ON THE DERIVATION AND ANALYSIS OF THE SPIN S XY-MODEL SUSCEPTIBILITY SERIES EXPANSIONS

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We have derived 10th order series expansions for the free energy and 9th order ones for the fluctuation of the order parameter in the spin- S XY-model on the f.c.c. lattice for $S = \frac{1}{2}, 1, \frac{3}{2}, 2, 3, 4, 6$ and 8. Confluent singularity analyses of the fluctuation series confirm the universality hypothesis, with critical exponent $\gamma_1 = 1.33 \pm 0.02$ and correction-to-scaling exponent $\Delta_1 = 0.6 \pm 0.2$. These values also agree with the results for $S = \infty$.

1. Introduction

One of the interesting and useful features of simple models is that they allow us to explicitly test theoretical predictions and hypotheses. The universality hypothesis¹⁾ in critical phenomena, in particular, can be tested by analysis of series expansions for different lattice models. Universality is already known to hold with respect to different lattice structures²⁾, Camp and Van Dyke³⁾ tested the universality of the Ising model with respect to spin value and Dekeyser and Rogiers⁴⁾ tested the universality hypothesis on the XY-model with respect to a magnetic field in the z -direction.

The purpose of the present paper is to investigate the dependence of the critical parameters of the XY-model with respect to spin value. This is an interesting way to test universality with respect to the quantum nature of a given model, since the XY-model becomes a classical model in the limit $S \rightarrow \infty$. Rogiers et al.⁵⁾ recently undertook a comprehensive comparison between the $S = \frac{1}{2}$ and classical XY-model. The present work is based on shorter series (9th order), but concentrates on the intermediate S -values.

When analysing the series expansions for thermodynamic quantities, it is usually essential to go beyond a simple power law behaviour, and to include confluent corrections to the main singularities. It has been shown by Wegner⁶⁾

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how such corrections can arise in the framework of the renormalization group approach. Several techniques have been elaborated for analysing such confluent singularities on series expansion⁷⁻⁹), and they have been successfully applied to the Ising^{3,7}) and Heisenberg¹⁰) models.

We have chosen to apply these methods of analysis to the series derived for the fluctuation of the order parameter in the XY-model on the f.c.c. lattice. The fluctuation series, which exhibits the same singularity as the susceptibility¹¹), has proved to be the series which gives the most precise values for the critical parameters. The f.c.c. lattice also leads in general to the best behaved series, since other cubic lattices have singularities on the negative real axis, which mask the effect of any confluent terms.

In section 2 the series are derived, and we present their analysis in section 3.

2. Derivation of the series

The XY-model is described by the Hamiltonian (1), normalized as shown:

$$\mathcal{H}/kT = -\frac{K}{S(S+1)} \sum_{\langle ij \rangle} (S_i^+ S_j^- + S_j^+ S_i^-), \quad (1)$$

where the sum runs over all nearest-neighbour pairs on the lattice, and S_i^+ and S_i^- are spin raising and lowering operators respectively for the spin- S states on the i th lattice site.

The standard technique for obtaining high-temperature series for this model has been derived by Betts et al.¹²⁻¹⁴). The partition function is expanded in powers of $\beta = 1/kT$ as

$$Z = \text{Tr} e^{-\beta \mathcal{H}} = \text{Tr}(1 - \beta \mathcal{H} + \beta^2 \mathcal{H}^2/2! - \beta^3 \mathcal{H}^3/3! + \dots). \quad (2)$$

With Hamiltonian (1), the coefficient of β^n in (2) is a sum over all possible products of n pairs of raising and lowering operators $S_i^+ S_j^-$ between nearest neighbour sites, in all possible orderings. Such a pair of operators may be represented by an arrow pointing from lattice site j to site i . The expansion (2) may thus be represented by the trace over the set of operator products that corresponds to the class of all ordered directed graphs built by such arrows on a given lattice. This means that one must construct all possible directed graphs of n arrows and compute the trace over all $n!$ possible orderings of the n corresponding operator pairs.

The trace corresponding to a particular ordered graph immediately factorizes into the product of the traces of the operators on the different lattice sites which span the graph (the operators on different lattice sites commute).

For the $S = \frac{1}{2}$ model in the absence of a magnetic field, such a single lattice site trace can take on only two values, namely unity if the operators on that site appear in the order $S^+S^-S^+S^- \dots S^+S^-$ or $S^-S^+S^-S^+ \dots S^-S^+$, and zero otherwise. For a general S , however, there are more operator orderings that give nonvanishing contributions. Whereas the rule (see refs. 4, 12–14) still applies that only those graphs which have an equal number of raising and lowering operators at every lattice site contribute, for general S it is no longer true that the trace vanishes when two or more raising or lowering operators follow each other. The eigenstates $|S_z\rangle$ of the operator S^z satisfy the relations

$$S^+|S_z\rangle = f^{1/2}(S_z)|S_z + 1\rangle, \quad (3a)$$

$$S^-|S_z\rangle = f^{1/2}(S_z - 1)|S_z - 1\rangle, \quad (3b)$$

with

$$f(S_z) = S(S + 1) - S_z(S_z + 1). \quad (4)$$

The trace of a product of m raising and m lowering operators on a given lattice site can then be written as

$$\sum_{S_z=-S}^{+S} \dots f^{n_2}(S_z - 2)f^{n_1}(S_z - 1)f^{n_0}(S_z)f^{n_1}(S_z + 1)f^{n_2}(S_z + 2) \dots, \quad (5)$$

where

$$\sum_k n_k = m; \quad (6)$$

the values of the n_k depend of course on the specific ordering of the raising and lowering operators. For all such operator products (up to $m = 5$), we have calculated these traces for a chosen set of spin values ($S = \frac{1}{2}, 1, \frac{3}{2}, 2, 3, 4, 6, 8$) and we have tabulated them together with an appropriate binary identification code for the corresponding operator ordering; in this way these traces could be easily retrieved at the next computational step.

The next step consists of generating all permutations of the operator pairs for all contributing graphs and combining their contributions with the appropriate lattice constants in order to obtain the thermodynamic functions.

This procedure was applied to the partition function on the f.c.c. lattice up to 10th order in K . Results are given in table I. We also derived the series for the fluctuation in the order parameter, defined as

$$Y' = \frac{1}{N} \sum'_{i,j} \langle S_i^+ S_j^- + S_i^- S_j^+ \rangle, \quad (7)$$

where the prime indicates that summation has been taken over $i \neq j$ only. We thus omitted the terms with $i = j$, which should be present in the full

TABLE I
Coefficients a_n from $1/N \log Z = \sum a_n K^n$, for several S -values¹⁹⁾

	n	a_n		n	a_n
$S = 1/2$	2	2.666666666666667	$S = 3$	2	2.666666666666667
	3	4.740740740740741		3	4.740740740740741
	4	8.493827160493827		4	15.021604938271605
	5	18.962962962962963		5	52.765432098765432
	6	60.096204846822131		6	203.3391185040545
	7	234.048224356043286		7	848.9237872835457
	8	974.9517741923735		8	3761.033872479681
	9	4168.888377072883		9	17459.18995541180
	10	18211.01339395487		10	84141.14458090645
$S = 1$	2	2.666666666666667	$S = 4$	2	2.666666666666667
	3	4.740740740740741		3	4.740740740740741
	4	12.777777777777778		4	15.204044444444444
	5	41.185185185185185		5	53.705362962962963
	6	146.43497942386826		6	208.3202127423869
	7	570.6238683127565		7	874.6668534386820
	8	2394.637514697232		8	3893.926232881513
	9	10617.64421957057		9	18154.42425006916
	10	49089.60760683363		10	87836.26781380978
$S = 3/2$	2	2.666666666666667	$S = 6$	2	2.666666666666667
	3	4.740740740740741		3	4.740740740740741
	4	14.025955555555556		4	15.347694633408919
	5	47.631591769547325		5	54.445284286554128
	6	177.094763878189300		6	212.2797961108674
	7	716.702626948818777		7	895.2674073012520
	8	3095.020053664542		8	4000.992374972840
	9	14050.64758364721		9	18717.98727974001
	10	66365.11903831037		10	90847.84254461963
$S = 2$	2	2.666666666666667	$S = 8$	2	2.666666666666667
	3	4.740740740740741		3	4.740740740740741
	4	14.567407407407407		4	15.402177640603567
	5	50.424362139917695		5	54.725880201288843
	6	191.169969821673525		6	213.7902223528516
	7	786.874532540771223		7	903.1569745571479
	8	3444.949868423792		8	4042.163819976678
	9	15825.29520352322		9	18935.50348091656
	10	75547.56916975950		10	92014.03755150228
			$S = \infty$	2	2.666666666666667
				3	4.740740740740741
				4	15.478518518518519
				5	55.119012345679012
				6	215.91461829175
				7	914.28233162565
				8	4100.3771946063
				9	19243.805996334
				10	93670.582940201
				11	469717.80397011

TABLE II

Coefficients b_n from $Y' = 1 + \sum b_n K^n$, for several S -values. For $S = \infty$, the coefficients are for the full fluctuation function Y , including the diagonal terms¹⁹).

n		b_n	n		b_n
$S = 1/2$	1	8.	$S = 3$	1	8.
	2	58.66666666666667		2	58.66666666666667
	3	388.74074074074074		3	427.90740740740741
	4	2514.96296296296296		4	3045.5
	5	16138.00823045267490		5	21383.12655213320
	6	102723.30681298582533		6	148827.0233008361
	7	649106.0237333151		7	1029290.844578495
	8	4078422.401323834		8	7084423.725503597
	9	25512171.60235545		9	48576513.98194566
$S = 1$	1	8.	$S = 4$	1	8.
	2	58.66666666666667		2	58.66666666666667
	3	414.44444444444444		3	429.00204444444444
	4	2864.6666666666667		4	3060.13893333333333
	5	19521.53456790122		5	21537.24380110123
	6	131774.0913580246		6	150269.2516097283
	7	883456.7621791087		7	1041882.222464038
	8	5892383.897030307		8	7189437.647648159
	9	39140309.94204867		9	49424266.69807662
$S = 3/2$	1	8.	$S = 6$	1	8.
	2	58.66666666666667		2	58.66666666666667
	3	421.93351111111111		3	429.86394557823129
	4	2965.44205432098765		4	3071.65885613504661
	5	20549.37140000658436		5	21658.88705724606
	6	141103.85564128395062		6	151410.7419069378
	7	962537.5097169165		7	1051875.656146356
	8	6533153.992727777		8	7273012.764369627
	9	44169070.89046708		9	50100772.47412802
$S = 2$	1	8.	$S = 8$	1	8.
	2	58.66666666666667		2	58.66666666666667
	3	425.18222222222222		3	430.19084362139918
	4	3009.01432098765432		4	3076.02657750342936
	5	21001.24841152263374		5	21705.09109353962
	6	145272.8967791494		6	151845.0413505582
	7	998429.5095522187		7	1055684.216648377
	8	6828408.287567920		8	7304916.674661030
	9	46520602.09702965		9	50359444.74547848
			$S = \infty$	1	8.
				2	59.733333333333333
				3	433.49333333333333
				4	3092.642539681184
				5	21815.30425598524
				6	152664.0328680401
				7	1062061.905099488
				8	7355052.383126519
				9	50751547.17632099

fluctuation function Y :

$$Y = \frac{1}{N} \left\langle \left(\sum_i (S_i^+ + S_i^-) \right)^2 \right\rangle. \quad (8)$$

We replaced the diagonal part of this term by the trivial value of unity, which it takes for $S = \frac{1}{2}$. An exact treatment of this diagonal part would have necessitated the generation of a new class of graphs. Furthermore, it can be seen that their omission cannot influence the singular behaviour of the function, since

$$\frac{1}{N} \sum_i \langle S_i^+ S_i^- + S_i^- S_i^+ \rangle \leq 2S(S+1), \quad (9)$$

which is finite. The series coefficients for this fluctuation function Y' are given in table II.

3. Analysis of the series

The initial analyses were carried out using the standard methods of series analysis¹⁵⁾, notably the method of Padé approximants applied to the logarithmic derivative of the series, and Neville tables to the ratios of successive terms. As observed for the Ising model^{3,7)} and as we considered likely for the XY-model, the estimates of γ decreased steadily with increasing spin value from the $S = \frac{1}{2}$ value of $\gamma \approx 1.333$ ¹⁶⁾, which of course is contrary to the universality hypothesis. As expected, the estimates of the critical point also decreased with increasing spin value. A summary of the results obtained from poles and residues of diagonal and off-diagonal Padé approximants to the logarithmic derivative of the series is shown in table III. Note that the

TABLE III
Estimates of K_c and γ for various spin values,
obtained from poles and residues of Padé ap-
proximants to $(d/dK) \log Y'$.

Spin value	Pole = K_c	Residue = γ
$\frac{1}{2}$	0.1660	1.34
1	0.1557	1.310
$\frac{3}{2}$	0.1528	1.302
2	0.1516	1.298
3	0.1505	1.292
4	0.1501	1.290
6	0.1497	1.288
8	0.1496	1.288
∞	0.14965	1.299

spin ∞ values¹⁷⁾ are slightly higher than those for $S = 8$, but this is an artefact of the longer $S = \infty$ series only.

A study of the ratios by Neville tables gives similar estimates of the critical exponent and critical point, so these are not shown here. One feature of these tables however is that none of the columns had settled down to a constant value, rather they were smoothly increasing or decreasing. Since the Neville tables provide a sequence of approximants such that the k th column of the table allows for additional correction terms proportional to n^{-k} (where n is the order of the coefficient), the observed behaviour suggests that if the dominant correction term is of the form n^{-k} , then k is not an integer. That is, a confluent singularity is indicated. Here we are using the term confluent singularity to mean that the function Ψ defined by

$$f(x) = \Phi(x) + (1 - x/x_c)^{-\gamma} \Psi(x) \quad (10)$$

is not analytic in the neighbourhood of x_c . If it were analytic, the correction terms assumed by the Neville table method would be appropriate.

To analyse for confluent singularities, there are essentially three distinct methods available. One is the method used by Saul et al.⁷⁾ in their analysis of the Ising model. In this method one considers the expansion of the function

$$g(x) = A(1 - x/x_c)^{-\gamma}(1 + B(1 - x/x_c)^{\Delta_1}) = \sum_{n \geq 0} a_n x^n. \quad (11)$$

The five unknowns A , B , γ , Δ_1 and x_c may be estimated by comparing successive quintuples of coefficients $a_k, a_{k+1}, \dots, a_{k+4}$ with the corresponding coefficients of the series under investigation. The five resulting equations are then solved for increasing values of k until there are no further series coefficients. These equations, while non-linear, require only the solution of a cubic equation, whereupon subsequent equations are quadratic or linear. If one or more of the critical parameters are known or assumed, biased estimates of the remaining parameters can clearly be obtained by solving the appropriately restricted set of equations. We shall refer to this method as the SWJ method.

The second method, due to Baker and Hunter⁸⁾, relies on the observation that if a function F can be written as the sum of a number (N) of confluent singularities, so that

$$F(x) = \sum_{i=1}^N A_i (1 - x/x_c)^{-\gamma_i}, \quad (12)$$

then introducing the variable $y = \ln(1 - x/x_c)$ leads to the function f defined by

$$f(y) = \sum_{i=1}^N A_i \sum_{j=0}^{\infty} \frac{(\gamma_i y)^j}{j!}. \quad (13)$$

Multiplying the j th coefficient of the transformed series by $j!$ and resumming gives a new function $\mathcal{F}$, defined by

$$\mathcal{F}(y) = \sum_{i=1}^{\infty} A_i / (1 - y\gamma_i) \quad (14)$$

which has simple poles at the exponents γ_i . Thus poles and residues of Padé approximants to $\mathcal{F}$ should give estimates of the exponents γ_i and the amplitudes A_i . Note that this is a biased method, in that the critical point has to be specified in order to effect the transformation.

The third method is an extension of the recurrence relation method introduced by Guttmann and Joyce⁹), and extended to handle confluent singularities by Rehr et al.¹⁸). This method worked well on the $S = \infty$ Ising model, but the number of terms available for that series (12) seems to be about the minimum that this method requires in order to give a sufficient number of estimates that the apparent rate of convergence may be assessed. For this reason, since our series have only 9 terms, we will not give any results obtained by this method.

The unbiased SWJ method was applied to the series with spin values between $S = 1/2$ and $S = 8$. This was not particularly informative, as a convergent series of estimates was only obtained for $S = \frac{1}{2}$. Even then, the convergence was limited to the dominant singularity, enabling us to estimate $\gamma = 1.33$ and $K_c = 0.1658$ for this value. The next singularity was not identifiable – this is in agreement with the analysis of Rogiers et al.⁵) who found no evidence of a confluent singularity for this series. For spin values $S \geq 1$, the estimate of the critical point was convergent to 3 significant digits, which were in agreement with those of the Padé approximants presented in table III, while those of the other parameters were too erratic to enable estimates to be made.

Turning now to the Baker–Hunter method of analysis, we find the situation considerably improved. Recalling that the Baker–Hunter method requires an estimate of the critical point as input, we chose a range of critical points around that estimated by the Padé approximants.

The series for $S = \frac{3}{2}$ gave exponent estimates that seemed rather better converged than those for other spin values. In table IV we give some results for this series, and in table V we give results for another spin value, $S = 4$, which are typical of the other S values with $S \geq 2$. For $S = \frac{3}{2}$ it can be seen that for $K_c = 0.15292$ the estimates of γ_1 are somewhat more erratic, even for higher order approximants, than those for the other values of K_c . As K_c increases, not only do the estimates of γ_1 get closer, but so too do those of γ_2 . Beyond $K_c = 0.15295$, the estimates of γ_2 become rather unrealistic. As is usual in series analysis, we use the apparent regularity in the sequences for γ_1

TABLE IV

Estimates of the two leading exponents derived by the Baker-Hunter method for the $S = \frac{3}{2}$ XY-model for a range of critical points.

	$K_c = 0.152\ 92$		$K_c = 0.152\ 93$		$K_c = 0.152\ 94$		$K_c = 0.152\ 95$		$K_c = 0.152\ 96$	
M/N	γ_1	γ_2	γ_1	γ_2	γ_1	γ_2	γ_1	γ_2	γ_1	γ_2
[5/2]	1.338	—	1.339	—	1.340	—	1.341	—	1.342	—
[4/3]	1.309	—	1.309	—	1.310	—	1.311	—	1.311	—
[3/4]	1.314	0.47	1.315	0.49	1.316	0.50	1.318	0.51	1.319	0.52
[2/5]	1.306	—	1.307	—	1.308	—	1.308	—	1.309	—
[6/2]	1.298	—	1.300	—	1.302	—	1.304	—	1.306	—
[5/3]	1.315	—	1.316	—	1.318	—	1.319	—	1.321	—
[4/4]	1.324	0.81	1.328	0.86	1.332	0.90	1.337	0.94	1.343	0.97
[3/5]	1.323	0.78	1.327	0.82	1.330	0.86	1.335	0.90	1.339	0.93
[6/3]	1.317	—	1.319	—	1.329	—	1.324	—	1.326	—
[5/4]	1.319	0.54	1.323	0.69	1.329	0.82	1.337	0.93	1.347	1.01
[4/5]	1.320	0.64	1.324	0.73	1.329	0.83	1.337	0.93	1.347	1.02
[3/6]	1.319	0.54	1.323	0.71	1.322	0.83	1.337	0.99	1.346	1.01

TABLE V

Estimates of the two leading exponents derived by the Baker-Hunter method for the $S = 4$ XY-model for a range of critical points.

	$K_c = 0.150\ 22$		$K_c = 0.150\ 28$		$K_c = 0.150\ 34$	
M/N	γ_1	γ_2	γ_1	γ_2	γ_1	γ_2
[5/2]	—	—	—	—	—	—
[4/3]	1.299	—	1.304	—	1.310	—
[3/4]	1.305	0.45	1.312	0.52	1.321	0.58
[2/5]	1.297	—	1.301	—	1.306	—
[6/2]	1.267	—	1.278	—	1.289	—
[5/3]	1.310	—	1.317	—	1.325	—
[4/4]	1.413	1.18	1.520	1.22	1.616	1.23
[3/5]	1.360	1.07	1.424	1.15	1.499	1.19
[6/3]	1.311	—	1.324	—	1.337	—
[5/4]	1.312	0.21	1.333	0.69	1.371	0.99
[4/5]	1.317	0.68	1.342	0.89	1.388	1.04
[3/6]	1.311	—	1.334	0.73	1.377	0.99

and γ_2 as a criterion for selecting the appropriate value of the critical parameters. In this way the value $K_c = 0.152\,94$ is selected, with exponent estimates $\gamma_1 = 1.33 \pm 0.02$, $\gamma_2 = 0.8 \pm 0.15$ and so the correction-to-scaling exponent, Δ_1 defined by (11), is given by $\Delta_1 = \gamma_1 - \gamma_2 = 0.53 \pm 0.17$.

For $S = 4$ the quality of the results shown is typical of all spin values between 2 and 8. That is, the $[M/N]$ Padé approximants with $M + N = 8$ give rather scattered estimates of γ_1 . For $M + N = 7$ the estimates are not widely scattered, but are rather insensitive to the choice of K_c , and give very little information about the second exponent γ_2 . Thus we are forced to draw almost all our conclusions from the $[M/N]$ Padé approximants with $M + N = 9$. In this situation, and pending the production of longer series, any conclusions drawn must be regarded as rather tentative. For $K_c \leq 0.150\,22$ one of the estimates of γ_2 disappears entirely. As K_c increases the estimates of γ_1 retain a similar degree of scatter, while those of γ_2 become closer. We therefore tentatively estimate $K_c = 0.150\,28$, $\gamma_1 = 1.33 \pm 0.02$, $\gamma_2 = 0.75 \pm 0.15$ and $\Delta_1 = \gamma_1 - \gamma_2 = 0.58 \pm 0.17$.

As stated, the results for $S = 4$ are typical of those for all values of $2 \leq S \leq 8$. Rather than give detailed tables of these, we summarise our estimates in table VI. It can be seen that a value of $\gamma_1 = 1.33 \pm 0.02$ for all values of spin is well supported, while the confluent exponent is estimated to be $\gamma_2 = 0.75 \pm 0.2$ for all spin values. Thus the correction-to-scaling exponent is estimated as $\Delta_1 = 0.6 \pm 0.2$ for all spin values greater than $\frac{1}{2}$. In agreement with the analysis of Rogiers et al.⁵⁾, we find no evidence of a confluent singularity for the $S = \frac{1}{2}$ XY-model.

In an attempt to improve the degree of confidence ascribable to these results, biased estimates using the SWJ method were obtained. First, using the critical point estimates of table VI, estimates of the remaining 4 parameters were sought. For $S = 1$ and $\frac{3}{2}$ these gave estimates of γ_1 around 1.33–1.35, but

TABLE VI
Summary of results of Baker–Hunter method of analysis,
showing the dominant and next-to-dominant exponent
estimates.

Spin value	γ_1	γ_2	K_c used
$\frac{1}{2}$	1.333 ± 0.005	—	0.165 79
1	1.33 ± 0.02	0.5–1.0	0.155 81
$\frac{3}{2}$	1.33 ± 0.02	0.80 ± 0.15	0.152 94
2	1.33 ± 0.02	0.75 ± 0.15	0.151 73
3	1.33 ± 0.02	0.73 ± 0.15	0.150 69
4	1.33 ± 0.02	0.75 ± 0.15	0.150 26
6	1.33 ± 0.02	0.75 ± 0.15	0.149 94
8	1.34 ± 0.02	0.80 ± 0.20	0.149 82

rather erratic estimates of Δ_1 . For $S \geq 2$, even the estimates of γ_1 were erratic, and no worthwhile estimate was obtainable. Next, exponent biased estimates were sought. With $\gamma_1 = \frac{4}{3}$ the convergence was moderately good for $S = 1$ and $S = \frac{3}{2}$, but poor for higher spin values. Trials with other values of γ_1 were no more successful.

Thus for these series we conclude that only the Baker-Hunter method gave useful results (quoted above), for spin values $1 \leq S \leq 8$ and these results, taken in conjunction with the $S = \frac{1}{2}$ estimates¹⁶⁾ of $\gamma_1 = 1.333 \pm 0.001$, and $S = \infty$ estimates⁵⁾ of $\gamma_1 = 1.333 \pm 0.01$, $\gamma_2 = 0.74 \pm 0.01$ and $\Delta_1 = 0.59 \pm 0.08$ give support to the view of spin independence of the critical exponents.

The critical point estimates in table III can be fitted by the expression $\Delta K_c(S) = K_c(S) - K_c(\infty) = 0.0123/S(S+1)$, with $K_c(\infty) = 0.14966$ to an accuracy of 0.16% in the estimates of $K_c(S)$. Given the estimates of critical points and critical exponents in table VI, we estimated the critical amplitudes by forming Padé approximants to $(\chi_0)^{1/\gamma_1}$ and evaluated these at $K = K_c$. Defining the critical amplitude by C_0^+ , where $\chi_0 \sim C_0^+(1 - K/K_c)^{-\gamma_1}$, the estimates of the critical amplitudes found are shown in table VII. Unfortunately our results are not sufficiently well converged to be able to estimate the amplitude of the confluent singularity.

TABLE VII
Amplitudes of the leading singularity obtained from Padé approximants to $(K_c - K)\chi^{1/\gamma}$ evaluated at $K = K_c$. γ was chosen to be $\frac{4}{3}$, and K_c was taken from table VI. The last figure quoted is uncertain.

Spin value	Amplitude C_0^+
$\frac{1}{2}$	1.010
1	0.875
$\frac{3}{2}$	0.826
2	0.810
3	0.793
4	0.782
6	0.777
8	0.775
∞	0.771

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