Mental Computation or Standard Algorithm? Children's Strategy Choices on Multi-Digit Subtractions

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Introduction

During the last decades, the variety, frequency, efficiency and flexibility of people's strategy use has been of major interest for both researchers and practitioners in the domain of mathematics education. An ever-growing number of researchers provided fine-grained analyses of people's strategy competencies in diverse mathematical domains. However, as argued by several researchers (cf. Kilpatrick, Swafford and Findell 2001; Siegler 2000; Verschaffel, Greer and De Corte 2007), most research attention has been spent on single-digit arithmetic. Despite increasing research interest in people's strategy performances on other, more complex mathematical tasks, such as multi-digit arithmetic (see, e.g., Blöte, Van der Burg and Klein 2001; Carpenter, Franke, Jacobs, Fennema and Empson 1998; Fuson et al. 1998; Hickendorff, van Putten, Verhelst and Heiser 2010; Peters, De Smedt, Torbeyns, Ghesquière and Verschaffel 2013; Thompson 2000), our understanding of people's strategy competencies in these other domains remains relatively limited. The present study aimed at deepening our insight into children's strategy competencies in multi-digit subtraction up to 1000, by systematically analyzing their choice for mental computation strategies versus the standard written algorithm, as well as the frequency, efficiency and flexibility with which they execute both types of strategies.

Strategy Use in Multi-Digit Subtraction

Since the end of the previous century, the acquisition of various strategies that can be applied insightfully, efficiently and flexibly on different types of mathematical tasks has become a major goal of elementary mathematics education worldwide (Baroody and Dowker 2003; Kilpatrick et al. 2001; Verschaffel et al. 2007). According to the adherents of the worldwide reform movement, mathematics instruction should no longer primarily focus on children's perfect mastery of some standard methods for solving particular kinds of mathematical problems. Rather, it should foster the development of children's disposition to solve mathematical tasks efficiently, creatively and flexibly (or adaptively) with a diversity of meaningfully acquired strategies.[1] Although reformers assume that promoting strategy variety and flexibility is a feasible and valuable goal across different levels of mathematical achievement, including lower ones (Baroody 2003; Moser Opitz 2001; Peltenburg, van den Heuvel-Panhuizen and Robitzsch 2012), this assumption has been largely grounded in rhetoric rather than in convincing research-based evidence (Author 2006; Geary 2003; Verschaffel, Luwel, Torbeyns and Van Dooren 2009). Stated otherwise, the feasibility of the worldwide changes in the elementary mathematical curricular goals and content, in particular for lower achieving children, still requires further research attention.

When confronted with multi-digit subtractions, children can use different types of strategies, including mental computation strategies and standard written algorithms (Kilpatrick et al. 2001; Verschaffel et al. 2007). According to the most common view among mathematics educators (Anghileri 1999; Buys 2001; Thompson 1999a), *mental computation strategies* can be defined as clever calculation methods, relying on one's understanding of the basic features of the number system and of arithmetic operations, a well-developed feeling for numbers and a sound knowledge of the elementary number facts. Although mental computation strategies are typically executed in children's heads -- thus without using paper and pencil -- mental calculators may write down their calculation steps and/or intermediate results during the solution process. Stated otherwise, mental computation strategies are strategies that require children to calculate *with* their head -- using their knowledge of numbers and operations -- rather than *in* their head -- without paper and pencil (Buys 2001). Children's mental computation strategies for multi-digit subtractions can be further classified into three basic categories (cf. Buys 2001; Verschaffel et al. 2007): (a) decomposition strategies, involving splitting off the hundreds, tens and units in both integers and subtracting them separately (e.g., 457 - 298 = __ ; 400 - 200 = 200, 50 - 90 = -40, 7 - 8 = -1, 200 - 40 - 1 = 159); (b) sequential strategies, consisting of subtracting down first by the hundreds, next by the tens and finally by the units of the second integer from the first un-split integer (e.g., 457 - 298 = __ ; 457 - 200 = 257, 257 - 90 = 167, 167 - 8 = 159); (c) varying strategies, referring to diverse clever strategies that involve the flexible adaptation of the numbers and operations in the problem on the basis of one's understanding of number relations and/or the properties of arithmetic operations. An example of a varying strategy is the compensation strategy, which can be applied efficiently on subtractions with a minuend unit value 1 or 2 and on subtractions with a subtrahend unit value 8 or 9 (e.g., respectively, 601 - 234 = (600 - 234) + 1 = 366 + 1 = 367, and 457 - 298 = 457 - (300 - 2) = 157 + 2 = 159).

By contrast, *standard written algorithms* are fixed and well-defined step-by-step procedures for solving multi-digit subtractions, involving operations with digits rather than the real magnitude of the numbers in the problem, such as calculating the difference between 5 and 3 (rather than 50 and 30) and between 4 and 2 (instead of 400 and 200) when solving 457 – 238 = __ . Although people who apply a standard algorithm normally rely on paper and pencil, it is -- at least in principle -- also possible to execute it in one's head without reliance on writing materials. As exemplified in Verschaffel et al. (2007), there exist different standard algorithms for multi-digit subtraction, varying in the number and detail of steps written down when applying the algorithm, but also in the very nature of the arithmetic operations to be performed and the mathematical principles on which they are based. Figure 1 provides an illustration of the standard algorithm for multi-digit subtraction as normally taught in Flanders (Belgium).

-- Insert Figure 1 here --

Although researchers and practitioners generally agree on the importance of strategy flexibility as a goal of elementary mathematics education, they agree less on what it exactly means to make *flexible* strategy choices (Heinze, Star and Verschaffel 2009; Verschaffel et al. 2009). Definitions of strategy flexibility range from rather basic ones wherein strategy flexibility is conceived as adapting one's strategy choices to some predefined task characteristics (e.g., answering 457 - 298 = __ via the compensation strategy as 457 - 300 + 2, since the subtrahend is very close to the next hundred), to more complex definitions that also incorporate subject characteristics (e.g., solving 457 - 298 = __ via the compensation strategy as this strategy is mastered best by the individual, i.e., leads faster to an accurate answer than the other strategies available in that individual's strategy repertoire) and even contextual variables (e.g., solving 457 - 298 = __ via the compensation strategy because the solver expects this strategy to be valued most by the teacher or the parents). In the present study, we defined and operationalized flexible strategy choices as strategy choices fitted to specific item and subject characteristics.

Previous Studies on Children's Use of Mental Computation Strategies Versus Standard Written Algorithms on Multi-Digit Subtractions

Although the place and value of mental computation strategies versus standard written algorithms in current elementary mathematics education curricula is heavily debated among adherents and critics of the reform movement (cf. Koninklijke

Nederlandse Akademie van Wetenschappen 2009; National Mathematics Advisory Panel 2008), it received, up to now, only scarce research interest.

A first series of studies, conducted mainly in the U.S. and the U.K., documented the effectiveness of reform-oriented approaches teaching and learning mental computation strategies and standard algorithms on multi-digit additions and subtractions (Carpenter et al. 1998; Fuson et al. 1997; Hiebert and Wearne 1996; Thompson 1999b 2000). Despite some differences in the concrete instructional programs and materials, in all studies, children from reform-based classrooms received ample instruction in understanding the basic properties and characteristics of the base-10 number system and were stimulated to invent and apply diverse mental computation strategies, using their number-facts knowledge and their understanding of multi-digit numbers. After prolonged instruction in multi-digit number concepts and clever computation strategies, the standard algorithms were introduced as procedures for efficiently solving multi-digit problems. By contrast, children from traditional classrooms were already confronted with the standard algorithms at the start of multi-digit instruction, with little or no attention to mental arithmetic. Fine-grained analyses of children's strategy competencies on multi-digit additions and subtractions revealed that children instructed in reform-based classrooms developed a rich diversity of insightful and clever mental computation strategies that were mastered well. On the other hand, the premature introduction of the standard algorithms in traditional classrooms resulted in a greater reliance on so-called "buggy procedures" (i.e., incorrect variations of the standard algorithm; Thompson 1999c) and in more systematic errors.

Next to these intervention studies, a second line of studies, mainly conducted by continental European researchers, focused on children's mental computation strategy use in reform- and more traditionally-oriented classrooms (Beishuizen 1993 1999;

Blöte, Klein and Beishuizen 2000; Blöte et al. 2001; Selter 1998). In these studies, children from reform-oriented classrooms received instruction in the efficient and flexible application of diverse mental computation strategies from the start of the teaching process, whereas children from more traditionally oriented classrooms started with practicing one specific mental computation strategy and were introduced into strategy variety and flexibility only towards the end of the instruction. These studies documented the effectiveness of introducing strategy variety and flexibility already at the start of the teaching process, as children instructed in reform-oriented classrooms flexibly applied diverse mental computation strategies, including clever varying strategies, on different types of multi-digit additions and subtractions from the first lessons on. By contrast, children from more traditionally-oriented classrooms mainly relied on the mental computation strategy that was taught as the default strategy to solve all types of multi-digit problems.

Taken together, the two above-mentioned lines of research converge to the conclusion that prolonged instruction in deep conceptual understanding of multi-digit numbers and in flexible mental computation enhances children's strategy competencies and performances in the domain. By contrast, premature introduction of the standard written algorithms rather leads to reliance on "buggy procedures", reflecting weak performance in and limited understanding of the domain. In line with these results, in reform-based documents worldwide, strong pleas are made for teaching children mental computation strategies *before* and *besides* the standard algorithms (cf. Department for Education and Employment 1999; Kilpatrick et al. 2001; National Council of Teachers of Mathematics 2003; Thompson 1999b; van den Heuvel-Panhuizen 2001). The claim is that early and prolonged instruction in mental computation strategies, for children of all achievement levels, (a) will lead to the insightful, efficient and flexible acquisition of these strategies for multi-digit problems, (b) will provide the necessary step-stones for the insightful introduction of the standard algorithms, and (c) will guarantee that learners will continue to use clever mental computation strategies to solve multi-digit problems with particular numerical features once the algorithms are taught.

However, to the best of our knowledge, this claim was never directly and systematically tested, except in a study by Selter (2001), focusing on the developmental changes in German children's strategy competencies before and after the introduction of the standard written algorithms for addition and subtraction at school. After explicit instruction in mental computation strategies during the first years of formal mathematics education, the children participating in Selter's (2001) study were taught the standard algorithms for multi-digit addition and subtraction in the second term of Grade 3. Contrasting the claim that prolonged instruction in mental computation strategies will support children's continued efficient and flexible use of this type of strategies, the results of this study revealed that, once the standard algorithms were introduced at school, German 3rd- and 4th-graders started to solve multi-digit problems very frequently, but also very *in*efficiently and *in*flexibly, with the standard algorithms. Children's overreliance on inefficiently executed standard algorithms was even observed on subtractions as 527 – 399 = __ that strongly invited the use of mental computation strategies, c.q., compensation.

Although Selter's (2001) study provided new and important insights into the development of children's mental computation versus standard algorithm use after early and prolonged instruction in mental computation, his findings are limited in three ways. First, in line with previous work on multi-digit addition and subtraction, strategy flexibility was defined on the basis of only the numerical characteristics of the items, thereby ignoring the influence of other variables as subject and/or context characteristics (Verschaffel et al. 2009). Second, children's strategy competencies were investigated in a "choice" condition only, allowing children to selectively assign strategies to problems on the basis of their individual subject and/or specific item characteristics, which might have resulted in questionable strategy efficiency data (Siegler and Lemaire 1997). Finally, the relation between children's general mathematical achievement level and (the development of) their strategy competencies was not addressed, leaving the above-mentioned questions about the feasibility of strategy variety and flexibility for children of the lowest mathematical achievement levels unanswered (Verschaffel et al. 2009). On the basis of these three weaknesses, we aimed at investigating the use of mental computation strategies and the standard written algorithm on multi-digit subtractions in children of different mathematical achievement levels, using a broader definition of strategy flexibility and applying the "choice/no-choice" method (Siegler and Lemaire 1997).

Research Questions

Starting from the claim that early and prolonged instruction in mental computation strategies (a) enhances children's efficient and flexible use of this type of strategies, (b) provides them the necessary tools to insightfully acquire the standard algorithms for multi-digit computation, and (c) stimulates the continued use of clever mental computation strategies after the introduction of the standard algorithms at school in children of all achievement levels, we aimed at addressing the following four research questions.

First, to what extent do children, of all achievement levels, use both mental computation strategies and the standard algorithm in the choice condition (= *Research question 1a*)? And how frequently do they apply the two different types of strategies on the subtractions from the choice condition (= *Research question 1b*)?

Second, do we observe efficiency differences between mental computation strategies and the standard algorithm on the different types of subtractions in the no-choice conditions, i.e., does the obligatory use of mental computation result in more accurate and faster answers than the standard algorithm on multi-digit subtractions that are assumed to evoke mental computation, but in less accurate and slower responses on subtractions that do not elicit clever mental computation strategy use (= *Research question 2a*)? And are these differences dependent on the achievement level of the children (= *Research question 2b*)?

Third, do children flexibly fit their strategy choices to the numerical characteristics of the items in the choice condition, i.e., do we observe a higher frequency of mental computation strategies on multi-digit subtractions that are assumed to evoke mental computation than on standard subtractions (= *Research question 3a*)? And do they flexibly take into account their strategy performance characteristics during the strategy choice process, i.e., do we observe a correlation between, on the one hand, children's frequency of mental computation strategy use in the choice condition and, on the other hand, the differences in their mastery of the two types of strategies in the no-choice conditions (= *Research question 3b*)? And does the flexible nature of children's strategy choices differ between children of different mathematical achievement level (= *Research question 3c*)?

Method

Participants

Fifty-eight 4th-graders participated in the study (33 boys; M_{Age} = 9 years 7 months, *SD* = 4 months). Children were recruited from three classrooms in two middle-income schools in Flanders (Belgium). All children had parental consent to participate in the study. We distinguished among three groups of children on the basis of their scores on the standardized tests for Mathematics Medio 4th Grade of the widely used Student Following System (Billiaert, Dudal, Grysolle and Van Dooren 2003). The group of *high* achievers consisted of all children scoring at or above the 75th percentile (n = 22), the group of *above-average* achievers received a score between the 51st and the 74th percentile (n = 14), and the group of *below-average* achievers scored below the 50th percentile on this test (n = 22). Table 1 describes the number, age, and mathematics achievement test score of the children per mathematics achievement group.

-- Insert Table 1 here --

Our analyses of the mathematical textbooks used in the participating classes and of individual interviews with the teachers of the participating children revealed that all children had received instruction in mental subtraction as well as in the standard algorithm, in line with what is typically the case in Flanders. They had received explicit instruction in multi-digit numbers and in mental computation strategies on multi-digit subtractions starting in 2nd grade, using base-10 structuring materials such as MAB-materials and the hundred square as concrete models for numbers and operations in this domain. Their teachers had focused on the sequential jump strategy, involving the sequential subtraction of hundreds (H), tens (T) and units (U) of the subtrahend from the minuend, as the standard mental computation strategy to solve subtractions up to 100 and up to 1000 in respectively 2nd and 3rd grade. Besides this main "default" strategy for mental subtraction, children had also been taught varying strategies including compensation. Instruction in the standard algorithm for multi-digit subtraction started halfway 3rd grade. As is the case in most Flemish mathematics handbooks, and in contrast to the guided reinvention approach to the teaching of algorithms developed and propagated by Realistic Mathematics Educators (Beishuizen and Anghileri 1998; van den Heuvel-Panhuizen 2001), the teachers did not stimulate and help the pupils to actively construct the standard algorithm gradually out of their available mental calculation strategies, but almost immediately presented the algorithm in its final shortest form (see Figure 1). After intensive practice of the standard algorithm in 3rd grade, children briefly rehearsed it during the first months of 4th grade, before moving to the algorithms for multiplication and division. Consequently, all participating children had been taught and intensively practiced the algorithm for multi-digit subtractions for (at least) one year, after intensive and prolonged instruction in mental computation strategies.

Materials

All children were individually offered three series of eight subtractions up to 1000 in one choice and two no-choice conditions (one series of eight subtractions per condition). We selected two item types, with four subtractions per item type in each series of subtractions. Items of the first type, *mental computation* or *MC-items*, could be efficiently solved using mental computation, and, more particularly, compensation (e.g., 963 – 499 = __; 601 – 126 = __). MC-items were defined on the basis of two characteristics: (a) the first or second term can be easily rounded to, respectively, the previous or next hundred, i.e., differs only 1 or 2 units from the previous or next hundred; (b) the other term differs at least 26 units from the previous or next hundred (i.e., TU with values 26 up to 74). Items of the second type, *standard algorithm* or *SA-items*, were assumed to evoke neither compensation nor any other varying strategy (e.g., 952 – 474 = __; 631 – 153 = __). SA-items were characterized by (a) both the minuend and the subtrahend are at least 26 units larger or smaller than the previous or next hundred (i.e., TU values 26 up to 74); (b) neither the minuend nor the subtrahend contains a unit value 5, 8 or 9. Both MC- and SA-items required carrying over between both T and U. To match the difficulty of the three item sets, we equated the mean size of the minuend and the subtrahend as well as the mean size of the differences across the series.

The items were ordered on the basis of four criteria: (a) the different item types are offered randomly; (b) H, T and U of the minuend are not repeated on subsequent trials; (c) H, T and U of the subtrahend are not repeated on subsequent trials; (d) the answer to each item cannot be easily deduced from the previous item.

As mentioned above, all children individually answered the three series of items in three different conditions. In the choice condition, they could choose between mental computation and the standard algorithm on each item on the basis of pictures (see Figure 2). Children were asked to solve the subtraction either with the type of strategy used by the boy or the girl. As was the case in their classroom, they were invited to write down the (most important) solution steps on the scrap paper between the pictures.

-- Insert Figure 2 here --

In the no-choice mental condition, all items had to be answered by mental computation. In the no-choice algorithm condition, the standard algorithm had to be used on all subtractions. To guarantee that the children in the two no-choice conditions would actually perform the required type of strategy, they were experimentally forced to use these strategies on the basis of the presentation of the items (they got either the picture with the boy or the picture with the girl, see Figure 2) and the accompanying instructions (to write down their solution steps).

Procedure

All children were tested individually in a quiet room at their school. All children started with the choice condition. The order of the no-choice conditions and which of the two strategies (mental computation or standard algorithm) was represented by which pictured figure (boy or girl) were counterbalanced across the children. On each trial, and in each condition, children's answer, response time, and (chosen or forced) strategy were registered. The speed of responding was registered with a stop watch, starting at the moment that the experimenter offered the item to the child and ending immediately after the child had stated the complete answer to the subtraction. No time limit was included. Children were instructed to solve all items as accurately and as fast as possible. They were also asked to report the strategy used by writing down the strategy and/or intermediate results during the solution process (in the same way as they were used to do in their regular mathematics classes). In case of unclear and/or incomplete notes, children were invited to provide a complete and clear verbal strategy report immediately after answering each problem.

Results

Preliminary analyses revealed that neither the order of the no-choice conditions nor the association between the strategies and the depicted figures influenced children's task performances, *p*s > .05. Therefore, we grouped the data from the two orders of no-choice conditions and the two strategy-figure associations in all further analyses.

Strategy Repertoire and Frequency in the Choice Condition

With respect to Research question 1a, our analyses revealed that about half of the children (46%) applied both mental computation and the standard algorithm at least once in the choice condition. Only one child (2%) answered all six items via mental computation; the other children (52%) solved all items with the standard algorithm. We observed no achievement group differences in strategy repertoire, $chi^2(4, n = 58) = 1.74$, $p > .05$. Thus, about half of the children in the three achievement groups (i.e., 45% of the high and below-average achievers and 50% of the above-average achievers) applied mental computation strategies as well as the standard algorithm, whereas almost all of the remaining children from each achievement group systematically chose for the standard algorithm.

Although about half of the children applied both mental computation and the standard algorithm to solve the subtractions in the choice condition, mental computation was used in a minority (only 21%) of the cases (cf. Research question 1b). We observed no differences in the frequency of mental computation strategy use between the three achievement groups, $F(2, 55) < 1$, indicating that high achievers applied this type of strategies with the same low frequency as above-average and below-average achievers, respectively, $M = 0.22$ ($SD = 0.29$), $M = 0.23$ ($SD = 0.27$) and $M = 0.18$ ($SD = 0.23$).

Strategy Efficiency in the No-Choice Conditions

To answer our second research question (cf. Research questions 2a and 2b), we analyzed the accuracy and speed of responding in the two no-choice conditions via repeated measurements ANOVA, with condition and item type as within-subjects variables and achievement group as between-subjects variable. Post-hoc analyses were corrected for multiple comparisons with Bonferroni adjustments. Table 2 shows the accuracy and speed of answering in the two no-choice conditions.

Accuracy data. For accuracy, we observed a main effect of Condition, $F(1, 52) = 20.46$, $p < .01$, and Item Type, $F(1, 52) = 6.17$, $p = .02$. Overall, the obligatory use of the standard algorithm led to more accurate answers than the obligatory use of mental computation, and the SA-items were answered more accurately than the MC-items, respectively $M = 0.76$ ($SD = 0.23$) and $M = 0.70$ ($SD = 0.25$). The Condition × Item Type interaction was not significant, $F(1, 52) = 1.72$, indicating that both the MC- and the SA-items were answered less accurately when children had to apply mental computation than when they were required to use the standard algorithm. Furthermore, the differences in children’s general mathematical achievement level were reflected by accuracy differences in the no-choice conditions, $F(2, 52) = 5.56$, $p < .01$. High achievers answered the subtractions more accurately than below-average achievers, respectively $M = 0.82$ ($SD = 0.14$) and $M = 0.61$ ($SD = 0.25$), but there were no accuracy differences between above-average achievers ($M = 0.78$, $SD = 0.20$) and the two other achievement groups. The Achievement Group × Condition interaction just failed to reach significance, $F(2, 52) = 2.86$, $p = .07$. As shown in Table 2, the obligatory use of the standard algorithm tended to result in more accurate answers than the obligatory use of mental computation for both high and below-average achievers, whereas above-average achievers tended to execute the two types of strategies with the same accuracy. The Achievement Group × Item Type interaction tended to reach significance, $F(2, 52) = 3.27$, $p = .05$, indicating that the accuracy differences between SA- and MC-items were only observed for below-average achievers; high and above-average achievers answered both item types with the same accuracy. The interaction between condition, item type, and achievement group was not significant, $F(2, 52) < 1$.

Speed data. Turning to the speed data, the obligatory use of the standard algorithm also resulted in faster responses than the obligatory use of mental computation, $F(1, 52) = 70.72$, $p < .01$. Children generally solved SA-items slower than MC-items, $F(1, 52) = 8.59$, $p < .01$, respectively $M = 39.02$s ($SD = 12.57$) and $M = 36.45$s ($SD = 10.46$). The Condition × Item Type interaction was significant, $F(1, 52) = 11.07$, $p < .01$. The obligatory use of the standard algorithm resulted in faster responses than the obligatory use of mental computation on both SA- and MC-items; but the speed differences between the two types of strategies were larger on the SA-items than on the MC-items. The inclusion of achievement group revealed a main effect of this subject factor on the speed of responding, $F(2, 52) = 5.14$, $p < .01$, but no significant additional interaction effects. High achievers answered the subtractions faster than the below-average achievers, respectively *M* = 32.49s (*SD* = 11.61) and *M* = 42.67s (*SD* = 8.74); above-average achievers solved the subtractions slower than the high achievers and faster than their below-average achieving peers, but the differences were too small to reach significance, *M* = 38.75s (*SD* = 10.22).

Summary. Taken together, contrary to the major claims of the reformers, on all item types and in all achievement groups, the obligatory use of mental computation strategies led to less accurate and slower responses than the standard algorithm. Of course, these results might be due to the kind of mental computation strategies children actually applied in the no-choice mental condition. A more detailed analysis of children's strategies in this no-choice mental condition indeed revealed that they mainly applied sequential jump and decomposition strategies, and only rarely mental strategies of the varying, c.q. compensation, type. More concretely, children answered more than 80% of the subtractions in the no-choice mental condition using sequential jump and decomposition strategies, and applied the compensation strategy on hardly 14% of the subtractions -- and, more specifically, on only about 20% of the MC-subtractions, which were assumed to strongly elicit the latter strategy.

-- Insert Table 2 here --

Strategy Selection in the Choice Condition

In line with previous studies using the choice/no-choice method (Author 2006; Siegler and Lemaire 1997), we assessed children's strategy flexibility in the choice condition with two different techniques. We first determined whether children took into account the numerical features of the subtractions by calculating the frequency of mental computation strategies and the standard algorithm on the two types of items in the choice condition (cf. Research question 3a). As mentioned in paragraph 3.1.1, children applied mental computation on only a minority, i.e., 21%, of the subtractions in the choice condition. A repeated measurements ANOVA on the frequency of mental computation strategy use in the choice condition, with item type as within-subjects variable and mathematics achievement group as between-subjects variable, revealed no significant difference in the frequency of mental computation between the two item types, $F(1, 55) = 2.08$: Not only the SA-items but also the MC-items were hardly answered with mental computation strategies, respectively, $M = 0.18$ ($SD = 0.26$) and $M = 0.23$ ($SD = 0.31$). The Item Type × Achievement Group interaction was not significant either, $F(2, 55) < 1$, indicating an overreliance on the standard algorithm for both MC- and SA-subtractions in all three groups (cf. Research question 3c). Taken together, these results indicate that children of all achievement levels did *not* fit their strategy choices to the numerical characteristics of the items.

Secondly, we assessed the flexibility of children's strategy choices on the basis of their strategy performance characteristics by correlating the frequency of mental computation in the choice condition with the accuracy and speed differences between the two types of strategies in the no-choice conditions (using participant as unit of analysis) (cf. Research question 3b). The correlation between the frequency of mental computation in the choice condition and the accuracy differences in the no-choice conditions was significant, $\rho(58) = 0.25$, $p = .02$, indicating that children took into account their individual accuracies in mental versus standard written computation (as assessed in the two no-choice conditions) during their strategy choice processes (in the choice condition). The correlation between mental computation frequency and speed of standard written versus mental computation strategies also reached significance, $\rho(58) = 0.28$, $p < .01$. In other words, children also adapted their strategy choices flexibly to the speed with which they were able to perform both types of strategies. However, we observed differences in the flexibility of children's strategy choices between the three achievement groups (Research question 3c). High achievers flexibly fitted their strategy choices in the choice condition to their strategy speed characteristics, $\rho(22) = 0.56$, $p < .01$, but not their strategy accuracy, $\rho(22) = 0.34$. By contrast, above-average achievers took into account their strategy accuracy, $\rho(14) = 0.67$, $p < .01$, but not their speed characteristics, $\rho(14) = 0.26$. For the group of below-average achievers, we found no significant correlation for accuracy, $\rho(22) = 0.02$, or speed, $\rho(22) = 0.34$. In sum, the results of our second set of flexibility analyses demonstrate that high and above-average achievers flexibly fitted their strategy choices to their strategy performance characteristics, i.e., respectively, the speed and the accuracy with which they mastered the different types of strategies. By contrast, below-average achievers did not take into account their strategy mastery during the strategy selection process, and (thus) did not fit their strategy choices to either the accuracy or the speed with which they could execute the different types of strategies.

Conclusion and Discussion

International efforts to reform elementary mathematics education stress the importance of stimulating children’s abilities to flexibly apply different types of strategies on mathematical tasks. As far as multi-digit computation is concerned, reform-based curricula worldwide no longer pay exclusive attention to the standard algorithms, but teach those algorithms besides and after instruction in various kinds of mental computation strategies. The claim is that children of all achievement levels will continue to efficiently apply the latter type of strategies after the introduction of the standard algorithms and develop a disposition to choose flexibly between these algorithms and mental computation strategies (e.g., Department for Education and Employment 1999; Kilpatrick et al. 2001; National Council of Teachers of Mathematics 2003; Thompson 1999; van den Heuvel-Panhuizen 2001). The present study aimed at empirically evaluating this claim by analyzing the extent to which and the efficiency and flexibility with which children of varying math achievement level, after prolonged instruction in mental computation strategies before the introduction of the standard algorithms at school, apply mental computation strategies versus the standard algorithm on subtractions up to 1000. Hereafter, we discuss our major findings as well as their theoretical, methodological and instructional implications.

Frequent and Efficient Use of Standard Algorithm

Our findings do *not* empirically support the above-mentioned claim about children’s continued frequent and efficient use of clever mental computation strategies after the introduction of the standard algorithm for multi-digit subtraction at school. By contrast, after one year of practice with this algorithm, about half of the children of all achievement levels *only* relied on the standard algorithm for multi-digit subtraction (cf. Research question 1a). Children of all achievement levels preferred the algorithm, even on subtractions such as 963 – 499 = __ that were especially included to evoke the compensation strategy (cf. Research question 1b). Moreover, children of all achievement levels more efficiently executed the standard algorithm than the mental computation strategies, again even on subtractions as 963 – 499 = __ that were assumed to be most easily and quickly solved via the compensation strategy (cf. Research questions 2a and 2b). Additional qualitative analyses of children’s mental computation strategies revealed that they hardly applied the expected clever compensation strategy but rather relied on less efficient decomposition and sequential strategies.

How can we explain children’s highly frequent and efficient use of the standard algorithm and their highly infrequent use of the clever compensation strategy, even on subtractions as 963 – 499 = __ ? The most plausible explanation for these findings refers to children’s instructional histories in this curricular domain. Although they all received ample instruction in multi-digit numbers and mental computation strategies for multi-digit subtraction in 2nd and 3rd grade, before the standard algorithm for multi-digit subtraction was introduced at school, their teachers most frequently used base-10 structuring materials to support their number and arithmetic instruction and mainly focused on the mastery of the sequential jump strategy to effectively solve multi-digit subtractions via mental computation. As such, the 2nd and 3rd grade teachers probably provided the necessary building blocks to acquire a good understanding of the properties of our decimal number system, thereby laying the foundations for an insightful acquisition of the standard algorithm by their pupils later on (Carpenter et al. 1998; Fuson et al. 1997; Hiebert and Wearne 1996). But they also constrained children's development of strategy variety in 2nd and 3rd grade, by primarily focusing on the sequential jump strategy during the first months of mental computation strategy instruction and paying attention to other, clever varying strategies, including compensation, only towards the end of this instruction and to a minor extent (Blöte et al. 2001). Moreover, in all participating classrooms, instruction emphasized the mastery of the standard algorithm for multi-digit subtraction from the middle of 3rd grade on. As a result of this strong instructional focus on the standard algorithm starting in 3rd grade, children presumably became gradually more efficient in this algorithm, while their mastery of mental computation in general, and compensation in particular, may have stagnated or even declined. This instructional focus on the standard algorithm and the socio-cultural classroom norms concerning the "prestige" of both kinds of methods (Yackel and Cobb 1996) probably also led them to construct the personal belief that the newly learnt algorithm was the superior way to subtract larger numbers, whatever the nature of the given numbers and/or their subjective mastery of both types of computation strategies. Evidently, this hypothetical explanation in terms of children's classroom practice and culture should be tested in future studies, comparing the strategy competencies of children instructed along the lines of reform-based curricula with those in more traditionally-oriented classrooms, before as well as after instruction in standard algorithms.

Flexibility in Mental Computation Strategies and Standard Algorithm Use

Turning to the flexibility results, we extended previous work on the flexible nature of children's strategy choices on multi-digit problems by defining and operationalizing strategy flexibility on the basis of not only item but also subject characteristics. Our findings clearly indicate that it is important to broaden the view on flexible strategy choices by incorporating more than only numerical task characteristics in the definition and operationalization of this concept (Heinze et al. 2009; Verschaffel et al. 2009). First, departing from a simple definition of strategy flexibility as fitting strategy choices to the numerical characteristics of the subtractions, children's strategy behavior needed to be characterized as highly *in*flexible as children most frequently relied on the standard algorithm on *all* types of subtractions (which is in line with Selter's [2001] results) (cf. Research question 3a). But using a more complex definition of strategy flexibility as using the strategy that best matches individual strategy performance characteristics, high and above-average achieving children's strategy choices were flexible. In other words, starting from the definition of strategy flexibility that also takes into account subject characteristics, these children's frequent reliance on the standard algorithm for multi-digit subtraction was flexible (cf. Research question 3b).

Methodologically, the choice/no-choice method allowed us to analyze the flexible nature of children's strategy choices departing from such a broadened definition of strategy flexibility by providing the necessary data to compare children's strategy behavior in the choice condition with the accuracy and speed of strategy execution in the different no-choice conditions (see also Author 2013; Luwel, Onghena, Torbeyns and Verschaffel 2010; Siegler and Lemaire 1997). But we were also confronted with a major constraint of this method, namely the tension between internally valid versus ecologically valid strategy data on tasks that can be solved with a rich diversity of strategies. In the present study, children were experimentally allowed to apply various mental computation strategies in both the choice and the no-choice mental condition. Although this variety in strategy use is an ecologically valid reflection of children's actual strategy behavior on multi-digit subtractions, it forced us to compare the frequency and efficiency of using the standard algorithm for multi-digit subtraction with these of a diversity of mental computation strategies, i.e., decomposition, sequential as well as varying strategies including compensation. It could be argued that, by restricting children's mental computation strategies to the application of *only* the compensation strategy and (thus) comparing the characteristics of the standard algorithm with these of that clever compensation strategy only, our results might have been different. More specifically, they might have provided more empirical support for the assumed continued efficient and flexible use of the clever compensation strategy on subtractions with specific numerical characteristics. It is a great challenge for future studies to find a proper balance between, on the one hand, allowing people to apply a reasonable range of available strategies (with a view to get ecologically valid results), and, on the other hand, restricting people's strategy choices somehow with a view to gather internally valid data.

Strategy Competencies of Children from Different Achievement Levels

To address the discussion about the feasibility of the changes in the curricular goals and content of elementary mathematics education proposed by the reform movement for children of lower achievement levels, we analyzed the occurrence, frequency, efficiency and flexibility of children's mental computation versus standard algorithm use in relation to their general math achievement level (cf. Research questions 1a, 1b, 2b, 3c). Although higher achieving children generally outperformed their lower achieving peers, i.e., solved multi-digit subtractions more accurately and faster than the latter, our results revealed hardly any differences in their strategy characteristics. More specifically, the occurrence, frequency and efficiency of mental computation versus standard algorithm use did not differ among the different mathematical achievement groups. We only observed differences in the flexibility of their strategy choices, the hardest and most advanced strategy competency they need to acquire. As discussed in more detail above, neither the high achievers, nor the above-average or the below-average achievers flexibly fitted their strategy choices to the numerical characteristics of the subtractions. But the high and above-average achievers took into account their individual mastery of the different types of strategies during the strategy choice process, whereas below-average achievers did not incorporate this characteristic in the selecting process. As the development of adaptive expertise is a major goal of current elementary mathematics curricula, for children of all achievement levels, future intervention studies specifically designed to stimulate the development of strategy flexibility in lower achieving children are needed (Verschaffel et al. 2009). The explicit comparison of the efficiency of different types of strategies on various types of problems has proved a highly effective intervention method in the domain of algebra, for students of different general mathematical achievement level and varying domain-specific knowledge and skill (Rittle-Johnson and Star 2009; Rittle-Johnson, Star and Durkin 2012). The applicability and the effectiveness of this comparison method in elementary mathematics education, in younger age groups and in children of different mathematical achievement levels, is an interesting venue for further research.

Notes

[1]As discussed in Verschaffel, Luwel, Torbeyns and Van Dooren (2009), the terms "flexibility" and "adaptivity" are used with different meanings in the international research literature. Surveying the literature, it seems that the term flexibility is primarily used to refer to *switching (smoothly) between different strategies,* whereas the term adaptivity puts more emphasis on *selecting the most appropriate strategy.* In the present study, we use these terms as synonyms, referring to children's ability to switch between different strategies taking into account task and/or individual strategy performance characteristics.

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Table 1.

Number, gender, mean age and mean mathematics achievement test score (SD between parentheses) per achievement group

Math achievement group	*n*	Gender		Age[a]	Math achievement score[b]
		m	f		
High	22	13	9	9y7m (3m)	51,82 (4,08)
Above-average	14	9	5	9y7m (4m)	43,36 (1,78)
Below-average	22	11	11	9y8m (4m)	29,36 (7,71)
All	58	33	25	9y7m (4m)	41,26 (11,29)

[a] Age is expressed in years, months (*SD* in months).

[b] The maximum score on the mathematics achievement test is 60. High achievers scored higher on the mathematics achievement test than above-average achievers, who, at their turn, outperformed below-average achievers, $F(2, 57) = 94.50$, $p < .01$.

Table 2.

Mean accuracy and speed of responding in the no-choice conditions per item type and achievement group (SD between parentheses)

Math Achievement Group	Item Type	Mental Computation Accuracy	Mental Computation Speed	Standard Algorithm Accuracy	Standard Algorithm Speed
High	MC	0.66 (0.33)	35.39 (17.22)	0.95 (0.10)	28.11 (8.33)
	SA	0.70 (0.31)	40.30 (19.59)	0.98 (0.07)	26.16 (7.43)
	Total	0.68 (0.30)	37.84 (17.28)	0.97 (0.06)	27.14 (7.75)
Above-Average	MC	0.77 (0.27)	41.10 (13.05)	0.77 (0.33)	31.33 (8.17)
	SA	0.73 (0.35)	50.28 (17.35)	0.82 (0.28)	31.85 (12.79)
	Total	0.76 (0.26)	45.93 (12.67)	0.80 (0.28)	31.59 (10.13)
Below-Average	MC	0.43 (0.32)	49.30 (13.01)	0.67 (0.27)	33.30 (5.92)
	SA	0.50 (0.40)	53.50 (15.91)	0.83 (0.25)	33.53 (8.06)
	Total	0.47 (0.34)	51.75 (13.88)	0.75 (0.22)	33.60 (5.87)
All	MC	0.60	42.05	0.80	30.86

		(0.34)	(15.75)	(0.27)	(7.68)
	SA	0.63	47.72	0.88	30.33
		(0.36)	(18.41)	(0.22)	(9.63)
	Tot	0.62	45.07	0.84	30.66
al		(0.32)	(15.97)	(0.21)	(8.16)

Note. Accuracy is expressed in proportion correct; speed is expressed in seconds.

Figure Captions

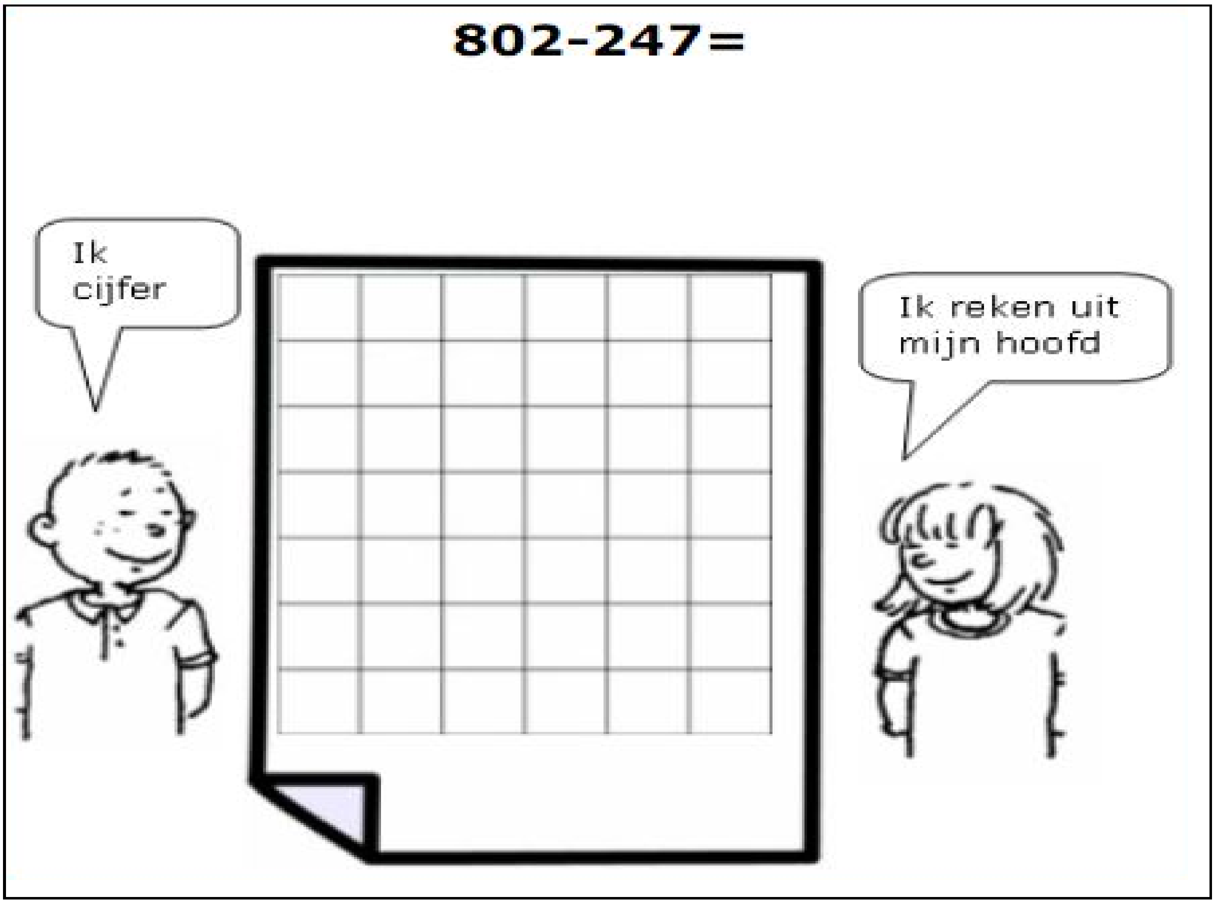
Figure 1. Standard algorithm for multi-digit subtraction as taught in Flanders (Belgium). Example item 482 - 299 = __

Figure 2. Example of an item offered in the choice condition. The boy states "I use the standard algorithm"; the girl says "I use mental computation".

17

3 7 12
4 8 2
- 2 9 9
1 8 3

Figure 1. Standard algorithm for multi-digit subtraction as taught in Flanders (Belgium). Example item 482 - 299 = __



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