

M-estimates of location for the robust central tendency of fuzzy data

Beatriz Sinova, María Ángeles Gil and Stefan Van Aelst

Abstract—The Aumann-type mean has been shown to possess valuable properties as a measure of the location or central tendency of fuzzy data associated with a random experiment. However, concerning robustness its behaviour is not appropriate. The Aumann-type mean is highly affected by slight changes in the fuzzy data or when outliers arise in the sample. Robust estimators of location, on the other hand, avoid such adverse effects. For this purpose, this paper considers the M-estimation approach and discusses conditions under which this alternative yields valid fuzzy-valued M-estimators. The resulting M-estimators are applied to a real-life example. Finally, some simulation studies show empirically the suitability of the introduced estimators.

Index Terms—fuzzy number-valued data, M-estimators, random fuzzy numbers, robust location of fuzzy data.

I. INTRODUCTION

IN the current information age, new types of data emerge and we have to face the new challenging problems associated with the analysis of these data. Among these new types of data, fuzzy number-valued outcomes are more and more used in real-life applications to model data from intrinsically imprecise attributes.

When analyzing fuzzy data from a statistical viewpoint, the interest is often focused on the location (central tendency) of the random mechanism that generated the data. More concretely, fuzzy data can be assumed to be measurements of a random fuzzy number (fuzzy random variable in Puri and Ralescu's sense [1]) and the statistical analysis may concern the location of the distribution of this fuzzy number-valued random element.

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Sinova and Gil are with the Department of Statistics, O.R. and D.M. of the University of Oviedo, 33071 Asturias, Spain, e-mail: magil@uniovi.es, (see <http://bellman.ciencias.uniovi.es/SMIRE>), Van Aelst is with the Department of Mathematics, KU Leuven (University of Leuven), 3001 Leuven, Belgium, and Sinova and Van Aelst are with the Department of Applied Mathematics, Computer Science and Statistics, Ghent University, 9000 Gent, Belgium.

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A well-known measure of the location of the distribution of a random fuzzy number is its Aumann-type mean (Puri and Ralescu [1]), which satisfies that

- it is well-defined under quite general conditions,
- it extends the mean value of a real-valued random variable,
- it preserves the main valuable properties of the real-valued mean (it is a 'linear' operator - i.e., it is equivariant under fuzzy arithmetic-based linear transformations-, it fulfils Fréchet's principle w.r.t. most of the L^2 metrics between fuzzy numbers, and it is coherent with the usual fuzzy arithmetic).

However, the Aumann-type mean also inherits from the mean of a real-valued random variable its high sensitivity to either the existence of outliers or small changes in the data. When looking for a more robust location measure, it seems convenient to follow successful approaches that were developed for other types of data. In this respect, this paper focuses on the well-known M-estimation approach.

To adapt the M-estimation approach to the case where the available data are fuzzy number-valued, two methodologies can be considered, namely:

- By using a suitable L^2 metric, fuzzy data can be represented as functional data resulting from (a convex cone within) certain Hilbert space-valued random elements (see, for instance, González-Rodríguez *et al.* [2] and Sinova *et al.* [3]). Consequently, we may adopt existing results and methods from Functional Data Analysis (more specifically, Kim and Scott's ideas and results [4], [5]) to the fuzzy case. This approach is valid whenever one can guarantee that the resulting measure cannot move out of the cone of the fuzzy data.
- *Ad hoc* fuzzy-valued M-estimators can be developed by using particular metrics based on specific representations of fuzzy numbers. For these representations there should exist sets of necessary and sufficient conditions which guarantee that the representation corresponds to a fuzzy number. Two *ad hoc* M-estimators have been introduced in Sinova *et al.* [6], [7] by using convenient L^1 metrics between fuzzy data. The corresponding fuzzy-valued M-estimators are quite easy to compute since their level sets can be expressed in terms of medians of real-valued random variables. It has also been shown that they preserve valuable properties of the median of a real-valued random variable, such as its robustness and strong consistency.

This paper aims to approach the problem by using M-estimators which have been proposed in the area of Functional Data Analysis. Some recent proposals in the context of robust nonparametric density estimation connect ideas from traditional kernel density estimation with standard M-estimation. These proposals can be adapted to Hilbert space-valued random elements and they can be shown to yield valid estimators when restricted to random fuzzy numbers.

The relevant concepts about fuzzy number-valued data and random fuzzy numbers, i.e. one-dimensional random fuzzy sets are recalled in Section II. In Section III M-estimators of location for random fuzzy numbers are formally introduced. Necessary and sufficient conditions guaranteeing their existence, as well as their expression as a weighted mean of the fuzzy-valued sample data are provided. This feature is crucial for fuzzy data to guarantee that the corresponding M-estimators are fuzzy-valued under the considered conditions. Properties of the M-estimators are examined in detail. The weighted mean expression for M-estimators allows us to extend the iteratively re-weighted least squares algorithm used in classical M-estimation for the computation of sample location M-estimates for fuzzy data, as is illustrated by means of an example. Although the conditions for the existence of the M-estimators cover the extension of several important classical M-estimators, some recently proposed *ad hoc* M-estimators do not satisfy these conditions. We recall these proposals here for reasons of comparison. The use of fuzzy-valued M-estimators of location is illustrated by means of a real-life example.

Section IV presents the results of several simulation studies that have been carried out to empirically compare the robustness behavior of the location M-estimators introduced in this paper to the *ad hoc* proposals. Some concluding remarks are gathered in Section V.

II. FUZZY DATA AND RANDOM FUZZY NUMBERS

Fuzzy data appear in several domains. They are mainly used to describe imprecise human assessments such as ratings, opinions, judgments or perceptions in a natural and easy-to-handle way. In particular, when conducting quality ratings, satisfaction valuations and many other surveys, it can often not be expected that responses are expressed as fixed values in a precise scale, since they are essentially imprecise (see e.g. De la Rosa de Sáa *et al.* [8] for a recent detailed discussion about this point).

Fuzzy number-valued data can be interpreted as a ‘level-wise’ extension of interval-valued data, where the levels add a certain gradualness to the imprecision of interval-valued data. Formally, they are defined as follows:

Definition II.1. $\mathcal{F}_c^*(\mathbb{R})$ is the space of the (bounded) fuzzy numbers, that is, the mappings $\tilde{U} : \mathbb{R} \rightarrow [0, 1]$ such that their α -levels

$$\tilde{U}_\alpha = \begin{cases} \{x \in \mathbb{R} : \tilde{U}(x) \geq \alpha\} & \text{if } \alpha \in (0, 1] \\ \text{cl}\{x \in \mathbb{R} : \tilde{U}(x) > 0\} & \text{if } \alpha = 0, \end{cases}$$

are nonempty compact intervals. $\tilde{U}(x)$ can be interpreted as the ‘degree of compatibility’ of x with \tilde{U} (or ‘degree of truth’ of the assertion “ x is \tilde{U} ”).

To perform statistics with fuzzy data, one of the key tools is given by the arithmetic to operate with these data. Most of the theoretical and practical studies with fuzzy data consider the usual and natural approach corresponding to Zadeh’s extension principle, which in this case extends level-wise the usual interval arithmetic.

Definition II.2. Let $\tilde{U}, \tilde{V} \in \mathcal{F}_c^*(\mathbb{R})$. The **sum** of \tilde{U} and \tilde{V} is defined as the fuzzy number $\tilde{U} + \tilde{V} \in \mathcal{F}_c^*(\mathbb{R})$ given for each $\alpha \in [0, 1]$ by

$$\begin{aligned} (\tilde{U} + \tilde{V})_\alpha &= \text{Minkowski sum of } \tilde{U}_\alpha \text{ and } \tilde{V}_\alpha \\ &= [\inf \tilde{U}_\alpha + \inf \tilde{V}_\alpha, \sup \tilde{U}_\alpha + \sup \tilde{V}_\alpha]. \end{aligned}$$

Let $\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})$ and $\gamma \in \mathbb{R}$. The **product** of \tilde{U} by the scalar γ is defined as the fuzzy number $\gamma \cdot \tilde{U} \in \mathcal{F}_c^*(\mathbb{R})$ given for each $\alpha \in [0, 1]$ by

$$(\gamma \cdot \tilde{U})_\alpha = \gamma \cdot \tilde{U}_\alpha = \begin{cases} [\gamma \cdot \inf \tilde{U}_\alpha, \gamma \cdot \sup \tilde{U}_\alpha] & \text{if } \gamma \geq 0 \\ [\gamma \cdot \sup \tilde{U}_\alpha, \gamma \cdot \inf \tilde{U}_\alpha] & \text{otherwise.} \end{cases}$$

A. Distances between fuzzy data

It is well-known that when $\mathcal{F}_c^*(\mathbb{R})$ is endowed with this arithmetic it determines a semi-linear but not a linear space. As a consequence from this fact, *there is no ‘difference operation’* between these values that is simultaneously well-defined and preserves the main properties of the difference between real values in connection with the sum.

Moreover, it should be pointed out that, although fuzzy numbers are formalized as $[0, 1]$ -valued functions, one cannot treat fuzzy data directly as if they were functional data in the way they are usually handled in Functional Data Analysis. This is due to the fact that the functional arithmetic on $\mathcal{F}_c^*(\mathbb{R})$ can yield elements that lie outside this space and thus the meaning of fuzzy data is lost.

These concerns have been substantially overcome by developing statistical methods for imprecise data based on suitable distances between fuzzy numbers. On one hand, distances allow to ‘translate’ the equality of fuzzy numbers into the distance between these values being equal to 0. On the other hand, appropriate distances also allow us to ‘identify’ fuzzy set-valued data with functional data through their so-called support function (see Puri and Ralescu [1]). Distances between fuzzy data have received a lot of attention in the literature. They have often been considered in connection with studies of similarity between fuzzy numbers, as well as for statistical purposes such as classification of fuzzy data or inferential statistics with fuzzy number-valued random elements.

In this paper, a parameterized and versatile family of distances has been chosen, although a few other distances will occasionally be mentioned. This family of distances has been introduced by Bertoluzza *et al.* [9] (see Casals *et al.* [10] for a recent review, and Trutschnig *et al.* [11] for its generalization to fuzzy vector-valued data). It is formalized as follows:

Definition II.3. Let $\theta \in (0, +\infty)$ and let φ be an absolutely continuous probability measure on $([0, 1], \mathcal{B}_{[0,1]})$ with the mass function being positive on $(0, 1)$. The **mid/spr-based L^2 distance** is defined as the mapping $D_\theta^\varphi : \mathcal{F}_c^*(\mathbb{R})$

$\times \mathcal{F}_c^*(\mathbb{R}) \rightarrow [0, +\infty)$ which associates with each pair \tilde{U}, \tilde{V} of elements from $\mathcal{F}_c^*(\mathbb{R})$, the value

$$D_\theta^\varphi(\tilde{U}, \tilde{V}) = \left[\int_{[0,1]} \left(\text{mid } \tilde{U}_\alpha - \text{mid } \tilde{V}_\alpha \right)^2 d\varphi(\alpha) + \theta \int_{[0,1]} \left(\text{spr } \tilde{U}_\alpha - \text{spr } \tilde{V}_\alpha \right)^2 d\varphi(\alpha) \right]^{1/2},$$

where $\text{mid } \tilde{U}_\alpha = (\inf \tilde{U}_\alpha + \sup \tilde{U}_\alpha)/2$ and $\text{spr } \tilde{U}_\alpha = (\sup \tilde{U}_\alpha - \inf \tilde{U}_\alpha)/2$.

The mid/spr-based metric has been shown to be very suitable for the development of statistical methodology for experimental fuzzy set-valued data. For instance, González-Rodríguez *et al.* [2] provides a detailed explanation of an approach to ANOVA with fuzzy data based on the functional data identification. The recent reviews of Blanco-Fernández *et al.* [12], [13], [14], [15] and Gil *et al.* [16] summarize most of these statistical developments.

In these distances, differences in center and shape are taken into account through the mid and spr functions, respectively. The parameter θ and the measure φ do not have a stochastic meaning. The parameter θ weighs the influence of the ‘deviation in shape’ between the fuzzy data with respect to the influence of their ‘deviation in center’. Note that the choice $\theta = 1$ corresponds to a generalization of the well-known distance ρ_2 by Diamond and Kloeden [17], which for any \tilde{U}, \tilde{V} from $\mathcal{F}_c^*(\mathbb{R})$ is given by

$$\begin{aligned} D_1^\varphi(\tilde{U}, \tilde{V}) &= \rho_2^\varphi(\tilde{U}, \tilde{V}) \\ &= \left[\frac{1}{2} \int_{[0,1]} \left(\inf \tilde{U}_\alpha - \inf \tilde{V}_\alpha \right)^2 d\varphi(\alpha) + \frac{1}{2} \int_{[0,1]} \left(\sup \tilde{U}_\alpha - \sup \tilde{V}_\alpha \right)^2 d\varphi(\alpha) \right]^{1/2}. \end{aligned}$$

Another interesting choice is associated with the value $\theta = 1/3$, since (see Casals *et al.* [10]) it can be equivalently expressed as

$$D_{1/3}^\varphi(\tilde{U}, \tilde{V}) = \sqrt{\int_{[0,1]} \int_{[0,1]} \left(\tilde{U}_\alpha^{[\eta]} - \tilde{V}_\alpha^{[\eta]} \right)^2 d\varphi(\alpha) d\ell(\eta)},$$

with $\ell =$ Lebesgue measure on $([0, 1], \mathcal{B}_{[0,1]})$ and $\tilde{U}_\alpha^{[\eta]} = \eta \cdot \sup \tilde{U}_\alpha + (1 - \eta) \cdot \inf \tilde{U}_\alpha$ for all $\eta \in [0, 1]$.

The mid/spr-based distances are L^2 metrics that, through the support function (Puri and Ralescu’s sense [1]), allow us to embed the space of fuzzy numbers into a convex cone of a Hilbert space by a Rådström-type isometry. More details can be found in Trutschnig *et al.* [11] and González-Rodríguez *et al.* [2] (see also Blanco-Fernández *et al.* [12] and Gil *et al.* [16] for recent reviews). In this way, despite the fact that fuzzy data should not be treated directly as functional data, they can be treated as functional data by considering the identification *via* the support function. As one can see later, this identification is crucial for the extension of M-estimators to fuzzy data.

B. Random Fuzzy Numbers

To formalize the statistical analysis of fuzzy data, an essential tool is the adaptation of a suitable model for the random mechanism generating fuzzy data. The notion of random fuzzy number in Puri and Ralescu’s sense [18], [1] is a soundly established and well-supported model within the probabilistic setting. This notion is based on the ideas of Fréchet [19] and Féron [20]. In this model, fuzziness is assumed to affect the nature of values/data and randomness is assumed to affect their generation. Random fuzzy numbers can be defined as follows:

Definition II.4. Let (Ω, \mathcal{A}, P) be a probability space modeling a random experiment. A mapping $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c^*(\mathbb{R})$ is said to be a **random fuzzy number** associated with the random experiment if, and only if, for each $\alpha \in [0, 1]$ the interval-valued mapping \mathcal{X}_α (where $\mathcal{X}_\alpha(\omega) = (\mathcal{X}(\omega))_\alpha$ for all $\omega \in \Omega$) is a random compact interval or equivalently, the real-valued functions $\inf \mathcal{X}_\alpha$ and $\sup \mathcal{X}_\alpha$ are random variables.

It should be highlighted that a random fuzzy number is a Borel-measurable mapping with respect to the Borel σ -field associated with several metrics like, for instance, the mid/spr-based distance (see, for instance, González-Rodríguez *et al.* [2] and Gil *et al.* [16]). As a consequence from the Borel-measurability, crucial concepts such as the *induced distribution of a random fuzzy number* or the *stochastic independence of random fuzzy numbers* can be immediately stated.

The best known measure of location of a random fuzzy number is the Aumann-type mean, which is given by

Definition II.5. Let \mathcal{X} be a random fuzzy number and assume that the expected values of the random variables $\inf \mathcal{X}_\alpha$ and $\sup \mathcal{X}_\alpha$ are finite. The **Aumann-type mean** of \mathcal{X} is the fuzzy number $\tilde{E}(\mathcal{X}) \in \mathcal{F}_c^*(\mathbb{R})$ such that for each $\alpha \in [0, 1]$

$$(\tilde{E}(\mathcal{X}))_\alpha = [E(\inf \mathcal{X}_\alpha), E(\sup \mathcal{X}_\alpha)].$$

Note that this notion, introduced by Puri and Ralescu [1], is coherent with the usual fuzzy arithmetic. In particular, if the possible values of a random fuzzy number \mathcal{X} are given by $\{\tilde{x}_1, \dots, \tilde{x}_m\}$ and $P(\mathcal{X} = \tilde{x}_i) = P(\{\omega \in \Omega : \mathcal{X}(\omega) = \tilde{x}_i\})$, then it holds that

$$\tilde{E}(\mathcal{X}) = P(\mathcal{X} = \tilde{x}_1) \cdot \tilde{x}_1 + \dots + P(\mathcal{X} = \tilde{x}_m) \cdot \tilde{x}_m.$$

That is,

$$\begin{aligned} &(\tilde{E}(\mathcal{X}))_\alpha \\ &= \left[\sum_{i=1}^m \inf(\tilde{x}_i)_\alpha \cdot P(\mathcal{X} = \tilde{x}_i), \sum_{i=1}^m \sup(\tilde{x}_i)_\alpha \cdot P(\mathcal{X} = \tilde{x}_i) \right]. \end{aligned}$$

The Aumann-type mean of a random fuzzy number is supported by Strong Laws of Large Numbers (see e.g. Colubi *et al.* [21]) and also by the Fréchet approach [19], since it is the fuzzy number minimizing the ‘mean square error’ of \mathcal{X} about a fuzzy number. That is,

$$\tilde{E}(\mathcal{X}) = \arg \min_{\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})} E \left[\left(D_\theta^\varphi(\mathcal{X}, \tilde{U}) \right)^2 \right]$$

for every θ and φ for which the expectation exists.

III. LOCATION M-ESTIMATES FROM FUZZY DATA

M-estimation is a well-established approach that yields robust estimators. M-estimators were first introduced by Huber [22] to estimate the location of real-valued data with the aim of limiting the influence of outliers on methods like least squares and maximum likelihood. For this purpose, the key idea consists of replacing the square of the ‘errors’ with a (usually less rapidly increasing) loss function applied to the errors of the data. In this respect, M-estimators were presented as intermediaries between the sample mean and median.

In this section, the notion of M-estimators is extended to the case of fuzzy number-valued data. Sufficient conditions on the loss function to guarantee that the M-estimation process leads to well-defined (i.e., fuzzy-valued) estimates are established. Following Huber’s ideas [22], [23], a location M-estimate in connection with a random fuzzy number is formalized as follows:

Definition III.1. *Let (Ω, \mathcal{A}, P) be a probability space and $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c^*(\mathbb{R})$ an associated random fuzzy number. Moreover, let ρ be an arbitrary continuous real-valued function (which will be referred to hereinafter as the loss function), and $(\mathcal{X}_1, \dots, \mathcal{X}_n)$ a simple random sample from \mathcal{X} . Then, the sample fuzzy M-estimator of location is the fuzzy number-valued statistic $\tilde{g}^M[(\mathcal{X}_1, \dots, \mathcal{X}_n)]$, given by*

$$\tilde{g}^M[(\mathcal{X}_1, \dots, \mathcal{X}_n)] = \arg \min_{\tilde{g} \in \mathcal{F}_c^*(\mathbb{R})} \frac{1}{n} \sum_{i=1}^n \rho(D_\theta^\varphi(\mathcal{X}_i, \tilde{g})),$$

if it exists.

The ρ -function in Definition III.1 generalizes the squared loss function in the least squares approach. Actually, it is usually assumed to vanish at 0 and to be non-decreasing.

If the whole class of fuzzy numbers $\mathcal{F}_c(\mathbb{R})$ (including unbounded ones, with non-compact 0-level) is considered, it is possible to prove that the M-estimator of location is Borel measurable, and therefore, a statistic.

Proposition III.1. *Let (Ω, \mathcal{A}, P) be a Polish probability space and $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c(\mathbb{R})$ an associated random fuzzy number. Moreover, let $(\mathcal{X}_1, \dots, \mathcal{X}_n)$ be a simple random sample from \mathcal{X} and ρ a continuous loss function. Then, the M-estimator of location is well-defined.*

It is very common in practice to fix a bounded referential in \mathbb{R} , as is the case for the fuzzy rating scale by Hesketh *et al.* [37]. When adopting this situation and working on $\mathcal{F}_c([a, b])$, for certain $a, b \in \mathbb{R}$ such that $a < b$, instead of $\mathcal{F}_c(\mathbb{R})$, Proposition III.1 also holds.

To show that Definition III.1 yields valid estimators for the location of fuzzy random variables, we have to show that the estimators are indeed fuzzy-valued statistics. This is an immediate corollary of the following representer theorem.

A. Representer theorem for M-estimates with fuzzy data

Kim and Scott [4], [5] recently proposed a robust non-parametric density estimator by combining a traditional kernel density estimator with ideas from standard M-estimation. An interesting contribution of Kim and Scott’s work lies in their

analysis of the conditions to ensure the existence of sample M-estimates of location. These conditions allow us to express the M-estimates as weighted linear combinations of the sample elements. This feature will be very convenient to assure that the M-estimates are indeed fuzzy number-valued statistics.

The novel representer theorem that will be stated in Theorem III-A.1 is an adaptation of Kim and Scott’s work [4], [5] to the fuzzy-valued case by means of the isometrical embedding mentioned before. It expresses the sample location M-estimate for fuzzy number-valued data as a convex linear combination of the sample components, $\tilde{x}_1, \dots, \tilde{x}_n$.

Theorem III-A.1. *Consider $\mathcal{F}_c^*(\mathbb{R})$ endowed with the metric D_θ^φ . Let $\tilde{\mathbf{x}}_n = (\tilde{x}_1, \dots, \tilde{x}_n)$ be a sample of independent observations from a random fuzzy number $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c^*(\mathbb{R})$ on a probability space (Ω, \mathcal{A}, P) . Moreover, let ρ be a continuous loss function which satisfies the assumptions*

- ρ is non-decreasing, $\rho(0) = 0$ and $\lim_{x \rightarrow 0} \rho(x)/x = 0$,
- Let $\phi(x) = \rho'(x)/x$ and $\phi(0) \equiv \lim_{x \rightarrow 0} \phi(x)$. Then $\phi(0)$ exists and it is finite.

Then, the sample M-estimate of location exists and it can be expressed as

$$\widehat{\tilde{g}^M[\tilde{\mathbf{x}}_n]} = \sum_{i=1}^n w_i \cdot \tilde{x}_i$$

with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$. Furthermore,

$$w_i \propto \phi(D_\theta^\varphi(\tilde{x}_i, \widehat{\tilde{g}^M[\tilde{\mathbf{x}}_n]})).$$

Many loss functions ρ that are used to compute M-estimators satisfy the above-mentioned conditions. One of the most well-known loss functions which fulfills the required conditions is the *Huber loss function* [23], which is given by

$$\rho_a(x) = \begin{cases} x^2/2 & \text{if } |x| \leq a \\ a(|x| - a/2) & \text{otherwise,} \end{cases},$$

with $a > 0$ a tuning parameter. Note that the Huber loss function is convex, but not strictly convex. Huber’s loss function yields a hybrid approach between squared and absolute error losses. Indeed, it corresponds to a parabola in the vicinity of 0 and increases linearly from the given level a onwards, so that one can put appropriate emphasis on large and small errors. Some discussions on the choice of the tuning parameter a can be found in e.g. Wang *et al.* [25] and Debruyne *et al.* [26].

Another well-known loss function is the *Hampel loss function* [27], which corresponds to

$$\rho_{a,b,c}(x) = \begin{cases} x^2/2 & \text{if } 0 \leq |x| < a \\ a(|x| - a/2) & \text{if } a \leq |x| < b \\ \frac{a(|x| - c)^2}{2(b-c)} + \frac{a(b+c-a)}{2} & \text{if } b \leq |x| < c \\ \frac{a(b+c-a)}{2} & \text{if } c \leq |x|, \end{cases}$$

where the nonnegative parameters $a < b < c$ allow us to control the degree of suppression of large errors. The smaller their values, the higher this degree.

The performance of M-estimators thus depends on the choice on the tuning parameter(s) that appear in both loss functions. To fix these tuning parameter(s), one can follow the ideas in Kim and Scott [5], that is,

- 1) Choose as starting solution an easily computable robust centrality measure (to ensure the robustness of the M-estimator unless we can assure that the M-estimator is unique);
- 2) Compute the distances between the fuzzy observations and this initial solution;
- 3) Set a equal to the median of these real-valued distances for both loss functions. For Hampel's loss function, set b and c equal to the 75th and the 85th percentiles of the obtained distances, respectively.

An example of the Huber loss function for $a = 2.36$ (used in Example III.1) is shown in Figure 1. Figure 1 displays the loss function $\rho_{2.36}$, its derivative $\rho'_{2.36}$ and $\phi_{2.36}$ (corresponding to the quotient of $\rho'_{2.36}$ and the identity function).

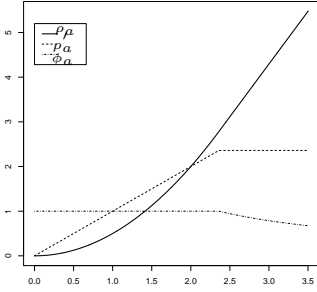


Fig. 1. Huber loss function $\rho_{2.36}$, its derivative $\rho'_{2.36}$ and $\phi_{2.36}$

To illustrate the Hampel loss function for the values of the tuning parameters in Example III.1 below, Figure 2 displays the loss function $\rho_{2.36,3.18,3.51}$, its derivative $\rho'_{2.36,3.18,3.51}$ and $\phi_{2.36,3.18,3.51}$.

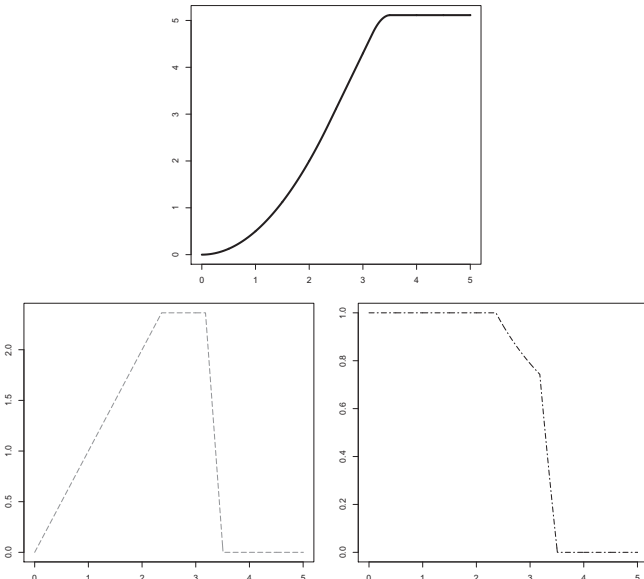


Fig. 2. Hampel loss function $\rho_{a,b,c}$ (top), its derivative $\rho'_{a,b,c}$ (bottom left) and $\phi_{a,b,c}$ (bottom right) with $a = 2.36$, $b = 3.18$ and $c = 3.51$

The following (artificial) example explains the algorithm to compute the M-estimates corresponding to the Huber loss function $\rho_{2.36}$ and to the Hampel loss function $\rho_{2.36,3.18,3.51}$ based on the L^2 metric $D_{1/3}^\ell$.

Example III.1. Consider a sample $\tilde{\mathbf{x}}_3 = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ with three trapezoidal/triangular fuzzy values given by $\tilde{x}_1 = \text{Tra}(0, 2, 3, 4)$, $\tilde{x}_2 = \text{Tri}(1, 1.5, 2)$ and $\tilde{x}_3 = \text{Tri}(3, 4, 5)$ (displayed in grey in Figure 3).

The parameter values are fixed to $a = 2.36$, $b = 3.18$ and $c = 3.51$. For the Huber loss function $\rho_{2.36}$, the corresponding M-estimate of location can then be determined by means of the following algorithm:

Step 1. Take as initial estimate $\tilde{g}_{(1)}^M$ ($k = 1$) a 'central' value. In this case, we have chosen the 1-norm median (see Subsection III-C) as initial robust location measure. Fix a tolerance $\varepsilon = 10^{-7}$.

Step 2. Update the weights

$$w_i^{(k)} = \frac{\phi(D_{1/3}^\ell(\tilde{x}_i, \tilde{g}_{(k)}^M))}{\sum_{j=1}^n \phi(D_{1/3}^\ell(\tilde{x}_j, \tilde{g}_{(k)}^M))}$$

and the estimate

$$\tilde{g}_{(k+1)}^M = \sum_{i=1}^n w_i^{(k)} \cdot \tilde{x}_i.$$

Step 3. Repeat Step 2 until

$$\frac{|J(\tilde{g}_{(k+1)}^M) - J(\tilde{g}_{(k)}^M)|}{J(\tilde{g}_{(k)}^M)} < \varepsilon.$$

The final estimate is given by the trapezoidal fuzzy number $\text{Tra}(1.33, 2.50, 2.83, 3.66)$, which has been displayed in black (straight line) in Figure 3.

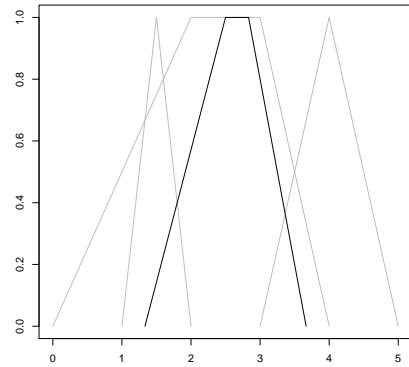


Fig. 3. A Huber (black line) M-estimate of location of the sample of the grey fuzzy data (using $D_{1/3}^\ell$)

Analogously, the corresponding M-estimate of location for the Hampel loss function $\rho_{2.36,3.18,3.51}$ can be calculated, which in this example is almost indistinguishable from the solution for the Huber loss function.

B. Properties of M-estimates satisfying the representer theorem

In this subsection, we state several interesting properties of the fuzzy-valued M-estimates when the conditions of the

representer theorem are satisfied. These properties have not been studied by Kim and Scott in the context of density estimation yet. Proofs can be found as the supplementary material.

In particular, the M-estimates are translation equivariant and under additional assumptions also scale (and, hence, symmetry) equivariant.

Proposition III-B.1. (*Translation equivariance*) Consider $\mathcal{F}_c^*(\mathbb{R})$ endowed with the metric D_θ^φ . Let $\tilde{\mathbf{x}}_n = (\tilde{x}_1, \dots, \tilde{x}_n)$ be a sample of independent observations from a random fuzzy number $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c^*(\mathbb{R})$ on a probability space (Ω, \mathcal{A}, P) . Moreover, let ρ be an arbitrary continuous loss function fulfilling the assumptions in Theorem III-A.1, such that the corresponding sample M-estimate of location is unique.

Let $\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})$. If $\tilde{g}^M[\tilde{\mathbf{x}}_n + \tilde{U}]$ is the location M-estimate based on $\tilde{\mathbf{x}}_n + \tilde{U}$, then it holds that

$$\tilde{g}^M[\tilde{\mathbf{x}}_n + \tilde{U}] = \tilde{g}^M[\tilde{\mathbf{x}}_n] + \tilde{U},$$

where $\tilde{g}^M[\tilde{\mathbf{x}}_n]$ is the location M-estimate based on $\tilde{\mathbf{x}}_n$, which satisfies

- i) $\tilde{g}^M[\tilde{\mathbf{x}}_n] = \sum_{i=1}^n w_i \cdot \tilde{x}_i$,
- ii) $w_i \propto \phi(D_\theta^\varphi(\tilde{x}_i, \tilde{g}^M[\tilde{\mathbf{x}}_n]))$,
- iii) $\sum_{i=1}^n w_i = 1$.

Proposition III-B.2. (*Scale equivariance*) Consider $\mathcal{F}_c^*(\mathbb{R})$ endowed with the metric D_θ^φ . Let $\tilde{\mathbf{x}}_n = (\tilde{x}_1, \dots, \tilde{x}_n)$ be a sample of independent observations from a random fuzzy number $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c^*(\mathbb{R})$ on a probability space (Ω, \mathcal{A}, P) . Moreover, let ρ be an arbitrary continuous loss function fulfilling the assumptions in Theorem III-A.1, such that the corresponding sample M-estimate of location is unique.

If ρ satisfies the condition $\phi(k \cdot x) \propto \phi(x)$ for every $k \in (0, \infty)$ (i.e., the coefficient of proportionality is independent of x), then for all $c \in \mathbb{R}$ it holds that

$$\tilde{g}^M[c \cdot \tilde{\mathbf{x}}_n] = c \cdot \tilde{g}^M[\tilde{\mathbf{x}}_n],$$

where $\tilde{g}^M[c \cdot \tilde{\mathbf{x}}_n]$ is the location M-estimate based on $c \cdot \tilde{\mathbf{x}}_n$, and $\tilde{g}^M[\tilde{\mathbf{x}}_n]$ is the location M-estimate based on $\tilde{\mathbf{x}}_n$, which satisfies

- i) $\tilde{g}^M[\tilde{\mathbf{x}}_n] = \sum_{i=1}^n w_i \cdot \tilde{x}_i$,
- ii) $w_i \propto \phi(D_\theta^\varphi(\tilde{x}_i, \tilde{g}^M[\tilde{\mathbf{x}}_n]))$,
- iii) $\sum_{i=1}^n w_i = 1$.

For instance, if ϕ is a power function, then the corresponding M-estimate is scale equivariant.

Note that the extra condition on the loss function in Theorem III-B.2 is a strong requirement. Most loss functions for M-estimates of location do not satisfy this property. Therefore, similarly as in the real setting (see e.g. Maronna *et al.* [28]) the resulting M-estimate, especially its robustness, may depend heavily on the considered measurement units. To avoid this drawback, we determine the values of the tuning parameters in the considered loss functions based on the distribution of the distances for the particular dataset, as already explained after Theorem III-A.1.

As an immediate consequence from the representer theorem for fuzzy data, one can trivially prove that the M-estimates of LR fuzzy numbers sharing L and R functions (see Dubois and Prade [29]) also fall in the LR family. Recall that \tilde{U} is said to be an LR fuzzy number $(b, c, b - a, d - c)_{LR}$ if

$$\tilde{U}(x) = \begin{cases} L\left(\frac{b-x}{b-a}\right) & \text{if } x \in [a, b) \\ 1 & \text{if } x \in [b, c] \\ R\left(\frac{x-c}{d-c}\right) & \text{if } x \in (c, d] \\ 0 & \text{otherwise} \end{cases}$$

where $a, b, c, d \in \mathbb{R}$, $a \leq b \leq c \leq d$, and $L, R : [0, 1] \rightarrow [0, 1]$ are continuous non-increasing functions such that $L(x) = R(y) = 1$ iff $x = y = 0$ and $L(x) = R(y) = 0$ iff $x = y = 1$.

Proposition III-B.3. Assume the conditions in Theorem III-A.1 are fulfilled. Assume also that $\tilde{\mathbf{x}}_n = (\tilde{x}_1, \dots, \tilde{x}_n)$ is a sample of independent observations such that all \tilde{x}_i 's are LR fuzzy numbers sharing the L and R functions. Then, the sample M-estimate of location is an LR fuzzy number with the same L and R functions. In particular, if all \tilde{x}_i 's are trapezoidal fuzzy numbers, then the sample M-estimate of location is trapezoidal as well.

Another interesting property when examining the adequacy of the location M-estimates based on fuzzy data as central tendency measures concerns their behavior in case of symmetrically distributed samples. In the real-valued case, a well-known result is that the median of a symmetric random variable coincides with the point of symmetry whenever it is unique. In the fuzzy-valued case, a similar result can be obtained for location M-estimates under certain assumptions.

Let $\tilde{\mathbf{x}}_n = (\tilde{x}_1, \dots, \tilde{x}_n)$ be a sample of independent observations from a random fuzzy number $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c^*(\mathbb{R})$ on a probability space (Ω, \mathcal{A}, P) . Then, the sample $\tilde{\mathbf{x}}_n$ is said to be symmetric about $c \in \mathbb{R}$ (see Sinova *et al.* [30]) if and only if $\tilde{\mathbf{x}}_n - c$ and $c - \tilde{\mathbf{x}}_n$ (or, equivalently, $2c - \tilde{\mathbf{x}}_n$ and $\tilde{\mathbf{x}}_n$) include exactly the same fuzzy data. We then have the following result:

Proposition III-B.4. Consider $\mathcal{F}_c^*(\mathbb{R})$ endowed with the metric D_θ^φ . Let $\tilde{\mathbf{x}}_n = (\tilde{x}_1, \dots, \tilde{x}_n)$ be a sample of independent observations from a random fuzzy number $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c^*(\mathbb{R})$ on a probability space (Ω, \mathcal{A}, P) . Moreover, let ρ be an arbitrary continuous loss function fulfilling the assumptions in Theorem III-A.1, such that the associate M-estimate is unique. If the sample $\tilde{\mathbf{x}}_n$ is symmetric about $c \in \mathbb{R}$, then the corresponding location M-estimate $\tilde{g}^M[c \cdot \tilde{\mathbf{x}}_n]$ is a symmetric fuzzy number about c .

In addition to these properties, it is also of importance to investigate the consistency of sample M-estimators as well as their robustness.

Strong consistency concerns the almost sure convergence of sample M-estimators to their population analogue, given by

$$\tilde{g}^M(\mathcal{X}) = \arg \min_{\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})} E \left[\rho \left(D_\theta^\varphi(\mathcal{X}, \tilde{U}) \right) \right].$$

In the literature, consistency has been formally guaranteed under certain conditions (see e.g. Vandermeulen and Scott [31]). However, those results do not include most of the commonly used loss functions. In order to cover, among others, the well-known Huber and Hampel loss functions, the following theorem is now stated under the assumption that the referential fuzzy numbers are defined on is bounded referential, as happens when using the fuzzy rating scale.

Theorem III-B.1. *Consider the metric space $(\mathcal{F}_c([a, b]), D_\theta^\lambda)$, with $a, b \in \mathbb{R}$, $a < b$ and $\theta > 0$. Let $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c^*(\mathbb{R})$ be a random fuzzy number associated with a probability space (Ω, \mathcal{A}, P) . Under any of the following assumptions:*

- ρ is non-decreasing, subadditive and unbounded,
- ρ is the Huber loss function (for any tuning parameter $a > 0$),
- ρ is the Hampel loss function (for any tuning parameters $c > b > a > 0$),

and whenever $\tilde{g}^M(\mathcal{X})$ exists and is unique, the sample M-estimator of location is a strongly consistent estimator of $\tilde{g}^M(\mathcal{X})$, i.e.,

$$\lim_{n \rightarrow \infty} D_\theta^\varphi(\tilde{g}^M([\widehat{\mathcal{X}}_1, \dots, \mathcal{X}_n]), \tilde{g}^M(\mathcal{X})) = 0 \quad \text{a.s. } [P].$$

We now investigate the robustness of location M-estimators for fuzzy data by means of the *finite sample breakdown point*, for short fsbp (Hampel [32], Donoho and Huber [33]). The fsbp is a measure of robustness that can be adapted to estimators taking values in general metric spaces (Cuevas *et al.* [34]). It represents the smallest fraction of observations in a sample that needs to be perturbed to make the (suitably chosen) distance between the M-estimates in the original and contaminated samples arbitrarily large.

The first result shows the importance of the translational equivariance of M-estimators as stated in Proposition III-B.1.

Theorem III-B.2. *Consider the metric space $(\mathcal{F}_c^*(\mathbb{R}), D_\theta^\varphi)$ and θ any nonnegative value. Let $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c^*(\mathbb{R})$ be a random fuzzy number associated with a probability space (Ω, \mathcal{A}, P) . Moreover, let ρ be an arbitrary continuous loss function fulfilling the assumptions in Theorem III-A.1, such that the corresponding sample M-estimate of location is unique. Then, the finite sample breakdown point of the corresponding location M-estimator is at most $\frac{1}{n} \lfloor \frac{n+1}{2} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function.*

Moreover, it is possible to prove that this upper bound for the finite sample breakdown point is attained under an additional sufficient condition on the loss function.

Theorem III-B.3. *Consider the metric space $(\mathcal{F}_c^*(\mathbb{R}), D_\theta^\varphi)$ and θ any nonnegative value. Let $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c^*(\mathbb{R})$ be a random fuzzy number associated with a probability space (Ω, \mathcal{A}, P) . Moreover, let ρ be a continuous loss function fulfilling the assumptions in Theorem III-A.1, such that the corresponding sample M-estimate of location is unique. Under any of these assumptions*

- ρ has linear upper and lower bounds;
- $\rho_{a,b,c}$ is a Hampel's loss function whose tuning parameters satisfy

$$\rho_{a,b,c} \left(\max_{1 \leq i, j \leq n} D_\theta^\varphi(\tilde{x}_i, \tilde{x}_j) \right) < \frac{n - 2 \lfloor \frac{n-1}{2} \rfloor}{n - \lfloor \frac{n-1}{2} \rfloor - 1} \cdot \rho_{a,b,c}(c);$$

the finite sample breakdown point of the corresponding location M-estimator is exactly $\frac{1}{n} \lfloor \frac{n+1}{2} \rfloor$.

Notice that Theorem III-B.3 includes Huber's loss function, which is linear for large values of its argument and thus can be upper and lower bounded by straight lines. This result indicates that the fsbp of both Huber and Hampel M-estimates (the latter for sufficient large choices of the tuning parameters) is equal to $\frac{1}{n} \lfloor \frac{n+1}{2} \rfloor$. In contrast, the fsbp of the sample Aumann-type mean has been shown to be equal to $\frac{1}{n}$ (see e.g. Sinova *et al.* [6], [7]). M-estimates thus yield a huge gain in robustness in comparison to the commonly used Aumann-type mean.

C. Related ad hoc estimators

The representer theorem guarantees the existence of sample M-estimates of location as well as their expression as random weighted means of the sample elements for loss functions that satisfy the needed conditions. While several common choices of the loss function ρ satisfy these conditions, there are also some interesting choices which do not satisfy these conditions. In particular, for the choice $\rho(x) = |x|$, which yields the median in the real-valued case, one cannot apply the results in Section III-A. However, Sinova *et al.* [6], [7] proposed median-type M-estimates for random fuzzy numbers. These M-estimates cannot be expressed as weighted means of the sample elements, but level-wise they can be expressed in terms of the classical medians of real-valued random variables characterizing the fuzzy number. This makes computation of the median-type M-estimators rather easy to accomplish in practice. In particular, Sinova *et al.* [6] introduced the 1-norm median as follows.

Definition III.2. *Let $\tilde{\mathbf{x}}_n = (\tilde{x}_1, \dots, \tilde{x}_n)$ be a sample of independent observations from a random fuzzy number $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c^*(\mathbb{R})$ on a probability space (Ω, \mathcal{A}, P) . The sample **1-norm median** of \mathcal{X} is defined as the fuzzy number $\widehat{\text{Me}}[\tilde{\mathbf{x}}_n] \in \mathcal{F}_c^*(\mathbb{R})$ such that for each $\alpha \in [0, 1]$*

$$\begin{aligned} \left(\widehat{\text{Me}}[\tilde{\mathbf{x}}_n] \right)_\alpha &= [\text{Me}\{\inf(\tilde{x}_1)_\alpha, \dots, \inf(\tilde{x}_n)_\alpha\}, \\ &\quad \text{Me}\{\sup(\tilde{x}_1)_\alpha, \dots, \sup(\tilde{x}_n)_\alpha\}] \end{aligned}$$

(where in case the real-valued median Me is non-unique the mid-point convention is considered).

The 1-norm median satisfies

$$\widehat{\text{Me}}[\tilde{\mathbf{x}}_n] = \arg \min_{\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})} \frac{1}{n} \sum_{i=1}^n \left(\rho_1(\tilde{x}_i, \tilde{U}) \right),$$

where for \tilde{U}, \tilde{V}

$$\begin{aligned} \rho_1(\tilde{U}, \tilde{V}) &= \frac{1}{2} \int_{[0,1]} \left| \inf \tilde{U}_\alpha - \inf \tilde{V}_\alpha \right| d\alpha \\ &\quad + \frac{1}{2} \int_{[0,1]} \left| \sup \tilde{U}_\alpha - \sup \tilde{V}_\alpha \right| d\alpha \end{aligned}$$

(see Diamond and Kloeden [17]).

In other words, the 1-norm median corresponds to an M-estimate associated with the loss function $\rho(x) = |x|$ and the L^1 distance ρ_1 .

Similarly, Sinova *et al.* [7] have introduced the φ -wabl/ldev/rdev median as follows.

Definition III.3. Let $\tilde{\mathbf{x}}_n = (\tilde{x}_1, \dots, \tilde{x}_n)$ be a sample of independent observations from a random fuzzy number $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c^*(\mathbb{R})$ on a probability space (Ω, \mathcal{A}, P) . Then the sample φ -wabl/ldev/rdev median of \mathcal{X} is defined as the fuzzy number $\widehat{\text{Me}}[\tilde{\mathbf{x}}_n] \in \mathcal{F}_c^*(\mathbb{R})$ such that for each $\alpha \in [0, 1]$

$$\begin{aligned} \left(\widehat{\text{M}}^\varphi[\tilde{\mathbf{x}}_n] \right)_\alpha &= \left[\text{Me}\{\text{wabl}^\varphi(\tilde{x}_1), \dots, \text{wabl}^\varphi(\tilde{x}_n)\} \right. \\ &\quad - \text{Me}\{\text{ldev}_{\tilde{x}_1}^\varphi(\alpha), \dots, \text{ldev}_{\tilde{x}_n}^\varphi(\alpha)\}, \\ &\quad \text{Me}\{\text{wabl}^\varphi(\tilde{x}_1), \dots, \text{wabl}^\varphi(\tilde{x}_n)\} \\ &\quad \left. + \text{Me}\{\text{rdev}_{\tilde{x}_1}^\varphi(\alpha), \dots, \text{rdev}_{\tilde{x}_n}^\varphi(\alpha)\} \right] \end{aligned}$$

(where in case the real-valued median Me is non-unique the mid-point convention is considered).

The φ -wabl/ldev/rdev median satisfies

$$\widehat{\text{M}}^\varphi[\tilde{\mathbf{x}}_n] = \arg \min_{\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})} \frac{1}{n} \sum_{i=1}^n \left(\mathcal{D}_1^\varphi(\tilde{x}_i, \tilde{U}) \right),$$

where for \tilde{U}, \tilde{V}

$$\begin{aligned} \mathcal{D}_1^\varphi(\tilde{U}, \tilde{V}) &= |\text{wabl}^\varphi(\tilde{U}) - \text{wabl}^\varphi(\tilde{V})| \\ &\quad + \frac{1}{2} \int_{[0,1]} |\text{ldev}_{\tilde{U}}^\varphi(\alpha) - \text{ldev}_{\tilde{V}}^\varphi(\alpha)| d\varphi(\alpha) \\ &\quad + \frac{1}{2} \int_{[0,1]} |\text{rdev}_{\tilde{U}}^\varphi(\alpha) - \text{rdev}_{\tilde{V}}^\varphi(\alpha)| d\varphi(\alpha), \end{aligned}$$

with $\text{wabl}^\varphi(\tilde{U}) = \int_{[0,1]} \text{mid } \tilde{U}_\alpha d\varphi(\alpha)$, $\text{ldev}_{\tilde{U}}^\varphi(\alpha) = \text{wabl}^\varphi(\tilde{U}) - \inf \tilde{U}_\alpha$, $\text{rdev}_{\tilde{U}}^\varphi(\alpha) = \sup \tilde{U}_\alpha - \text{wabl}^\varphi(\tilde{U})$.

In other words, the φ -wabl/ldev/rdev median corresponds to an M-estimate associated with the loss function $\rho(x) = |x|$ and the L^1 distance \mathcal{D}_1^φ .

Since these *ad hoc* M-estimates have similar properties as the M-estimates obtained through the representer theorem, we make an empirical comparison in Section IV to investigate finite-sample differences.

D. An illustrative real-life application

We now consider a real-life example with fuzzy-valued data and compute the location M-estimates in Subsections III-A and III-C.

Example III.2. In 2011, TIMSS (Trends in International Mathematics and Science Study) and PIRLS (Progress in International Reading Literacy Study) assessments have joined for the first time to assess the fourth grade students in three fundamental curricular areas: mathematics, science, and reading.

The questionnaires in these studies are in a standard format, to be answered in accordance with a 4-point Likert scale, namely DISAGREE A LOT, DISAGREE A LITTLE, AGREE A LITTLE and AGREE A LOT. As indicated in De la Rosa de

Sáa *et al.* [8], only limited conclusions can be obtained when the Likert scale responses are encoded by consecutive integer numbers.

On the other hand, the fuzzy rating scale has been introduced (Hesketh *et al.* [37]) as an approach that allows to combine a free-response format with a fuzzy valuation. In the fuzzy rating scale, along a continuous line between two end-points

- a respondent first selects or draws a ‘representative position/interval’ of the respondent rating (i.e., the set of points which she/he considers to be fully compatible with such a rating),
- the respondent then also indicates ‘latitudes of acceptance’ on either side by determining the highest and lowest possible positions for the respondent rating (i.e., the set of points which she/he considers to be compatible to some extent with such a rating).

The fuzzy rating scale allows to explore and exploit more information than the Likert-scale. Moreover, from a statistical perspective the fuzzy rating scale is usually also more informative than the usual fuzzy linguistic approach (see e.g. Herrera *et al.* [38], Dong *et al.* [39], Rodríguez *et al.* [40] for some recent contributions to this approach), as shown by De la Rosa de Sáa *et al.* [8].

It can be argued that fuzzy rating scale-based questionnaires are not as user-friendly as Likert scales. However, minor training is usually sufficient to explain non-experts how to respond on a fuzzy rating scale (see Hesketh *et al.* [37]). To show how this fuzzy rating scale works by means of a simple example, some of the items for the Student questionnaire TIMSS/PIRLS (see http://timss.bc.edu/timss2011/downloads/T11_StuQ_4.pdf) have been adapted in accordance with this scale, and the questionnaire has been conducted on the fourth grade students of the Colegio San Ignacio in Oviedo-Asturias (Spain). Note that the questions have not been modified with respect to the original questionnaire, but only the fuzzy rating scale has been added as a second way to respond. The students received instructions (in Spanish) on how to answer the questions (see <http://carleos.epv.uniovi.es:8080/>). We remark that this example accurately follows the spirit of Zadeh, who has coined it as the “*precisiation of the imprecise*” (see Zadeh [41]). Surprisingly, no answer has been slightly more frequent with the Likert than with the fuzzy rating scale.

We now analyze data from one of the considered items (a different and more detailed analysis of datasets from the whole adapted questionnaire can be found in Gil *et al.* [42]), referring to the degree of agreement with the statement that studying mathematics is harder than any other subject. The paper-and-pencil format corresponding to this question is graphically displayed in Figure 4.

Data from the 68 fourth grade students from Colegio San Ignacio have been collected. The results are shown in the plot on the left in Figure 5. The outputs for the four location M-estimates discussed in the previous section are displayed in the plot on the right in Figure 5.

Mathematics in school

Mathematics

How much do you agree with these statements about learning mathematics?

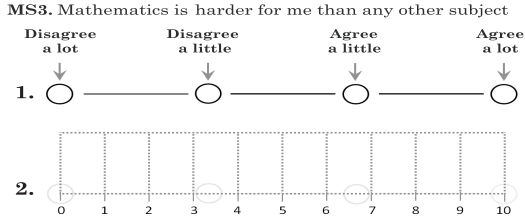


Fig. 4. An item about mathematics to fill out in the double-response questionnaire

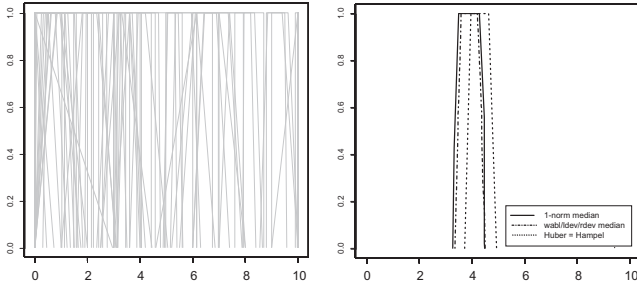


Fig. 5. Sample fuzzy data and location M-estimates of the 68 fuzzy rating scale-based responses to Question MS3

It can be seen that Hampel and Huber M-estimates, which are based on the $D_{1/3}^\ell$ metric and use the 1-norm median as initial solution, coincide.

It can be easily checked that the conditions for the representer theorem are satisfied for the Huber and Hampel M-estimates in this example, whence (as shown in Proposition III-B.3) they preserve the trapezoidal shape of the sample data.

This appealing property does not hold for the two median-type M-estimates, as can be seen from Figure 5.

On the other hand, differences in the behavior of the L^1 -type medians and the M-estimates in Figure 5 are mainly due to the influence of the corresponding loss functions. In this way, Huber and Hampel loss functions ‘penalize’ less (more) the small (large) distances than the absolute value.

It is interesting to comment that after integer encoding of the Likert-type responses, the medians for item MS3 correspond to AGREE A LITTLE (6.6̂). Hence, as already pointed out by De la Rosa de Sáa *et al.* [8], the fuzzy rating scale responses offer richer nuances and expressiveness.

In the literature one can find several arguments to employ triangular or trapezoidal fuzzy numbers or approximations preserving exactly or approximately ambiguity, expected interval, etc. (see, for instance, Pedrycz [44], Grzegorzewski [45], [46], [47], Grzegorzewski and Pasternak-Winiarska [48], Ban *et al.* [49], Lubiano *et al.* [50], and others). However, we emphasize that to compute the location M-estimation of fuzzy responses by means of the approaches in this paper, there is no need for fuzzy data to be trapezoidal/triangular (see e.g. Kosiński *et al.* [43] for some examples of other fuzzy data).

IV. EMPIRICAL COMPARATIVE ANALYSIS OF M-ESTIMATES WITH FUZZY DATA

While it has been shown that the Aumann-type mean is not robust (Sinova *et al.* [6], [7]), the M-estimators do show robust behavior in terms of fsbp as discussed in Section III. In this section we empirically compare the performance and robustness of the fuzzy-valued M-estimators in more detail. In particular, we consider the M-estimators based on the Huber and Hampel loss functions using the $D_{1/3}^\ell$ and D_1^ℓ distances, as well as the 1-norm and ℓ -wabl/rldev/rdev medians. For each of the estimators, the mean squared error of the estimator is approximated by simulation in different settings.

The general scheme of the simulations is as follows:

STUDY 1 (mid-points of the 1-levels generated from a symmetric distribution)

Based on four real-valued random variables, trapezoidal fuzzy data are generated according to $\mathcal{X} = \text{Tra}(X_1 - X_2 - X_3, X_1 - X_2, X_1 + X_2, X_1 + X_2 + X_4)$. Each sample contains a fraction c_p of contaminated observations, where $c_p = 0, 0.1, 0.2$, or 0.4 .

In the first case (CASE 1) the variables X_i are independent. In particular,

- $X_1 \sim \mathcal{N}(0, 1)$ and $X_2, X_3, X_4 \sim \chi_1^2$ for the regular observations.
- $X_1 \sim \mathcal{N}(0, 3) + C_D$ and $X_2, X_3, X_4 \sim \chi_4^2 + C_D$ for the contaminated observations.

In the second case (CASE 2) dependence between the variables X_i is introduced as follows.

- $X_1 \sim \mathcal{N}(0, 1)$ and $X_2, X_3, X_4 \sim 1/(X_1^2 + 1)^2 + 0.1 \cdot \chi_1^2$ for the non-contaminated subsample (with χ_1^2 independent of X_1),
- $X_1 \sim \mathcal{N}(0, 3) + C_D$ and $X_2, X_3, X_4 \sim 1/(X_1^2 + 1)^2 + 0.1 \cdot \chi_1^2 + C_D$ for the contaminated subsample (with χ_1^2 independent of X_1).

The constant C_D , which determines the distance between the distribution of the regular and contaminated observations, equals 0, 1, 5, 10, or 100. This yields 16 different settings for each case.

STUDY 2 (mid-points of the 1-levels generated from an asymmetric distribution)

For CASE 3, the trapezoidal fuzzy data are generated according to $\mathcal{X} = \text{Tra}(X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)})$, with $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq X_{(4)}$ the order statistics of X_1, X_2, X_3, X_4 , which are distributed as

- $X_1, X_2, X_3, X_4 \sim \text{Beta}(5, 1)$ for the regular observations.
- $X_1, X_2, X_3, X_4 \sim \text{Beta}(1, C_D + 1)$ for the contaminated observations.

Trapezoidal fuzzy data are generated as in STUDY 1, but now X_i are distributed as

- $X_1 \sim \text{Beta}(5, 1)$, $X_2 \sim \text{Uniform}[0, \min\{X_1, 1 - X_1\}]$, $X_3 \sim \text{Uniform}[0, X_1 - X_2]$ and $X_4 \sim \text{Uniform}[0, 1 - X_1 - X_2]$ for the regular observations.

TABLE I
M-ESTIMATE SHOWING THE LOWEST ESTIMATED MEAN SQUARED ERROR FOR THE SIMULATION DESIGNS IN STUDIES 1 AND 2

c_p	C_D	STUDY 1 - CASE 1				STUDY 1 - CASE 2				STUDY 2 - CASE 3				STUDY 2 - CASE 4			
		ρ_1	\mathcal{D}_1^ℓ	$D_{1/3}^\ell$	ρ_2	ρ_1	\mathcal{D}_1^ℓ	$D_{1/3}^\ell$	ρ_2	ρ_1	\mathcal{D}_1^ℓ	$D_{1/3}^\ell$	ρ_2	ρ_1	\mathcal{D}_1^ℓ	$D_{1/3}^\ell$	ρ_2
0.1	0	HAMP*	HAMP*	HAMP*	HAMP*	1-n	1-n	1-n	1-n	HAMP*	HAMP*	HAMP*	HAMP*	HAMP	HAMP	HAMP	HAMP
0.1	1	HAMP	HAMP	HAMP	HAMP	1-n	1-n	HAMP*	1-n	HAMP	HAMP	HAMP	HAMP	HAMP	HAMP	HAMP	HAMP
0.1	5	HAMP	HAMP	HAMP	HAMP	1-n	1-n	1-n	1-n	HAMP	HAMP	HAMP	HAMP	HAMP	HAMP	HAMP	HAMP
0.1	10	HAMP	HAMP	HAMP	HAMP	1-n	1-n	1-n	1-n	HAMP	HAMP	HAMP	HAMP	HAMP	1-n	1-n	HAMP
0.1	100	HAMP	HAMP	HAMP	HAMP	1-n	1-n	HAMP	1-n	HAMP	HAMP	HAMP	HAMP	HAMP*	HAMP*	HAMP*	HAMP*
0.2	0	HAMP*	HAMP*	HAMP*	HAMP*	1-n	1-n	1-n	1-n	HAMP	HAMP	HAMP	HAMP	HAMP	HAMP*	HAMP*	HAMP*
0.2	1	HAMP*	HAMP*	HAMP*	HAMP*	1-n	HAMP*	HAMP*	HAMP*	HAMP	HAMP	HAMP	HAMP	HAMP	HAMP	1-n	1-n
0.2	5	HAMP	HAMP	HAMP	HAMP	1-n	1-n	1-n	1-n	HAMP	HAMP	HAMP	HAMP	HAMP	HAMP	HAMP	HAMP
0.2	10	HAMP	HAMP	HAMP	HAMP	1-n	1-n	1-n	1-n	HAMP	HAMP	HAMP	HAMP	HAMP*	HAMP*	HAMP*	HAMP*
0.2	100	HAMP	HAMP	HAMP	HAMP	1-n	1-n	1-n	1-n	HAMP	HAMP	HAMP	HAMP	HAMP*	HAMP*	HAMP*	HAMP*
0.4	0	HAMP*	w/l/r	w/l/r	HAMP*	1-n	1-n	1-n	1-n	HAMP	HAMP	1-n	HAMP	HAMP*	HAMP*	HAMP*	HAMP*
0.4	1	HAMP*	HAMP*	HAMP*	HAMP*	1-n	1-n	1-n	1-n	HAMP	HAMP	HAMP	HAMP	HAMP*	HAMP*	HAMP*	HAMP
0.4	5	HAMP*	w/l/r	w/l/r	w/l/r	1-n	1-n	1-n	1-n	HAMP	HAMP	HAMP	HAMP	HAMP	w/l/r	w/l/r	HAMP
0.4	10	HAMP*	HAMP*	HAMP*	w/l/r	1-n	1-n	1-n	1-n	HAMP	HAMP	HAMP	HAMP	w/l/r	w/l/r	w/l/r	HUB
0.4	100	HAMP*	HAMP*	w/l/r	w/l/r	1-n	1-n	1-n	1-n	HAMP	1-n	1-n	HAMP	w/l/r	w/l/r	w/l/r	HUB

- $X_1 \sim \text{Beta}(1, C_D + 1)$, $X_2 \sim \min\{X_1, 1 - X_1\} \cdot \text{Beta}(1, C_D + 1)$, $X_3 \sim (X_1 - X_2) \cdot \text{Beta}(1, C_D + 1)$ and $X_4 \sim (1 - X_1 - X_2) \cdot \text{Beta}(1, C_D + 1)$ for the contaminated observations.

The same values of c_p and C_D as in STUDY 1 are considered.

For each of the four cases above, the simulation study was performed according to the following steps.

Step 1. To determine the population analog of the M-estimates for each case, $N = 1000$ samples of size $n = 10000$, consisting of only regular fuzzy-number observations ($c_p = C_D = 0$) have been generated. The population target for each M-estimator is then taken as the Monte Carlo average of the corresponding M-estimates in these N samples.

Step 2. For each case and each combination of c_p and C_D , we then generated $N = 1000$ random samples and calculated the corresponding M-estimates. Based on these M-estimates, the mean squared error w.r.t. the population targets has been computed for each location measure, using four different metrics, namely ρ_1 , \mathcal{D}_1^ℓ , $D_{1/3}^\ell$, and ρ_2 .

The results of our comparative analysis are summarized in Table I, which lists for each setting the M-estimator that yielded the lowest estimated mean squared error. We used the abbreviations HAMP = M-estimator with Hampel loss function, HUB = M-estimator with Huber loss function, 1-n = 1-norm median, w/l/r = φ -wabl/ldev/rdev median. The notation * distinguishes the Hampel and Huber M-estimators computed by means of the D_1^ℓ distance. The detailed results from the simulations can be found at <http://bellman.ciencias.uniovi.es/SMIRE/Archivos/SimulationsMestimates.pdf>.

From Table I it becomes clear that there does not exist a uniformly most appropriate M-estimator of location. However, the M-estimator based on Hampel's loss function seems to be the most suitable estimator in most of the situations when the trapezoidal fuzzy data are assumed to be generated from independent real random variables. In the case that these variables are dependent, the 1-norm median seems to be most successful when mid-points are generated from a symmetric distribution,

whereas there is a high diversity for asymmetrically distributed mid-points.

Remark IV.1. In addition to the empirical advantages of the M-estimators based on Hampel's loss function, it should be highlighted that thanks to the representer theorem, the M-estimates of important classes of fuzzy data such as trapezoidal data are guaranteed to give a location estimate that belongs to the same class. This is because the estimates are expressible as weighted averages. This result is not generally true for *ad hoc* robust measures, which complicates their interpretation as measure of the location of the fuzzy data.

V. CONCLUDING REMARKS

One of the advantages of the new location M-estimators for random fuzzy numbers introduced in this paper lies in the fact that the recently established representer theorem in the context of robust density estimation could be easily extended to the case of fuzzy -number data, as shown in Section III. Although the M-estimates based on the representer theorem have been developed for the D_θ^φ metric, other L^2 metrics between fuzzy numbers allowing to embed $\mathcal{F}_c^*(\mathbb{R})$ into a convex cone of a Hilbert space are also worthwhile to consider. For instance, in most of the examples and simulations in this paper we have also considered the L^2 metric based on the wabl/ldev/rdev representation introduced in Sinova *et al.* [3], but the conclusions were essentially the same.

It should be pointed out that, in spite of the fact that the mathematical apparatus behind the M-estimates in Subsections III-A and III-B involves the Hilbert space-valued setting, the particular case of fuzzy data is definitely much simpler and easier-to-follow. Furthermore, due to the representer theorem, the computations reduce to the calculus of weighted means. The main practical difficulty concerns the calculus of the involved weights, but this iterative procedure can be implemented in \mathbb{R} [55] or similar tools.

Finally, an important remaining problem is to adapt the M-estimates of location with unknown dispersion to the fuzzy-valued case. This would formally overcome the lack of scale equivariance for the location M-estimators. Therefore, the tuning constant(s) in the loss functions would not need to be

determined from the data anymore. We hope to solve this issue in the future, when robust dispersion measures for fuzzy data are also developed.

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Beatriz Sinova received her BSc and MSc degrees in Mathematics in 2009 and 2010, respectively, from the University of Oviedo, Spain. In 2014, she has received a Joint PhD degree in Mathematics and Statistics from the Universities of Oviedo and Gent. Her current research interests include the development of new statistical techniques for the location of imprecise-valued data, as well as their application to real-life problems. She is involved in some research projects regarding the statistical analysis of interval-, fuzzy- and functional-valued data.



María Ángeles Gil is full professor of Statistics and O.R. at the University of Oviedo (Spain) since 1992. Her research was initially focused on the measurement and statistical developments concerning the amount of information and inequality of random variables. Since 1983, her research interests have been mostly related to the statistical analysis of imprecise data (mainly set-valued and fuzzy set-valued). On this topic she heads the research group SMIRE (<http://bellman.ciencias.uniovi.es>) and she is a member of the Core SMPS Group steering the

biennial conferences on Soft Methods in Probability and Statistics. She was co-editor-in-chief of the journal *TEST* (2005-2008), and is currently associate/area editor for *Computational Statistics and Data Analysis*, *Fuzzy Sets and Systems*, *Information Sciences*, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *International Transactions in Operational Research*, and *TEST*. She has been awarded with the Silver Medal of the Principality of Asturias in 2014 and is an IFSA fellow since 2015.



Stefan Van Aelst Stefan Van Aelst received the B.Sc. and M.Sc. degrees in mathematics from the University of Antwerp, Belgium, in 1997 and the Ph.D. degree (also in mathematics) from the University of Antwerp, Belgium, in 2000. He has been assistant and associate professor in statistics at Ghent University. Since 2013 he is associate professor in statistics in the department of Mathematics at the University of Leuven (KULeuven) and part-time associate professor at Ghent University his research is mainly focused on robust statistics, covering estimation, inference, bootstrap methods, model selection, the analysis of fuzzy

data and high-dimensional data analysis.