IDENTIFIABILITY OF TENSOR RANK DECOMPOSITIONS

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1. Extended abstract

The tensor rank decomposition is an expression of a tensor $\mathfrak{A} \in \mathbb{P}^{n_1} \otimes \cdots \otimes \mathbb{P}^{n_d}$ of the form

$$\mathfrak{A} = \sum_{i=1}^{r} \mathbf{a}_i^1 \otimes \mathbf{a}_i^2 \otimes \cdots \otimes \mathbf{a}_i^d,$$

where $\mathbf{a}_i^k \in \mathbb{P}^{n_k}$ [6, 7]. In the above expression, r is assumed to be the least integer value for which $\mathfrak{A}$ can be expressed as a linear combination of rank-1 tensors. The value r is called the *rank* of the tensor. The problem that we consider in this computation presentation regards the number of distinct rank decompositions that a tensor admits. Kruskal [9, 8] demonstrated that if the rank is less than some linear expression in the dimensions of the tensor, then the rank decomposition is unique. Strassen [11] proved that often such expressions are *unique*, modulo the trivial scaling of the vectors and the order of the terms in the above expression, provided that the rank is not too large and the tensor is sufficiently general. A tensor that admits only one rank decomposition is said to be *identifiable*.

Determining the number of decompositions of a generic tensor of rank r can be reformulated as a geometrical question. Rank-1 tensors are namely points on the Segre variety, i.e., the image of the Segre embedding. With some abuse of notation, we write the Segre variety as $\mathcal{S} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_d}$. Then, tensors of rank r form a dense open subset of the r-secant variety of the Segre variety $\mathcal{S}$, i.e.,

$$\sigma_r(\mathcal{S}) = \overline{\bigcup_{p_1,\ldots,p_r \in \mathcal{S}} \langle p_1, \ldots, p_r \rangle}.$$

The problem we consider is the following: How many r-planes $\langle p_1, \ldots, p_r \rangle$ intersect in some known point $p = p_1 + p_2 + \ldots p_r$? That is, what is the r-secant order [2]? For general points $p_i \in \mathcal{S}$, a simple computational procedure can be devised for distinguishing between the case where there is precisely *one* decomposition and the case where there are more decompositions. The idea consists of testing whether the Segre variety is not r-tangentially weakly defective [3, 1]. The main proposition [4] can be summarized briefly as follows.

Proposition 1. *Let $\mathcal{S}$ be a nondefective Segre variety, and let r be less than the generic rank of $\mathbb{P}^{n_1} \otimes \cdots \otimes \mathbb{P}^{n_d}$. Take a general $(p_1, p_2, \ldots, p_r) \in \mathcal{S}^{\times r}$ and a general point p in their span. If the r-secant order of $\mathcal{S}$ is not one, then through each $p_1, p_2, \ldots, p_r$ there will pass at least one curve that is contained in*

$$\mathcal{C}_r = \{q \in \mathcal{S} \mid \mathrm{T}_q\mathcal{S} \subset \mathrm{M} = \langle \mathrm{T}_{p_1}\mathcal{S}, \mathrm{T}_{p_2}\mathcal{S}, \ldots, \mathrm{T}_{p_r}\mathcal{S} \rangle\} \subset \mathcal{S}.$$

Different curves may pass through different p_i's.

The contrapositive of this proposition is that $\dim \mathcal{C}_r = 0$ entails that the r-secant order is one, provided that the $p_i \in \mathcal{S}$ are general. The zero-dimensionality of the r-tangential contact locus $\mathcal{C}_r$ can be tested through a straightforward linear algebra procedure that we explained in [4]. The essentials of our approach can be summarized as follows:

1. Compute the tangent space to the r-secant variety of $\mathcal{S}$ by invoking Terracini's lemma [12], i.e., $\mathrm{M} = \langle \mathrm{T}_{p_1}\mathcal{S}, \mathrm{T}_{p_2}\mathcal{S}, \ldots, \mathrm{T}_{p_r}\mathcal{S} \rangle$.
2. Compute the kernel of M.
3. Each of the basis vectors in the kernel specifies an equation in the ideal of the linear space M. Derive each of these equations twice, i.e., compute their Hessian.
4. Stack all of the obtained Hessians.
5. Evaluate the stacked Hessian in each of the points p_i and verify that the rank of the Hessian is maximal, i.e., equal to the dimension of the Segre variety $\mathcal{S}$. If it is, then the r-secant variety is identifiable.

In this computation presentation, we will show the computational benefit of adopting a straightforward procedure based on classic linear algebra, comparing a straightforward Macaulay2 implementation with a special-purpose C++ code. We demonstrate that reductions in the execution times of up to two orders of magnitude can be obtained. It is because of these computational savings that we were able to prove that in every tensor space—with a small number of exceptions—of affine dimension less than $17,500$ the general tensor admits exactly one decomposition. The current state of progress towards proving identifiability in tensor spaces of dimension between $17,500$ and $20,000$ can be monitored live at: `http://people.cs.kuleuven.be/~nick.vannieuwenhoven/CompletedCases.html`.

The procedure that we proposed in [4] can be applied correctly to any smooth point of the r-secant variety of $\mathcal{S}$. In particular, the procedure allows one to verify the identifiability of *specific* points, provided that one can prove their smoothness using, e.g., standard flattenings or Young flattenings [10]. We will show some examples where our procedure can prove identifiability that fall outside of the scope of other state-of-the-art techniques [5] for proving identifiability of specific points.

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