Similar to the results for the radius of 0.5 m, the cut-on frequency is lowered to 201 Hz, while a new cut-on frequency is created at 1161 Hz.

Figure 9: Dispersion curves for the curved host structure with radius $a=1$ m with TVAs for the unit cell shown in Figure 1 for propagation directions $\theta = 0, \pi/4, \pi/2$ (only propagating parts of waves are presented)

Fig. (10) shows the dispersion curves calculated for a radius of 0.2 m. For this radius, the curved host structure has a cut-on frequency at 1145 Hz. Contrary to the previous results, this frequency lies within the targeted stopband zone, possibly influencing the stopband creation as infinite flat plate behaviour occurs after this frequency. The dispersion relations show a lowered cut-on frequency at 834 Hz and a new cut-on frequency after the stopband at 1397 Hz. From the dispersion curves, stopband behaviour is still clearly observed. With decreasing radius, a more gradual transition between coupled and uncoupled modes can be observed, making it more difficult to identify the behaviour of the flexural wave. The width of the stopband for propagation directions $\theta=0$ and $\pi/4$ shows a slight increase as compared to the results of the infinite flat panel. For circumferential wave propagation, this frequency zone becomes 1000 – 1159 Hz, while for the helical wave propagation, the stopband lies between 1010 Hz and 1192 Hz. For these propagation directions $\theta=0$ and $\pi/4$, onset of the flexural mode of the host structure occurs at 0 Hz and 539 Hz, respectively; consequently, a stopband can be obtained for these directions while for the axial propagation direction, the onset of the flexural mode of the host structure occurs at 1052 Hz, which is within the targeted stopband zone. From Fig. (10), a large stopband is observed between 1001 Hz and 1346 Hz for the waves propagating in the axial direction. It is noted, however, that before this stopband, an additional zone of flexural waves now starts at 795 Hz, significantly lower than the 1052 Hz of the host structure. The stopband is larger in this case, but there could be efficient radiation due to the flexural waves at lower frequencies as compared to the host structure.

Figure 10: Dispersion curves for the curved host structure with radius $a = 0.2$ m with TVAs for the unit cell shown in Figure 1 for propagation directions $\theta = 0, \pi/4, \pi/2$ (only propagating parts of waves are presented)

Figure 11: Dispersion curves for the curved host structure with radius $a = 0.1$ m with TVAs for the unit cell shown in Figure 1 for propagation directions $\theta = 0, \pi/4, \pi/2$ (only propagating parts of waves are presented)

Table 2: Cut-on frequencies (analytical and numerical) for host structure, host structure with equivalent mass and for host structure with resonant additions, and stopband frequencies for axial wave propagation

Radius [m]	$f_{c,host}$ [Hz]	$f_{r,\rho_{eq}}$ [Hz]	$f_{c,1,MM}$ [Hz]	$f_{c,2,MM}$ [Hz]	Stopband in the axial direction [Hz]
Infinite (flat panel)	-	-	-	-	1001-1157
1	229	201.5	201	1161	1000-1160
0.5	458	403	396	1179	1000-1175
0.2	1145	1007.5	834	1397	1001-1346
0.1	2289	2015	984	2368	-

Decreasing the radius to 0.1 m provides the dispersion curves shown in Fig. (11). For this radius, the cut-on frequency of the host structure lies far above the targeted frequency zone at 2289 Hz. For the circumferential and helical propagation directions $\theta = 0$ and $\pi/4$, the dispersion curves show strong coupled behaviour in this frequency zone. Stopband zones are found for the flexural waves propagating in $\theta = 0$ and $\pi/4$ directions. As the cut-on frequency of the flexural wave in the axial direction is above the stopband zone, no clear discontinuity is observed in the curves corresponding to this direction. This does not influence the existence of an overall stopband zone for flexural waves in the targeted frequency range.

Table 2 summarizes the results for panels with various radii. In this table, $f_{c,host}$, $f_{r,\rho_{eq}}$, $f_{c,1,MM}$ and $f_{c,2,MM}$ are the ring frequency of the host structure, the ring frequency of host structure with equivalent mass, cut-on frequency of the structure with TVA and the additional cut-on frequency of the longitudinal mode which shows up with the addition of TVAs, respectively. The cut-on frequency is given for every radius. Using Eq. (20), the expected cut-on frequency $f_{r,\rho_{eq}}$ is calculated for the host structure with an equivalent mass density of the structure with resonant additions. This can be compared with the lowered cut-on frequency $f_{c,1,MM}$ for the curved panel with TVA additions, showing that for smaller radii, the dynamic mass effect of the relative motion between host structure and TVAs becomes more important. An increase in the stopband for the axial wave propagation is noted with decreasing radius, for cut-on frequencies below the targeted stopband zone. When the cut-on frequency lies within the frequency zone of interest, the stopband is preceded by an additional zone of propagative flexural waves but the stopband covers a wider range of frequency compared to other cases.

3.3. Cylindrically curved locally resonant metamaterial panels

Following from the TVA representation of the metamaterial described in section 3.2, in the following section, the behaviour of a locally resonant metamaterial with resonant cells for which the flat counterparts have already been experimentally validated is investigated. Note that, when dealing with acoustic radiation efficiency, flexural mode of vibration is of high importance; therefore, the resonant cell is designed to have an outspoken flexural resonance which introduces a stopband in the dispersion curves of the flexural mode when added to the host structure in the subwavelength scale.

3.3.1. Design of curved resonant unit cells

Following from the research of Claeys et al. , a resonant cell is introduced to the host structure. The resonant cell consists of a mass connected to the host structure with two legs. The unit cells and the resulting geometries are given in Figs. (12) and (13), while the dimensions of the resonant cells are given in Figs. (14) and (15). For the case of the cylindrically curved unit cell, the curvature of the resonator is kept equal to that of the host structure as also shown in Fig. (14). The circumferential angular length of the unit cell is considered as the ratio of the length of the planar unit cell which is 12.5 mm and the radius of the panel. Various amounts of radii are considered for the cylindrically curved metamaterial panel through the following sections.

The material properties of the metamaterial are those of a polyamide and are given in Table 1. As explained before, the resonant cell is designed to introduce a local flexural resonance to the vibration of the host structure.

This resonance can be found at 1101 Hz for a cylindrically curved panel of radius 0.1 m; while for all of the resonators with various radii which are considered in this paper, the changes in this frequency is below 1 Hz, in the case of the flat panel this resonance is found at 1098.4 Hz. This eigenmode is evaluated using COMSOL Multiphysics® and is shown in Fig. (16). Fixed boundary conditions are assumed at the end of the legs of the resonator. The mesh consists of 7664 quadratic tetrahedron solid elements with the maximum element size of 0.7 mm forming 35304 degrees of freedom to ensure the convergence of the first mode of vibration with an error less than 0.1%. Fig. (16) shows the deformed configuration of the resonator at the stated flexural resonance. The wireframe represents the undeformed state of the resonator while the solid form shows the deformed configuration representing an outspoken flexural mode.

Figure 12: Schematic of the infinite cylindrically curved panel (left) and its representative unit cell (right).

Figure 13: Schematic of the infinite plate (left) and its representative unit cell (right).

Figure 14: Geometrical specifications of the curved resonant cells (all dimensions are in mm).

Figure 15: Geometrical specifications of the flat resonant cells (all dimensions are in mm).

Figure 16: First flexural mode in the resonator at frequency of 1101.1 Hz.

3.3.2. Dispersion curves of the cylindrically curved locally resonant panel

To calculate the dispersion curves, the unit cell is discretized with almost 20000 quadratic tetrahedron solid elements with a maximum element size of 0.7 mm forming approximately 100000 degrees of freedom to ensure the convergence of the results in the frequency range of 0-1500 Hz. Considering the change in radius, the number of elements and therefore, degrees of freedom slightly vary in different case studies.

3.3.2.1. Wave propagation in the host structure

Similar to section 3.2.1, to better understand the effect of resonant cells on free wave propagation through the current cylindrically curved panel, first, the dispersion curves of the host structure without the resonators (as shown in Fig. (17)) are investigated.

Figure 17: Schematic of the infinite host structure (left) and its representative unit cell (right).

Figs. (18) and (19) show the dispersion curves of waves propagating in a host structure of radius 0.5 m, at three different wave propagation directions of $\theta=0, \pi/4$ and $\pi/2$ as circumferential, helical and axial directions, respectively. As can be seen from Fig. (18), below a cut-on frequency at 380 Hz which is the

counterpart of the so-called ring frequency in closed cylindrical shells, only two types of waves propagate and the third wave cuts on after this frequency. While in the axial direction of wave propagation this frequency indicates the onset of the flexural mode, in other directions of wave propagation, this frequency represents the cut-on frequency of the longitudinal mode. The cut-on frequencies are calculated by setting the wavenumber equal to zero in the quadratic eigenvalue problem of Eq. (18) and solving for frequency while limits of the stopband zone are considered as the intersection of the attenuated complex curves existing in stopband zone with the real axis. The full complex solutions of the dispersion relation for the same host structure in the range of 0-1500 Hz are given in Fig. (19). For the sake of clarity and considering that the dispersion curves are symmetric with respect to the real axis, Fig. (19) shows only the on-going waves. As it can be seen in the circumferential and axial directions, dispersion curves of different modes are distinguishable, and the cut-on frequency indicates the on-set of the longitudinal and flexural modes, respectively. In the circumferential direction, the flexural mode is showing up in two pairs of propagating and evanescent waves while in the axial direction, below the cut-on frequency, the flexural mode is highly attenuated and only propagates after this frequency. In the direction of $\theta = \pi/4$, at lower frequencies only the shear mode is distinguishable while the other modes are coupled at lower values of μ while they decouple as μ (and frequency) increases, resulting in the propagation of the flexural mode while the longitudinal waves propagate only after the cut-on frequency at 380 Hz. In this case, the flexural mode starts to propagate at $k_\phi = 1/a$ which corresponds to the bending mode in a closed cylindrical shell.

These results are in agreement with those given in the literature about cylindrical shells and those presented in section 3.2, considering that below the cut-on frequency, waves have a coupled nature while at higher frequencies different modes completely decouple .

Figure 18: Dispersion curves of the propagating waves in $\theta = 0, \pi/4$ and $\pi/2$ directions in the unit cells depicted in Fig. (17) with a radius of 0.5 m.

Figure 19: Dispersion curves of all the existing waves in $\theta = 0, \pi/4$ and $\pi/2$ directions in the unit cells depicted in Fig. (17) with a radius of 0.5 m.

3.3.2.2. Stopband in cylindrically curved metamaterial panel

The existence of a stopband zone in the flexural mode of vibration due to the current configuration of the resonant cell has been already proven for the case of a planar host structure by Claeys et al.. The goal of this section is to extend the findings of previous researches for planar metamaterial shown in Fig. (13) to the case of cylindrical panels shown in Fig. (12). To this end, Figs. (18) and (19) can be used as a reference case to compare with the results obtained for the case of the cylindrically curved locally resonant metamaterial panel.

Figure 20: Dispersion curves of the propagating waves in $\theta = 0, \pi/4$ and $\pi/2$ directions in the unit cells depicted in Fig. (12) with a radius of 0.5 m.

Fig. (20) depicts the dispersion curves of the waves propagating in $\theta = 0, \pi/4$ and $\pi/2$ directions in the infinite cylindrically curved panel of radius 0.5 m as shown in Fig. (12). As it can be seen, at frequencies lower than the local resonance, the overall shapes of the dispersion curves are similar to those obtained for the host structure while at frequencies around this resonance and in the dispersion curves of the flexural mode, a stopband is formed in the frequency range of 1000-1170 Hz and in this range, the flexural waves cannot propagate due to the destructive interference between motion of the host structure and motion of the local resonators. Figs. (21)-(23) show the complex wave propagation constants for the same unit cell in the circumferential, $\theta = \pi/4$ and axial directions of wave propagation, respectively. As it can be seen from the

complex solutions, flexural waves after the cut-on frequency and below the stopband zone are either purely propagative or evanescent while at the stopband zone they become highly attenuated complex waves and then reappear after the stopband as propagative/evanescent waves with lower propagation constants compared to the zone before the stopband. The right figures show all the possible complex solutions in the frequency range of 0-1500 Hz. The stopbands can be detected as the complex branches in these figures.

Figure 21: Dispersion curves of all the existing waves in $\theta = 0$ directions in the unit cells depicted in Fig. (12) with a radius of 0.5 m.

Figure 22: Dispersion curves of all the existing waves in $\theta = \pi/4$ directions in the unit cells depicted in Fig. (12) with a radius of 0.5 m.

Figure 23: Dispersion curves of all the existing waves in $\theta = \pi/2$ directions in the unit cells depicted in Fig. (12) with a radius of 0.5 m.

Considering the high importance of the cut-on frequencies of the waves propagating in the cylindrically curved panel, and as predicted in section 3.2 that radius plays an important role on determining this cut-on frequency. Similar to section 3.2, the dispersion curves of the flat unit cell shown in Fig. (13) are given in Fig.

(24) as a reference case. This configuration theoretically describes a cylindrically curved unit cell when the radius tends to infinity. The dispersion curves show a clear stopband between the frequencies of 1000 and 1155 Hz in the wave propagation of the flexural mode. As expected, in this case all of the waves are distinguishable as no coupling exists between different types of propagating waves.

Figure 24: Dispersion curves of the propagating waves in $\theta = 0, \pi/4$ and $\pi/2$ directions in the unit cells depicted in Fig. (13).

Figs. (25), (26) and (28) show the dispersion curves of waves propagating in a cylindrically curved locally resonant panel of radius 1, 0.2 and 0.1 m, respectively. As it can be seen from Fig. (25), the behaviour of the cylindrically curved panel of radius 1 m is quite similar to that of the flat panel, with a stopband zone in the frequency range of 1000-1155 Hz. In addition, except for the very low frequencies below the cut-on frequency at 160 Hz, waves of different modes are decoupled in this case. This behaviour can be explained due to the fact that the cut-on frequencies of all of the waves propagating in this frequency range are below the stopband zone which means the cylindrically curved unit cell is behaving similar to a planar unit cell in the frequency range of interest.

Figure 25: Dispersion curves of the propagating waves in $\theta = 0, \pi/4$ and $\pi/2$ directions in the unit cells depicted in Fig. (12) with a radius of 1 m.

In the case of $a = 0.2$ m, the cut-on frequency of the host structure is close to the stopband zone at 948 Hz while this frequency is lower in presence of the resonators at 720 Hz as can be seen in Fig. (26). In this case,

in the axial direction of wave propagation, a clear gap is observed in the dispersion curves of the flexural mode in the range of 1010-1250 Hz, while in the direction of wave propagation of $\theta = 0$ and $\pi/4$ due to the coupling of waves, the upper limit of the gap is more difficult to detect due to the addition of an extra cut-on frequency above the stopband zone as explained in section 3.2. In the upper limit of the stopband zone and in the lower wavenumbers, longitudinal and flexural waves are coupled causing the stopband zone to be difficult to detect as the non-flexural portion of the coupled wave is still able to propagate; but this does not affect the efficiency of the metamaterial as the flexural part of the waves are not appearing in this zone.

Figure 26: Dispersion curves of the propagating waves in $\theta = 0, \pi/4$ and $\pi/2$ directions in the unit cells depicted in Fig. (12) with a radius of 0.2 m.

To further understand this coupled behaviour, Fig. (27) shows the full complex solution of all of the existing waves in the circumferential direction of wave propagation. As it can be seen, the expected stopband behaviour is observed; as the discontinuity in the dispersion curves of the flexural waves is visible and flexural waves are showing up as highly attenuated complex solutions in the frequency range of the stopband. However, the longitudinal mode is propagating in the stopband zone. The longitudinal wave is switched into the flexural wave at the upper limit of the stopband and later on fully cuts-on at the frequency of 1280 Hz. This is in agreement with findings of section 3.2 that in the cylindrically curved resonant metamaterial of radius 0.2 m, a second cut-on frequency of the longitudinal mode can be detected at higher frequencies.

In the case of radius 0.1 m, as shown in Fig. (28) and similar to section 3.2.2, in the axial direction, no discontinuity related to the stopband behaviour is observed which is due to the fact that the cut-on frequency of the flexural mode is above the stopband zone at 1875 Hz. In the $\theta = 0$ and $\pi/4$ directions where the flexural mode is cut-on below the stopband zone, stopbands can be detected in the dispersion curves. Moreover, in the axial direction and in the vicinity of the expected stopband zone, an additional dispersion curve appears due to wave propagation in the resonators.

Figure 27: Dispersion curves of all the existing waves in circumferential direction in the unit cells depicted in Fig. (12) with a radius of 0.2 m.

Figure 28: Dispersion curves of the propagating waves in $\theta = 0, \pi/4$ and $\pi/2$ directions in the unit cells depicted in Fig. (12) with a radius of 0.1 m.

For the sake of comparison, Fig. (29) provides the dispersion curves corresponding to the waves propagating in different directions for cylindrical metamaterial panels of different radii, $a = \infty, 1, 0.5, 0.2$ and 0.1 m. It is observed that for the case of a flat panel, the stopband zone is similar for all directions of wave propagation and the curves corresponding to $\theta=0$ and $\pi/2$ are exactly the same; this is expected based on the symmetry of the geometry. This symmetry does not hold for a curved unit cell, therefore, dispersion curves for these directions deviate from each other. The deviation between dispersion curves of waves propagating in axial and circumferential direction increases as the radius decreases. For the case of highly curved unit cells, i.e. $a=0.1$ and 0.2 m, the flexural waves interact with longitudinal waves; this behavior can be clearly noticed in the dispersion curves of the flexural waves above the stopband where it can be seen that the longitudinal wave transitions to a flexural wave. Moreover, while in the case of a flat panel and after the stopband, the flexural wave cuts on with a wavenumber equal to zero, in curved cases, these waves cut-on with a non-zero wavenumber due to their transition from a longitudinal mode.

Figure 29: Dispersion curves of the propagating waves in the unit cells depicted in Fig. (12) with $a = \infty, 1, 0.5, 0.2, 0.1$ m with various directions of wave propagation.

Fig. (30) compares dispersion curves related to flexural waves propagating in metamaterials of radius $a = \infty, 1, 0.5, 0.2$ and 0.1 m for different directions of wave propagation. As it can be seen, decreasing the radius to lower values deviates the dispersion curves from those of the flat panel and this deviation increases as θ increases, reaching its biggest difference in the axial direction where the onset of the flexural mode is close to the ring frequency of the host structure.

Figure 30: Dispersion curves of the propagating waves in $\theta = 0, \pi/4$ and $\pi/2$ directions in the unit cells depicted in Fig. (12) with various radii.

The cut-on frequencies of the locally resonant cylindrically curved panels of various radii and their host structures together with their stopbands in different directions are given in table 3. The conclusions drawn from the previous figures can once again be observed in this table. As it can be seen, The stopband in the axial direction of wave propagation is wider for the panels with cut-on frequencies close to the stopband zone while

for panels with cut-on frequencies above this zone, no clear stopband is observed, but the addition of the local resonance has caused the cut-on frequency of the flexural mode to shift up as compared to that of the host structure. Finally, it is once again clear that with increase in the radius, the behaviour of the cylindrically curved panel tends to that of its planar counterpart. The stopband behaviour in the axial direction can be of more importance when dealing with full cylinders as this direction of wave propagation is the counterpart of the breathing mode which is the most dominant wave when it comes to acoustic radiation.

Table 3: cut-on frequencies and the stopband zone of flexural waves

Radius [m]	Cut-on frequency of the host structure [Hz]	Cut-on frequency [Hz]	Stopband in circumferential direction [Hz]	Stopband in the direction of $\theta = \pi/4$ [Hz]	Stopband in axial direction [Hz]
Infinite (flat panel)	-	-	1000-1155	1000-1155	1000-1155
1	191	160	1000-1155	1000-1155	1000-1155
0.5	380	315	1010-1160	1000-1160	1000-1165
0.2	948	716	1010-1170	1010-1180	1010-1280
0.1	1875	1912	1010-1200	1010-1270	-

Conclusion

In this paper, wave propagation in an infinite cylindrically curved locally resonant panel is investigated. The resonant cells are distributed periodically on a subwavelength scale, allowing the application of unit cell modelling through Bloch-Floquet theorem. Following this method, dispersion curves of the stated structure are obtained to analyse the stopband behavior. For better understanding of wave propagation in cylindrically curved metamaterial panels, both an academic TVA representation and a more realistic metamaterial design are discussed. It is observed that similar to the case of a planar metamaterial, introduction of the resonant structures to the curved host structure creates a stopband zone in the dispersion curves in which free wave propagation is not possible. In addition, it is observed that the radius of the cylindrically curved panel has a significant effect on the stopband behavior due to its effect on the cut-on frequencies of propagating waves. This effect is more significant for the waves propagating in the axial direction where the cut-on frequency denotes the onset of the pure flexural mode. It is shown that curved host structures with a larger radius with cut-on frequencies below the stopband zone behave identically to their planar counterparts while if the cut-on frequency is in the vicinity of the local resonance the stopband becomes wider; especially, in the case of the waves propagating in the axial direction. For smaller radii, where the stopband is below the cut-on frequency of the flexural wave in the axial direction, no clear discontinuity is observed in the dispersion curves of the waves propagating in the axial direction. Stopband behaviour is still obtained considering that the addition of the local resonances creates

stopband zones in other directions of wave propagation in which flexural wave is cut-on. In conclusion, due to the effects of the cut-on frequency on stopband zone in cylindrically curved panels, the efficiency of the metamaterial is highly related to the behavior of the host structure, although this relevance is understood and therefore can be predicted.

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References

- [1] L. Fritschi, L. Brown, R. Kim, D. Schwela and S. Kephelopous, "Burden of disease from environmental noise: Quantification of healthy life years lost in Europe," WHO Regional Office for Europe, 2011.
- [2] F. Lemoult, N. Kaina, M. Fink and G. Lerosey, "Wave propagation control at the deep sub-wavelength scale in metamaterials," *Nature Physics*, vol. 9, pp. 55-60, 2012.
- [3] Z. Liu, X. Zhang, Y. Mao, Y. Zhu, Z. Yang, C. Chan and P. Sheng, "Locally resonant sonic materials," *Science*, vol. 289, no. 5485, p. 1734–1736, 2000.
- [4] H. Reda, Y. Rahali, J. Ganghoffer and H. Iakiss, "Wave propagation in 3D viscoelastic auxetic and textile materials by homogenized continuum micropolar models," *Composite structures*, vol. 141, pp. 328-345, 2016.
- [5] X. Zhou, X. Liu and G. Hu, "Elastic metamaterials with local resonances: an overview," *Theoretical and Applied Mechanics Letters*, vol. 2, no. 041001, 2012.
- [6] C. Claeys, K. Vergote, P. Sas and W. Desmet, "On the potential of tuned resonators to obtain low-frequency vibrational stop bands in periodic panels," *Journal of Sound and Vibration*, vol. 332, no. 6, p. 1418–1436, 2013.
- [7] C. Claeys, P. Sas and W. Desmet, "On the acoustic radiation efficiency of local resonance based stop band materials," *Journal of Sound and Vibration*, vol. 333, no. 14, pp. 3203-3213, 2014.
- [8] P. Pai, H. Peng and S. Jiang, "Acoustic metamaterial beams based on multi-frequency vibration absorbers," *International Journal of Mechanical Sciences*, vol. 79, p. 195–205, 2014.
- [9] H. Peng and P. Pai, "Acoustic metamaterial plates for elastic wave absorption and structural vibration suppression," *International Journal of Mechanical Sciences*, vol. 89, p. 350–361, 2014.
- [10] C. Claeys, E. Deckers, B. Pluymers and W. Desmet, "A lightweight vibro-acoustic metamaterial demonstrator: Numerical and experimental investigation," *Mechanical Systems and Signal Processing*, Vols. 70-71, pp. 853-880, 2016.
- [11] C. Goffaux, J. Sánchez-Dehesa, A. Yeyati, P. Lambin, A. Khelif, J. Vasseur and B. Djafari-Rouhani, "Evidence of fano-like interference phenomena in locally resonant materials," *Physical Review Letters*, vol. 88, no. 22, p. 225502, 2002.
- [12] U. Fano, "Effects of Configuration Interaction on Intensities and Phase Shifts," *Physical Review Letters*, vol. 124, no. 1866, 1961.
- [13] C. Claeys, "Design and analysis of resonant metamaterials for acoustic insulation," Ph.D. dissertation, KU Leuven, Leuven, April 2014.
- [14] H. Peng, P. Pai and H. Deng, "Acoustic multi-stopband metamaterial plates design for broadband elastic wave absorption and vibration suppression," *International Journal of Mechanical Sciences*, vol. 103, pp. 104-114, 2015.
- [15] J. Mead, "Wave propagation in continuous periodic structures: Research contributions from Southampton,"

Journal of Sound and Vibration, vol. 190, no. 3, pp. 495-524, 1996.

[16] F. Bloch, "Über die Quantenmechanik der Electron in Kristallgittern," *Zeitschrift für Physik*, vol. 52, pp. 550-600, 1928.

[17] G. Floquet, "Sur les équations diff Trentielles linéaires a coefficients périodiques," *An-nales de l'Ecole Normale Supérieure*, vol. 12, p. 47-88, 1883.

[18] L. Brillouin, *Wave propagation in periodic structures*, New York: Dover Publication, 1953.

[19] M. Collet, M. Ouisse, M. Ruzzene and M. Ichchou, "Floquet-Bloch decomposition for the computation of dispersion of two-dimensional periodic, damped mechanical systems," *International Journal of Solids and Structures*, vol. 48, p. 2837-2848, 2011.

[20] K. Graff, *Wave Motion in Elastic Solids*, New York: Dover Publications, 1991.

[21] A. Leissa, *Vibration of Shells*, Washington D.C.: U.S. GPO, 1973.

[22] D. Gazis, "Three dimensional investigation of the propagation of waves in hollow circular cylinders. I. Analytical foundation," *The Journal of Acoustical Society of America*, vol. 31, p. 568-573, 1959.

[23] D. Gazis, "Three-dimensional investigation of the propagation of waves in hollow circular cylinders. II. Numerical result," *The Journal of Acoustical Society of America*, vol. 31, p. 573-578, 1959.

[24] P. Naghdi and R. Cooper, "Propagation of elastic waves in cylindrical shells, including the effects of transverse shear and rotatory inertia," *The Journal of Acoustical Society of America*, vol. 28, p. 56-63, 1956.

[25] I. Mirsky and G. Herrmann, "Nonaxially symmetric motions of cylindrical shells," *The Journal of Acoustical Society of America*, vol. 29, p. 1116-1123, 1957.

[26] V. Tyutekin, "Helical waves of an elastic cylindrical shell," *Acoustical Physics*, vol. 50, p. 273-277, 2004.

[27] C. Pany and S. Parthan, "Axial wave propagation in long periodic curved panels," *Journal of Vibration and Acoustics*, vol. 125, p. 24-30, 2003.

[28] E. Manconi and B. Mace, "Wave characterization of cylindrical and curved panels using a finite element method," *The Journal of Acoustical Society of America*, vol. 125, no. 1, pp. 154-163, 2009.

[29] H. Fassbender and D. Kressner, "Structured Eigenvalue Problems," *GAMM Mitteilungen*, vol. 29, no. 2, pp. 297-318, 2006.

[30] F. Tisseur and K. Meerbergen, "The quadratic eigenvalue problem," *SIAM Rev.*, vol. 43, no. 2, p. 235-286, 2001.

[31] "Introduction to COMSOL Multiphysics [Computer software]," Retrieved from <http://cdn.comsol.com/documentation/5.0.1.276/IntroductionToCOMSOLMultiphysics.pdf>, 2015.

[32] F. Fahy, *Sound and structural vibration*, London: Academic Press, 1987.

[33] E. Manconi and B. Mace, "Modelling wave propagation in two-dimensional structures using a wave/finite element technique," ISVR Technical Memorandum 966, Southampton, 2007.

[34] B. Banerjee, *An Introduction to Metamaterials and Waves in Composites*, London: Taylor and Francis, 2011.

Highlights

- Dispersion curves are calculated based on the direct solution of curved unit cell models.
- Both infinite cylindrically curved panels with spring-mass systems and realizable curved structures with resonant structures are studied
- The effects of radius on stopband behaviour for flexural waves are investigated
- Existence of flexural stopbands is shown for various propagation directions where flexural wave motion is cut on.