Correlation kernels for sums and products of random matrices

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Abstract

Let X be a random matrix whose squared singular value density is a polynomial ensemble. We derive double contour integral formulas for the correlation kernels of the squared singular values of GX and TX, where G is a complex Ginibre matrix and T is a truncated unitary matrix. We also consider the product of X and several complex Ginibre/truncated unitary matrices. As an application, we derive the precise condition for the squared singular values of the product of several truncated unitary matrices to follow a polynomial ensemble. We also consider the sum $H + M$ where H is a GUE matrix and M is a random matrix whose eigenvalue density is a polynomial ensemble. We show that the eigenvalues of $H + M$ follow a polynomial ensemble whose correlation kernel can be expressed as a double contour integral. As an application, we point out a connection to the two-matrix model.

1 Introduction

Eigenvalues and singular values of sums and products of random matrices have been studied extensively over the last decade. The focus of the research has been the global density of the eigenvalues and singular values of the sum and product of large random matrices. As the dimension of random matrices

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tends to infinity, the macroscopic eigenvalue density of sums and products of random matrices can be described under fairly general conditions using free probability techniques [31]. However, the free probability method can only give global asymptotic results. For exact correlation functions of the eigenvalues and singular values of the sums and products of random matrices, until recently there were only a limited number of cases in which results were known. Notable cases are the Gaussian unitary ensemble with external source [9], which is the sum of a Gaussian unitary matrix (GUE) with a deterministic matrix; and the complex Wishart ensemble [5], which is the product of a complex Ginibre matrix with a deterministic matrix. Another interesting result is known for the sum of complex null Wishart matrices, in the form of the multiple Laguerre minor process with a fixed time [1]. It is remarkable that all the examples mentioned above are *polynomial ensembles*, a special kind of determinantal point processes, and that their correlation kernels all have double contour integral representations.

Recently, the correlation functions of the (squared) singular values are found for more types of products of random matrices, for instance for products of complex Ginibre matrices and products of truncated unitary Haar distributed matrices, as well as their inverses, see [3], [28], [17], [27], [25], and [19]. For a comprehensive survey of the current developments, see [2] and references therein. In all the cases listed above, the squared singular value densities are polynomial ensembles, and the correlation kernels can be expressed as double contour integrals.

Polynomial ensembles arise naturally in the study of probability densities for eigenvalues and singular values of random matrices. Their structure is preserved under certain operations on random matrices, such as multiplication by a complex Ginibre matrix [27] and multiplication by a truncated unitary Haar distributed matrix [25]. Those random matrix operations thus induce transformations of polynomial ensembles [26]. In this paper, we obtain explicit transformation formulas for the correlation kernels for a number of such transformations.

One feature of our results is that the correlation kernels obtained in this paper are all in the double contour integral form. Double contour integral formulas are very suitable to derive asymptotic results on local statistics, by the classical saddle point method. Numerous applications of double contour formulas can be found in literature. Two recent examples are [28] for the discovery of a family of new hard-edge universality and [29] for the proof of bulk and soft-edge universality.

A polynomial ensemble [27] is a probability density function for n particles $x_1, \ldots, x_n$ on the real line of the form

$$\frac{1}{Z_n} \Delta_n(x) \det \left[f_{k-1}(x_j) \right]_{j,k=1}^n \tag{1.1}$$

with given functions $f_0, \dots, f_{n-1}$. Here we use

$$\Delta_n(x) = \prod_{1 \le j < k \le n} (x_k - x_j) = \det \left[x_j^{k-1} \right]_{j,k=1}^{n} \tag{1.2}$$

to denote the Vandermonde determinant for the n-tuple $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and Z_n is a normalization constant. It is a polynomial ensemble on $E \subset \mathbb{R}$ if $f_k(x) = 0$ for every $x \in \mathbb{R} \setminus E$ and every $k = 0, \dots, n-1$, since then the particles are in E (with probability one).

The polynomial ensemble (1.1) is a determinantal point process and its correlation kernel can be written in the form

$$k_n(x, y) = \sum_{j=0}^{n-1} p_j(x) q_j(y), \tag{1.3}$$

where p_j is a polynomial of degree j and q_j belongs to the linear span of $f_0, \dots, f_{n-1}$ for every $j = 0, \dots, n-1$, in such a way that they satisfy the biorthogonality conditions

$$\int_{-\infty}^{\infty} p_j(x) q_k(x) dx = \delta_{j,k}, \qquad \text{for } j, k = 0, \dots, n-1. \tag{1.4}$$

We call such a set of p_j's and q_j's a *biorthogonal system* associated with the polynomial ensemble (1.1). We sometimes refer to the q_j's as *dual functions*. The p_j's and q_j's are not unique, but the sum $\sum_{j=0}^{n-1} p_j(x) q_j(y)$ in (1.3) is. If we require that p_j is a monic polynomial, and that q_j is in the linear span of $f_0, \dots, f_j$ for every j, then the biorthogonal system is unique. However, such a biorthogonal system does not always exist, and, if it exists, it may not be the most convenient one to work with. The unique monic polynomial p_n of degree n which is orthogonal to $f_0, \dots, f_{n-1}$, i.e. $\int_{-\infty}^{\infty} p_n(x) f_k(x) dx = 0$ for $k = 0, \dots, n-1$, is the average characteristic polynomial

$$p_n(x) = \mathbb{E} \left(\prod_{j=1}^{n} (x - x_j) \right), \tag{1.5}$$

where $\mathbb{E}$ denotes the average over particles $x_1, \dots, x_n$ in the polynomial ensemble (1.1).

In [26] a number of transformations which preserve the structure of a polynomial ensemble are given with transformation formulas for the functions f_k in (1.1), but without explicit formulas for the correlation kernels. In this paper, we give transformation formulas for the correlation kernels, biorthogonal systems and average characteristic polynomials for three such transformations. The polynomial ensembles which we consider are joint probability densities for:

(i) the eigenvalues of the sum $H+M$ of a GUE matrix H with a Hermitian random matrix M,

(ii) the squared singular values of the product GX of a complex Ginibre matrix G with a random matrix X,

(iii) the squared singular values of the product TX of a truncation of a Haar distributed unitary matrix with a random matrix X.

Here and throughout the paper, we understand M and X as follows.

Assumption 1.1. M is an $n \times n$ Hermitian random matrix whose eigenvalues $x_1, \dots, x_n$ follow a polynomial ensemble expressed by (1.1). We let $p_k(x)$ and $q_k(y)$ $(k = 0, 1, \dots, n-1)$ be a biorthogonal system, $k_n(x, y)$ be the correlation kernel, and $p_n(x)$ be the average characteristic polynomial for the polynomial ensemble.

Assumption 1.2. X is an $\ell \times n$ $(\ell \geq n)$ complex matrix whose squared singular values $x_1, \dots, x_n$, that is, eigenvalues of X^*X, follow a polynomial ensemble expressed by (1.1). We let $p_k(x)$ and $q_k(y)$ $(k = 0, 1, \dots, n-1)$ be the biorthogonal system, $k_n(x, y)$ be the correlation kernel, and $p_n(x)$ be the average characteristic polynomial for the polynomial ensemble.

Note that although notations f_k, p_k, q_k, k_n and p_n have different meanings in Assumptions 1.1 and 1.2, since M and X will not appear together in our paper, no confusion will occur.

In cases (ii) and (iii), transformation formulas were obtained in [27] and [25] for the joint probability densities of the squared singular values of GX and TX in terms of the density for X. We will prove a similar transformation formula in case (i). Our main focus is on the correlation kernels. We will express, in case (i), the eigenvalue correlation kernel of $H + M$ in terms of the eigenvalue correlation kernel of M, and in case (ii) (resp. case (iii)) the squared singular value correlation kernels of GX (resp. TX), in terms of the squared singular value correlation kernel of X. In addition, we will give transformation formulas for biorthogonal systems and for the average characteristic polynomials.

2 Statement of main results

2.1 Addition of a GUE matrix

Up to an overall scale factor, a GUE matrix is a Hermitian random matrix H with diagonal entries in standard real normal distribution, upper-diagonal entries in standard complex normal distribution, and all upper-triangular and diagonal entries independent. The joint probability density function for

the eigenvalues $x_1, \ldots, x_n$ of H is given by

$$\frac{1}{Z_n}\Delta_n(x)^2 \prod_{j=1}^{n} e^{-\frac{x_j^2}{2}}, \qquad Z_n = (2\pi)^{\frac{n}{2}} \prod_{k=1}^{n} k!, \tag{2.1}$$

which is the polynomial ensemble (1.1) in the case where $f_k(x) = x^k e^{-x^2/2}$. If we add a GUE matrix to a random matrix with eigenvalues in a polynomial ensemble, we are led to a transformed polynomial ensemble for the eigenvalues of the sum.

Theorem 2.1. *Let H be an $n \times n$ GUE matrix and M be the random matrix from Assumption 1.1, independent of H. The density of the eigenvalues $y_1, \ldots, y_n$ of $H + M$ is given by*

$$\frac{1}{Z'_n}\Delta_n(y) \det\left[F_{k-1}(y_j)\right]_{j,k=1}^{n}, \tag{2.2}$$

for some constant Z'_n, where

$$F_k(y) = \int_{-\infty}^{\infty} f_k(t) e^{-\frac{1}{2}(y-t)^2} dt, \qquad k = 0, \ldots, n-1. \tag{2.3}$$

We will prove Theorem 2.1 in Section 3.1.

Remark 2.2. The functions F_k in the transformed polynomial ensemble (2.2) are convolutions of the functions f_k from the original polynomial ensemble (1.1) with the Gaussian density $e^{-x^2/2}$.

We now describe the effect of this transformation on the biorthogonal system, on the correlation kernel, and on the average characteristic polynomial. Since the eigenvalue density of $H + M$ is the polynomial ensemble (2.2), the associated correlation kernel K_n can be written in the form

$$K_n(x, y) = \sum_{k=0}^{n-1} P_k(x) Q_k(y), \tag{2.4}$$

where P_k is a polynomial of degree k and Q_k is in the linear span of $F_0, \ldots, F_{n-1}$, such that

$$\int_{-\infty}^{\infty} P_j(x) Q_k(x) dx = \delta_{j,k}, \qquad \text{for } j, k = 0, \ldots, n-1. \tag{2.5}$$

We express the quantities K_n, P_k, and Q_k in the transformed polynomial ensemble in terms of their counterparts k_n, p_k, and q_k in the original polynomial ensemble for M.

Theorem 2.3. *Let H be an $n \times n$ GUE random matrix and M be the random matrix from Assumption 1.1, independent of H, as in Theorem 2.1. Then,*

(a) *a biorthogonal system in the transformed polynomial ensemble defined by* (2.2) *and* (2.3) *is given by*

$$P_k(x) = \frac{1}{\sqrt{2\pi}i} \int_{-i\infty}^{+i\infty} p_k(s) e^{\frac{1}{2}(x-s)^2} ds, \tag{2.6}$$

$$Q_k(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q_k(t) e^{-\frac{1}{2}(y-t)^2} dt, \tag{2.7}$$

(b) *the correlation kernel for the eigenvalues of* $H + M$ *is given by*

$$K_n(x,y) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \int_{-\infty}^{\infty} dt\, k_n(s,t)\, e^{\frac{1}{2}((x-s)^2-(y-t)^2)}, \tag{2.8}$$

(c) *the average characteristic polynomial* P_n *of* $H + M$ *is given by* (2.6) *with* $k = n$.

The proof of Theorem 2.3 will be given in Section 3.2.

Remark 2.4. In the integral formula (2.8), we use an extension of the kernel $k_n(s,t)$ for s in the complex plane. We can do this because $k_n(s,t)$ is polynomial in s for any $t \in \mathbb{R}$.

Remark 2.5. The integral formulas (2.7) and (2.3) for Q_k and F_k can be recognized as Weierstrass transforms [22, Chapter VIII] of q_k and f_k (up to the prefactors; the usual definition of the Weierstrass transform also has $1/4$ instead of $1/2$ in the exponent); P_k given in (2.6) is the inverse Weierstrass transform of p_k, up to the prefactor, which is chosen such that P_k is monic if p_k is monic.

As an application of Theorem 2.3, we derive in Section 3.3 a double contour integral formula (3.22) for the correlation kernel of the eigenvalues of the sum of a GUE matrix and an arbitrary random unitary invariant matrix. This formula is suitable for the analysis of limiting local statistics, and we investigate the analytic consequences in a separate paper.

2.2 Multiplication with a Ginibre matrix

A complex Ginibre matrix is a rectangular matrix whose entries are independent and identically distributed in the standard complex normal distribution. We define the complex Ginibre matrix G as follows.

Assumption 2.6. Let $\nu \geq 0$ and $\ell \geq n$. G is an $(n+\nu) \times \ell$ complex Ginibre matrix.

In the case $\ell = n$, the joint probability density for the squared singular values $x_1, \ldots, x_n$ of G is given by

$$\frac{1}{Z_n} \Delta_n(x)^2 \prod_{j=1}^{n} x_j^{\nu} e^{-x_j}, \qquad x_j \geq 0, \qquad Z_n = \prod_{k=1}^{n} k!\Gamma(k+\nu). \tag{2.9}$$

This is a polynomial ensemble on $[0,\infty)$ defined by the functions $f_k(x) = x^{\nu+k}e^{-x}$, $k = 0, 1, \ldots, n-1$.

Left multiplication of a random matrix by a complex Ginibre matrix induces a transformation of polynomial ensembles on $[0,\infty)$: the following result was proved by Kuijlaars and Stivigny [27] (based on ideas from [3, 4]).

Proposition 2.7. *Let G be the complex Ginibre matrix from Assumption 2.6 and X be the random matrix from Assumption 1.2, independent of G. Then the squared singular values $y_1, \ldots, y_n$ of $Y = GX$ follow a polynomial ensemble on $[0,\infty)$ given by*

$$\frac{1}{Z'_n}\Delta_n(y)\,\det\left[F_{k-1}(y_j)\right]_{j,k=1}^{n}, \tag{2.10}$$

for some constant Z'_n, where

$$F_k(y) = \int_0^\infty t^\nu e^{-t} f_k\left(\frac{y}{t}\right)\frac{dt}{t}, \qquad y > 0. \tag{2.11}$$

We complement the above result with transformation formulas for the biorthogonal system, for the correlation kernel, and for the average characteristic polynomial.

Theorem 2.8. *Let G be the complex Ginibre matrix from Assumption 2.6 and X be the random matrix from Assumption 1.2, independent of G, as in Proposition 2.7. Then,*

(a) a biorthogonal system in the transformed polynomial ensemble defined by (2.10) *and* (2.11) *is given by*

$$P_k(x) = \frac{1}{2\pi i}\oint_\Sigma s^{-\nu}e^s p_k\left(\frac{x}{s}\right)\frac{ds}{s}, \tag{2.12}$$

$$Q_k(y) = \int_0^\infty t^\nu e^{-t} q_k\left(\frac{y}{t}\right)\frac{dt}{t}, \tag{2.13}$$

where Σ is a simple counter-clockwise oriented contour encircling the origin,

(b) the correlation kernel K_n for the squared singular values of GX is given by

$$K_n(x,y) = \frac{1}{2\pi i}\oint_\Sigma \frac{ds}{s}\int_0^\infty \frac{dt}{t}\,k_n\left(\frac{x}{s},\frac{y}{t}\right)\left(\frac{t}{s}\right)^\nu e^{s-t}, \tag{2.14}$$

(c) the average characteristic polynomial P_n of X^*G^*GX is equal to

$$P_n(x) = \frac{(n+\nu)!}{2\pi i}\oint_\Sigma s^{-\nu}e^s p_n\left(\frac{x}{s}\right)\frac{ds}{s}. \tag{2.15}$$

The proof of Theorem 2.8 will be given in Section 5.1.

Remark 2.9. The functions Q_k and F_k can be recognized as Mellin convolutions of f_k and q_k with the Gamma density $t^\nu e^{-t}$. We will show that P_k defined in (2.12) can also be written in an alternative way: if $p_k(x) = \sum_{j=0}^k a_j x^j$, then we have

$$P_k(x) = \sum_{j=0}^{k} \frac{a_j}{(j+\nu)!} x^j. \tag{2.16}$$

Thus the factor $(n+\nu)!$ in (2.15) makes $P_n(x)$ monic.. The transformation formula (2.16) for P_k is reminiscent of similar transforms in [18, Formulas (3.16) and (3.43)]. We can also write (2.14) as

$$K_n(x,y) = \frac{1}{2\pi i} \left(\frac{y}{x}\right)^\nu \oint_\Sigma \frac{ds}{s} \int_0^\infty \frac{dt}{t}\, k_n(s,t) \left(\frac{s}{t}\right)^\nu e^{\frac{x}{s}-\frac{y}{t}}. \tag{2.17}$$

2.3 Multiplication with a truncated unitary matrix

The truncated unitary matrix T is defined as follows.

Assumption 2.10. Let $\nu \geq 1$, $m \geq \ell \geq n$ and $\mu = m - n - \nu \geq 1$. Let U be an $m \times m$ Haar distributed random unitary matrix. T is the $(n+\nu) \times \ell$ truncation of U.

In the case where $\ell = n$ and $\mu \geq n$, the joint probability density for the squared singular values $x_1, \ldots, x_n$ of T is given by [16, Section 3.8.3]

$$\frac{1}{Z_n} \Delta_n(x)^2 \prod_{j=1}^n x_j^\nu (1-x_j)^{m-2n-\nu}, \qquad 0 \leq x_j \leq 1,$$

$$Z_n = \prod_{k=1}^n \frac{k!\Gamma(k+\nu)\Gamma(k+m-2n-\nu)}{\Gamma(k+m-n)}. \tag{2.18}$$

The following result is due to Kieburg, Kuijlaars and Stivigny [25].

Proposition 2.11. *Let T be the truncated unitary matrix from Assumption 2.10 and X be the random matrix from Assumption 1.2, independent of T. Then the squared singular values $y_1, \ldots, y_n$ of $Y = TX$ follow a polynomial ensemble on $[0,\infty)$ given by*

$$\frac{1}{Z'_n} \Delta_n(y) \det\left[F_{k-1}(y_j)\right]_{j,k=1}^n, \tag{2.19}$$

for some constant Z'_n, where

$$F_k(y) = \int_0^1 t^\nu (1-t)^{\mu-1} f_k\left(\frac{y}{t}\right) \frac{dt}{t}, \qquad y > 0. \tag{2.20}$$

We have the counterpart of Theorem 2.8 as follows.

Theorem 2.12. *Let T be the truncated unitary matrix from Assumption 2.10 and X be the random matrix from Assumption 1.2, independent of T, as in Proposition 2.11. Then,*

(a) a biorthogonal system in the transformed polynomial ensemble defined by (2.19) *and* (2.20) *is given by*

$$P_k(x) = \frac{\mu}{2\pi i} \oint_\Sigma s^{-\nu}(1-s)^{-\mu-1} p_k\left(\frac{x}{s}\right) \frac{ds}{s}, \tag{2.21}$$

$$Q_k(y) = \int_0^1 t^\nu (1-t)^{\mu-1} q_k\left(\frac{y}{t}\right) \frac{dt}{t}, \tag{2.22}$$

where Σ is a simple counter-clockwise contour around 0 but not containing 1,

(b) the correlation kernel for the squared singular values of TX is given by

$$K_n(x,y) = \frac{\mu}{2\pi i} \oint_\Sigma \frac{ds}{s} \int_0^1 \frac{dt}{t} \left(\frac{t}{s}\right)^\nu k_n\left(\frac{x}{s}, \frac{y}{t}\right) (1-s)^{-\mu-1} (1-t)^{\mu-1}, \tag{2.23}$$

(c) the average characteristic polynomial P_n of X^*T^*TX is equal to

$$P_n(x) = \frac{\mu!(n+\nu)!}{2\pi i (n+\nu+\mu)!} \oint_\Sigma s^{-\nu}(1-s)^{-\mu-1} p_n\left(\frac{x}{s}\right) \frac{ds}{s}. \tag{2.24}$$

We will prove Theorem 2.12 in Section 6.1.

Remark 2.13. The functions Q_k and F_k are again Mellin convolutions of q_k and f_k, but now with the Beta density $t^\nu(1-t)^{\mu-1}$ supported on $[0,1]$. We will show that P_k in (2.21) can alternatively be written as

$$P_k(x) = \frac{1}{(\mu-1)!} \sum_{j=0}^{k} \frac{(j+\nu+\mu)!}{(j+\nu)!} a_j x^j, \tag{2.25}$$

if $p_k(x) = \sum_{j=0}^k a_j x^j$. Thus the factor $\mu!(n+\nu)!/(n+\nu+\mu)!$ in (2.24) makes $P_n(x)$ monic. The formula (2.25) is analogous to (2.16), and is reminiscent of similar formulas in [18], see Remark 2.9. We can also write (2.23) as

$$K_n(x,y) = \frac{\mu}{2\pi i} \left(\frac{y}{x}\right)^\nu \oint_{\Sigma'} \frac{ds}{s-x} \int_y^\infty \frac{dt}{t-y} k_n(s,t) \left(\frac{s}{t}\right)^{\nu+\mu} \left(\frac{t-y}{s-x}\right)^\mu, \tag{2.26}$$

where Σ' is a simple counter-clockwise contour encircling both 0 and x.

2.4 Generalization

Proposition 2.7, Theorem 2.8, Proposition 2.11, and Theorem 2.12 have an obvious structural similarity. The new functions F_k in (2.11) and (2.20) are given as Mellin convolutions of f_k with a fixed function φ where $\varphi(t) = t^\nu e^{-t}$ in the case of multiplication with a complex Ginibre matrix as in Proposition 2.7 and $\varphi(t) = t^\nu(1-t)^{\mu-1}\chi_{[0,1]}(t)$ in the case of multiplication with a truncated unitary matrix as in Proposition 2.11. The two Theorems 2.8 and 2.12 are obtained as special cases of the following lemma which deals with a general function φ.

Lemma 2.14. *Let $p_0, \dots, p_{n-1}$, $q_0, \dots, q_{n-1}$ be a biorthogonal system for the polynomial ensemble (1.1) on $[0,\infty)$. Suppose*

$$\frac{1}{Z_n'}\Delta_n(y)\,\det\left[F_{k-1}(y_j)\right]_{j,k=1}^n \tag{2.27}$$

is a polynomial ensemble on $[0,\infty)$ with

$$F_k(y) = \int_0^\infty \varphi(t) f_k\left(\frac{y}{t}\right)\frac{dt}{t}, \qquad y > 0, \tag{2.28}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a given non-negative function with finite non-zero moments. Define

$$b_j = \left[\int_0^\infty t^j \varphi(t)\,dt\right]^{-1} > 0, \qquad j = 0, 1, \dots, n-1. \tag{2.29}$$

(a) Then a biorthogonal system for (2.27) is given by polynomials $P_0, \dots, P_{n-1}$, and functions $Q_0, \dots, Q_{n-1}$ where for $k = 0, 1, \dots, n-1$,

$$P_k(x) = \sum_{j=0}^k a_{j,k} b_j x^j \qquad \text{if} \quad p_k(x) = \sum_{j=0}^k a_{j,k} x^j \tag{2.30}$$

and

$$Q_k(y) = \int_0^\infty \varphi(t) q_k\left(\frac{y}{t}\right)\frac{dt}{t}, \qquad y > 0. \tag{2.31}$$

(b) Let ψ be given by

$$\psi(x) = \sum_{j=-\infty}^{\infty} b_j x^j \tag{2.32}$$

where b_j is given by (2.29) for $j = 0, \dots, n-1$ and otherwise arbitrary. Assume that the Laurent series (2.32) converges in the annulus $\{r_\psi < |x| < R_\psi\}$. Then the polynomial P_k has the alternative representation

$$P_k(x) = \frac{1}{2\pi i}\oint_\Sigma \psi(s) p_k\left(\frac{x}{s}\right)\frac{ds}{s}, \tag{2.33}$$

where Σ is a closed, positive oriented contour lying in the annulus and encircling the origin once.

(c) *Under the same assumptions as in (b), if $k_n(x,y)$ is the correlation kernel for the polynomial ensemble* (1.1)*, then the correlation kernel $K_n(x,y)$ for the transformed ensemble* (2.27) *is given by*

$$K_n(x,y) = \frac{1}{2\pi i} \oint_\Sigma \frac{ds}{s} \int_0^\infty \frac{dt}{t} \psi(s)\varphi(t) k_n\left(\frac{x}{s}, \frac{y}{t}\right). \tag{2.34}$$

We prove Lemma 2.14 in Section 4 and we obtain Theorems 2.8 and 2.12 from it as easy consequences.

2.5 Extensions

Lemma 2.14 can be applied to other situations as well, in particular to products with a number of Ginibre matrices or a number of truncated unitary matrices. We first consider the product of a random matrix X satisfying Assumption 1.2 and several complex Ginibre matrices as follows.

Assumption 2.15. Let $\nu_0, \nu_1, \dots, \nu_r$ be non-negative integers. Let $G_1, G_2, \dots, G_r$ be independent Ginibre matrices where G_j is of size $(n+\nu_j) \times (n+\nu_{j-1})$ for $j = 1, 2, \dots, r$.

By applying Proposition 2.7 repeatedly, one easily sees that the squared singular values of $G_r G_{r-1} \cdots G_1 X$ follow a polynomial ensemble, as shown in [27]. It fits in the framework of Lemma 2.14 with the function

$$\varphi = \varphi_r * \cdots * \varphi_1, \tag{2.35}$$

where $\varphi_j(t) = t^{\nu_j} e^{-t}$ for $j = 1, \dots, r$, and $*$ denotes the Mellin convolution. Then φ is a Meijer G-function, $\varphi(t) = G^{r,0}_{0,r}\left(\begin{smallmatrix} - \\ \nu_1, \dots, \nu_r \end{smallmatrix} \middle| t \right)$, and we use Meijer G-functions freely in our results concerning products with several matrices. For their definition and properties, see [30], [32], and [6].

Then we have, from Lemma 2.14:

Corollary 2.16. *Let $G_1, \dots, G_r$ be the complex Ginibre matrices from Assumption 2.15 and let X be the random matrix from Assumption 1.2 with $\ell = n + \nu_0$, independent of $G_1, \dots, G_r$. Then the squared singular value density of the product $G_r \cdots G_1 X$ is a polynomial ensemble on $[0,\infty)$ with biorthogonal system consisting of polynomials*

$$P_k(x) = \frac{1}{2\pi i} \oint_\Sigma G^{1,1}_{1,r+1}\left(\begin{matrix} 0 \\ 0, -\nu_1, \dots, -\nu_r \end{matrix} \middle| -s \right) p_k\left(\frac{x}{s}\right) \frac{ds}{s} \tag{2.36}$$

where Σ is a simple closed counter-clockwise oriented contour around 0, and dual functions

$$Q_k(y) = \int_0^\infty G^{r,0}_{0,r}\left(\begin{matrix} - \\ \nu_1, \dots, \nu_r \end{matrix} \middle| t \right) q_k\left(\frac{y}{t}\right) \frac{dt}{t}. \tag{2.37}$$

Furthermore, the correlation kernel for the transformed polynomial ensemble is

$$K_n(x,y) = \frac{1}{2\pi i}\oint_\Sigma \frac{ds}{s}\int_0^\infty \frac{dt}{t} G^{1,1}_{1,r+1}\left(\begin{matrix} 0 \\ 0,-\nu_1,\dots,-\nu_r \end{matrix}\middle| -s\right) \times G^{r,0}_{0,r}\left(\begin{matrix} - \\ \nu_1,\dots,\nu_r \end{matrix}\middle| t\right) k_n\left(\frac{x}{s},\frac{y}{t}\right). \quad (2.38)$$

The proof of Corollary 2.16 will be given in Section 5.2.

Similarly, we consider the product of the random matrix X and several truncated unitary matrices as follows.

Assumption 2.17. Let $\nu_0,\nu_1,\dots,\nu_r$ be non-negative integers. Let $m_j \geq n+\nu_{j-1}$ and $m_j - n - \nu_j = \mu_j \geq 1$ for all $j = 1,2,\dots,r$. Let $U_1,\dots,U_r$ be independent Haar distributed random unitary matrices of size m_j for $j = 1,\dots,r$. T_j are the $(n+\nu_j)\times(n+\nu_{j-1})$ truncations of U_j for $j = 1,2,\dots,r$.

By applying Proposition 2.11 repeatedly, we see that the squared singular value density of $T_rT_{r-1}\cdots T_1X$ is a polynomial ensemble, as shown in [25]. Then we have the following result as another consequence of Lemma 2.14.

Corollary 2.18. *Let $T_1,\dots,T_r$ be the truncated unitary matrices from Assumption 2.17 and X be the random matrix from Assumption 1.2 with $\ell = n+\nu_0$, independent of $T_1,\dots,T_r$. Then the squared singular value density of the product $T_r\cdots T_1X$ is a polynomial ensemble on $[0,\infty)$ with the biorthogonal system consisting of polynomials*

$$P_k(x) = \frac{1}{2\pi i}\oint_\Sigma G^{1,r+1}_{r+1,r+1}\left(\begin{matrix} 0,-\nu_1-\mu_1,\dots,-\nu_r-\mu_r \\ 0,-\nu_1,\dots,-\nu_r \end{matrix}\middle| -s\right) p_k\left(\frac{x}{s}\right)\frac{ds}{s} \quad (2.39)$$

where Σ is a simple counter-clockwise contour around 0 lying inside the unit disk, and dual functions

$$Q_k(y) = \int_0^1 G^{r,0}_{r,r}\left(\begin{matrix} \nu_1+\mu_1,\dots,\nu_r+\mu_r \\ \nu_1,\dots,\nu_r \end{matrix}\middle| t\right) q_k\left(\frac{y}{t}\right)\frac{dt}{t}. \quad (2.40)$$

Furthermore, the correlation kernel for the transformed polynomial ensemble is

$$K_n(x,y) = \frac{1}{2\pi i}\oint_\Sigma \frac{ds}{s}\int_0^1 \frac{dt}{t} G^{1,r+1}_{r+1,r+1}\left(\begin{matrix} 0,-\nu_1-\mu_1,\dots,-\nu_r-\mu_r \\ 0,-\nu_1,\dots,-\nu_r \end{matrix}\middle| -s\right) \times G^{r,0}_{r,r}\left(\begin{matrix} \nu_1+\mu_1,\dots,\nu_r+\mu_r \\ \nu_1,\dots,\nu_r \end{matrix}\middle| t\right) k_n\left(\frac{x}{s},\frac{y}{t}\right). \quad (2.41)$$

We will prove Corollary 2.18 in Section 6.2.

We remark that Corollaries 2.16 and 2.18 can also be proved by repeated application of Theorems 2.8 and 2.12. However, this would lead to cumbersome manipulations of integrals with Meijer G-functions, which is avoided by the use of Lemma 2.14.

We can use Corollary 2.18 to study the squared singular values of a product of the form $T_r \cdots T_1$. To do so, we can proceed in two ways. We can either start with a truncated unitary matrix T_1 whose squared singular values are a determinantal point process and then apply Corollary 2.18 with $X = T_1$, which is multiplied by $r-1$ truncated unitary matrices $T_r \cdots T_2$. In this way we recover the result of [25, Corollary 2.6 and Proposition 2.7], as we discuss briefly in Remark 6.2. (In Section 5.3 we consider an analogous problem, the squared singular values of $G_r \cdots G_1$, in more detail and recover the result of [28, Proposition 5.1].)

The squared singular values of T_1 are a determinantal point process only if $\mu_1 \geq n$, in which case it is the Jacobi ensemble (2.18) with exponents ν_1 and $m_1 - 2n - \nu_1 = \mu_1 - n \geq 0$. If $\mu_1 < n$, then T_1 has a singular value at 1 of multiplicity $\geq n - \mu_1$, and the squared singular values are not a determinantal process in the usual, non-degenerate sense.

However, we may use Corollary 2.18 in a second way, by first taking a limit where the random matrix X approaches an $n \times n$ deterministic matrix with distinct singular values, say

$$A = \operatorname{diag}(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}), \quad a_j \in (0, \infty) \text{ are distinct.} \tag{2.42}$$

Then the squared singular values of $T_r \cdots T_1 A$ are a degenerate form of a polynomial ensemble, and we discuss this in Section 5.4. in the context of products with complex Ginibre matrices.

In a further step we can take the limit $A \to I$, which in the context of Corollary 2.18 leads to the final main result of this paper. It answers a question posed in [25, Section 7], about the precise conditions on truncated unitary matrices so that the squared singular values of $T_r \cdots T_1$ are a determinantal point process.

Theorem 2.19. *Let $T_1, \ldots, T_r$ be defined as in Assumption 2.17 with $\nu_0 = 0$. Then the squared singular values of $T_r \cdots T_1$ are a determinantal point process if and only if*

$$n \leq \sum_{j=1}^{r} \mu_j. \tag{2.43}$$

If (2.43) does not hold, then $T_r \cdots T_1$ has a singular value at 1 of multiplicity $\geq n - \sum_{j=1}^{r} \mu_j$.

If (2.43) holds, then the determinantal point process is a polynomial ensemble on $[0,1]$ with biorthogonal system consisting of polynomials

$$\begin{aligned} P_k(x) &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{l=1}^{r} \frac{(j+\nu_l+\mu_l)!}{(j+\nu_l)!} x^j \\ &= k! G^{0,r+1}_{r+1,r+1} \left(\begin{matrix} k+1, -\nu_1 - \mu_1, \ldots, -\nu_r - \mu_r \\ 0, -\nu_1, \ldots, -\nu_r \end{matrix} \middle| x \right) \end{aligned} \tag{2.44}$$

and dual functions

$$Q_k(y) = \frac{1}{k!} G^{r+1,0}_{r+1,r+1}\left(\begin{matrix} -k, \nu_1+\mu_1, \dots, \nu_r+\mu_r \\ 0, \nu_1, \dots, \nu_r \end{matrix} \middle| y \right), \tag{2.45}$$

and the correlation kernel is given by

$$\begin{aligned} K_n(x,y) &= \frac{1}{(2\pi i)^2} \int_C ds \int_\gamma dt \prod_{j=0}^{r} \frac{\Gamma(s+1+\nu_j)\Gamma(t+1+\nu_j+\mu_j)}{\Gamma(t+1+\nu)\Gamma(s+1+\nu_j+\mu_j)} \frac{x^t y^{-s-1}}{s-t} \\ &= -\int_0^1 G^{0,r+1}_{r+1,r+1}\left(\begin{matrix} -\nu_0-\mu_0, \dots, -\nu_r-\mu_r \\ -\nu_0, \dots, \nu_r \end{matrix} \middle| ux \right) \\ &\qquad \times G^{r+1,0}_{r+1,r+1}\left(\begin{matrix} \nu_0+\mu_0, \dots, \nu_r+\mu_r \\ \nu_0, \dots, \nu_r \end{matrix} \middle| uy \right) du \end{aligned} \tag{2.46}$$

where $\nu_0 = 0$ and $\mu_0 = -n$. The contour C is a positively oriented Hankel contour in the left-half of the complex s-plane that starts and ends at $-\infty$ and encircles $(-\infty, -1]$, and γ is a closed contour around $[0, n]$ that is disjoint from C.

We will prove Theorem 2.19 in Section 6.3.

Remark 2.20. In a recent paper [23], Imamura and Sasamoto derived a polynomial ensemble related to the O'Connell–Yor directed random polymer model. The biorthogonal system of the polynomial ensemble, see [23, Formulas (2.1) and (2.3)], has similarities with our formulas (2.6) and (2.7).

Remark 2.21. In this paper we consider the product $G_r G_{r-1} \cdots G_1 X$ or $T_r T_{r-1} \cdots T_1 X$ with the parameter r fixed. It is possible to consider the correlation of the squared singular values of all the products $G_r G_{r-1} \cdots G_1 X$ or $T_r T_{r-1} \cdots T_1 X$ with $r = 0, 1, 2, \dots$, see the recent paper [33] by Strahov.

Outline

In Section 3, we prove Theorem 2.1 and Theorem 2.3 about the eigenvalues of the sum of a GUE matrix with another Hermitian random matrix. We will also discuss some concrete consequences of those results. In Section 4, we prove Lemma 2.14. In Section 5, we apply Lemma 2.14 to prove the results in Theorem 2.8 on multiplication with a complex Ginibre matrix and corollaries on multiplication with several Ginibre matrices. In Section 6, we apply Lemma 2.14 to prove the results on multiplication with a truncated unitary matrices stated in Theorem 2.12, and the corollaries on multiplication with several truncated matrices. At last we prove Theorem 2.19.

3 Addition of a GUE matrix

3.1 Proof of Theorem 2.1

Let H be an $n \times n$ GUE matrix, and temporarily let M be a fixed $n \times n$ Hermitian matrix with eigenvalues $x_1, \ldots, x_n$. Then the eigenvalue density of H is given by (2.1) and the random matrix $Y = H + M$ has distribution

$$\frac{1}{Z_n} e^{-\frac{1}{2}\operatorname{Tr}(Y-M)^2} dY, \qquad Z_n = (2\pi)^{\frac{n}{2}} \prod_{k=1}^{n} k!, \tag{3.1}$$

which is the distribution for a random matrix Y in the GUE with external source M [9]. It is known from [7], [9] and [24] that the eigenvalues $y_1, \ldots, y_n$ of $Y = H + M$ are distributed according

$$P(y; x) = \frac{1}{\widehat{Z}_n \Delta_n(x)} \Delta_n(y) \det\left[e^{-\frac{1}{2}(y_j - x_k)^2}\right]_{j,k=1}^{n}, \qquad \widehat{Z}_n = n!(2\pi)^{\frac{n}{2}}, \tag{3.2}$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Now, let M be the random matrix satisfying Assumption 1.1. We obtain the joint probability density function of the eigenvalues $y_1, \ldots, y_n$ of $H + M$ by integrating (3.2) over the eigenvalues $x_1, \ldots, x_n$ of M. If their density is the polynomial ensemble (1.1), we obtain that the density of the eigenvalues $y_1, \ldots, y_n$ of $H + M$ is (Z_n is defined in (1.1), the polynomial ensemble of the eigenvalues of M)

$$\begin{aligned} P(y) &= \int_{\mathbb{R}^n} P(y; x) \Delta_n(x) \det\left[f_{k-1}(x_j)\right]_{j,k=1}^{n} dx_1 \cdots dx_n \\ &= \frac{\Delta_n(y)}{Z_n \widehat{Z}_n} \int_{\mathbb{R}^n} \det\left[f_{k-1}(x_j)\right]_{j,k=1}^{n} \det\left[e^{-\frac{1}{2}(y_j - x_k)^2}\right]_{j,k=1}^{n} dx_1 \cdots dx_n. \end{aligned} \tag{3.3}$$

By the Andréief formula, see e.g. [11], we have

$$P(y) = \frac{1}{Z'_n} \Delta_n(y) \det\left(\int_{-\infty}^{\infty} f_{k-1}(x) e^{-\frac{1}{2}(y_j - x)^2} dx\right)_{j,k=1}^{n}, \tag{3.4}$$

with $Z'_n = n! Z_n \widehat{Z}_n$, which is the polynomial ensemble defined by (2.2) and (2.3). □

3.2 Proof of Theorem 2.3

We assume that p_k and q_k for $k = 0, 1, \ldots, n-1$, are the biorthogonal system as in Assumption 1.1 associated to M. In other words, we assume that for $k = 0, \ldots, n-1$, p_k is a polynomial of degree k, q_k is in the linear span of $f_0, \ldots, f_{n-1}$, and they satisfy (1.4).

Denote the Weierstrass transform [22] of a function φ by

$$\mathcal{W}\varphi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(t) e^{-\frac{1}{2}(y-t)^2} dt. \tag{3.5}$$

The integral transform

$$\mathcal{W}^{-1}\Phi(x) = \frac{1}{\sqrt{2\pi}i} \int_{-i\infty}^{+i\infty} \Phi(s) e^{\frac{1}{2}(x-s)^2} ds \tag{3.6}$$

is the inverse Weierstrass transform: we have $\left(\mathcal{W} \circ \mathcal{W}^{-1}\right) \Phi = \Phi$ for a large class of functions. We will need this inversion property only for polynomials Φ. A property of the Weierstrass transform is that it sends monic polynomials to monic polynomials of the same degree.

Formulas (2.6) and (2.7) can now be written as

$$P_k(x) = \mathcal{W}^{-1} p_k(x), \qquad Q_k(y) = \mathcal{W} q_k(y). \tag{3.7}$$

Since q_k is in the linear span of $f_0, \ldots, f_{n-1}$, Q_k is in the linear span of $\mathcal{W}f_0, \ldots, \mathcal{W}f_{n-1}$, and thus also in the linear span of $F_0, \ldots, F_{n-1}$ since $F_j = \sqrt{2\pi}\mathcal{W}f_j$ by (2.3).

By (3.7), (3.5) and Fubini's theorem, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} P_j(x) Q_k(x) dx &= \int_{-\infty}^{\infty} \mathcal{W}^{-1} p_j(x)\, \mathcal{W} q_k(x) dx \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{W}^{-1} p_j(x) e^{-\frac{1}{2}(t-x)^2} dx \right) q_k(t) dt. \end{aligned} \tag{3.8}$$

The expression within parentheses is the Weierstrass transform of $\mathcal{W}^{-1}p_j$ and is thus equal to $p_j(t)$. Hence

$$\int_{-\infty}^{\infty} P_j(x) Q_k(x) dx = \int_{-\infty}^{\infty} p_j(x) q_k(x) dx = \delta_{j,k} \tag{3.9}$$

by (1.4). This proves part (a) of the theorem.

Next, we substitute (2.6) and (2.7) into (2.4) and use (1.3). This gives (2.8) and proves part (b).

Formula (3.9) is also valid for $j = n$ and $k = 0, 1, \ldots, n-1$. Thus P_n is a monic polynomial of degree n which is orthogonal to $Q_0, \ldots, Q_{n-1}$. This implies that P_n is the average characteristic polynomial in the ensemble (2.2) and we have proven part (c) of the theorem. □

3.3 Example: GUE plus a unitary invariant random matrix

In this subsection we assume that the random matrix M from Assumption 1.1 is defined by the unitary invariant probability measure [10]

$$\frac{1}{C_n} e^{-\operatorname{Tr} V(M)} dM, \qquad C_n = \int e^{-\operatorname{Tr} V(M)} dM, \tag{3.10}$$

where V is a real-valued function such that the integral defining C_n is convergent and which is allowed to depend on n. This is a random matrix ensemble which is invariant under unitary conjugation. If $V(x) = x^2/2$, it is simply the GUE. The joint probability density function for the eigenvalues of M is the polynomial ensemble with f_k given by $f_k(x) = x^k e^{-V(x)}$ in (1.1). For the biorthogonal systems, it is convenient to let p_k be the monic degree k orthogonal polynomial with respect to the weight $e^{-V(x)}$ on the real line and

$$q_k(x) = \frac{1}{h_k} p_k(x) e^{-V(x)}, \qquad h_k = \int_{-\infty}^{\infty} p_k(x)^2 e^{-V(x)} dx. \tag{3.11}$$

The correlation kernel k_n can be written as

$$k_n(x,y) = e^{-V(y)} \sum_{j=0}^{n-1} \frac{1}{h_j} p_j(x) p_j(y), \tag{3.12}$$

and by the Christoffel-Darboux formula we also have

$$k_n(x,y) = \frac{1}{h_{n-1}} e^{-V(y)} \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x-y}. \tag{3.13}$$

Theorem 2.3 applies to this case and we obtain the formula

$$K_n(x,y) = \frac{1}{2\pi i h_{n-1}} \int_{-i\infty}^{+i\infty} ds \int_{-\infty}^{\infty} dt \, \frac{p_n(s)p_{n-1}(t) - p_n(t)p_{n-1}(s)}{s-t} \times \quad e^{-V(t)} e^{\frac{1}{2}((s-x)^2 - (t-y)^2)}, \tag{3.14}$$

for the eigenvalue correlation kernel of the matrix $Y = H + M$, where H is a GUE matrix of size n and M has the distribution (3.10). We show how to rewrite this formula in a way that may be suitable for asymptotic analysis.

For a large class of potentials V (in particular, for n-dependent potentials of the form $V(x) = nV_0(x)$ with V_0 independent of n), large n asymptotics for $p_n(z)$ and $p_{n-1}(z)$ are known for z anywhere in the complex plane. However, it is not straightforward to apply saddle point techniques on the integrals in (3.14), especially because one has to integrate over the real line, where the zeros of the orthogonal polynomials p_n, p_{n-1} are, and where the integrand is oscillatory. Therefore, we derive an alternative expression for the correlation

kernel, which involves Cauchy transforms of the orthogonal polynomials and which avoids integration over the real line.

Define a 2×2 matrix-valued function (which is the solution to the Riemann-Hilbert problem for orthogonal polynomials [10], [15])

$$Y(z) = \begin{pmatrix} p_n(z) & \frac{1}{2\pi i} \int_{-\infty}^{\infty} p_n(s) e^{-V(s)} \frac{ds}{s-z} \\ -2\pi i h_{n-1}^{-1} p_{n-1}(z) & -h_{n-1}^{-1} \int_{-\infty}^{\infty} p_{n-1}(s) e^{-V(s)} \frac{ds}{s-z} \end{pmatrix}. \tag{3.15}$$

The entries in the first column are defined everywhere in the complex plane, the Cauchy transforms in the second column are defined for $z \in \mathbb{C} \setminus \mathbb{R}$. We can express the correlation kernel k_n in (3.13) in terms of Y. It is straightforward to check that [10, Chapter 8]

$$k_n(s,t) = \frac{-1}{2\pi i(s-t)} e^{-V(t)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y^{-1}(s) Y(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.16}$$

for $s \in \mathbb{C}$, $t \in \mathbb{R}$. Writing $Y_+(t)$ (resp. $Y_-(t)$) for the limit of $Y(z)$ as z approaches $t \in \mathbb{R}$ from the upper half plane (resp. lower half plane), we have the relation

$$Y_+(t) = Y_-(t) \begin{pmatrix} 1 & e^{-V(t)} \\ 0 & 1 \end{pmatrix}, \qquad t \in \mathbb{R}. \tag{3.17}$$

It follows that

$$e^{-V(t)} Y(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (Y_+(t) - Y_-(t)) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad t \in \mathbb{R}, \tag{3.18}$$

and that

$$k_n(s,t) = \frac{-1}{2\pi i(s-t)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y^{-1}(s) \left(Y_+(t) - Y_-(t)\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.19}$$

for $t \in \mathbb{R}$ and for any $s \in \mathbb{C}$. In (2.8), we need to integrate $k_n(s,t)$ in t over the real line. This integral can be deformed to a contour Γ which consists of a curve in the lower half plane oriented from left to right (for example $\Gamma = \mathbb{R} - i\delta$ for some $\delta > 0$), and its complex conjugate in the upper half plane oriented from left to right. We obtain, for any s which is not on Γ,

$$\begin{aligned} \int_{-\infty}^{\infty} k_n(s,t) e^{-\frac{1}{2}(y-t)^2} dt = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{s-\zeta} \begin{pmatrix} 0 & 1 \end{pmatrix} Y^{-1}(s) Y(\zeta) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\frac{1}{2}(y-\zeta)^2} d\zeta \\ + \begin{cases} e^{-\frac{1}{2}(y-s)^2} & \text{if } s \text{ is inside } \Gamma, \\ 0 & \text{if } s \text{ is outside } \Gamma. \end{cases} \end{aligned} \tag{3.20}$$

The last term is the residue contribution from the pole at $\zeta = s$. In (2.8), the integration over the imaginary axis can be changed to any vertical line C as contour of integration for s. Let $w_\pm$ be the intersection points of C

with Γ, with w_+ in the upper half plane and $w_- = \overline{w}_+$. (We assume there are only two intersection points.)

Then the contribution from the additional term to $K_n(x,y)$ is

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{w_-}^{w_+} e^{-\frac{1}{2}(y-s)^2} e^{\frac{1}{2}(x-s)^2} ds \\
&\qquad = \frac{1}{2\pi i(y-x)} e^{\frac{1}{2}(x^2-y^2)} \left(e^{(y-x)w_+} - e^{(y-x)w_-} \right) \\
&\qquad = \frac{1}{\pi(y-x)} e^{\frac{1}{2}(x^2-y^2)} e^{(y-x)\operatorname{Re} w_+} \sin\left((y-x)\operatorname{Im} w_+\right),
\end{aligned} \tag{3.21}$$

where x and y are real.

We thus obtain the alternative expression for the eigenvalue correlation kernel K_n of $H+M$,

$$\begin{aligned}
K_n(x,y) &= \frac{1}{\pi(y-x)} e^{\frac{1}{2}(x^2-y^2)} e^{(y-x)\operatorname{Re} w_+} \sin\left((y-x)\operatorname{Im} w_+\right) \\
&+ \frac{1}{(2\pi i)^2} \int_C ds \int_\Gamma \frac{d\zeta}{s-\zeta} \begin{pmatrix} 0 & 1 \end{pmatrix} Y^{-1}(s) Y(\zeta) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\frac{1}{2}((x-s)^2-(y-\zeta)^2)},
\end{aligned} \tag{3.22}$$

in which integration over the real line is avoided. This formula may be more convenient for asymptotic analysis than (3.14), but we do not aim to investigate this here.

Remark 3.1. The random matrix $H+M$ appears in the two-matrix model, which is defined as a measure on the space of pairs (M_1, M_2) of Hermitian $n \times n$ matrices,

$$\frac{1}{C_n} e^{-\operatorname{Tr}(W_1(M_1)+W_2(M_2)-M_1M_2)} dM_1 dM_2, \tag{3.23}$$

for certain functions W_1 and W_2 such that the above defines a probability distribution. If $W_1(M_1) = M_1^2/2$, as observed in [13], the probability measure (3.23) can be written as

$$\frac{1}{C_n} e^{-\operatorname{Tr}\left(\frac{1}{2}(M_1-M_2)^2+W_2(M_2)-\frac{1}{2}M_2^2\right)} d(M_1-M_2) dM_2. \tag{3.24}$$

This implies that M_2 and $M_1 - M_2$ are independent random matrices: $M_1 - M_2$ is a GUE matrix and M_2 is a random matrix from a unitary invariant one-matrix model with probability distribution (3.10), with

$$V(M) = W_2(M) - \frac{1}{2}M^2. \tag{3.25}$$

In other words, the matrix M_1 in the two-matrix model (3.23) then takes the form $M_1 = H + M$, where H is a GUE matrix and M is a random matrix from the ensemble (3.10).

Remark 3.2. If V in (3.10) is a polynomial, the eigenvalue correlation kernel K_n for $H+M$ can be expressed in terms of multiple orthogonal polynomials, since we can interpret $H+M$ as the matrix M_1 in the two-matrix model, but the asymptotic analysis of those multiple orthogonal polynomials using Riemann-Hilbert techniques is hard in general, in particular if the degree of V is large. The case where V is a quartic symmetric polynomial has been studied in detail in [14]. Our results yield an alternative expression for the correlation kernel of the eigenvalues of $H+M$, for a general potential V, which does not involve multiple orthogonal polynomials but only usual orthogonal polynomials and contour integrals thereof.

4 Proof of Lemma 2.14

Part (a) From the definitions (2.28) and (2.31) it is clear that Q_k belongs to the linear span of $F_0, \ldots, F_{n-1}$, since q_k belongs to the linear span of $f_0, \ldots, f_{n-1}$. It is also clear from (2.30) that P_k is a polynomial of degree k for every $k = 0, \ldots, n-1$.

To verify the biorthogonality we define an operator $\mathcal{L}$ acting on polynomials p by

$$\mathcal{L}p(x) = \sum_{j=0}^{\deg(p)} a_j b_j x^j \qquad \text{where} \quad p(x) = \sum_{j=0}^{\deg(p)} a_j x^j \tag{4.1}$$

is arbitrary while b_j are defined in (2.29), and an operator $\mathcal{M}$ acting on functions q on $[0,\infty)$ by

$$\mathcal{M}q(y) = \int_0^\infty \varphi(t) q\left(\frac{y}{t}\right) \frac{dt}{t}, \qquad y > 0. \tag{4.2}$$

Then $P_k = \mathcal{L}p_k$ and $Q_k = \mathcal{M}q_k$ by (2.30) and (2.31). We prove the identity

$$\int_0^\infty \mathcal{L}p(x)\, \mathcal{M}q(x)\, dx = \int_0^\infty p(x)q(x)\, dx \tag{4.3}$$

and then the biorthogonality follows from

$$\int_0^\infty P_j(x) Q_k(x)\, dx = \int_0^\infty p_j(x) q_k(x)\, dx = \delta_{j,k} \tag{4.4}$$

since $P_j = \mathcal{L}p_j$, $Q_k = \mathcal{M}q_k$ for $j,k = 0, 1, \ldots, n-1$.

To prove (4.3) we calculate by Fubini's theorem for $k = 0, 1, \ldots$

$$\int_0^\infty x^k\, \mathcal{M}q(x)\, dx = \int_0^\infty \varphi(t) \left(\int_0^\infty x^k q\left(\frac{x}{t}\right) dx \right) \frac{dt}{t}. \tag{4.5}$$

We substitute $x = tu$ in the inner integral and obtain

$$\begin{aligned}\int_0^\infty x^k \, \mathcal{M}q(x)\, dx &= \int_0^\infty t^k \varphi(t) dt \int_0^\infty u^k q(u) du \\ &= b_k^{-1} \int_0^\infty x^k q(x) dx. \end{aligned} \tag{4.6}$$

By the definition (4.1) we have $\mathcal{L}x^k = b_k x^k$, and thus (4.6) implies that (4.3) holds if $p(x) = x^k$ for $k = 0, 1, \ldots$. By linearity (4.3) holds for every polynomial p.

Part (b) The transformation (2.30) is the Hadamard (or termwise) product of p_k with the function ψ. The Hadamard product of two convergent Laurent series $a(x) = \sum_{j=-\infty}^{\infty} a_j x^j$ and $b(x) = \sum_{j=-\infty}^{\infty} b_j x^j$ has a well-known (and easy to prove) contour integral representation, namely

$$\sum_{j=-\infty}^{\infty} a_j b_j x^j = \frac{1}{2\pi i} \oint_{\Sigma_r} a(s) b\left(\frac{x}{s}\right) \frac{ds}{s}, \tag{4.7}$$

where Σ_r is the circle of radius r around the origin with positive orientation. Here we assume that the Laurent series for $a(x)$ converges for $r_a < |x| < R_a$, and the Laurent series for $b(x)$ converges for $r_b < |x| < R_b$. Then formula (4.7) is valid for $r_a r_b < |x| < R_a R_b$ and $r \in (r_a, R_a) \cap (\frac{|x|}{R_b}, \frac{|x|}{r_b})$.

The formula (2.33) follows from (4.7) and the definitions (2.30) and (2.32).

Part (c) The correlation kernel for (2.27) is

$$K_n(x, y) = \sum_{k=0}^{n-1} P_k(x) Q_k(y). \tag{4.8}$$

We insert the integral representations (2.33) and (2.31) for P_k and Q_k, and interchange the sum with the integrals. Then (2.34) follows because of (1.3). □

5 Multiplication with complex Ginibre matrices

5.1 Proof of Theorem 2.8

Theorem 2.8 follows immediately from Lemma 2.14. In the situation of Proposition 2.7 we have

$$\varphi(t) = t^\nu e^{-t} \tag{5.1}$$

with ν a non-negative integer. Then the moments (2.29) are

$$b_j = \left[\int_0^\infty t^{j+\nu} e^{-t} dt\right]^{-1} = [(j+\nu)!]^{-1}, \tag{5.2}$$

and for the function ψ from (2.32) we take

$$\psi(x) = \sum_{j=-\nu}^{\infty} \frac{x^j}{(j+\nu)!} = x^{-\nu} e^x. \tag{5.3}$$

Then parts (a) and (b) of Lemma 2.14 give the formulas (2.12) and (2.13) for the biorthogonal system in Theorem 2.8(a), and Theorem 2.8(c) is also derived in the same way. Lemma 2.14(c) gives the transformed kernel (2.34) in Theorem 2.8(b). □

5.2 Proof of Corollary 2.16

Assume X and $G_1, \ldots, G_r$ are as in Corollary 2.16. Applying Proposition 2.7 r times, see also [27], we find that the squared singular value density of $Y = G_r \cdots G_1 X$ is a polynomial ensemble with functions F_k that are the Mellin convolution of f_k with

$$\varphi = \varphi_r * \varphi_{r-1} * \cdots * \varphi_1 \tag{5.4}$$

where $\varphi_j(t) = t^{\nu_j} e^{-t}$ for $j = 1, \ldots, r$, and $*$ denotes the Mellin convolution. Thus φ is a Meijer G-function

$$\varphi(t) = G^{r,0}_{0,r}\left(\begin{matrix} - \\ \nu_1, \ldots, \nu_r \end{matrix} \middle| t \right) \tag{5.5}$$

which has the moments

$$b_j^{-1} = \int_0^\infty t^j \varphi(t) dt = \prod_{k=1}^{r} (j+\nu_k)!. \tag{5.6}$$

Now we apply Lemma 2.14. We take for ψ

$$\psi(s) = \sum_{j=0}^{\infty} \frac{s^j}{\prod_{k=1}^r (j+\nu_k)!}, \tag{5.7}$$

which is a generalized hypergeometric function

$$\psi(s) = \frac{1}{\prod_{k=1}^r \nu_k!}\, {}_1F_r\left(\begin{matrix} 1 \\ \nu_1+1, \ldots, \nu_r+1 \end{matrix} \middle| s \right) \tag{5.8}$$

and it is also a Meijer G-function

$$\psi(s) = G^{1,1}_{1,r+1}\left(\begin{matrix} 0 \\ 0, -\nu_1, \ldots, -\nu_r \end{matrix} \middle| -s \right). \tag{5.9}$$

Then Corollary 2.16 is a direct consequence of Lemma 2.14. □

5.3 Products of Ginibre matrices

We consider the product

$$Y_r = G_r G_{r-1} \cdots G_1 \tag{5.10}$$

of independent complex Ginibre matrices where G_j are defined in Assumption 2.15 with $\nu_0 = 0$. It was shown in [28] that the squared singular value density of Y_r is a polynomial ensemble with correlation kernel

$$K_n(x,y) = \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2}+i\mathbb{R}} dv \oint_\gamma du \prod_{j=0}^{r} \frac{\Gamma(v+\nu_j+1)}{\Gamma(u+\nu_j+1)} \frac{\Gamma(u-n+1)}{\Gamma(v-n+1)} \frac{x^u y^{-v-1}}{v-u}, \tag{5.11}$$

where $\nu_0 = 0$ and γ encircles $0, 1, \ldots, n$ once in positive direction and lies to the right of $-\frac{1}{2}+i\mathbb{R}$, and Γ is the Euler Gamma function. For $r = 1$, this is the correlation kernel for the eigenvalues in the complex Wishart Ensemble $G_1^* G_1$ with parameter ν_1.

We show how to obtain (5.11) from Theorem 2.8 by induction on r.

5.3.1 Base step: Case $r = 1$

The $r = 1$ case of (5.11) differs from the well known double contour integral formula for the complex Wishart ensemble ([16, Section 5.8])

$$K_n(x,y) = \frac{1}{(2\pi i)^2} \oint_\Sigma du \oint_\Gamma dv \frac{e^{xu} v^{n+\nu_1} (u-1)^n}{e^{yv} u^{n+\nu_1} (v-1)^n} \frac{1}{u-v}, \tag{5.12}$$

where Σ and Γ are disjoint closed contours both oriented counterclockwise such that Σ encloses 0 and Γ encloses 1. The $r = 1$ case of (5.11) is less well known, but it also appeared in the literature as a special case of the *multiple Laguerre minor process* in [1, Theorem 3(c)] with the fixed time n and special choice of $\alpha_k = n + \nu_1 - k$ for $k = 1, \ldots, n$, see also [21] for its relation to the Laguerre Muttalib–Borodin model. To be precise, the relation is

$$K_n(x,y)|_{r=1} = \frac{y^{\nu_1}}{x^{\nu_1}} K(n,y;n,x)\Big|_{\alpha_k = n+\nu_1-k \text{ for } k=1,\ldots,n}, \tag{5.13}$$

where K is the correlation kernel defined in [1, Theorem 3(c)], and the factor y^{ν_1}/x^{ν_1} is to conjugate the kernel into the $\sum_{k=0}^{n-1} p_k(x) q_k(y)$ form. The choice of contour in [1, Theorem 3(c)] is different from that in (5.11), but they are equivalent by the residue theorem. In comparison, the better known formula (5.12) is a special case of the *Wishart minor process* stated in [1, Theorem 3(b)], which was originally studied in [8], [12] and [20]. Both of the two correlation kernels are constructed by $\sum_{k=0}^{n-1} p_k(x) q_k(y)$ with $p_k(x) = L_k^{(\nu_1)}(x)$ and $q_k(y) = k!\Gamma(k+\nu_1+1)^{-1} L_k^{(\nu_1)}(y) y^{\nu_1} e^{-y}$, by expressing $p_k(x)$ and $q_k(y)$

into contour integral forms, and then summing them up via a telescoping trick. The difference lies in the fact that different contour integral formulas for p_k and q_k are used.

5.3.2 Induction step

Let $r \geq 2$, and assume that (5.11) with r replaced by $r-1$ is the correlation kernel for the squared singular values of $Y_{r-1} = G_{r-1} \cdots G_1$. Then $Y_r = G_r Y_{r-1}$, and using (5.11) (with $r-1$ instead of r) for k_n in the formula (2.14) with $\nu = \nu_r$, we obtain a quadruple integral on $-1/2 + i\mathbb{R}$, γ, Σ, and $[0, \infty)$ for the new correlation kernel. We then first deform the closed contour Σ to a Hankel contour L that comes from $-\infty$ in the lower half-plane, loops around the negative real axis and goes to $-\infty$ in the upper half-plane. We can then interchange the order of integration. The s- and t-integrals are evaluated explicitly as

$$\frac{1}{2\pi i} \int_L s^{-\nu_r - u - 1} e^s ds = \frac{1}{\Gamma(u + \nu_r + 1)}, \tag{5.14}$$

$$\int_0^\infty t^{\nu_r + v} e^{-t} dt = \Gamma(v + \nu_r + 1), \tag{5.15}$$

by the integral representations for the Gamma function and its reciprocal. Then the result is a double contour integral on $-1/2 + i\mathbb{R}$ and γ, that is precisely (5.11) with parameter r.

5.4 Limiting case of Corollary 2.16

We may also approach the calculation of the squared singular values of $Y_r = G_r \cdots G_1$ as a limiting case of Corollary 2.16.

In Corollary 2.16 we assume $\nu_0 = 0$ and let the random matrix X approach a fixed $n \times n$ matrix with distinct squared singular values $a_1, \ldots, a_n$, or without loss of generality, $X \to A$ as defined in (2.42). This is a limiting case of a polynomial ensemble with functions f_{k-1} that approach the Dirac delta functions $\delta(x - a_k)$ for every $k = 1, \ldots, n$. A limiting biorthogonal system is given by the polynomials

$$p_k(x) = \prod_{j=1}^{k} (x - a_j), \qquad k = 0, 1, \ldots, n-1 \tag{5.16}$$

with $p_0(x) = 1$ and dual functions q_k that are given as a $(k+1) \times (k+1)$ determinant involving Dirac delta functions

$$q_k(y) = \frac{1}{\Delta_{k+1}(a)} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \delta(y - a_1) & \delta(y - a_2) & \cdots & \delta(y - a_{k+1}) \end{vmatrix}. \tag{5.17}$$

Alternatively, we can use, instead of p_k and q_k, the Lagrange interpolating polynomials

$$\widetilde{p}_k(x) = \prod_{j \neq k} \frac{x - a_j}{a_k - a_j} \tag{5.18}$$

and the dual functions $\widetilde{q}_k(y) = \delta(y - a_k)$, for $k = 1, \ldots, n$. Note that $\widetilde{p}_k$ has degree $n - 1$ for every k. The biorthogonality

$$\int_0^\infty p_j(x) q_k(x) dx = \int_0^\infty \widetilde{p}_j(x) \widetilde{q}_k(x) dx = \delta_{j,k} \tag{5.19}$$

is easy to verify for both systems.

The correlation kernel for this degenerate ensemble is

$$\begin{aligned} k_n(x,y) &= \sum_{k=0}^{n-1} p_k(x) q_k(y) = \sum_{k=1}^{n} \widetilde{p}_k(x) \widetilde{q}_j(y) \\ &= \left(\prod_{j=1}^{n} (x - a_j) \right) \sum_{k=1}^{n} \left(\prod_{j \neq k} \frac{1}{a_k - a_j} \right) \frac{\delta(y - a_k)}{x - a_k} \end{aligned} \tag{5.20}$$

and with this expression for $k_n(x,y)$ we calculate (2.38) by first changing variables $s \mapsto x/s$, $t \mapsto y/t$, and then evaluating the t-integral which results in

$$\begin{aligned} K_n(x,y) = \frac{1}{2\pi i} \oint_\Sigma \frac{ds}{s} G^{1,1}_{1,r+1} \left(\begin{matrix} 0 \\ 0, -\nu_1, \ldots, -\nu_r \end{matrix} \middle| -\frac{x}{s} \right) \left(\prod_{j=1}^{n} (s - a_j) \right) \\ \times \sum_{k=1}^{n} \left(\prod_{j \neq k} \frac{1}{a_k - a_j} \right) \frac{1}{a_k (s - a_k)} G^{r,0}_{0,r} \left(\begin{matrix} - \\ \nu_1, \ldots, \nu_r \end{matrix} \middle| \frac{y}{a_k} \right). \end{aligned} \tag{5.21}$$

The summation can be written as a contour integral, by the residue theorem, and we obtain the following result.

Corollary 5.1. *Let $G_1, \ldots, G_r$ be the Ginibre random matrices defined in Assumption 2.15 with $\nu_0 = 0$, and let A be the deterministic matrix defined in* (2.42)*. Then the correlation kernel of the squared singular values of $G_r G_{r-1} \cdots G_1 A$ is*

$$\begin{aligned} K_n(x,y) = \frac{1}{(2\pi i)^2} \oint_\Sigma \frac{ds}{s} \oint_{C_a} \frac{du}{u} G^{1,1}_{1,r+1} \left(\begin{matrix} 0 \\ 0, -\nu_1, \ldots, -\nu_r \end{matrix} \middle| -\frac{x}{s} \right) \\ \times G^{r,0}_{0,r} \left(\begin{matrix} - \\ \nu_1, \ldots, \nu_r \end{matrix} \middle| \frac{y}{u} \right) \left(\prod_{j=1}^{n} \frac{s - a_j}{u - a_j} \right) \frac{1}{s - u}, \end{aligned} \tag{5.22}$$

where Σ is a closed, positive oriented contour around 0, and C_a is a closed contour, disjoint from Σ, in the right half-plane encircling each a_j once in the positive direction.

The above derivation of (5.22) was done under the assumption that the a_j's are mutually distinct. However in (5.22) we can easily let some or all the a_j's come together, and the expression is valid for every A with non-zero squared singular values $a_1, \ldots, a_n$.

In the limiting case where all $a_j \to 1$ and so $A = I$ we obtain

$$K_n(x,y) = \frac{1}{(2\pi i)^2} \oint_\Sigma \frac{ds}{s} \oint_\Gamma \frac{du}{u} G^{1,1}_{1,r+1}\left(\begin{matrix} 0 \\ 0, -\nu_1, \ldots, -\nu_r \end{matrix} \middle| -\frac{x}{s} \right) \times G^{r,0}_{0,r}\left(\begin{matrix} - \\ \nu_1, \ldots, \nu_r \end{matrix} \middle| \frac{y}{u} \right) \frac{(s-1)^n}{(u-1)^n} \frac{1}{s-u} \tag{5.23}$$

with Σ and Γ disjoint closed contours where Σ encloses 0 and Γ encloses 1. This is an alternative expression for (5.11), which for $r = 1$ reduces to (5.12).

6 Multiplication with truncated unitary matrices

6.1 Proof of Theorem 2.12

Theorem 2.12 follows from Lemma 2.14 in the same way as Theorem 2.8 does.

In the situation of Theorem 2.12 we have

$$\varphi(t) = \begin{cases} t^\nu (1-t)^{\mu-1} & \text{for } 0 < t < 1, \\ 0 & \text{otherwise,} \end{cases} \tag{6.1}$$

with integers $\nu \geq 0$ and $\mu \geq 1$. Then

$$b_j^{-1} = \int_0^1 t^{j+\nu}(1-t)^{\mu-1} dt = \frac{(\mu-1)!(j+\nu)!}{(j+\nu+\mu)!}. \tag{6.2}$$

We take for ψ

$$\begin{aligned} \psi(x) &= \sum_{j=-\nu}^{\infty} b_j x^j = \frac{1}{(\mu-1)!} \sum_{j=-\nu}^{\infty} \frac{(j+\nu+\mu)!}{(j+\nu)!} x^j \\ &= \mu x^{-\nu}(1-x)^{-\mu-1}, \qquad \text{for } 0 < |x| < 1. \end{aligned} \tag{6.3}$$

Then the statements of Theorem 2.12 follow immediately from Lemma 2.14. □

6.2 Proof of Corollary 2.18

The proof is similar to that of Corollary 2.16. Similar to the function φ in (5.4), we take here

$$\varphi = \varphi_r * \cdots * \varphi_1 \tag{6.4}$$

with now

$$\varphi_k(t) = \begin{cases} \frac{1}{(\mu_k-1)!} t^{\nu_k}(1-t)^{\mu_k-1}, & \text{for } 0 < t < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

Then φ is supported on $[0,1]$ with moments

$$b_j^{-1} = \int_0^1 t^j \varphi(t) dt = \prod_{k=1}^{r} \frac{(j+\nu_k)!}{(j+\nu_k+\mu_k)!}. \tag{6.6}$$

We can write φ as a Meijer G-function

$$\varphi(t) = G^{r,0}_{r,r}\left(\begin{matrix} \nu_1+\mu_1, \ldots, \nu_r+\mu_r \\ \nu_1, \ldots, \nu_r \end{matrix} \middle| t \right). \tag{6.7}$$

Furthermore, analogous to (5.7), we take here

$$\psi(s) = \sum_{j=0}^{\infty} b_j s^j = \sum_{j=0}^{\infty} \left(\prod_{k=1}^{r} \frac{(j+\nu_k+\mu_k)!}{(j+\nu_k)!} \right) s^j, \tag{6.8}$$

which is a generalized hypergeometric series

$$\psi(s) = \prod_{k=1}^{r} \frac{(\nu_k+\mu_k)!}{\nu_k!}\, {}_{r+1}F_r\left(\begin{matrix} 1, \nu_1+\mu_1+1, \ldots, \nu_r+\mu_r+1 \\ \nu_1+1, \ldots, \nu_r+1 \end{matrix} \middle| s \right) \tag{6.9}$$

and also a Meijer G-function

$$\psi(s) = G^{1,r+1}_{r+1,r+1}\left(\begin{matrix} 0, -\nu_1-\mu_1, \ldots, -\nu_r-\mu_r \\ 0, -\nu_1, \ldots, -\nu_r \end{matrix} \middle| -s \right). \tag{6.10}$$

Then Corollary 2.18 is a direct consequence of Lemma 2.14(c). □

6.3 Proof of Theorem 2.19

6.3.1 Proof of the necessity of (2.43)

The necessity of the condition (2.43) follows from the following linear algebra lemma.

Lemma 6.1. *Suppose that*

$$d = n - \sum_{j=1}^{r} \mu_j \geq 1. \tag{6.11}$$

Then the product $Y_r = T_r \cdots T_1$, where $T_1, \ldots, T_r$ are defined in Assumption 2.17 with $\nu_0 = 0$, has a singular value at 1 of multiplicity $\geq d$.

Note that although $T_1, \ldots, T_r$ are random, Lemma 6.1 uses only their matrix structure and the result and argument are deterministic.

Proof. Each T_j is the truncation of a unitary matrix U_j, which we partition as

$$U_j = \begin{pmatrix} T_j & * \\ S_j & * \end{pmatrix} \tag{6.12}$$

where S_j has size $\mu_j \times (n+\nu_{j-1})$, and $*$ denotes a block that is not important for our present purpose. Thus

$$\operatorname{rank}(S_j) \leq \mu_j. \tag{6.13}$$

Note that

$$U_j^* U_j = \begin{pmatrix} T_j^* T_j + S_j^* S_j & * \\ * & * \end{pmatrix} \tag{6.14}$$

and therefore, since U_j is unitary,

$$T_j^* T_j + S_j^* S_j = I. \tag{6.15}$$

Below we denote $Y_j = T_j T_{j-1} \cdots T_1$. Taking $j = 1$, we find by (6.13) and (6.15) that

$$\operatorname{rank}(I - Y_1^* Y_1) = \operatorname{rank}(S_1^* S_1) \leq \mu_1. \tag{6.16}$$

For $1 \leq j < r$ we have $Y_{j+1} = T_{j+1} Y_j$ and therefore by (6.15)

$$I - Y_{j+1}^* Y_{j+1} = I - Y_j^* T_{j+1}^* T_{j+1} Y_j = I - Y_j^* Y_j + Y_j^* S_{j+1}^* S_{j+1} Y_j. \tag{6.17}$$

Using (6.13) and elementary properties of the rank, we then get from this

$$\operatorname{rank}(I - Y_{j+1}^* Y_{j+1}) \leq \operatorname{rank}(I - Y_j^* Y_j) + \mu_{j+1}. \tag{6.18}$$

The two inequalities (6.16) and (6.18), and the assumption (6.11) of the lemma, lead to

$$\operatorname{rank}(I - Y_r^* Y_r) \leq \sum_{j=1}^{r} \mu_j = n - d \tag{6.19}$$

with $d \geq 1$. Since $Y_r^* Y_r$ has size $n \times n$, we obtain

$$\dim \ker(I - Y_r^* Y_r) \geq d. \tag{6.20}$$

Every vector v in the kernel of $I - Y_r^* Y_r$ is a right singular vector of Y_r with singular value 1. Thus Y_r has at least d singular values at 1. □

6.3.2 Proof of (2.44), (2.45), and (2.46) under condition (2.43)

First we consider a limiting degenerate case of Corollary 2.18, with $\nu_0 = 0$ and the matrix X replaced by the deterministic matrix A defined in (2.42). Then similar to Corollary 5.1 that is proved in Section 5.4, the biorthogonal system and the correlation kernel for the squared singular values of $T_r \cdots T_1 A$ are expressed by (2.39), (2.40) and (2.41) with the degenerate form of q_k and k_n containing Dirac δ-functions. The p_k's are polynomials and they are not degenerate.

Here we take the biorthogonal system (5.16) and (5.17) for the degenerate polynomial ensemble of the squared singular values of A. Then since $\int_0^\infty \varphi(t)\delta(y/t - a_k)dt/t = \varphi(y/a_k)/a_k$, it follows from (2.40), that

$$Q_k(y) = \frac{1}{\Delta_{k+1}(a)} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(y/a_1)/a_1 & \varphi(y/a_2)/a_2 & \cdots & \varphi(y/a_{k+1})/a_{k+1} \end{vmatrix} \tag{6.21}$$

where φ is defined in (6.7). On the other hand, P_k is obtained simply by plugging (5.16) into (2.39).

Now we take the limit where $A \to I$, so that all a_j tend to 1. Then P_k becomes the Hadamard product of (6.10) with

$$p_k(x) = (x-1)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} x^j \tag{6.22}$$

which is

$$P_k(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} b_j x^j, \qquad b_j = \prod_{l=1}^{r} \frac{(j+\nu_l+\mu_l)!}{(j+\nu_l)!} \tag{6.23}$$

as also given in (2.44).

The limit $a_j \to 1$ in (6.21) has to be done with more care. Note that each $\varphi_j(x)$ is zero for $x > 1$, real analytic on $(0,1)$, with the behaviour $\varphi_j(x) \sim \Gamma(\mu_j)^{-1}(1-x)^{\mu_j - 1}$ as $x \to 1_-$. Then inductively, for $\varphi^{(j)} = \varphi_j * \varphi_{j-1} * \cdots * \varphi_1$, it is zero for $x > 1$ and

$$\begin{aligned} \varphi^{(j)}(x) &= \varphi_j * (\varphi_{j-1} * \cdots * \varphi_1)(x) \\ &= \int_0^\infty \varphi_j\left(\frac{x}{t}\right) (\varphi_{j-1} * \cdots * \varphi_1)(t) \frac{dt}{t} \\ &\sim c_j \int_x^1 \left(1 - \frac{x}{t}\right)^{\mu_j - 1} (1-t)^{\mu_1 + \cdots + \mu_{j-1} - 1} \frac{dt}{t} \\ &\sim c_j' (1-x)^{\mu_1 + \cdots + \mu_j - 1}, \qquad \text{as } x \to 1_-, \end{aligned} \tag{6.24}$$

for some non-zero constants c_j, c_j'. Thus $\varphi = \varphi^{(r)}$ has only $\mu_1 + \cdots + \mu_r - 1$ derivatives on $(0, \infty)$.

However, since $n \leq \sum_{j=1}^{r} \mu_j$ and $k \leq n-1$, we have that φ is k times differentiable on $(0, \infty)$ and then it follows that the limit of (2.40) for all $a_j \to 1$ can be evaluated as

$$\begin{aligned} Q_k(y) &= \frac{1}{k!} \left(\frac{d}{da}\right)^k \Bigg|_{a=1} \left[\frac{1}{a}\varphi\left(\frac{y}{a}\right)\right] \\ &= \frac{(-1)^k}{k!} \left(\frac{d}{dy}\right)^k \left[y^k \varphi(y)\right). \end{aligned} \tag{6.25}$$

The second equality can be proved by induction on k. Recall that φ is the Meijer G-function (6.7) and then by elementary properties of Meijer G-functions, we find that Q_k is given by (2.45).

Then (2.44) and (2.45) give a biorthogonal system for the squared singular values of $T_r \cdots T_1$, which follow therefore a polynomial ensemble. Having the biorthogonal system (2.44) and (2.45) we can proceed as in the proof of Proposition 2.7 in [25] to find the expression (2.46) for the correlation kernel. □

Remark 6.2. In case $\mu_1 \geq n$, we can also use Theorem 2.12 to prove (2.46) by induction on r. The argument is similar to the proof of (5.11) given in Section 5.3 and we do not give the details. We only mention that instead of (5.14) and (5.15) we now use

$$\frac{\mu_r}{2\pi i} \int_L s^{-\nu_r - u - 1} (1-s)^{-\mu_r - 1} ds = \frac{\Gamma(u + \nu_r + \mu_r + 1)}{\Gamma(\mu_r)\Gamma(u + \nu_r + 1)}, \tag{6.26}$$

$$\int_0^1 t^{\nu_r + v} (1-t)^{\mu_r - 1} dt = \frac{\Gamma(\mu_r)\Gamma(v + \nu_r + 1)}{\Gamma(v + \nu_r + \mu_r + 1)}, \tag{6.27}$$

to obtain the desired result. Note that the integral in (6.26) is equivalent to [32, 5.12.9], while (6.27) is the familiar Beta integral.

Remark 6.3. For $r = 1$ the kernel is the correlation kernel of a Jacobi unitary ensemble [16, Section 3.8.3], [34]. The double contour integral formula (2.46) with $r = 1$ for this kernel is not well-known, and it was first described, to the best knowledge of the authors, in [1, Theorem 3(d)] as a special case of the *Jacobi–Piñeiro minor process*. (See also [21] for its relation to the Jacobi Muttalib–Borodin model.)

In the notation of [1], we take fixed time n, $\alpha_k = n + \nu_1 - k$, for $k = 1, \ldots, n$ and $M' = \mu_1$. Then

$$K_n(x,y)|_{r=1} = \frac{y^{\nu_1}}{x^{\nu_1}} K(n, y; n, x) \Bigg|_{\alpha_k = n + \nu_1 - k \text{ for } k = 1, \ldots, n,\ M' = \mu_1}, \tag{6.28}$$

analogous to (5.13), where K is the correlation kernel defined in [1, Theorem 3(d)].

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