Interactively Illustrating the Context-Sensitivity of Aristotelian Diagrams

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Abstract. This paper studies the logical context-sensitivity of Aristotelian diagrams. I propose a new account of measuring this type of context-sensitivity, and illustrate it by means of a small-scale example. Next, I turn toward a more large-scale case study, based on Aristotelian diagrams for the categorical statements with subject negation. On the practical side, I describe an interactive application that can help to explain and illustrate the phenomenon of context-sensitivity in this particular case study. On the theoretical side, I show that applying the proposed measure of context-sensitivity leads to a number of precise yet highly intuitive results.

Keywords: Aristotelian diagram, context-sensitivity, background logic, syllogistics, information visualization.

1 Introduction

Aristotelian diagrams are compact visual representations of the elements of some logical or conceptual field, and the logical relations holding between them. Without a doubt, the oldest and most widely known example is the so-called 'square of oppositions' [32]. The history of Aristotelian diagrams is well-documented: their origins can be traced back to the logical works of Aristotle, and they have been used extensively by medieval and modern authors such as William of Sherwood [23], John Buridan [36], John N. Keynes [22], George Boole and Gottlob Frege [33]. In contemporary research, Aristotelian diagrams have been used in various subbranches of logic, such as modal logic [4], intuitionistic logic [29], epistemic logic [24], dynamic logic [9] and deontic logic [28], and also even in metalogical investigations [12]. Furthermore, because of the ubiquity of the logical relations that they visualize, these diagrams are also often used in fields outside of pure logic, such as cognitive science [2,30,34], linguistics [1,17,41,43], philosophy [27,44], law [20,31,45] and computer science [10,13,15]. In sum, then, it seems fair to conclude that Aristotelian diagrams have come to serve "as a kind of *lingua franca*" [19, p. 81] for a highly interdisciplinary community of researchers who are all concerned, in some way or another, with logical reasoning.

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Logical geometry systematically investigates Aristotelian diagrams as objects of independent interest (regardless of their role as lingua franca), for example, in terms of their information content [42]. One of the major insights to come out of these investigations is that Aristotelian diagrams are *context-sensitive*: the exact details of an Aristotelian diagram are highly dependent on the precise logical system with respect to which this diagram is constructed.[1] Although this logical context-sensitivity has numerous and far-reaching consequences, it seems to be relatively unknown—or at least insufficiently appreciated—by contemporary researchers working on Aristotelian diagrams.

The main aim of this paper is therefore to further illustrate and study the context-sensitivity of Aristotelian diagrams. We will consider a single 8-formula fragment (consisting of the categorical statements with subject negation), and study the Aristotelian diagrams that it gives rise to in various logical systems. This context-sensitivity can be concretely illustrated by means of an online available application, which allows users to define their own logical system (by selecting the axioms they want to 'activate'), and instantaneously shows them how their choices affect the resulting Aristotelian diagram. Although this application was primarily developed for broadly pedagogical purposes, it has also played an important heuristic role in more theoretical investigations, for example, concerning the relation between logical strength and context-sensitivity.

It might be objected that the terms 'context' and 'context-sensitive' are used here in a highly abstract fashion, since the relevant contexts here are logical systems, which can be seen as mere lists of axioms. Indeed, the context-sensitivity of an Aristotelian diagram (with respect to the background logic that is being used) seems to be of a fundamentally different nature than the more canonical cases of context-sensitivity, such as the deictic words 'I', 'you', 'now', 'here', etc. (with respect to context of utterance), or the words 'to know' and 'knowledge' (with respect to epistemic standards) [38]. However, this objection fails to take into account that the acceptance or rejection of a certain axiom is often itself the manifestation of a substantial position in some philosophical or empirical debate. Consider, for example, the formulas Kp and $\neg KKp$ (where $K\varphi$ stands for 'the agent knows that φ'). The Aristotelian relation holding between these formulas depends on the background logic: they are contradictory in the system $\mathsf{S4}$, but subcontrary in the system T. However, these two systems only differ from each other with respect to whether the positive introspection principle for knowledge ($K\varphi \to KK\varphi$) is accepted as an axiom, and thus reflect different positions in the epistemological debate on the nature of knowledge [47].

The paper is organized as follows. Section 2 introduces some basic notions that will be used throughout the paper, and proposes a new account of measuring the logical context-sensitivity of Aristotelian diagrams. The next three sections deal with a single fragment of 8 formulas, and the Aristotelian diagrams it gives

[1] Strictly speaking, the term 'context-sensitive' does not apply to the Aristotelian diagram itself, but to the fragment of formulas occurring in that diagram. Throughout this paper, however, I will be using this term both in a strict sense (as applying to fragments of formulas) and in a looser sense (as applying to Aristotelian diagrams).

rise to under various logical systems. First, Sect. 3 introduces the fragment and the various logical systems, and discusses their conceptual and historical importance. Next, Sect. 4 describes the interactive application that was developed to illustrate the context-sensitivity of this 8-formula fragment. Finally, Sect. 5 shows how the context-sensitivity measure proposed in Sect. 2 can be applied to the 8-formula fragment, and analyzes the results of this application. To conclude, Sect. 6 wraps things up, and mentions some questions for further research.

2 Measuring Logical Context-Sensitivity

We begin by introducing the central notions that will be studied in this paper:

Definition 1. *Let* S *be a logical system, which is assumed to have connectives expressing classical negation* ($\neg$)*, conjunction* ($\wedge$) *and implication* ($\rightarrow$)*, and a model-theoretic semantics* ($\models$)*. The* Aristotelian relations for S *are defined as follows: two formulas* φ *and* ψ *are said to be*

S-contradictory	*iff*	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	*and*	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$,
S-contrary	*iff*	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	*and*	$\mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi)$,
S-subcontrary	*iff*	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	*and*	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$,
in S-subalternation	*iff*	$\mathsf{S} \models \varphi \rightarrow \psi$	*and*	$\mathsf{S} \not\models \psi \rightarrow \varphi$.

Definition 2. *Let* S *be a logical system as specified in Definition 1 and let* $\mathcal{F}$ *be a fragment of* S*-contingent and pairwise non-*S*-equivalent formulas that is closed under negation.*[2] *An* Aristotelian diagram for $\mathcal{F}$ in S *is a diagram that visualizes an edge-labeled graph* $\mathcal{G}$*. The vertices of* $\mathcal{G}$ *are the formulas of* $\mathcal{F}$*, and the edges of* $\mathcal{G}$ *are labeled by the Aristotelian relations holding between those formulas, i.e. if* $\varphi, \psi \in \mathcal{F}$ *stand in some Aristotelian relation in* S*, then this is visualized according to the code in Fig. 1(a).*

Definition 1 is a formalized version of the traditional perspective on the Aristotelian relations, according to which two formulas are, for example, contrary iff they cannot be true together, but can be false together. Note that the seemingly *absolute* statement "φ and ψ can be false together" corresponds to the statement "there exists an S-model that satisfies $\neg\varphi \wedge \neg\psi$" (formally: $\mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi)$), which refers to the logical system S, and is thus *logic-dependent*. The restrictions made in Definition 2 (S-contingent, pairwise non-equivalent, closed under negation) are motivated by historical as well as technical reasons (see [42, Subsection 2.1] for details). Figure 1(b) shows a typical example of an Aristotelian diagram, viz. a square for 4 formulas from classical propositional logic (CPL), and Fig. 1(c) shows this square's Boolean closure, i.e. the Aristotelian diagram that consists of all contingent Boolean combinations of formulas in the square.

In its theoretical study of Aristotelian diagrams, logical geometry makes extensive use of *bitstrings*. Bitstrings are representations of formulas that allow us

[2] So for all distinct $\varphi, \psi \in \mathcal{F}$, it holds that $\mathsf{S} \not\models \varphi$, $\mathsf{S} \not\models \neg\varphi$, $\mathsf{S} \not\models \varphi \leftrightarrow \psi$, and there exists a $\varphi' \in \mathcal{F}$ such that $\mathsf{S} \models \varphi' \leftrightarrow \neg\varphi$.

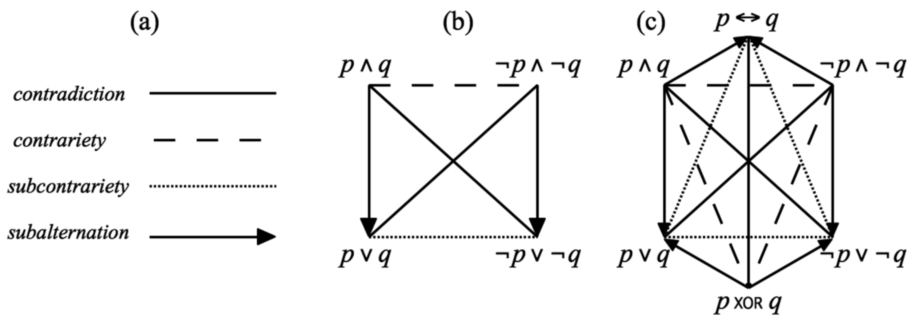

Fig. 1. (a) Code for visualizing the Aristotelian relations, (b) an Aristotelian square in CPL, and (c) its Boolean closure.

to easily determine the Aristotelian relations holding between these formulas. A systematic technique for assigning bitstrings to any given finite fragment $\mathcal{F}$ of formulas in any logical system S is described in detail in [11]; here we will focus on those aspects that are relevant for our current purposes. We define the partition $\Pi_{\mathsf{S}}(\mathcal{F}) := \{\bigwedge_{\varphi\in\mathcal{F}} \pm\varphi\} - \{\bot\}$ (where $+\varphi = \varphi$ and $-\varphi = \neg\varphi$),[3] and note that every formula $\varphi \in \mathcal{F}$ is S-equivalent to a disjunction of elements of $\Pi_{\mathsf{S}}(\mathcal{F})$, viz. $\varphi \equiv_{\mathsf{S}} \bigvee\{\alpha \in \Pi_{\mathsf{S}}(\mathcal{F}) \mid \mathsf{S} \models \alpha \to \varphi\}$.[4] The number $|\Pi_{\mathsf{S}}(\mathcal{F})|$ is the number of bit positions, i.e. the *bitstring length*, that is required to represent the formulas of $\mathcal{F}$ by means of bitstrings. If $\mathcal{D}$ is an Aristotelian diagram for the fragment $\mathcal{F}$ in the system S, then the Boolean closure of $\mathcal{D}$ contains $2^{|\Pi_{\mathsf{S}}(\mathcal{F})|} - 2$ formulas. Consider, for example, the fragment $\mathcal{F} := \{p\wedge q, \neg p\wedge\neg q, p\vee q, \neg p\vee\neg q\}$ of CPL-formulas, and its Aristotelian diagram, which is the square in Fig. 1(b). It can be shown that $\Pi_{\mathsf{CPL}}(\mathcal{F}) = \{p \wedge q, \neg p \wedge \neg q, p \text{ XOR } q\}$, and hence, the Boolean closure of the square in Fig. 1(b) should be a diagram containing $2^{|\Pi_{\mathsf{CPL}}(\mathcal{F})|} - 2 = 2^3 - 2 = 6$ formulas, which is exactly the hexagon in Fig. 1(c).

If a fragment $\mathcal{F}$ contains only S-contingent and pairwise non-S-equivalent formulas, and is closed under negation, then the relation between the fragment's size (i.e. $|\mathcal{F}|$) and the bitstring length required to represent it (i.e. $|\Pi_{\mathsf{S}}(\mathcal{F})|$) can be characterized as follows: $\lceil\log_2(|\mathcal{F}|+2)\rceil \leq |\Pi_{\mathsf{S}}(\mathcal{F})| \leq 2^{\frac{|\mathcal{F}|}{2}}$ [11, Subsect. 3.3]. Defining the *n-range* to be the set $R_n := \{x \in \mathbb{N} \mid \lceil\log_2(n+2)\rceil \leq x \leq 2^{\frac{n}{2}}\}$, this can trivially be reformulated as follows: if $\mathcal{F}$ contains the formulas appearing in some Aristotelian diagram, then $|\Pi_{\mathsf{S}}(\mathcal{F})| \in R_{|\mathcal{F}|}$ (informally: $\mathcal{F}$ can be represented by bitstrings of length $\ell \in R_{|\mathcal{F}|}$). Furthermore, it can be shown that all values in the n-range R_n will be 'needed' at some point, in the sense that for every $\ell \in R_n$, there exists a fragment/logic pair whose representation requires bitstrings of length exactly ℓ. Formally:

[3] The set $\Pi_{\mathsf{S}}(\mathcal{F})$ is called a 'partition' because its elements are (i) jointly exhaustive ($\mathsf{S} \models \bigvee \Pi_{\mathsf{S}}(\mathcal{F})$), and (ii) mutually exclusive ($\mathsf{S} \models \neg(\alpha\wedge\beta)$ for distinct $\alpha, \beta \in \Pi_{\mathsf{S}}(\mathcal{F})$).

[4] The bitstring representation of φ is meant to keep track which formulas of $\Pi_{\mathsf{S}}(\mathcal{F})$ enter into this disjunction. For example, if $\Pi_{\mathsf{S}}(\mathcal{F}) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, then φ is represented by the bitstring 1011 iff $\varphi \equiv_{\mathsf{S}} \alpha_1 \vee \alpha_3 \vee \alpha_4$.

for all $\ell \in R_n$, there exists a fragment $\mathcal{F}$ (such that $|\mathcal{F}| \leq n$) and there exists a logical system S such that $|\Pi_\mathsf{S}(\mathcal{F})| = \ell$.

Note that in order to reach every $\ell \in R_n$, the statement above allows us to choose specific values for both the 'fragment parameter' and the 'logical system parameter' (cf. the existential quantification over $\mathcal{F}$ as well as S). This observation leads to the following proposal to measure the context-sensitivity of a given Aristotelian diagram/fragment with respect to a set $\mathcal{S}$ of logical systems.

Proposal. The logical context-sensitivity of a given fragment $\mathcal{F}$ with respect to some set $\mathcal{S}$ of logical systems is positively correlated to the number of values in the $|\mathcal{F}|$-range that are reached if

1. the 'fragment parameter' is fixed to $\mathcal{F}$, and
2. the 'logical system parameter' varies within $\mathcal{S}$.

This proposal has two limiting cases:

– $\mathcal{F}$ is *minimally* context-sensitive with respect to $\mathcal{S}$.
  This means that for all logical systems $\mathsf{S}, \mathsf{T} \in \mathcal{S}$, it holds that $|\Pi_\mathsf{S}(\mathcal{F})| = |\Pi_\mathsf{T}(\mathcal{F})|$. This is equivalent to there being some $\ell \in R_{|\mathcal{F}|}$ such that for all logical systems $\mathsf{S} \in \mathcal{S}$, it holds that $|\Pi_\mathsf{S}(\mathcal{F})| = \ell$. Informally: by fixing the fragment parameter to $\mathcal{F}$, only a *single* value in the $|\mathcal{F}|$-range is reached.
– $\mathcal{F}$ is *maximally* context-sensitive with respect to $\mathcal{S}$.
  This means that for all $\ell \in R_{|\mathcal{F}|}$, there exists a logical system $\mathsf{S} \in \mathcal{S}$ such that $|\Pi_\mathsf{S}(\mathcal{F})| = \ell$. Informally: even though the fragment parameter is fixed to $\mathcal{F}$, varying the logical system parameter within $\mathcal{S}$ suffices to reach *all* values in the $|\mathcal{F}|$-range. In other words, all bitstring lengths that might theoretically be necessary to represent fragments of the same size as $\mathcal{F}$, are already needed to represent $\mathcal{F}$ itself, under the different logical systems in $\mathcal{S}$.[5]

To illustrate this account of context-sensitivity, we will consider the case of 4-formula fragments, i.e. the case of Aristotelian *squares*. Note that the 4-range is $R_4 = \{3, 4\}$, which means that every Aristotelian square (regardless of the formulas it contains, regardless of the logical system in which it is constructed) can be represented by bitstrings of length either 3 or 4. Now consider the specific 4-formula fragment $\mathcal{F}^\dagger := \{all(A, B), some(A, B), all(A, \neg B), some(A, \neg B)\}$. Informally, these formulas read as "all As are B", "some As are B", "all As are not B" and "some As are not B", respectively,[6] and they can be interpreted in various 'reasonable' logical systems, such as FOL (contemporary first-order logic) and SYL (i.e. $\mathsf{FOL} + \exists x Ax$).[7] It is shown in [11, Sect. 4] that

[5] Note the subtly different quantification patterns corresponding to these two limiting cases: minimal context-sensitivity corresponds to $\exists \ell \in R_{|\mathcal{F}|} : \forall \mathsf{S} \in \mathcal{S} : |\Pi_\mathsf{S}(\mathcal{F})| = \ell$, while maximal context-sensitivity corresponds to $\forall \ell \in R_{|\mathcal{F}|} : \exists \mathsf{S} \in \mathcal{S} : |\Pi_\mathsf{S}(\mathcal{F})| = \ell$.

[6] As is well-known, in the language of first-order logic, these formulas can be formalized as $\forall x(Ax \to Bx)$, $\exists x(Ax \wedge Bx)$, $\forall x(Ax \to \neg Bx)$ and $\exists x(Ax \wedge \neg Bx)$, respectively.

[7] Later in the paper, I will have more to say about when exactly a logical system can be considered 'reasonable' for a given fragment.

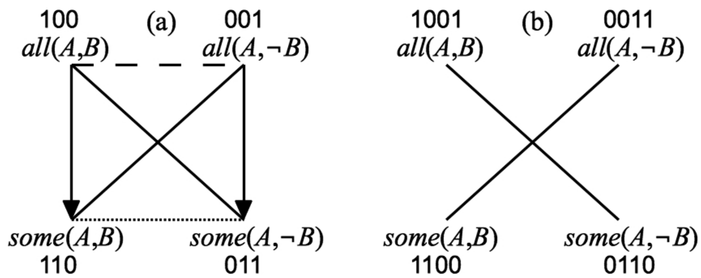

Fig. 2. (a) 'Classical' square for $\mathcal{F}^\dagger$ in SYL, (b) 'degenerated' square for $\mathcal{F}^\dagger$ in FOL. Each square is decorated with both $\mathcal{F}^\dagger$ and its bitstring representation.

- $\Pi_{\mathsf{FOL}}(\mathcal{F}^\dagger) = \{all(A,B) \wedge some(A,B), some(A,B) \wedge some(A,\neg B), all(A,\neg B) \wedge some(A,\neg B), all(A,B) \wedge all(A,\neg B)\}$,
- $\Pi_{\mathsf{SYL}}(\mathcal{F}^\dagger) = \{all(A,B), some(A,B) \wedge some(A,\neg B), all(A,\neg B)\}$,

and hence $|\Pi_{\mathsf{FOL}}(\mathcal{F}^\dagger)| = 4$ and $|\Pi_{\mathsf{SYL}}(\mathcal{F}^\dagger)| = 3$. This shows that for all $\ell \in R_4 = \{3,4\}$, there exists a logical system $\mathsf{S} \in \mathcal{S}^\dagger := \{\mathsf{FOL}, \mathsf{SYL}\}$ such that $|\Pi_{\mathsf{S}}(\mathcal{F}^\dagger)| = \ell$, and hence, the fragment $\mathcal{F}^\dagger$ is maximally context-sensitive with respect to $\mathcal{S}^\dagger$. This context-sensitivity can also clearly be seen in the Aristotelian diagrams themselves: in SYL, the fragment $\mathcal{F}^\dagger$ gives rise to a 'classical' square of opposition, which is shown in Fig. 2(a) and can be represented by bitstrings of length 3, whereas in FOL, the same fragment gives rise to a 'degenerated' square or "X of opposition" [3, p. 13], which is shown in Fig. 2(b) and can be represented by bitstrings of length 4.

3 Categorical Statements and Subject-Negation

At the end of the previous section, I introduced the fragment $\mathcal{F}^\dagger$, and showed it to be maximally context-sensitive with respect to the reasonable logical systems in $\mathcal{S}^\dagger$. The next three sections of this paper will be devoted to studying and illustrating the context-sensitivity of a larger fragment $\mathcal{F}^\ddagger$ (which includes $\mathcal{F}^\dagger$ itself) with respect to a larger set of logical systems $\mathcal{S}^\ddagger$ (which includes $\mathcal{S}^\dagger$ itself). I start by introducing the fragment $\mathcal{F}^\ddagger$ and the logical systems in $\mathcal{S}^\ddagger$.

The statements in the original fragment $\mathcal{F}^\dagger$ are *categorical statements*, which are of the form *quantifier*(*subject*, *predicate*). They are among the oldest sentences to be studied from a logical perspective [32], and traditionally, they are classified according to their *quantity* and *quality*. With respect to quantity, we distinguish between *universal* and *particular* statements, whose quantifiers are *all* and *some*, respectively. With respect to quality, we distinguish between *affirmative* and *negative* statements, whose predicates are of the form B and $\neg B$, respectively. The traditional classification according to quality is thus exclusively based on whether the statements' *predicates* are negated. Over the course of history, however, logicians have also become interested in the effects of *subject negation* [7,8,16,21,22,25,35,37], thereby obtaining the new statements

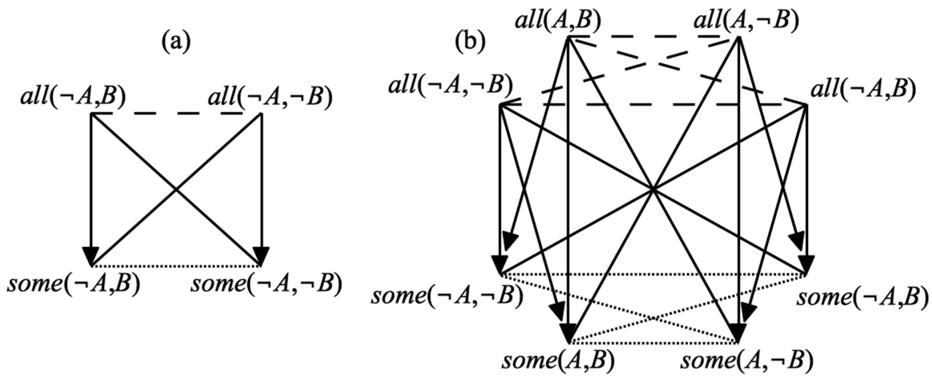

Fig. 3. (a) Aristotelian square for $\mathcal{F}^\ddagger - \mathcal{F}^\dagger$ in $\mathsf{FOL}(\{A2\})$, (b) Aristotelian octagon for $\mathcal{F}^\ddagger$ in $\mathsf{FOL}(\{A1, A2, A3, A4\})$.

$all(\neg A, B), some(\neg A, B)$, $all(\neg A, \neg B), some(\neg A, \neg B)$. The 8-formula fragment $\mathcal{F}^\ddagger$ is defined to contain exactly these 4 new statements, together with the 4 original statements of $\mathcal{F}^\dagger$. It thus trivially holds that $\mathcal{F}^\dagger \subseteq \mathcal{F}^\ddagger$.

The 4 new statements can themselves be used to construct a second Aristotelian square. Figure 3(a) shows this square, as constructed in the logical system $\mathsf{FOL}(\{A2\})$, which will be described below. Note that this square is classical (i.e. not degenerated) iff the underlying logic contains $\exists x \neg Ax$ as an axiom, which is analogous to the first square (for $\mathcal{F}^\dagger$) being classical iff the underlying logic contains $\exists x Ax$ as an axiom. More interestingly, we can also consider Aristotelian diagrams for the entire 8-formula fragment $\mathcal{F}^\ddagger$. Some authors have proposed an *octagon* [16,21,22], while others have made use of a *cube* [8,25,37]. In the current paper, the fragment $\mathcal{F}^\ddagger$ will always be visualized by means of an octagon. Figure 3(b) shows this octagon, as constructed in the logical system $\mathsf{FOL}(\{A1, A2, A3, A4\})$, which will also be described below. Because of the logical context-sensitivity of Aristotelian diagrams, other logical systems will lead to other versions of this octagon. I will therefore now introduce the various logical systems in which the Aristotelian octagon for $\mathcal{F}^\ddagger$ will be constructed.

The logical systems that will be relevant for our purposes all consist of first-order logic (FOL), with additional axioms coming from the set $\mathcal{AX} := \{A1, A2, A3, A4, A5, A6\}$, which contains the following statements:

$A1$ $\exists x Ax$	$A3$ $\exists x Bx$	$A5$ $\exists x \neg(Ax \leftrightarrow Bx)$
$A2$ $\exists x \neg Ax$	$A4$ $\exists x \neg Bx$	$A6$ $\exists x \neg(Ax \leftrightarrow \neg Bx)$

For any set $\mathcal{A} \subseteq \mathcal{AX}$, let $\mathsf{FOL}(\mathcal{A})$ be the logical system that is obtained by adding the formulas in $\mathcal{A}$ as axioms to FOL. We will be interested in the set of logical systems $\mathcal{S}^\ddagger := \{\mathsf{FOL}(\mathcal{A}) \mid \mathcal{A} \subseteq \mathcal{AX}\}$. Note that all the statements in $\mathcal{AX}$ are independent of each other, in the sense that there exist no $\mathcal{A} \subseteq \mathcal{AX}$ and $\alpha \in \mathcal{AX} - \mathcal{A}$ such that α is derivable in $\mathsf{FOL}(\mathcal{A})$ (in which case we would have $\mathsf{FOL}(\mathcal{A}) = \mathsf{FOL}(\mathcal{A} \cup \{\alpha\})$); this means that the logical systems in $\mathcal{S}^\ddagger$ are all distinct from each other, and hence $\mathcal{S}^\ddagger$ contains exactly $|\wp(\mathcal{AX})| = 2^{|\mathcal{AX}|} = 2^6 = 64$ distinct logical systems. The weakest system in $\mathcal{S}^\ddagger$ is $\mathsf{FOL}(\emptyset)$, i.e. FOL itself, while the strongest system in $\mathcal{S}^\ddagger$ is $\mathsf{FOL}(\mathcal{AX})$. Finally, note that since $\mathsf{FOL} = \mathsf{FOL}(\emptyset)$ and $\mathsf{SYL} = \mathsf{FOL}(\{A1\})$, it holds that $\mathcal{S}^\dagger = \{\mathsf{FOL}, \mathsf{SYL}\} \subseteq \mathcal{S}^\ddagger$.

All the logical systems in $\mathsf{S}^\ddagger$ are 'reasonable' to a certain degree, in the sense that all of their axioms have been defended by various logicians in relation to substantial philosophical and psychological debates. To begin with, note that all systems in $\mathcal{S}^\ddagger$ are extensions of the system FOL of first-order logic, which is itself by far the most widely used logical system today. Next, the statements $A1$–$A4$ can all be seen as (partial) interpretations of the traditional *existential import* principle. According to its most cautious interpretation [5,40], this principle states that the predicate occuring in the first argument position of a categorical statement should not have an empty extension, which is captured by $A1$. However, another interpretation is that *all* predicates should have non-empty extensions, regardless of whether they occur in the categorical statement's first or second argument position [39]; this means that both $A1$ and $A3$ should be accepted as axioms. Furthermore, based on psychological considerations, authors such as Seuren [39] have argued that just as a predicate's extension should not be allowed to be empty, it should not be allowed to encompass the entire universe either; this means that $A2$ and/or $A4$ should be accepted as axioms. The most liberal interpretation of the existential import principle, then, which is held by authors such as Keynes [22], Johnson [21] and Hacker [16], takes this principle to state that all of $A1$, $A2$, $A3$ and $A4$ should be accepted as axioms.

Finally, the statements $A5$ and $A6$ have been defended by Reichenbach [37]. Informally, the former states that the predicates A and B should not be perfect *synonyms*, while the latter states that A and B should not be perfect *antonyms*. More precisely, these statements impose a strict correlation between syntactic differences and semantic differences. For example, $A5$ states that since there is a syntactic difference between the predicates A and B (viz. they are symbolized using different letters), there should also be a semantic difference between them (viz. they should have different extensions). A similar principle is at work in Wittgenstein's version of predicate logic, in which distinct variables are taken to have distinct values [46,48].[8] From a more empirical perspective, principles such as $A5$ and $A6$ seem to be related to linguistic work on language evolution and language acquisition, in which it is often assumed that humans have an (innate) tendency to avoid perfect synonyms as much as possible [6,18,26].

4 An Interactive Illustration

Earlier research on the logical context-sensitivity of Aristotelian diagrams has focused on (families of) logics such as epistemic logic [14] and metalogic [12]. In

[8] For example, in this system, the sentence 'there are at least two As' would not be formalized as $\exists x \exists y(Ax \wedge Ay \wedge x \neq y)$, but simply as $\exists x \exists y(Ax \wedge Ay)$: the syntactic difference between the variables x and y suffices to indicate that there is also a semantic difference between them, i.e. that they have distinct values.

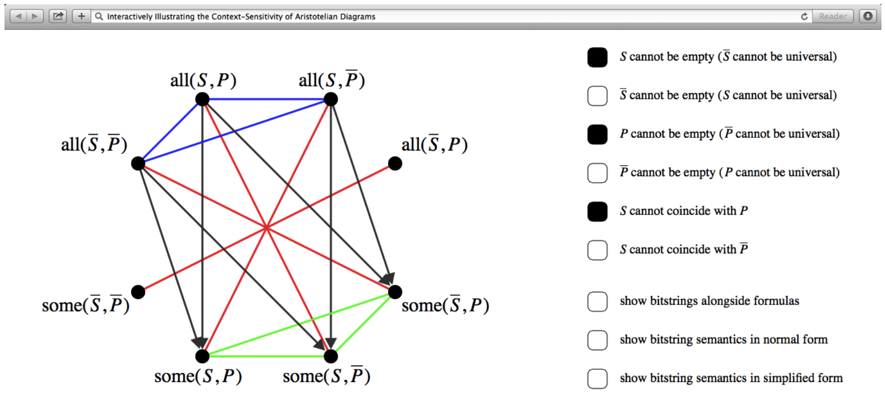

Fig. 4. Screenshot: the Aristotelian diagram for $\mathcal{F}^{\ddagger}$ in the system $\mathsf{FOL}(\{A1, A3, A5\})$.

contrast to these more advanced case studies, the fragment $\mathcal{F}^{\ddagger}$ and the logics in $\mathcal{S}^{\ddagger}$ presented in this paper are quite elementary, which renders them particularly suitable for explaining the phenomenon of context-sensitivity to a broader audience. In order to support and facilitate this pedagogical goal, an interactive application has been created and made available online, at the following location: `http://www.logicalgeometry.org/octagon_context.html`.

The application was developed using the XML-based Scalable Vector Graphics format (for the graphical aspects) and JavaScript (for the interactivity). The user interface has been kept very simple: the screen is vertically divided into a left half and a right half. The left half shows the Aristotelian diagram for $\mathcal{F}^{\ddagger}$, based on the logical system that is currently 'activated'. The right half contains 6 'axiom buttons' and 3 'auxiliary buttons'. The former correspond exactly to the statements in $\mathcal{AX}$, and each of them can be activated or deactivated. In this way, the user can select any of the logical systems $\mathsf{FOL}(\mathcal{A}) \in \mathcal{S}^{\ddagger}$, by activating exactly the axiom buttons corresponding to the statements in $\mathcal{A}$. As the user activates or deactivates a particular axiom (and thus goes from one logical system to another one), she can immediately observe the effects of this change on the Aristotelian diagram for $\mathcal{F}^{\ddagger}$ on the left half of the screen. For example, the screenshot in Fig. 4 shows that the user has activated the axiom buttons corresponding to $A1$, $A3$ and $A5$, and hence the application shows the Aristotelian diagram for $\mathcal{F}^{\ddagger}$ in the system $\mathsf{FOL}(\{A1, A3, A5\})$.

In addition to the 6 axiom buttons, the right half of the screen also contains 3 auxiliary buttons. The first button allows the user to decide whether the bitstring representations of the $\mathcal{F}^{\ddagger}$-formulas should be shown next to these formulas. Recall that the bitstring representation of a given formula depends on the logical system that is currently activated; for example, a single formula might correspond to a bitstring of length 6 in one logical system, and to a bitstring of length 9 in another logical system. Consequently, if the 'show bitstrings' button is switched on and the user activates or deactivates a particular axiom

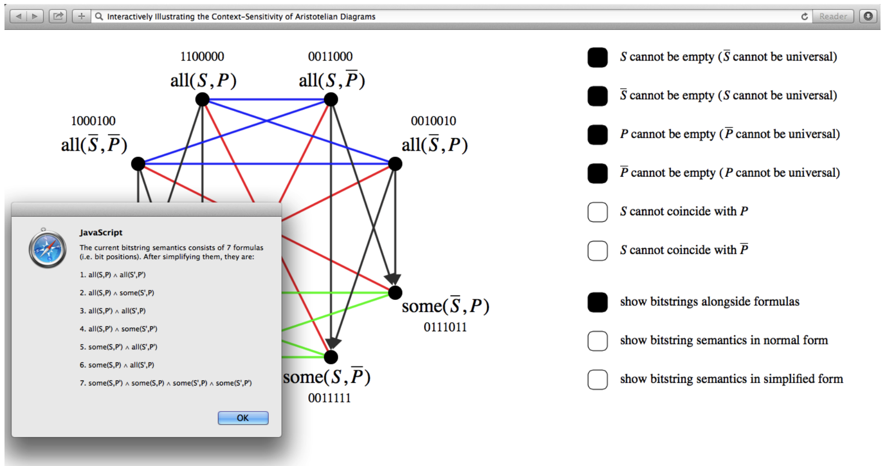

Fig. 5. Screenshot: the Aristotelian diagram for $\mathcal{F}^\ddagger$ in the system $\mathsf{FOL}(\{A1, A2, A3, A4\})$, its bitstring representation, and a popup window showing the formulas of $\Pi_{\mathsf{FOL}(\{A1,A2,A3,A4\})}(\mathcal{F}^\ddagger)$ (in simplified form).

(and thus goes from one logical system to another one), this will not only affect the Aristotelian diagram for $\mathcal{F}^\ddagger$ itself, but also the bitstring representations of the formulas in that diagram. In this way, the user can easily experiment with the various logical systems in $\mathcal{S}^\ddagger$, and explore which systems give rise to which kinds of bitstrings, etc. Finally, there are two buttons that launch popup windows showing the *bitstring semantics*, i.e. the formulas in $\Pi_{\mathsf{S}}(\mathcal{F}^\ddagger)$, with respect to which the bitstrings for $\mathcal{F}^\ddagger$ are defined (where $\mathsf{S} \in \mathcal{S}^\ddagger$ is the logical system that is currently activated). Recall from Sect. 2 that these formulas are formally defined as conjunctions of (negations of) the formulas in $\mathcal{F}^\ddagger$; however, in many cases, these long conjunctions can be simplified to equivalent, but much shorter formulas. One button shows the bitstring semantics in normal form (i.e. as the original conjunctions), while the other one shows it in simplified form.

For example, the screenshot in Fig. 5 shows that the user has activated the axiom buttons corresponding to $A1$, $A2$, $A3$ and $A4$, and hence the application shows the Aristotelian diagram for $\mathcal{F}^\ddagger$ in the system $\mathsf{FOL}(\{A1, A2, A3, A4\})$—also see Fig. 3(b). Furthermore, the 'show bitstrings' button has been switched on, so the formulas of $\mathcal{F}^\ddagger$ are shown together with their bitstring representations. Since $|\Pi_{\mathsf{FOL}(\{A1,A2,A3,A4\})}(\mathcal{F}^\ddagger)| = 7$, these are bitstrings of length 7. Finally, clicking the 'show bitstring semantics in simplified form' button has launched a popup window showing the 7 (simplified) formulas in $\Pi_{\mathsf{FOL}(\{A1,A2,A3,A4\})}(\mathcal{F}^\ddagger)$.

5 Theoretical Analysis

Making use of the application described in the previous section, it can easily be observed that each of the 64 logical systems in $\mathcal{S}^\ddagger$ gives rise to a different Aristotelian diagram for $\mathcal{F}^\ddagger$.[9] This can be explained by noting that the octagon for $\mathcal{F}^\ddagger$ contains exactly 6 squares,[10] and each square corresponds exactly to a statement in $\mathcal{AX}$: the square is classical (i.e. not degenerated) iff the corresponding statement is an axiom in the logical system with respect to which the octagon is defined. For example, the octagon for $\mathcal{F}^\ddagger$ with respect to $\mathsf{FOL}(\{A1, A2, A3, A4\})$ in Fig. 3(b) contains 4 classical squares (viz. those corresponding to $A1$, $A2$, $A3$ and $A4$) and 2 degenerated ones (viz. those corresponding to $A5$ and $A6$). Furthermore, note that this precise correspondence between the 6 squares inside the octagon for $\mathcal{F}^\ddagger$ and the 6 statements $A1$–$A6$ further corroborates the claim that $A1$–$A6$ are the most natural axioms to consider when studying $\mathcal{F}^\ddagger$.

The fact that each logical system in $\mathcal{S}^\ddagger$ gives rise to a different Aristotelian diagram for $\mathcal{F}^\ddagger$ is already a powerful illustration of the context-sensitivity of $\mathcal{F}^\ddagger$ with respect to $\mathcal{S}^\ddagger$. To assess this context-sensitivity in a mathematically more precise way, we will now make use of the account proposed in Sect. 2. Since $|\mathcal{F}^\ddagger| = 8$, we are interested in the 8-range, which is easily calculated to be $R_8 = \{4, 5, \ldots, 15, 16\}$. Recall that this intuitively means that every possible Aristotelian octagon (regardless of the formulas it contains, regardless of the logical system in which it is constructed) can be represented by bitstrings of length between 4 and 16 (inclusive). Again making use of the application described in the previous section, we obtain the following table:[11]

S	ℓ_S	S	ℓ_S	S	ℓ_S	S	ℓ_S	S	ℓ_S	S	ℓ_S	S	ℓ_S	S	ℓ_S
∅	16	13	10	46	10	235	9	234	8	356	8	1356	7	1234	7
1	12	14	10	35	10	246	9	146	8	456	8	1346	7	23456	6
2	12	15	10	36	10	134	8	236	8	3456	7	1345	7	13456	6
3	12	16	10	12	9	135	8	245	8	2456	7	1256	7	12456	6
4	12	23	10	34	9	123	8	156	8	2356	7	1246	7	12356	6
5	12	24	10	56	9	124	8	256	8	2346	7	1245	7	12346	6
6	12	25	10	136	9	125	8	345	8	2345	7	1236	7	12345	6
45	10	26	10	145	9	126	8	346	8	1456	7	1235	7	123456	5

For example, the cell on the uppermost row and rightmost column of this table tells us that if we are working in the system $\mathsf{FOL}(\{A1, A2, A3, A4\})$, the fragment $\mathcal{F}^\ddagger$ can be represented by bitstrings of length 7 (also see Fig. 5).

Comparing this table with the 8-range, several observations can be made. First of all, note that even though the logical systems in $\mathcal{S}^\ddagger$ all give rise to different Aristotelian diagrams for $\mathcal{F}^\ddagger$, it is *not* the case that they all give rise to different bitstring lengths (which is impossible, since $|\mathcal{S}^\ddagger| = 64 > 13 = |R_8|$). It turns out that only 8 bitstring lengths are required to represent $\mathcal{F}^\ddagger$ under the various systems in $\mathcal{S}^\ddagger$, viz. 16, 12, 10, 9, 8, 7, 6 and 5. Since $\frac{8}{|R_8|} = \frac{8}{13} \approx 0.62$, this means that of all the bitstring lengths that might theoretically be necessary to represent an arbitrary 8-formula fragment with respect to an arbitrary logical

[9] We already encountered a similar situation in Sect. 2, where it was shown that each system in $\mathcal{F}^\dagger$ gives rise to a different Aristotelian square for $\mathcal{F}^\dagger$; see Fig. 2.

[10] An Aristotelian octagon can be seen as consisting of 4 pairs of contradictory formulas (PCDs), and a square as 2 PCDs. The number of squares inside an octagon thus equals the number of ways in which one can select 2 PCDs out of 4 (without replacement), which is $\binom{4}{2} = \frac{4!}{2!2!} = 6$.

[11] For reasons of space, a logical system such as $\mathsf{FOL}(\{A1, A2, A3\})$ is abbreviated as '123', and the bitstring length $|\Pi_\mathsf{S}(\mathcal{F}^\ddagger)|$ as ℓ_S.

system, about 62% is already necessary to represent the particular fragment $\mathcal{F}^\ddagger$ with respect to the particular logical systems in $\mathcal{S}^\ddagger$.

In terms of extreme values, we see that the highest value in the 8-range is reached, i.e. there is a system $\mathsf{S} \in \mathcal{S}^\ddagger$ such that $|\Pi_\mathsf{S}(\mathcal{F}^\ddagger)| = 16$, viz. $\mathsf{S} = \mathsf{FOL}(\emptyset)$. By contrast, the lowest value in the 8-range is not reached, i.e. there is no $\mathsf{S} \in \mathcal{S}^\ddagger$ such that $|\Pi_\mathsf{S}(\mathcal{F}^\ddagger)| = 4$. In other words, even though there exist some 8-formula fragment $\mathcal{F}$ and some logical system S such that $|\Pi_\mathsf{S}(\mathcal{F})| = 4$, we cannot take $\mathcal{F} = \mathcal{F}^\ddagger$ and $\mathsf{S} \in \mathcal{S}^\ddagger$.[12] Note, however, that the second lowest value in the 8-range is reached by some system in $\mathcal{S}^\ddagger$, since $|\Pi_{\mathsf{FOL}(\mathcal{AX})}(\mathcal{F}^\ddagger)| = 5$. The weakest logical system $\mathsf{FOL}(\emptyset)$ thus yields the highest bitstring length (16), while the strongest logical system $\mathsf{FOL}(\mathcal{AX})$ yields the lowest (attainable) bitstring length (5). This suggests an inverse correlation between logical strength and bitstring length: stronger logical systems yield shorter bitstrings. The intuitive explanation of this inverse correlation is based on the fact that bitstring length is itself positively correlated to the size of the Boolean closure (cf. Sect. 2): a stronger logical system can prove more formulas in the Boolean closure of $\mathcal{F}^\ddagger$ to be equivalent to each other, so this Boolean closure will contain fewer formulas (up to logical equivalence), which in turn means that the bitstrings will be shorter.

In order to make this inverse correlation more precise, note that for logical systems $\mathsf{S} \in \mathcal{S}^\ddagger$, the logical strength of S can be taken to be simply the number of statements in $\mathcal{AX}$ that are axioms of S. In other words, for $\mathcal{A}, \mathcal{B} \subseteq \mathcal{AX}$, we say that $\mathsf{FOL}(\mathcal{A})$ is stronger than $\mathsf{FOL}(\mathcal{B})$ iff $|\mathcal{A}| > |\mathcal{B}|$. The inverse correlation between logical strength and bitstring length can now be expressed as follows:

$$\text{for all } \mathcal{A}, \mathcal{B} \subseteq \mathcal{AX}\text{: if } |\mathcal{A}| < |\mathcal{B}|, \text{ then } |\Pi_{\mathsf{FOL}(\mathcal{A})}(\mathcal{F}^\ddagger)| \geq |\Pi_{\mathsf{FOL}(\mathcal{B})}(\mathcal{F}^\ddagger)| \quad (\textsc{InvCor})$$

The truth of (InvCor) can be checked by means of the table provided above. Furthermore, note that we almost have an even stricter version of this inverse correlation principle, in the sense that for almost all sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{AX}$, the comparison operator $\geq$ in the consequent of (InvCor) can be replaced by $>$. As can be verified by means of the table provided above, the only counterexamples to this stricter claim involve $\mathcal{A} \in \{\{A1, A2\}, \{A3, A4\}, \{A5, A6\}\}$ and $\mathcal{B} \in \{\{A1, A3, A6\}, \{A1, A4, A5\}, \{A2, A3, A5\}, \{A2, A4, A6\}\}$, in which case we have $|\mathcal{A}| = 2 < 3 = |\mathcal{B}|$ and yet $|\Pi_{\mathsf{FOL}(\mathcal{A})}(\mathcal{F}^\ddagger)| = 9 = |\Pi_{\mathsf{FOL}(\mathcal{B})}(\mathcal{F}^\ddagger)|$.

6 Conclusion

This paper has studied the logical context-sensitivity of Aristotelian diagrams, focusing on the fragment $\mathcal{F}^\ddagger$ of categorical statements with subject negation, and the set $\mathcal{S}^\ddagger$ of logical systems based on the axioms in $\mathcal{AX}$. I have described an interactive application that can help to illustrate the context-sensitivity of

[12] There certainly do exist systems S such that $|\Pi_\mathsf{S}(\mathcal{F}^\ddagger)| = 4$. This is the case, for example, for the system S^* that is obtained by adding to $\mathsf{FOL}(\mathcal{AX})$ the additional axiom $all(A,B) \vee all(A,\neg B) \vee all(\neg A, B) \vee all(\neg A, \neg B)$. Note, however, that $\mathsf{S}^* \notin \mathcal{S}^\ddagger$, and, more importantly, S^* is far less reasonable than any of the systems in $\mathcal{S}^\ddagger$.

$\mathcal{F}^\ddagger$ with respect to $\mathcal{S}^\ddagger$. On the theoretical side, I have proposed a new account to measure the context-sensitivity of Aristotelian diagrams, and shown that it leads to precise yet highly intuitive results in the case of $\mathcal{F}^\ddagger$ and $\mathcal{S}^\ddagger$. In future work, this account will be applied to other fragments and logical systems.

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