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**A Branch-and-Bound Procedure for the Generalized
Resource-Constrained Project Scheduling Problem**

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**A BRANCH-AND-BOUND PROCEDURE FOR THE
GENERALIZED RESOURCE-CONSTRAINED PROJECT
SCHEDULING PROBLEM**

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ABSTRACT

In this paper a branch-and-bound procedure is described for scheduling project activities subject to precedence diagramming type of precedence relations, ready times, due dates, and variable multiple resource availability constraints, where the objective is to minimize project duration. The procedure is based on a depth-first solution strategy in which nodes in the solution tree represent resource and precedence feasible partial schedules. Branches emanating from a parent node correspond to exhaustive and minimal combinations of activities, the delay of which resolves resource conflicts at each parent node. A precedence based lower bound and several dominance rules are introduced in order to restrict the growth of the solutions tree. The procedure has been programmed in the C language. Extensive computational experience is reported.

The specific resource allocation problem addressed in this paper is the *generalized resource-constrained project scheduling problem (GRCPSP)*, in which it is assumed that a project activity is subject to technological precedence diagramming type of precedence constraints (finish-start, finish-finish, start-start and start-finish) and cannot be interrupted once begun (no job preemption allowed). Ready times and due dates can be specified for all activities in the project. Resources are assumed to be available per period in variable amounts and are demanded by an activity in constant amounts throughout the duration of the activity. The objective is to schedule the activities subject to ready times, due dates, precedence relations and resource constraints in order to minimize the total project duration. As such the problem discussed in this paper is an extension of the classical *resource-constrained project scheduling problem (RCPSP)* which involves the minimization of the project duration subject to the classical finish-start precedence constraints and constant resource availability constraints.

To the best of our knowledge, we are not aware of any open literature that deals with resource-constrained project scheduling problems under the realistic assumptions of other than finish-start precedence constraints, ready times and/or due dates. For the case of variable resource availabilities, Simpson (1991) developed a serial and parallel implicit enumeration algorithm as an extension of the solution procedure presented by Talbot and Patterson (1978) for the classical **RCPSP**.

The purpose of this paper is to describe a new efficient branch-and-bound procedure for the **GRCPSP**. The procedure is a major extension of the branch-and-bound procedure developed by Demeulemeester and Herroelen (1991) for the **RCPSP**. It is based on a depth-first solution strategy in which nodes in the solution tree represent resource and precedence feasible partial schedules. Branches emanating from a parent node correspond to exhaustive and minimal combinations of activities, the delay of which resolves resource conflicts at each parent node. The procedure has

been programmed in the C language and runs on personal computers. Extensive computational experience on two problem sets indicate the procedure to outperform the solution procedures presented by Simpson (1991), both in terms of computational efficiency and effectiveness.

The paper is organized as follows. In Section 1, the **GRCPS**P is formally defined and a review of the literature is presented. In Section 2, the general concepts of the branch-and-bound solution procedure are introduced. We will prove some theorems that restrict the number of possibilities that have to be explored during the search process. A formal description of the procedure is given together with an illustration of the basic algorithmic steps on a small numerical example. Computational results obtained on two problem sets are presented in Section 3. The first problem set has been constructed by Simpson (1991) and was based on the well-known standard Patterson problem set for the **RCPS**P. The second set is new and is based on variations of one of the most difficult problems in the Patterson set. An indication is given on the impact of adding the various additional constraints to the **RCPS**P. The last section is reserved for overall conclusions.

1. THE GENERALIZED RESOURCE-CONSTRAINED PROJECT SCHEDULING PROBLEM (GRCPSP)

The classical *resource-constrained project scheduling problem (RCPSP)* is commonly based on the following assumptions :

- (a) A project consists of different activities which are represented in the activity-on-the-node format (i.e. a directed, acyclic graph in which the nodes represent the activities and where the arcs denote the precedence constraints). Two dummy activities are introduced : activity 1 represents the start activity of the project and is a (direct or indirect) predecessor of every other activity in the project, while activity n denotes the end activity of the project and is a (direct or indirect) successor of every other activity in the project.
- (b) The activities are related by a set of finish-start precedence relations with a time lag of 0, implying that no activity can be started before all its predecessors have completed.
- (c) No ready times or due dates are imposed on any of the project activities.
- (d) Each activity i ($i = 1, \dots, n$) has a constant duration d_i (setup times are negligible or are included in the fixed duration).
- (e) Each activity i requires a constant number of units r_{ik} of a renewable resource type k ($k = 1, \dots, K$). The resource requirements r_{ik} are known constants over the processing interval of the activity.
- (f) The availability of the renewable resource type k , a_k , is also a known constant throughout the project duration interval.
- (g) No activity can be interrupted once begun (activity preemption is not allowed).
- (h) The objective is to complete the project as soon as possible without violating any resource or precedence constraints.

This problem can be conceptually formulated as follows :

$$\text{minimize } f_n \quad [1]$$

$$\text{subject to : } f_i \leq f_j - d_j \quad \text{for all } (i, j) \in H \quad [2]$$

$$f_1 = 0 \quad [3]$$

$$\sum_{i \in S_t} r_{ik} \leq a_k \quad k = 1, \dots, K \text{ and } t = 1, \dots, f_n \quad [4]$$

where : n = the number of activities in the project

K = the number of resource types

f_i = the finish time of activity i

d_i = the duration of activity i

H = the set of pairs of activities indicating finish-start precedence relations

S_t = the set of activities in progress during time interval $]t - 1, t]$

$= \{ i \mid f_i - d_i < t \leq f_i \}$

r_{ik} = the amount of resource type k that is required by activity i

a_k = the total availability of resource type k

Reviews of the **RCPSP** can be found in Herroelen (1972) and Davis (1966, 1973). Numerous researchers have tried to solve this problem using integer linear programming (Bowman 1959, Brand, Meyer and Schaffer 1964, Elmaghraby 1967, McCoy 1972, Moodie and Mandeville 1966, Patterson and Huber 1974, Patterson and Roth 1976, Pritsker, Watters and Wolfe 1969, Wiest 1964), but this solution approach proved to be inefficient. Carruthers and Battersby (1966), Petrovic (1968) and Shackleton (1973), among others, tried to use dynamic programming in order to solve resource-constrained problems, but the 'dimensionality curse' precludes the use of

dynamic programming as a practical, efficient solution method. However, it can be proven that in order to minimize the project duration it is only necessary to consider **semi-active timetabling** (French 1982). Timetabling is called semi-active if in the resulting schedule there does not exist an activity which could be started earlier without violating any of the precedence or resource constraints. Consequently, numerous enumerative (branch-and-bound) procedures for solving this problem have been developed (Balas 1970, Bell and Park 1990, Carlier and Latapie 1991, Christofides, Alvarez-Valdes and Tamarit 1987, Davis and Heidorn 1971, Demeulemeester and Herroelen 1991, Johnson 1967, Schrage 1970, Stinson 1976, Stinson, Davis and Khumawala 1978, Talbot and Patterson 1978). Patterson (1984) compared the performance of the exact procedures of Davis and Heidorn (1971), Talbot and Patterson (1978) and Stinson (1976, 1978). On a problem set of 110 problems it was found that the procedure of Stinson was the only one capable of solving all 110 problems, and this in a very small average CPU-time (0.82 seconds on an Amdahl 470/V8). Demeulemeester and Herroelen (1991) provide computational evidence that their procedure outperforms Stinson's by a factor of 11.6. Computational experience also indicates that the procedures of Christofides, Alvarez-Valdes and Tamarit (1987), of Bell and Park (1990) and of Carlier and Latapie (1991) are outperformed on the Patterson problem set by both the procedure of Demeulemeester and Herroelen and that of Stinson. Consequently, the procedure of Demeulemeester and Herroelen (1991) seems to be the fastest exact solution method that is capable of solving the **RCPSP**.

In this paper some of the assumptions of the **RCPSP** will be relaxed :

- (a) **Precedence diagramming** is introduced in the following way. Instead of finish-start relations with a lag of 0, four different types of precedence relations are defined (Moder, Phillips and Davis 1983, page 95) :

SS_{ij} : denotes a start-to-start constraint, and is equal to the minimum number of time units that must be completed on the preceding activity i prior to the start of the successor j .

FF_{ij} : denotes a finish-to-finish constraint, and is equal to the minimum number of time units that must remain to be completed on the successor j after the completion of the predecessor i .

FS_{ij} : denotes a finish-to-start constraint, and is equal to the minimum number of time units that must transpire from the completion of the predecessor i prior to the start of the successor j . (Note : This is the only type of precedence relation used in the classical activity-on-the-node scheme, with $FS_{ij} = 0$).

SF_{ij} : denotes a start-to-finish constraint, and is equal to the minimum number of time units that must transpire from the start of the predecessor i to the completion of the successor j .

The solution procedure that will be described in the next section is applicable only if no activity is allowed to start before one of its predecessors has started (this would for instance be possible if a precedence relation $SS_{ij} < 0$ was specified). However, this is not a real restriction in project scheduling practice : the notions 'predecessor' and 'successor' already imply that a successor is not allowed to start before one of its predecessors.

- (b) **Ready times g_i** and **due dates h_i** can be specified for all activities i in the project. The ready time g_i indicates that activity i cannot be started before time g_i , while a due date h_i signifies that activity i should not finish later than at time h_i .
- (c) **Resource availability a_{kt}** is now assumed to be **variable** over the project horizon and thus a_{kt} denotes the availability of resource type k during period $]t - 1, t]$.

This **generalized resource-constrained project scheduling problem (GRCPSP)** can be expressed in mathematical terms as follows :

$$\text{minimize } f_n \quad [5]$$

$$\text{subject to : } f_i - d_i + SS_{ij} \leq f_j - d_j \quad \text{for all } (i, j) \in H_1 \quad [6]$$

$$f_i - d_i + SF_{ij} \leq f_j \quad \text{for all } (i, j) \in H_2 \quad [7]$$

$$f_i + FS_{ij} \leq f_j - d_j \quad \text{for all } (i, j) \in H_3 \quad [8]$$

$$f_i + FF_{ij} \leq f_j \quad \text{for all } (i, j) \in H_4 \quad [9]$$

$$f_1 = 0 \quad [10]$$

$$f_i - d_i \geq g_i \quad i = 1, \dots, n \quad [11]$$

$$f_i \leq h_i \quad i = 1, \dots, n \quad [12]$$

$$\sum_{i \in S_t} r_{ik} \leq a_{kt} \quad k = 1, \dots, K \text{ and } t = 1, \dots, f_n \quad [13]$$

where : H_1 = the set of pairs of activities indicating start-start relations with a lag of SS_{ij}

H_2 = the set of pairs of activities indicating start-finish relations with a lag of SF_{ij}

H_3 = the set of pairs of activities indicating finish-start relations with a lag of FS_{ij}

H_4 = the set of pairs of activities indicating finish-finish relations with a lag of FF_{ij}

g_i = the ready time of activity i

h_i = the due date of activity i

a_{kt} = the availability of resource type k during period $[t - 1, t]$

It should be observed that although we are referring to the single project resource-constrained project scheduling problem, the problem statement easily allows the introduction of *multiple projects*. As will be demonstrated in the numerical example of section 2.7, the different projects can be combined into one single project allowing the direct application of a single project procedure.

To the best of our knowledge, we are not aware of any open literature that deals with resource-constrained project scheduling problems in which precedence diagramming, ready times and/or due dates are introduced. The case of variable resource availabilities received somewhat more attention. Simpson (1991) measured the decrease in computation time for the **RCPSP** when multiprocessor architectures were introduced. He used a parallel algorithm (simulated on a single processor computer system) which required each processor to execute a serial implicit enumeration procedure to solve the **RCPSP**. The serial procedure implemented by each processor was based on the integer programming algorithm developed by Talbot and Patterson (1978). Simpson took advantage of the extendability of the Talbot and Patterson procedure to devise a serial and parallel implicit enumeration algorithm for the nonconstant resource availability problem.

In order to gain computational experience on problems where resource availability fluctuates over time, Simpson modified the Patterson test problems in the following way. For each resource type required by a problem, availability levels were reduced at random intervals in the scheduling horizon between period 1 and the period equalling the sum of the activity durations. The number of occurrences of a resource reduction was established as a function of the length of this scheduling horizon. The number of occurrences were uniformly distributed with a mean of one occurrence every 20 periods. The duration of each occurrence of a resource reduction was uniformly distributed between one and five periods. A mild resource reduction consisted of a one unit decrease in the resources available and was assigned a

probability of 0.75. A moderate resource reduction of two units occurred with a probability of 0.25.

The serial procedure was able to optimally solve 97 out of the 110 modified problems, when the procedure was allowed 600 CPU seconds on an IBM 3090. The average computation time for these 97 problems was 100.85 seconds with a standard deviation of 199.62 seconds. The parallel procedure could solve 98 out of those 110 problems and this in an average CPU time of 96.63 seconds with a standard deviation of 195.90 seconds. The computation time for the parallel procedure was obtained through the use of a special program which simulated the effect of parallelism on a single processor computer. The computation time of the simulated parallel program is then obtained by dividing the total solution time by the number of processors that were simulated (Simpson 1991). In the next section we describe a new branch-and-bound procedure for the **GRCPS** that outperforms the Simpson procedures on the problem set described above.

2. SOLUTION METHODOLOGY

The branch-and-bound procedure to be described in this section is a major extension of the procedure developed by Demeulemeester and Herroelen (1991) for the classical **RCPSP**. It is based on a depth-first solution strategy in which nodes in the solutions tree represent resource and precedence feasible partial schedules. Branches emanating from a parent node correspond to exhaustive and minimal combinations of activities, the delay of which resolves resource conflicts at each parent node. Before a formal description of the branch-and-bound procedure can be given, a number of definitions are in order. In addition we have to describe the ways in which the suggested procedure bypasses possible anomalies caused by the introduction of precedence diagramming and the ways in which it copes with ready times and due dates. Last but not least, a number of theorems are presented which will allow for a drastic reduction in the search of the branch-and-bound solution tree.

2.1 Conversion of precedence relations

The introduction of precedence diagramming can entail some anomalies. De Wit and Herroelen (1990) discuss an anomaly which is caused by the procedure used by commercial software packages to deal with precedence diagramming. Another anomaly is of a more fundamental nature and is explained in Moder et al. (1983). This anomaly, which causes project duration to increase when activity durations decrease, is avoided by converting all precedence relations into finish-start relations. This conversion is always possible because the nonpreemption assumption immediately links the start of an activity to its finish. For any two activities that are linked by at least one precedence relation, the following conversion formula is applied :

$$FS'_{ij} = \max \{ SS_{ij} - d_i; SF_{ij} - d_i - d_j; FS_{ij}; FF_{ij} - d_j \} \quad [14]$$

The logic behind Eq. [14] is apparent and demonstrated in Figure 1. The SS_{ij} -relation, for instance, can easily be converted into an FS_{ij} -relation by deducting d_i from its lag-value.

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Insert Figure 1

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2.2 Ready times and due dates

The **ready time** g_i of an activity i can easily be transformed into a finish-start relation between the dummy start activity 1 (which starts and finishes at time 0) and activity i :

$$FS'_{1i} = \max \{g_i; FS_{1i}\} \quad [15]$$

Coping with **due dates** h_j is somewhat more involved. For every activity j a latest allowable start time l_j has to be computed such that whenever this activity j is delayed to start later than l_j , the due date of this activity or of one of its direct or indirect successors is exceeded even if all subsequent activities were scheduled as soon as possible without considering the resource constraints. Consequently, if during the branch-and-bound procedure an activity j is assigned an early start time s_j that exceeds its latest allowable start time l_j , backtracking can occur as no feasible solution can be found by continuing the search from this partial schedule.

2.3 The search process

The description of the search process can depart from the following definitions. Let A denote the set of all activities $i = 1, \dots, n$. S_t has already been defined as the set of activities in progress during time interval $[t - 1, t]$, while the partial

schedule PS_t denotes the set of all activities that have been assigned a finish time at time t . We define the set of unscheduled activities NS_t to comprise all activities that have not been assigned a finish time at time t : $NS_t = A - PS_t$. The set of finished activities F_t is defined as the set of all activities that belong to PS_t and that have been assigned a finish time that is smaller than or equal to t (those activities have actually been completed at time t) : $F_t = PS_t - S_t$. U_t then denotes the set of unfinished activities and is defined as $U_t = A - F_t$. We also define the cutset C_t as the set of all unscheduled activities whose predecessors all belong to the partial schedule PS_t . The eligible set E_t denotes the set of all activities that belong to the cutset C_t and that can start at time t . As in most cases the precise time instant at which these sets are defined will be clear from the context, the subscripts will be omitted for simplicity of notation.

The search process starts by initialising S , PS and C to be empty sets and by adding the dummy start activity 1 to S and PS with a finish time $f_1 = 0$. All activities i that have activity 1 as a single predecessor are added to the cutset C and are assigned an early start time, based on the precedence relations FS'_{1i} (which include the ready times). The next decision point m is then computed as the smallest early start time of any activity in the cutset. The activities in the cutset that can start at time m are added to the eligible set E . All activities in S that complete before time m are deleted from S and all activities in E are scheduled : they are added to S and PS and are assigned a finish time that equals the sum of the decision point m and the duration of the activity involved. We also immediately update the cutset. If due to the resource constraints it is impossible to schedule all activities in S concurrently, a resource conflict occurs. Such a conflict will produce a new branching in the branch-and-bound tree at level p : the branches describe ways to resolve the resource conflict; i.e., decisions about which combinations of activities are to be delayed.

We define a **delaying alternative** D_q as a set of activities that belong to S , the delay of which would resolve the resource conflict that occurred at level p of the solutions tree and for which it holds that if an activity belongs to D_q all its direct and indirect successors that belong to S are also included in D_q . In order to simplify the construction process of the delaying alternatives a precedence relation is added for every activity that can be partially overlapped with one of its indirect successors. As such, only the direct successors need to be examined in order to satisfy the second condition. The **delaying set** $D(p)$ then consists of all possible delaying alternatives D_q that resolve the resource conflict at level p of the branching tree. For each delaying alternative D_q the **delaying point** w_q is computed as the earliest time at which either the resource availability changes, or an activity that belongs to $(S - D_q)$ finishes, or one of the unscheduled activities that has no predecessor in D_q could finish if all unscheduled activities were scheduled as soon as possible. A **precedence-based lower bound** L_q is then calculated by adding the maximal remaining critical path length of any of the activities that belong to D_q to the delaying point w_q . The delaying alternative with the smallest lower bound is chosen (ties are broken arbitrarily) and these activities are removed from S and PS (as well as all completed successors of one of these activities). All other delaying alternatives are stored for backtracking purposes. The cutset is updated and the process of constructing the eligible set, adding it to S and PS and branching whenever resource conflicts occur is repeated either until a solution to the problem is found or until it can be shown that by branching from this node only infeasible solutions or solutions which would not be better than another one (which could be constructed along a different path) could be generated. When this happens the procedure backtracks.

2.4 Semi-active timetabling

As should be clear from the description given above, partial schedules PS_t correspond to the set of activities that have been assigned a finish time at time t . The

following theorem provides the proof that the partial schedules may be constructed by semi-active timetabling. Semi-active timetabling (French 1982) means that each activity is started as soon as it can within the precedence and resource constraints.

THEOREM 1. In order to solve the GRCPSP it is sufficient to construct the partial schedules by semi-active timetabling.

PROOF. See Appendix.

2.5 Dominance rules

Three dominance rules are used to prune the search tree. A first dominance rule is related to the construction of the delaying set $D(p)$ at level p of the branching tree. We call a delaying alternative $D_q \in D(p)$ minimal if it does not contain other delaying alternatives $D_v \in D(p)$ as a subset.

THEOREM 2. In order to resolve a resource conflict it is sufficient to consider only minimal delaying alternatives.

PROOF. See Appendix.

A second dominance rule is based on the definition of a cutset and will be referred to as the cutset dominance rule.

THEOREM 3. Consider a cutset C_m at time m which contains the same activities as a cutset C_k , which was previously saved during the search of another path in the search tree. If time k was not greater than time m , if all activities in progress at time k did not finish later than the maximum of m and the finish time of the corresponding activities in PS_m and if the earliest possible

start time of every activity in C_k is smaller than or equal to the earliest start time of the corresponding activity in C_m , then the current partial schedule PS_m is dominated.

PROOF. See Appendix.

A third and last dominance rule is called the left-shift dominance rule and considers the delaying alternatives that were used at the current level p and the previous level $p - 1$ of the solutions tree.

THEOREM 4. Consider the partial schedule PS_m at level p of the solutions tree which starts activity i at time t . If the delay of the delaying alternative that was selected at the previous level $p - 1$ of the solutions tree forced this activity to become eligible at time t , and if this activity can be left-shifted without violating the precedence or resource constraints (because activities that were in progress were delayed by the current delaying alternative at level p), then the partial schedule PS_m is dominated.

PROOF. See Appendix.

These three dominance rules considerably reduce the number of schedules that have to be considered as well as the computation times involved. In the sequel of this section the branch-and-bound procedure is formally stated and demonstrated on a small problem example.

2.6 The branch-and-bound procedure

The detailed algorithmic steps of the branch-and-bound algorithm are described below. The save operation performed in step 2 below should be

distinguished from the store operation performed in step 4. The save operation saves cutset information needed in order to apply the cutset dominance rule. The store operation saves information which is restored during backtracking.

Step 1 : Initialisation.

- Let $T = 9999$ be an upper bound on the project duration.
- Set the level of the branch-and-bound tree $p = 0$.
- Initialize $m = m' = 0$.
- For all activities i for which a ready time g_i is specified, put $FS_{1i} = \max \{g_i; FS_{1i}\}$.
- For every activity i compute the latest allowable start time l_i based on its due date h_i and the due dates of all direct or indirect successors of activity i .
- Compute FS'_{ij} as indicated in Eq. [14].
- For every activity i compute the remaining critical path length q_i based on all finish-start relations.
- Schedule the dummy start activity : $f_1 = 0$, $PS = \{1\}$ and $S = \{1\}$.
- Compute the lower bound at level 0 : $LB(0) = q_1$.
- Put all immediate successors of activity 1, that have no other predecessors, in the cutset : $C = \{x \mid x \text{ has activity 1 as a single predecessor}\}$ and set the early start times s_x of the cutset activities x equal to FS'_{1x} .

Step 2 : Augmentation.

- If the dummy end activity n belongs to the partial schedule PS , the schedule is completed. Update the schedule length $T = f_n$. If T equals $LB(0)$, then STOP (with the optimal solution), else decrease the branching level $p = p - 1$ and go to step 6 (backtrack).
- Compute the next decision point m as the smallest early start time of all cutset activities : $m = \min \{s_x \mid x \in C\}$.
- For all activities $j \in S$ for which $f_j \leq m$ update the set of activities in progress : $S = S - \{j\}$.

- Check if the current cutset is dominated by a previously saved cutset. If it is dominated, go to step 6 (backtrack), else save the current cutset C with the corresponding early start times, the decision point m at which it is saved and the activities in progress as well as their finish times.
- Construct the eligible set : $E = \{x \in C \mid s_x = m\}$.

Step 3 : Scheduling.

- Temporarily put all eligible activities in progress : for all $i \in E$ set $f_i = m + d_i$, $PS = PS + \{i\}$ and $S = S + \{i\}$.
- Update the cutset : $C = C - E + \{x \mid x \text{ is an immediate successor of an activity } i \in E \text{ and all predecessors of } x \text{ belong to } PS\}$ and set the early start times of these new cutset activities : $s_x = \max \{f_y + FS'_{yx} \mid (y, x) \in H\}$.
- For each resource type k check if $\sum_{i \in S} r_{ik} \leq a_{km}$. If there is at least one resource

type k for which the sum of the resource requirements of all activities in progress exceeds the resource availability at time m , we have a resource conflict : go to step 4, else go to step 2.

Step 4 : Resource conflict.

- Update the branch level of the search tree : $p = p + 1$.
- Determine for each resource type k how many units have to be released in order to resolve the resource conflict : for each resource type k compute $c_k = \sum_{i \in S} r_{ik} - a_{km}$.
- Define the delaying set $D(p) = \{D_q \mid D_q \text{ is a subset of } S, \sum_{i \in D_q} r_{ik} \geq c_k \text{ for each resource type } k, \text{ all successors of an activity } i \in D_q \text{ that are in progress also belong to } D_q \text{ and } D_q \text{ does not contain another } D_v \in D(p) \text{ as a subset}\}$.
- For every $D_q \in D(p)$ determine the minimum of the next change in the resource availability, the earliest finish time of all activities that remain in progress if D_q is

- delayed and the earliest possible finish time of all unscheduled activities that have no predecessor in D_q . Assign this value to w_q .
- Compute the precedence-based lower bound L_q for every delaying alternative D_q as the sum of w_q and the maximum of all remaining critical path lengths q_i for each activity $i \in D_q$.
 - Select the $D_b \in D(p)$ with the smallest L_b (ties are broken arbitrarily) : the subscript b denotes the best delaying alternative left at level p .
 - Update the delaying set : $D(p) = D(p) - D_b$.
 - Set $LB(p) = \max \{LB(p - 1), L_b\}$. If $LB(p) \geq T$, decrease the branching level $p = p - 1$ and go to step 6.
 - Store the activity completion times f_i , the partial schedule PS, the set S of activities in progress, the set C of cutset activities with their early start times s_x and the decision points m and m' .

Step 5 : Delaying.

- Branch into a new node. Define DS as the set of all activities that started earlier than at time m' but must be delayed : $DS = \{i \in D_b \mid f_i - d_i < m'\}$.
- Delay the delaying alternative : $PS = PS - D_b$ and $S = S - D_b$. Remove from PS all activities that are an immediate successor of one of the activities $i \in D_b$ and that belong to the partial schedule PS.
- Update the cutset : $C = C + D_b - \{y \mid x \in D_b \text{ and } (x, y) \in H\}$.
- For all $i \in D_b$, if $i \in C$, update the early start times : $s_i = w_b$.
- If an activity $i \in D_b$ has been assigned an early start time s_i that is larger than its latest allowable start time l_i , go to step 6 (backtrack).
- Update $m' = m$.
- If DS is not empty, invoke the left-shift dominance rule as follows. If the inclusion of an activity i in the delaying alternative, that was chosen at the previous level $p - 1$ of the solutions tree forced activity i to become eligible at time m' , if activity i is started at time m' and if delaying the activity set DS would allow activity i to be

left-shifted without causing a resource conflict, then this schedule is dominated : go to step 6 (backtrack).

- Go to step 2.

Step 6 : Backtracking.

- If the branching level $p = 0$, then STOP.
- If no delaying alternatives are left at this level (i.e., if $D(p) = \emptyset$), set $p = p - 1$ and repeat step 6.
- Select the delaying alternative $D_b \in D(p)$ with the smallest lower bound L_b left at this level (ties are broken arbitrarily) and update the delaying set $D(p) = D(p) - D_b$.
- Compute the lower bound $LB(p) = \max \{LB(p - 1), L_b\}$. If $LB(p) \geq T$, decrease the branching level : $p = p - 1$ and repeat step 6.
- Restore the activity completion times f_i , the partial schedule PS, the set S of activities in progress, the set C of cutset activities with their early start times s_x and the decision points m and m'. Go to step 5.

2.7 Numerical example

In this section a small example will be solved in order to demonstrate the different steps of the branch-and-bound procedure that was described in Section 2.6. The example is a multi-project scheduling problem and is represented in Figure 2. The numbers inside the nodes denote the activity numbers, the ones above each node signify the activity durations and the numbers below each node denote the resource requirements for the two different resource types. The type of precedence relation and the associated time lag are indicated on each arc of the network.

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Insert Figure 2

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In Figure 2 it can be seen that 3 different projects have to be scheduled. A first project consists of activities 2 to 6 and has an assumed ready time of 5 and a due date of 20 (the ready time is defined for activity 2 : $g_2 = 5$, while the due date is defined for activity 6 : $h_6 = 20$). A second project, consisting of activities 7 to 10, has a ready time of 0 and a due date of 15, while a third project consists of activities 11 to 13 and has a ready time of 0 and a due date of 25. It is assumed that the resource availabilities are varying over time. For ease of reference these values are indicated in Table I.

=====

Insert Table I

=====

In the first step of the branch-and-bound procedure all ready times and precedence relations are converted into finish-start precedence relations FS'_{ij} . For ease of reference these values are shown in Figure 3. Observe the extra arc that is added between activities 2 and 4 as these activities can be partially overlapped ($FS'_{24} = -1$). As mentioned before, the extra arc is added in order to simplify the task of generating delaying alternatives.

=====

Insert Figure 3

=====

The different steps of the branch-and-bound procedure will now be demonstrated on this numerical example :

Step 1 : Let $T = 9999$; $m = m' = p = 0$. Convert all ready times and precedence relations (see Figure 3). Compute the latest allowable start times : $l_{14} = 9999$; $l_{13} = 21$; $l_{12} = 18$; $l_{11} = 12$; $l_{10} = 9$; $l_9 = 5$; $l_8 = 6$; $l_7 = 3$; $l_6 = 18$; $l_5 = 11$; $l_4 = 13$; $l_3 = 12$; $l_2 = 5$; $l_1 = 0$. Compute the remaining critical path length :

$q_{14} = 0$; $q_{13} = 4$; $q_{12} = 7$; $q_{11} = 13$; $q_{10} = 6$; $q_9 = 10$; $q_8 = 9$; $q_7 = 12$; $q_6 = 2$; $q_5 = 9$; $q_4 = 7$; $q_3 = 8$; $q_2 = 15$; $q_1 = 20$. Schedule the dummy start activity 1 : $f_1 = 0$; $PS = \{1\}$; $S = \{1\}$. The lower bound at level 0 of the search tree is computed as $LB(0) = 20$. Construct the cutset $C = \{2, 7, 11\}$ and compute the early start times of all cutset activities : $s_2 = 5$; $s_7 = s_{11} = 0$.

Step 2 : Update $m = 0$ and $S = \emptyset$. C is not dominated : save the cutset. Construct the eligible set $E = \{7, 11\}$.

Step 3 : Temporarily put all eligible activities in progress : $PS = \{1, 7, 11\}$; $S = \{7, 11\}$; $f_7 = 2$; $f_{11} = 4$. Update the cutset $C = \{2, 8, 9, 12\}$ and assign early start times to the activities that are added to the cutset $s_8 = 3$; $s_9 = 2$; $s_{12} = 6$. Since $3 \leq 6$ and $3 \leq 5$, no resource conflict occurs.

Step 2 : $m = 2$; $S = \{11\}$; C is not dominated : save; $E = \{9\}$.

Step 3 : $PS = \{1, 7, 9, 11\}$; $S = \{9, 11\}$; $f_9 = 7$. $C = \{2, 8, 12\}$. $1 \leq 6$ and $4 \leq 5$: no resource conflict.

Step 2 : $m = 3$; $S = \{9, 11\}$; C not dominated : save; $E = \{8\}$.

Step 3 : $PS = \{1, 7, 8, 9, 11\}$; $S = \{8, 9, 11\}$; $f_8 = 7$. $C = \{2, 10, 12\}$; $s_{10} = 6$. $4 \leq 6$ and $5 \leq 5$: no resource conflict.

Step 2 : $m = 5$; $S = \{8, 9\}$; C not dominated : save; $E = \{2\}$.

Step 3 : $PS = \{1, 2, 7, 8, 9, 11\}$; $S = \{2, 8, 9\}$; $f_2 = 9$. $C = \{3, 5, 10, 12\}$; $s_3 = 7$; $s_5 = 11$. $5 \leq 6$ and $4 \leq 5$: no resource conflict.

Step 2 : $m = 6$; $S = \{2, 8, 9\}$; C not dominated : save; $E = \{10, 12\}$.

Step 3 : $PS = \{1, 2, 7, 8, 9, 10, 11, 12\}$; $S = \{2, 8, 9, 10, 12\}$; $f_{10} = 12$; $f_{12} = 11$. $C = \{3, 5, 13\}$; $s_{13} = 9$. $8 > 6$ and $8 > 5$: a resource conflict occurs.

Step 4 : Update the level of the search tree $p = 1$. In order to resolve the current resource conflict, 2 units of resource type 1 and 3 units of resource type 2 must be released : $c_1 = 2$; $c_2 = 3$. Define the delaying set $D(1) = \{\{10\}\}$.

For the single delaying alternative $D_1 = \{10\}$, the delaying point is computed as $w_1 = 7$, which is equal to the minimum of the next change in the resource availability (7), the earliest finish time of all activities that

remain in progress if D_1 is delayed (activities 8 and 9 finish at time 7) and the earliest possible finish time of all unscheduled activities that have no predecessor in D_1 (activity 3 could finish at time 10). Compute the precedence-based lower bound $L_1 = 7 + 6 = 13$.

Select D_1 : $D(1) = \emptyset$; $LB(1) = \max \{20, 13\} = 20 < T$; store the present situation.

Step 5 : Branch into node 1 of the solutions tree (see Figure 4). Compute $DS = \emptyset$ and delay the delaying alternative : $PS = \{1, 2, 7, 8, 9, 11, 12\}$ and $S = \{2, 8, 9, 12\}$.

Update the cutset $C = \{3, 5, 10, 13\}$, the early start time of the delayed activity : $s_{10} = 7 \leq l_{10}$ and $m' = 6$.

Step 2 : $m = 7$; $S = \{2, 12\}$; C not dominated : save; $E = \{3, 10\}$.

Step 3 : $PS = \{1, 2, 3, 7, 8, 9, 10, 11, 12\}$; $S = \{2, 3, 10, 12\}$; $f_3 = 10$; $f_{10} = 13$.

$C = \{4, 5, 13\}$; $s_4 = 8$. $6 \leq 7$, but $7 > 5$: a resource conflict occurs.

Step 4 : $p = 2$; $c_1 = -1$ and $c_2 = 2$: $D(2) = \{\{3\}, \{10\}\}$.

$D_2 = \{3\}$; $w_2 = 9$; $L_2 = 17$.

$D_3 = \{10\}$; $w_3 = 9$; $L_3 = 15$.

Select D_3 : $D(2) = \{\{3\}\}$; $LB(2) = 20 < T$; store.

Step 5 : Branch into node 2 of the solutions tree (see Figure 4). $DS = \emptyset$; $PS = \{1, 2, 3, 7, 8, 9, 11, 12\}$; $S = \{2, 3, 12\}$. $C = \{4, 5, 10, 13\}$; $s_{10} = 9 \leq l_{10}$; $m' = 7$.

Step 2 : $m = 8$; $S = \{2, 3, 12\}$; C not dominated : save; $E = \{4\}$.

Step 3 : $PS = \{1, 2, 3, 4, 7, 8, 9, 11, 12\}$; $S = \{2, 3, 4, 12\}$; $f_4 = 13$. $C = \{5, 10, 13\}$.

$5 \leq 7$ and $5 \leq 5$: no resource conflict.

Step 2 : $m = 9$; $S = \{3, 4, 12\}$; C not dominated : save; $E = \{10, 13\}$.

Step 3 : $PS = \{1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13\}$; $S = \{3, 4, 10, 12, 13\}$; $f_{10} = 15$;

$f_{13} = 13$. $C = \{5\}$. $6 \leq 7$, but $9 > 5$: a resource conflict occurs.

Step 4 : $p = 3$; $c_1 = -1$ and $c_2 = 4$: $D(3) = \{\{4, 10\}, \{10, 13\}, \{3, 4, 13\}, \{4, 12, 13\}\}$.

$D_4 = \{4, 10\}$; $w_4 = 10$; $L_4 = 17$.

$D_5 = \{10, 13\}$; $w_5 = 10$; $L_5 = 16$.

$D_6 = \{3, 4, 13\}$; $w_6 = 11$; $L_6 = 19$.

$D_7 = \{4, 12, 13\}$; $w_7 = 10$; $L_7 = 17$.

Select D_5 : $D(3) = \{\{4, 10\}, \{3, 4, 13\}, \{4, 12, 13\}\}$; $LB(3) = 20 < T$; store.

Step 5 : Branch into node 3 of the branch-and-bound tree. $DS = \emptyset$; $PS = \{1, 2, 3, 4, 7, 8, 9, 11, 12\}$; $S = \{3, 4, 12\}$. $C = \{5, 10, 13\}$; $s_{10} = 10 > l_{10}$: **backtrack**.

Step 6 : Select D_4 at level 3 : $D(3) = \{\{3, 4, 13\}, \{4, 12, 13\}\}$; $LB(3) = 20 < T$; restore the situation.

Step 5 : Branch into node 4 of the search tree. $DS = \emptyset$; $PS = \{1, 2, 3, 7, 8, 9, 11, 12, 13\}$; $S = \{3, 12, 13\}$. $C = \{4, 5, 10\}$; $s_4 = 10 \leq l_4$; $s_{10} = 10 > l_{10}$: **backtrack**.

Step 6 : Select D_7 at level 3 : $D(3) = \{\{3, 4, 13\}\}$; $LB(3) = 20 < T$; restore.

Step 5 : Branch into node 5 of the solutions tree. $DS = \{12\}$; $PS = \{1, 2, 3, 7, 8, 9, 10, 11\}$; $S = \{3, 10\}$. $C = \{4, 5, 12\}$; $s_4 = 10 \leq l_4$; $s_{12} = 10 \leq l_{12}$; $m' = 9$. No left-shift dominance.

Step 2 : $m = 10$; $S = \{10\}$; C not dominated : save; $E = \{4, 12\}$.

Step 3 : $PS = \{1, 2, 3, 4, 7, 8, 9, 10, 11, 12\}$; $S = \{4, 10, 12\}$; $f_4 = 15$; $f_{12} = 15$. $C = \{5, 13\}$; $s_{13} = 13$. $4 \leq 7$ and $5 \leq 5$: no resource conflict.

Step 2 : $m = 11$; $S = \{4, 10, 12\}$; C not dominated : save; $E = \{5\}$.

Step 3 : $PS = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12\}$; $S = \{4, 5, 10, 12\}$; $f_5 = 15$. $C = \{6, 13\}$; $s_6 = 18$. $7 \leq 7$, but $6 > 5$: resource conflict.

Step 4 : $p = 4$; $c_1 = 0$ and $c_2 = 1$; $D(4) = \{\{4\}, \{5\}, \{10\}, \{12\}\}$.

$D_8 = \{4\}$; $w_8 = 12$; $L_8 = 19$.

$D_9 = \{5\}$; $w_9 = 12$; $L_9 = 21$.

$D_{10} = \{10\}$; $w_{10} = 12$; $L_{10} = 18$.

$D_{11} = \{12\}$; $w_{11} = 12$; $L_{11} = 19$.

Select D_{10} : $D(4) = \{\{4\}, \{5\}, \{12\}\}$; $LB(4) = 20 < T$; store.

Step 5 : Branch into node 6 of the branch-and-bound tree. $DS = \{10\}$; $PS = \{1, 2, 3, 4, 5, 7, 8, 9, 11, 12\}$; $S = \{4, 5, 12\}$. $C = \{6, 10, 13\}$; $s_{10} = 12 > l_{10}$: **backtrack**.

Step 6 : Select D_8 at level 4 : $D(4) = \{\{5\}, \{12\}\}$; $LB(4) = 20 < T$; restore.

Step 5 : Branch into node 7 of the branch-and-bound tree. $DS = \emptyset$; $PS = \{1, 2, 3, 5, 7, 8, 9, 10, 11, 12\}$; $S = \{5, 10, 12\}$. $C = \{4, 13\}$; $s_4 = 12 \leq l_4$; $m' = 11$.

Step 2 : $m = 12$; $S = \{5, 10, 12\}$; C not dominated : save; $E = \{4\}$.

Step 3 : $PS = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12\}$; $S = \{4, 5, 10, 12\}$; $f_4 = 17$. $C = \{6, 13\}$; $s_6 = 18$. $7 \leq 7$ and $6 \leq 6$: no resource conflict.

Step 2 : $m = 13$; $S = \{4, 5, 10, 12\}$; C not dominated : save; $E = \{13\}$.

Step 3 : $PS = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13\}$; $S = \{4, 5, 10, 12, 13\}$; $f_{13} = 17$. $C = \{6\}$. $8 > 7$ and $8 > 6$: resource conflict.

Step 4 : $p = 5$; $c_1 = 1$ and $c_2 = 2$: $D(5) = \{\{10\}, \{13\}, \{4, 5\}\}$.

$D_{12} = \{10\}$; $w_{12} = 15$; $L_{12} = 21$.

$D_{13} = \{13\}$; $w_{13} = 15$; $L_{13} = 19$.

$D_{14} = \{4, 5\}$; $w_{14} = 15$; $L_{14} = 22$.

Select D_{13} : $D(5) = \{\{10\}, \{4, 5\}\}$; $LB(5) = 20 < T$; store.

Step 5 : Branch into node 8 of the solutions tree. $DS = \emptyset$; $PS = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12\}$; $S = \{4, 5, 10, 12\}$. $C = \{6, 13\}$; $s_{13} = 15 \leq l_{13}$; $m' = 13$.

Step 2 : $m = 15$; $S = \{4\}$; C not dominated : save; $E = \{13\}$.

Step 3 : $PS = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13\}$; $S = \{4, 13\}$; $f_{13} = 19$. $C = \{6\}$. $2 \leq 7$ and $3 \leq 6$: no resource conflict.

Step 2 : $m = 18$; $S = \{13\}$; C not dominated : save; $E = \{6\}$.

Step 3 : $PS = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$; $S = \{6, 13\}$; $f_6 = 20$. $C = \{14\}$; $s_{14} = 20$. $4 \leq 5$ and $3 \leq 5$: no resource conflict.

Step 2 : $m = 20$; $S = \emptyset$; C not dominated : save; $E = \{14\}$.

Step 3 : $PS = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$; $S = \{14\}$; $f_{14} = 20$. $C = \emptyset$. $0 \leq 5$ and $0 \leq 5$: no resource conflict.

Step 2 : Activity 14 is scheduled : $T = 20 = LB(0)$: **stop with optimal solution.**

The branch-and-bound tree for this small problem is represented in Figure 4. It should be noticed that no cutset dominance occurs for this problem as most of the nodes that were visited lie on a single path in the solutions tree. In general, increased benefit results from dominance rules and lower bounds whenever the size of the problem increases. One should also observe that at level 1 of the solutions tree there is only one possible delaying alternative. In the absence of precedence diagramming

constraints, the delaying sets at every level of the solutions tree would contain at least two delaying alternatives.

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Insert Figure 4

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The optimal solution that was found is shown in Figure 5. Figure 5a exhibits the resource profile for the first resource type, while Figure 5b displays the resource profile for the second resource type.

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Insert Figure 5

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The computational results of the branch-and-bound procedure will be presented in the next section.

3. COMPUTATIONAL EXPERIENCE

In this section, two problem sets will be used to gain computational experience with the branch-and-bound solution method that was described in the previous section. The first set has been constructed by Simpson (1991) and was based on the Patterson problem set which is generally accepted as the standard problem set for the **RCPS**P. The modifications to that set have already been described in Section 1 of this paper. These problems are used to test the influence of the introduction of variable resource availability on the efficiency of the solution methodology. The second problem set is new and is based on problem 72, which is experienced by the procedure of Demeulemeester and Herroelen (1991) to be the most difficult problem in the Patterson problem set. Through the addition and removal of constraints within the scope of the **GRCPS**P, an indication is obtained of the influence of these additional constraints on the computation time required.

The computational experience with the problem set of Simpson is described in Table II. It should be realized that Simpson's solution procedures were run on a mainframe computer, whereas our procedure was tested on a personal computer running under DOS. While the serial and parallel procedures of Simpson were able to solve 97, respectively 98 problems out of 110 within a time limit of 600 CPU seconds, our procedure solved all 110 problems in a very short computation time (all computation times were less than 15 seconds). There also exists a large difference in the average computation times between the three methods : while the procedures of Simpson used an average of 100.85 to solve 97 problems and 96.63 seconds to solve 98 problems optimally, our procedure required an average computation time of 1.2326 seconds to solve all problems to optimality. The standard deviation for all three procedures typically amounts to twice the average computation time. A comparison with the results obtained by the procedure of Demeulemeester and

Herroelen (1991) on the Patterson problem set with constant resource availabilities, shows that the required computation time increases by a factor of about six.

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Insert Table II

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The results in Table II clearly demonstrate the efficiency of the branch-and-bound procedure when variable resource availabilities are introduced. A small experiment has been conducted in order to test the computational impact of the addition of precedence diagramming type of precedence relations, ready times and due dates. The problem set is based on variations of Patterson's difficult problem 72. This problem involves the scheduling of 27 activities subject to constant availability constraints on three resource types. The results for this problem set are presented in Table III.

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Insert Table 3

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The standard problem referred to in row 1 of Table III is constructed as follows. Problem 72 of the Patterson problem set is copied into the correct format for the **GRCPS**P. No ready times or due dates were introduced, but 10 resource intervals were specified, one occurring every 5 time units. The resource availability for each of the three resource types during one of these resource intervals was randomly chosen from the interval [6, 9]. Precedence diagramming constraints were introduced as follows. For each pair of activities (i,j) in the original problem, the corresponding finish-start relationship with a time lag of zero is changed into a finish-start relation with a time lag FS_{ij} . This time lag was randomly chosen from the uniformly distributed interval $[-d_i, d_j]$. As indicated in Table III, this problem could be solved to optimality in 1.59 seconds. The addition of due dates for activities 18 and 22 (see row

2 in Table III) results in a much faster solution. This effect is quite normal as the introduction of due dates restricts the feasible solution space and therefore also the time needed to search this solution space for an optimal solution. The introduction of two ready times (row 3 in Table III) or an increase in the resource availability in one resource interval (row 4 in Table III) does not have much impact upon the computation time. Allowing more overlapping between different activities (three of the positive FS_{ij} -values were made negative), a very significant increase in the required computation time is observed. This result was to be expected : increased overlapping of activities increases the number of resource conflicts, which is directly related to the number of levels in the branch-and-bound tree and, hence, the size of the solutions tree. The process of changing due dates, ready times, resource availability and time lags of the precedence relations is then repeated once more, leading to similar observations as shown in the bottom five rows in Table III.

The computational experience reported in this section allows for a number of conclusions. The variable resource availability problem can be solved in a computation time, which is about six times higher than the time needed by the exact procedure of Demeulemeester and Herroelen (1991), the current best solution procedure for solving the RCPSP. Moderate changes in the ready times or in the resource availabilities do not have a significant impact on the computation times. The introduction of due dates decreases the solution time enormously. Allowing activity overlaps (negative FS_{ij} -values) causes a strong increase in the computation times required.

4. CONCLUSIONS

In this paper a new depth-first branch-and-bound algorithm was described for the generalized resource-constrained project scheduling problem (GRCPSP) which involves the scheduling of a project to minimize its total duration, subject to precedence constraints of the precedence diagramming type, ready times, due dates and variable resource availability constraints. Computational experience gained with the algorithm on a personal computer was very promising. On Simpson's modified 110 Patterson test problems with variable resource availabilities, an optimal solution could be found in an average CPU time of 1.2329 seconds and a standard deviation of 2.3133 seconds on an IBM PS/2 Model 70 A21 with a 25 MHz processor and a coprocessor. As such it seems to be more efficient and effective than the current best exact solution procedure reported in the literature. Compared to computational experience gained with the algorithm on the classical resource-constrained project scheduling problem (RCPSP), computation times seem to increase by a factor of about six.

Experience gained with modified versions of one of the most difficult Patterson problems, seems to indicate that moderate changes in ready times or resource availabilities do not have significant impact on the computation time. The introduction of activity due dates, however, has a strong beneficial impact on the required computational effort, while the introduction of activity overlaps (negative finish-start time lags) leads to a considerable increase in the CPU time required.

It would be tempting to consider possible extensions of the branch-and-bound algorithm to even more general problem types, namely the case of variable resource requirements and the case of generalized precedence constraints (Elmaghraby and Kamburowski 1989). In the latter case, maxima are imposed on the precedence lag, indicating that the start or finish of a successor should not occur more than a certain

period later than the start or finish of its corresponding predecessor. In both cases, however, the use of semi-active timetabling no longer guarantees the optimal solution. This fact forces a branch-and-bound procedure to make scheduling decisions at every discrete time instant (see e.g. the procedure of Davis and Heidorn 1971), with the result that the solution tree grows unwieldy even for moderately sized projects.

APPENDIX

Proof of Theorem 1 (based on French 1982, pages 27 - 28)

Consider an optimal schedule which has not been constructed by semi-active timetabling. Then there is at least one activity which could be started earlier. Of all such activities choose one with the earliest finish time (ties are broken arbitrarily). Retimetable this activity to start as early as possible without violating any precedence or resource constraints. In this new schedule no finish time has increased and thus the schedule length remains unchanged.

Repeat this process of retimabling activities, which could have started earlier, until there are no such activities left. Note that, because we always pick an activity which has the earliest finish time, no activity can be retimabled more than once. There are a finite number of activities so this retimabling process must terminate. The final schedule is the result of semi-active timetabling, since no activity could be started earlier. Moreover, since the schedule length did not increase at any stage, the final schedule is at least as good as the original. Hence, there exists an optimal schedule that can be found by semi-active timetabling.

Proof of Theorem 2

Let $(z_0, \dots, z_p, \dots, z_k)$ be the chain of nodes in the solutions tree that is leading to an optimal schedule G . With each such node z_p at level p of the solutions tree corresponds a delaying alternative D_{z_p} , a delaying point w_{z_p} and a set of constraints Ω_{z_p} . For node z_0 the set Ω_{z_0} equals the original precedence constraints, the ready times and due dates as well as the resource constraints. For all other nodes z_p the set Ω_{z_p} is obtained by adding to $\Omega_{z_{p-1}}$ the additional constraints that all activities in D_{z_p}

cannot start before the corresponding delaying point w_{z_p} . Let f_j^* denote the finish time of activity j in the optimal schedule G .

Let node z_q be the first node in the chain for which the delaying alternative D_{z_q} is non-minimal. If no such z_q exists, we are done.

Otherwise, let time m be the decision point at which the resource conflict occurred. Finish times f_j are associated with all scheduled activities (i.e., $j \in (PS_m - D_{z_q})$) as well as early start times s_i for all unscheduled activities (i.e., $i \in (NS_m + D_{z_q})$). For each activity $i \in D_{z_q}$ we have $s_i = w_{z_q}$ and for every other unscheduled activity the early start time is dependent upon the precedence relations. By definition of a minimal delaying alternative, there exists at least one activity x that can be removed from D_{z_q} : $D_{v_q} = D_{z_q} - \{x\}$ such that D_{v_q} is a valid delaying alternative, occurring at node v_q at level q of the search tree. Now two possibilities have to be considered :

- (a) In case neither activity x nor any of the successors of activity x without direct or indirect predecessors in $(NS_m + D_{v_q})$ could finish before the delaying point w_{z_q} , w_{z_q} would also be the delaying point for the delaying alternative D_{v_q} (i.e., $w_{v_q} = w_{z_q}$). As $D_{v_q} = D_{z_q} - \{x\}$, we have that the added constraints at node v_q are a subset of those added at node z_q and hence Ω_{v_q} is a subset of Ω_{z_q} . As the schedule G is feasible to Ω_{z_q} , G is also feasible to Ω_{v_q} and therefore can be found by continuing the branching process from v_q . Hence, node z_q can be fathomed.
- (b) If either activity x or any of the successors of activity x without direct or indirect predecessors in $(NS_m + D_{v_q})$ could finish before the delaying point w_{z_q} , w_{v_q} would equal the smallest finish time of one of these activities and thus $w_{v_q} < w_{z_q}$. As $D_{v_q} = D_{z_q} - \{x\}$ and $w_{v_q} < w_{z_q}$, we have that the added constraints at node v_q are less strict than the constraints that were added at node z_q and therefore Ω_{v_q} is less restrictive than Ω_{z_q} . Thus the optimal schedule G ,

which by assumption is feasible to Ω_{z_q} , is also feasible to Ω_{v_q} and therefore can be found by continuing the branching process from v_q . Hence, node z_q can be fathomed.

Now we have proven that an optimal solution can be found if branching takes place from a delaying alternative that has one activity less than the first non-minimal delaying alternative, encountered at node z_q of the solutions tree. The process of removing an activity from this non-minimal delaying alternative can be continued until a minimal delaying alternative D_{b_q} is obtained at node b_q . If branching continues from b_q , a new optimal schedule can be found along the chain $(z_0, \dots, z_{q-1}, b_q, b_{q+1}, \dots, b_{k'})$. Along this chain all delaying alternatives at nodes $z_1, \dots, b_q$ are minimal by construction, while the delaying alternatives at nodes $b_{q+1}, \dots, b_{k'}$ could be non-minimal. The procedure of removing activities from a non-minimal delaying alternative until a minimal delaying alternative is obtained can again be performed at that level where the first non-minimal delaying alternative occurred along the new chain. This action can then be repeated until all delaying alternatives are minimal. Hence, it is proven that optimal schedules can be obtained by only considering minimal delaying alternatives.

Proof of Theorem 3

Let G be an optimal schedule with finish times f_i^* generated by completing a partial schedule PS_m for which the cutset C_m satisfies the hypotheses of the theorem. Then there exists a cutset C_k which was previously generated on a different path in the search tree and which corresponds to a partial schedule $PS_k = PS_m$. Moreover, $k \leq m$ and for all $j \in PS_k$ we have $f_{jk} \leq \max \{m, f_{jm}\}$ where f_{jk} denotes the finish time of activity j in PS_k and f_{jm} denotes the finish time of activity j in PS_m . By assumption, we also have for all $i \in C_k = C_m$ that $s_{ik} \leq s_{im}$ where s_{ik} denotes the early start time of activity i in C_k and s_{im} denotes the early start time of activity i in

C_m . As G is generated by completing PS_m it holds for all activities $j \in PS_m$ that $f_{jm} \leq f_j^*$.

Let J denote the set of activities j for which $f_j^* - d_j > m$. Construct a new schedule G' as follows : for all activities $j \in J$ set $f'_j = f_j^*$ and for all activities $i \in (A - J)$ set $f'_i = f_{ik}$. G' is a feasible schedule : no precedence relation is violated as, by hypothesis, for all predecessors i of activities $j \in C_m$ it holds that $f'_i = f_{ik} \leq \max\{m, f_{im}\} \leq \max\{m, f_i^*\}$ and as for all cutset activities $j \in C_m$ we have that $s_{jk} \leq s_{jm} \leq f_j^* - d_j$. In addition no resource constraint is violated as PS_k and G were feasible and because for all activities $j \in PS_k$, it holds that $f_{jk} \leq \max\{m, f_{jm}\} \leq \max\{m, f_j^*\}$ while all activities $i \in J$ are scheduled in G' after both m and the maximal value of $f_j^* + FS_{ji}$ for any of its predecessors j . G' has the same schedule length as G and thus G' , which could be obtained by branching from the node at which PS_k was constructed, is also optimal. Therefore PS_m is dominated.

Proof of Theorem 4

Assume a branching level p at which the assumptions of the theorem are satisfied. Then we have the following situation. At time $t' < t$ the delaying alternative $D_{z_{p-1}} = \{i\} + X$ (with X an arbitrary subset of the activities in progress at time t' , except activity i) was delayed until the delaying point $w_{z_{p-1}} = t$. At time $t'' \geq t$ the delaying alternative $D_{z_p} = J + Y$ (with J a set of activities that were in progress at time t' and Y an arbitrary subset of the activities in progress at time t'' , except J) is delayed until the delaying point $w_{z_p} > t = w_{z_{p-1}}$. $D_{z'_{p-1}} = J + X$ is a valid (not necessarily minimal) delaying alternative at time t' and for this delaying alternative $w_{z'_{p-1}} \leq w_{z_{p-1}}$. The nodes z_{p-1} and z'_{p-1} are descendants from the same node z_{p-2} with the corresponding set of solutions $\Omega_{z_{p-2}}$. In node z'_{p-1} all solutions are deleted where the activities in $J + X$ start earlier than at time instant $w_{z'_{p-1}}$. At node z_{p-1} , all solutions are deleted where activities in X start earlier than at time instant $w_{z_{p-1}} \geq w_{z'_{p-1}}$ and at node z_p

all solutions are deleted where activities in J start earlier than at time $w_{z_p} > w_{z_{p-1}} \geq w_{z'_{p-1}}$. Therefore, all solutions that are deleted from $\Omega_{z_{p-2}}$ in node z'_{p-1} are a subset of the solutions that are deleted from $\Omega_{z_{p-2}}$ in nodes z_{p-1} and z_p . Thus, we have that Ω_{z_p} is a subset of $\Omega_{z'_{p-1}}$ and node z_p can be fathomed as any solution found by branching from z_p can also be found by branching from z'_{p-1} .

The fact that $D_{z'_{p-1}} = J + X$ isn't necessarily minimal is no objection to the proof of this theorem. If $D_{z'_{p-1}}$ is a non-minimal delaying alternative, then a minimal delaying alternative $D_{z''_{p-1}}$, which is a subset of $D_{z'_{p-1}}$, can easily be constructed. As it can be proven (see proof of theorem 2) that $\Omega_{z'_{p-1}}$ is a subset of $\Omega_{z''_{p-1}}$ and as we have proven in the previous paragraph that Ω_{z_p} is a subset of $\Omega_{z'_{p-1}}$, we have that Ω_{z_p} is a subset of $\Omega_{z''_{p-1}}$ and node z_p is dominated by a node z''_{p-1} with a minimal delaying alternative.

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Figure captions

Figure 1 : Conversion of precedence relations

Figure 2 : Network of problem example

Figure 3 : Network of problem example with converted precedence relations

Figure 4 : Branch-and-bound tree for problem example

Figure 5 : Resource profiles of optimal solution

Figure 1 : Conversion of precedence relations

Figure 2 : Network of problem example

Figure 3 : Network of problem example with converted precedence relations

Figure 4 : Branch-and-bound tree for problem example

Figure 5a : Resource profile for type 1

Figure 5b : Resource profile for type 2

Figure 5 : Resource profiles of optimal solution

Table captions

Table I : Resource availabilities for example problem

Table II : Computational experience on the Simpson problem set

Table III : The impact of additional constraints on computational efficiency

Time interval	Availability type 1	Availability type 2
0 - 7	6	5
7 - 12	7	5
12 - 18	7	6
18 - end	5	5

Table I : Resource availabilities for example problem

	Variable resource availabilities			Constant res. availabilities
Methodology	Simpson (serial)	Simpson (parallel)	Demeulemeester and Herroelen	Demeulemeester and Herroelen *
Computer used	IBM 3090	IBM 3090	IBM PS/2 70/A21	IBM PS/2 70/A21
Problems solved	97	98	110	110
Avg. comp. time	100.85	96.63	1.2329	0.215
Standard deviation	199.62	195.90	2.3133	0.314

*Results taken from Demeulemeester and Herroelen (1991)

Table II : Computational experience on the Simpson problem set

Problem specification	Computation time
Standard problem	1.59
$h_{18} = 27; h_{22} = 35$	0.27
$g_4 = 3; g_7 = 10$	0.27
Increase resource availability	0.33
Allow more overlapping	10.11
$h_{10} = 24; h_{13} = 23$	0.39
$g_{10} = 22; g_{13} = 19$	0.16
Lower resource availability	0.22
Allow more overlapping	0.39
$h_{22} = \text{infinity}$	3.30

Table III : The impact of additional constraints on computational efficiency