SERIES EXPANSIONS FOR A GENERALIZED XY HAMILTONIAN

II. TRICRITICAL BEHAVIOUR OF THE TAKAGI MODEL

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The high-temperature series expansions for the square of the fluctuation in the order parameter, for the specific heat, the concentration and the concentration susceptibility for the Takagi model on an f.c.c. lattice are analysed in the field variables. We obtain for the critical exponent $\gamma = \frac{4}{3}$ and the specific heat is possibly logarithmically divergent. If we take the tricritical point to be determined by $\gamma_t = 1$, the tricritical exponents α_t , λ_t and ω_t are found to be consistent with $\frac{1}{2}$.

1. Introduction

The existence of tricritical points, where two competing order parameters simultaneously become critical, was pointed out by Griffiths¹⁾ in 1970. He referred explicitly to ^3He – ^4He mixtures. This tricritical behaviour is also present in various other systems, like metamagnets (FeCl_2)²⁾ and in structural phase transitions (NH_4Cl)³⁾.

The scaling hypothesis for tricritical phenomena was formulated by Riedel⁴⁾, Chang *et al.*⁵⁾ and by Griffiths⁶⁾. Using the Ginzburg criterion, Bausch⁷⁾ showed that for $d > 3$ tricritical exponents are classical, with possible logarithmic corrections for $d = 3$. This last feature was observed earlier by Riedel and Wegner^{8,9)} with the renormalization-group methods.

Models for ^3He – ^4He mixtures were formulated by Blume, Emery and Griffiths (BEG)¹⁰⁾, by Takagi¹¹⁾ and by Cheng and Schick (CS)¹²⁾. The BEG model is a classical spin-1 Ising-like model. The order parameter is introduced in an artificial way and has dimension 1. The two other models treat the superfluid transition quantum-mechanically, the order parameter being two-dimensional. The difference

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between the Takagi and CS models lies in the statistics of the ^3He particles. In the Takagi model the ^3He particles are treated classically, while the CS model takes the quantum behaviour of the ^4He particles into account. As far as high-temperature behaviour is concerned, these two models should essentially be the same, since Fermi statistics is not important at high temperatures. Series expansions for these models were made and analysed by Saul, Wortis and Stauffer¹³⁾ for the BEG model and by Plischke, Elliott and Betts^{14,15)} for the CS model. The tricritical exponents found on the BEG model are in good agreement with the gaussian fixed-point values. For the CS model, the series are not well behaved and extrapolation to tricritical behaviour is very difficult.

The analysis of the high-temperature series for the Takagi model will be presented in this paper. In the second section, we briefly resume the Takagi model and the physical meaning of the parameters. We derived the series expansions for a hamiltonian of this kind in a previous paper¹⁶⁾. In the third section, we analyse the square of the fluctuation in the order parameter. The series for some other physical quantities are studied in the last section.

2. The Takagi model

The model proposed by Takagi¹¹⁾ for ^3He - ^4He mixtures treats the ^3He particles classically, and the ^4He particles quantum-mechanically. If we ignore the attractive interactions between the particles, we can write the hamiltonian of the model as

$$\mathcal{H} = -J \sum_{\langle ij \rangle} (a_i^\dagger a_j + a_i a_j^\dagger) - \tilde{\mu}_4 \sum_i n_{4i} - \tilde{\mu}_3 \sum_i n_{3i}. \quad (1)$$

The $a_i^\dagger$ and a_i are the creation and annihilation operators for a ^4He particle at the i th lattice site. The summation over $\langle ij \rangle$ runs over all nearest-neighbour pairs. $\tilde{\mu}_4$ and $\tilde{\mu}_3$ are the effective chemical potentials, which differ from the real chemical potentials by an additive term, arising from the kinetic energy of those particles. n_{4i} is the number operator for ^4He particles, and n_{3i} is a classical variable, which is zero or one, corresponding to occupation by zero or one ^3He particle of the i th lattice site, respectively.

The repulsion between the particles is simulated by allowing at most one particle at every lattice site. This restriction can be incorporated in the model by making the following identifications:

$$a_i^\dagger = 2^{-\frac{1}{2}} S_i^z S_i^+, \quad (2a)$$

$$a_i = 2^{-\frac{1}{2}} S_i^- S_i^z, \quad (2b)$$

$$n_{4i} = \frac{1}{2} S_i^z (S_i^z + 1), \quad (2c)$$

$$n_{3i} = \frac{1}{2} S_i^z (S_i^z - 1). \quad (2d)$$

In these expressions S_i^+ (or S_i^-) is the spin raising (or lowering) operator for spin 1. The three eigenstates of S_i^z correspond to an occupation of the i th lattice site by a ^4He atom (eigenvalue +1), to an empty site (eigenvalue 0) or to occupation by a ^3He atom (eigenvalue -1). The identification (2a) guarantees that we cannot create a ^4He particle on the i th site, if this site is already occupied, nor can we annihilate a ^4He particle if there was not one originally. One also verifies that

$$a_i^\dagger a_i^\dagger = a_i a_i = 0, \quad (3)$$

which prevents us from creating (annihilating) a ^4He particle twice at the same lattice site. The formulas (2) transform the hamiltonian (1) into a spin-1 hamiltonian, which is a special case of the general hamiltonian studied by us¹⁶⁾ in a previous paper, with the identification

$$H_1 = \frac{1}{2} (\bar{\mu}_4 - \bar{\mu}_3) \quad \text{and} \quad H_2 = \frac{1}{2} (\bar{\mu}_4 + \bar{\mu}_3). \quad (4)$$

The XY model gives a fairly good description of the superfluid transition in ^4He , as has been shown by Betts and Lothian¹⁷⁾. The Takagi model may be considered as an extension of the XY model to include classical ^3He impurities, and it has also a two-dimensional order parameter. Moreover, it allows for a variable density, a feature not exhibited by the BEG model. It is believed that the quantum effects of ^3He particles are not important in the high-temperature region [$T > T_c(x_3)$, where x_3 is the concentration of ^3He]. Therefore it is hoped that, along the high-temperature side of the phase diagram, the Takagi model will give a satisfactory description of the ^3He - ^4He mixture.

The high-temperature series expansions for Y , the square of the fluctuations in the order parameter, and for the free energy of such a hamiltonian were derived in a previous paper¹⁰⁾. We now analyse these series.

3. Analysis of the Y series

We have analysed the Y series for $\bar{\mu}_4 = 0$, and for different values of $\frac{1}{2}\beta(\bar{\mu}_4 - \bar{\mu}_3) = \beta H_1 = -2\beta\Delta$ with the ratio method and with the Padé approximant method¹⁸⁾. We will give the detailed analysis for $\beta H_1 = 5, 2, 0, -0.25, -0.50, -1.00$ and -1.05 . The results for other values of $\beta\bar{\mu}_4$ are similar to those for $\bar{\mu}_4 = 0$.

3.1. Ratio method

In fig. 1 we give the ratio plots for several values of βH_1 . The ratios are well behaved, and we can make rather accurate estimates of the critical temperatures. In table I we display the Neville tables for the series for the same values of βH_1 .

TABLE I

Neville tables for the Y series at $\bar{\mu}_4 = 0$.									
$\beta H_1 = 5$					$\beta H_1 = 2$				
$n \backslash m$	0	1	2	3	$n \backslash m$	0	1	2	3
6	4.7739				6	4.7362			
7	4.7391	4.5307			7	4.7018	4.4949		
8	4.7123	4.5241	4.5045		8	4.6751	4.4889	4.4707	
9	4.6915	4.5251	4.5283	4.5760	9	4.6546	4.4899	4.4934	4.5389
10	4.6747	4.5241	4.5203	4.5015	10	4.6380	4.4889	4.4848	4.4646
$\beta H_1 = 0$					$\beta H_1 = -0.25$				
$n \backslash m$	0	1	2	3	$n \backslash m$	0	1	2	3
6	3.3872				6	2.8931			
7	3.3633	3.2195			7	2.8732	2.7540		
8	3.3459	3.2243	3.2386		8	2.8593	2.7618	2.7851	
9	3.3328	3.2278	3.2404	3.2440	9	2.8490	2.7662	2.7817	2.7749
10	3.3221	3.2261	3.2190	3.1690	10	2.8405	2.7646	2.7582	2.7033
$\beta H_1 = -0.50$					$\beta H_1 = -0.75$				
$n \backslash m$	0	1	2	3	$n \backslash m$	0	1	2	3
6	2.3724				6	1.8824			
7	2.3570	2.2642			7	1.8717	1.8074		
8	2.3470	2.2774	2.3170		8	1.8661	1.8268	1.8848	
9	2.3398	2.2819	2.2975	2.2583	9	1.8620	1.8290	1.8369	1.7411
10	2.3339	2.2807	2.2761	2.2264	10	1.8587	1.8299	1.8333	1.8251
$\beta H_1 = -1.00$					$\beta H_1 = -1.05$				
$n \backslash m$	0	1	2	3	$n \backslash m$	0	1	2	3
6	1.4626				6	1.3884			
7	1.4588	1.4365			7	1.3867	1.3767		
8	1.4568	1.4428	1.4616		8	1.3851	1.3740	1.3659	
9	1.4555	1.4447	1.4515	1.4312	9	1.3843	1.3777	1.3908	1.4405
10	1.4549	1.4500	1.4711	1.5169	10	1.3842	1.3836	1.4073	1.4459

The first column ($m = 0$) gives the ratios of successive coefficients; the other three columns give estimates of T_c . For $m = 1$ we obtain the linear intercepts, or the ratio-method estimates of the critical temperature. We suppose the last estimate ($n = 10, m = 1$) to be the best conjecture for T_c .

Fig. 1. Ratio plots for the Y series of the Takagi model on the f.c.c. lattice for $\bar{\mu}_4 = 0$ and several values of βH_1 . The scale of the ordinates is only valid for $\beta H_1 = 5$ and 2. The other curves were shifted upwards with the indicated value of Δ . The slopes of these curves become horizontal as γ approaches 1.0.

With these estimates of the critical temperature, we can determine the exponent γ . The exponent estimates for three values of T_c are given in table II. The middle value of kT_c/J is the best estimate, the other two are marginal values. By inspection of table II one can estimate the errors on the exponents.

We have also determined the exponents with the confluent singularity method of Saul^{13,19}). We used the temperature biased equations to determine the exponents

and amplitudes. The results of this method are displayed in table III, for several values of $\beta H_1 \leq 0$. The estimates of the exponents agree very well with those obtained by the ratio method.

TABLE II

Exponent γ estimates from the ratio method, biased by three different values of K_c^{-1} . The central columns correspond to our best estimate for K_c^{-1} . The two other columns give the exponent estimates for marginal values of K_c^{-1} .

$\beta H_1 = 5$				$\beta H_1 = 2$			
n	kT_c/J			n	kT_c/J		
	4.520	4.524	4.528		4.484	4.489	4.494
7	1.3394	1.3329	1.3264	7	1.3399	1.3318	1.3236
8	1.3403	1.3329	1.3255	8	1.3410	1.3317	1.3225
9	1.3414	1.3331	1.3249	9	1.3423	1.3319	1.3215
10	1.3423	1.3332	1.3240	10	1.3434	1.3319	1.3204
$\beta H_1 = 0$				$\beta H_1 = -0.25$			
n	kT_c/J			n	kT_c/J		
	3.220	3.225	3.230		2.760	2.766	2.772
7	1.3115	1.3001	1.2888	7	1.2872	1.2714	1.2556
8	1.3128	1.2999	1.2870	8	1.2878	1.2698	1.2519
9	1.3152	1.3008	1.2864	9	1.2901	1.2699	1.2498
10	1.3171	1.3011	1.2852	10	1.2917	1.2694	1.2472
$\beta H_1 = -0.50$				$\beta H_1 = -0.75$			
n	kT_c/J			n	kT_c/J		
	2.272	2.282	2.292		1.820	1.830	1.840
7	1.2618	1.2300	1.1984	7	1.1988	1.1594	1.1205
8	1.2642	1.2279	1.1920	8	1.2025	1.1577	1.1133
9	1.2685	1.2279	1.1876	9	1.2075	1.1571	1.1074
10	1.2723	1.2273	1.1827	10	1.2129	1.1571	1.1019
$\beta H_1 = -1.00$				$\beta H_1 = -1.05$			
n	kT_c/J			n	kT_c/J		
	1.438	1.448	1.458		1.372	1.382	1.392
7	1.1014	1.0523	1.0040	7	1.0750	1.0238	0.9734
8	1.1047	1.0487	0.9935	8	1.0765	1.0180	0.9604
9	1.1094	1.0465	0.9844	9	1.0807	1.0149	0.9502
10	1.1177	1.0478	0.9789	10	1.0891	1.0161	0.9442

TABLE III

Exponent γ estimates with the temperature-biased confluent singularity method of Saul^{13,19}).

βH_1 K_c^{-1} n	0	-0.50	-0.75	-1.00	-1.05	-1.10
	3.226	2.281	1.830	1.450	1.384	1.320
5	1.2981	1.2359	1.1678	1.0352	1.0117	0.9906
6	1.2970	1.2307	1.1570	1.0271	4.5757	0.9887
7	1.2976	1.2319	1.1567	1.0114	0.9879	0.9702
8	1.2979	1.2318	1.1571	1.0340	1.0016	0.9766

For $\beta H_1 > 2$ the exponent estimates are fairly constant and close to $\frac{4}{3}$. For smaller values of βH_1 the exponent decreases and reaches a value of approximately 1 for $\beta H_1 = -1.05$. This variation of the exponent with βH_1 is in disagreement with universality, but can be understood from the scaling hypothesis for a series of finite length (the crossover). Since we do not dispose of low-temperature series for the Takagi model, the exact determination of the tricritical point is extremely difficult. The fact, however, that the exponent γ goes through the classical value 1 in the neighbourhood of $\beta H_1 = -1.05$, strongly suggests that the tricritical point for $\bar{\mu}_4 = 0$ will indeed be given by approximately this value of βH_1 . Our estimate for the tricritical point from this criterion is then $\beta H_1 = -1.05 \pm 0.05$ and $T = 1.38 \pm 0.07$.

The amplitudes, as estimated by the ratio method and the confluent singularity method of Saul, decrease with decreasing βH_1 , but do not become zero at the tricritical point determined above. This might be due to the presence of a confluent singularity, which we could not determine, probably because our series is too short.

3.2. Padé approximant method

The Padé approximants to the logarithmic derivative of the series give estimates of the critical temperature and the exponent. These are listed in table IV. The results are less consistent than those obtained from the ratio method. We see that the exponent γ first increases from 1.33 to 1.40 (for $\beta H_1 = -0.5$). For lower values of βH_1 the exponent decreases, and becomes smaller than 1 for $\beta H_1 = -1.05$.

The confluent singularity method of Baker and Hunter²⁰), biased with the critical temperatures determined from the ratio method, gives estimates of γ which are consistent with the ratio method results (see table V).

To obtain better estimates of T_c , we have raised the series to several powers $1/\gamma$, with $\gamma = \frac{4}{3}$, $\gamma_{RM} \pm 0.01$ and γ_{RM} (γ_{RM} is the best ratio-method estimate of γ). These estimates of the critical temperatures agree very well with the ratio-method

TABLE IV

Critical temperature and exponent γ obtained from the Padé approximants to the logarithmic derivative of the Y series. The upper numbers give kT_c/J , the lower numbers give the exponent.

$\beta H_1 = 5$				$\beta H_1 = 2$			
$D \backslash N$	4	5	6	$D \backslash N$	4	5	6
3	4.532 1.318	4.524 1.337	4.522 1.342	3	4.497 1.318	4.488 1.337	4.486 1.342
4	4.518 1.353	4.521 1.345		4	4.482 1.353	4.485 1.335	
5	4.522 1.341			5	4.487 1.341		
$\beta H_1 = 0$				$\beta H_1 = -0.25$			
$D \backslash N$	4	5	6	$D \backslash N$	4	5	6
3	4.505 0.003	3.209 1.375	3.215 1.358	3	2.713 1.540	2.746 1.389	2.749 1.374
4	3.209 1.376	3.211 1.370		4	2.740 1.416	2.749 1.377	
5	3.218 1.344			5	2.753 1.355		
$\beta H_1 = -0.50$				$\beta H_1 = -0.75$			
$D \backslash N$	4	5	6	$D \backslash N$	4	5	6
3	3.232 0.028	2.262 1.397	2.264 1.383	3	1.830 1.225	1.815 1.357	1.817 1.339
4	2.256 1.439	2.264 1.387		4	1.813 1.380	1.817 1.333	
5	2.266 1.367			5	1.820 1.309		
$\beta H_1 = -1.00$				$\beta H_1 = -1.05$			
$D \backslash N$	4	5	6	$D \backslash N$	4	5	6
3	1.421 1.534	1.447 1.086	1.459 0.942	3	1.350 1.688	1.383 0.999	1.397 0.834
4	1.451 1.025	1.470 0.822		4	1.388 0.928	1.414 0.675	
5	1.102 -1.217			5	1.405 0.756		

TABLE V

Exponent γ estimates obtained from the confluent singularity method of Baker and Hunter²⁰). This method gives estimates biased by K_c^{-1} , for which we took the ($n = 10$, $m = 1$) values of the Neville tables in table I.

$\beta H_1 = 5$					$\beta H_1 = 0$				
$D \backslash N$	2	3	4	5	$D \backslash N$	2	3	4	5
2	1.287	1.119			2	1.290	1.291		
3	1.336	1.331	1.316		3	1.291	1.290	1.299	
4	1.331	1.338	1.329	1.338	4	1.365	1.299	1.300	1.299
5		1.329	1.332	1.331	5		1.300	1.299	1.301
6			1.331		6			1.301	
$\beta H_1 = -0.50$					$\beta H_1 = -0.75$				
$D \backslash N$	2	3	4	5	$D \backslash N$	2	3	4	5
2	1.223	1.240			2	1.147	1.184		
3	1.241	1.213	1.244		3	1.184	1.188	1.184	
4	1.295	1.244	1.232	1.233	4	1.188	1.184	1.160	1.154
5		1.213	1.233	1.232	5		1.122	1.154	1.158
6			1.225		6			1.176	
$\beta H_1 = -1.00$					$\beta H_1 = -1.05$				
$D \backslash N$	2	3	4	5	$D \backslash N$	2	3	4	5
2	1.032	1.104			2	1.005	1.085		
3	1.106	1.052	1.000		3	1.087	1.024	0.997	
4	0.948	1.002	1.031	1.009	4	0.879	0.997	1.031	0.968
5		*	1.010	1.029	5		0.934	0.970	1.020
6			*		6			0.930	

estimates, as can be seen from table VI, where we display the results for $\gamma = \frac{4}{3}$ and γ_{RM} , and for three values of βH_1 (0, -0.5 and -1).

The amplitudes determined with the PA method agree with the ratio-method estimates.

3.3. Phase diagram

The consistency of the results derived by both methods, RM and PA, gives us a firm belief in their correctness. In fig. 2 we have drawn the critical temperature as

TABLE VI

Estimates of the critical temperatures from the poles of the Padé approximants to the series raised to the power γ^{-1} . The left-hand part of the table gives the estimates for $\gamma = \frac{4}{3}$, the right-hand part is for $\gamma = 1.30, 1.23$ and 1.02 for $\beta H_1 = 0, -0.5$ and -1 , respectively.

$\beta H_1 = 0$									
$D \backslash N$	3	4	5	6	$D \backslash N$	3	4	5	6
3	3.2263	3.2238	3.2207	3.2205	3	3.2432	3.2392	3.2287	3.2283
4	3.2190	3.2203	3.2203		4	3.2261	3.2277	3.2269	
5	3.2204	3.2203			5	3.2278	3.2272		
6	3.2203				6	3.2269			

$\beta H_1 = -0.5$									
$D \backslash N$	3	4	5	6	$D \backslash N$	3	4	5	6
3	3.7652	2.2836	2.2710	2.2709	3	3.2074	2.3106	2.2860	3.4445
4	2.2662	2.2702	2.2706		4	2.2800	2.2846	2.2838	
5	2.2708	2.2706			5	2.2851	2.2840		
6	2.2706				6	2.2831			

$\beta H_1 = -1$									
$D \backslash N$	3	4	5	6	$D \backslash N$	3	4	5	6
3		1.4321	1.4324	1.4756	3	1.4604	1.4531	1.4510	1.4523
4	1.7933	1.4325	1.4320		4	1.5810	1.4518	1.4525	
5	1.4328	1.4336			5	1.4527	1.4532		
6	1.4339				6	1.4532			

function of the difference in chemical potentials. The critical temperature decreases with decreasing H_1 . When H_1 is approximately equal to $-1.45J$, the curve starts hooking backwards. If we apply the criterion that $\gamma_t = 1$ to determine T_t , we find $(H_1/J)_t = -1.45 \pm 0.02$ and $T_t = 1.38 \pm 0.07$.

In the upper right corner of fig. 2 a detail of the curve in the neighbourhood of the tricritical point is given. The dashed curve is $T_c(H_1)$ as determined from the PA to the Y series, raised to the power $\frac{4}{3}$, while the full curve gives the results from the unbiased RM and PA analyses. The slope of the curve at the tricritical point is probably infinite, in agreement with the mean-field calculations¹¹).

Fig. 2. Critical temperature for the Takagi model on the f.c.c. lattice plotted as function of H_1 . Detail in inset: dashed curve from PA to the Y series raised to the power 0.75; full curve from unbiased RM and PA method.

4. Other series

The specific heat can be determined from the free energy as

$$C_{\beta H_1, \beta H_2} = \frac{K^2}{kN} \frac{\partial^2}{\partial K^2} \log Z. \quad (5)$$

One can also obtain ϱ , the density of the mixture, x_3 , the concentration of ^3He and χ_3 , the concentration susceptibility. We may write

$$\varrho = \langle (S^z)^2 \rangle, \quad (6)$$

$$x_3 = \frac{1}{2} (1 - \langle S^z \rangle / \langle (S^z)^2 \rangle) \quad (7)$$

and

$$\chi_3 = \frac{\partial x_3}{\partial \beta H_1}, \quad (8)$$

where

$$\langle S^z \rangle = \frac{1}{N} \frac{\partial}{\partial \beta H_1} \log Z \quad (9)$$

and

$$\langle (S^z)^2 \rangle = \frac{1}{N} \frac{\partial}{\partial \beta H_2} \log Z. \quad (10)$$

In this way we can derive from $\log Z$ the series for the above-mentioned physical quantities. We have analysed the specific heat, $\partial x_3 / \partial K$ and χ_3 .

The ratio method, applied to these series, gives estimates of the critical temperature that are not consistent; these estimates are in general different from the estimates obtained from the Y series. As an illustration we give the ratio plots for the

Fig. 3. Ratio plots for the specific-heat series of the Takagi model for $\mu_4 = 0$. Curves are shifted upwards with value of Δ .

specific-heat series for some values of βH_1 in fig. 3. The PA to the logarithmic derivative do not have consistent poles and residues either. Therefore we determined the exponents in the following way: we raise the series (or the n th derivative of the series, with $n = 1$ or 2) to several powers α^{-1} [or $(\alpha + n)^{-1}$] and we calculate the poles and residues of the PA to the new series. This exponent, for which the inverse of the smallest positive pole matches best with the critical temperature as determined from the Y series, is taken to be the best exponent estimate. The amplitude of the singularity is obtained from the residues in these poles.

We suppose that the singular forms of specific heat, concentration and susceptibility are

$$C_{\beta H_1, \beta H_2} = A (1 - K/K_c)^{-\alpha}, \quad (11)$$

$$x_3 = x_{3r} + B (1 - K/K_c)^\omega, \quad (12)$$

$$\chi_3 = C (1 - K/K_c)^{-\lambda}. \quad (13)$$

The results of this analysis are given in table VII and (for the exponents only) in fig. 4. The specific-heat exponent is around zero for high values of βH_1 , allowing for a logarithmic divergence of this quantity. The α exponent estimate increases with decreasing βH_1 , and becomes 0.7 for $\beta H_1 = -1.05$. Comparing these estimates with the estimates for the exponents that Hunter and Baker¹⁸⁾ obtained for a singularity with a logarithmic factor, we conclude that the specific-heat exponent α might be $\frac{1}{2}$ with a logarithmic factor included in the singular form. At the tricritical point we find for $\omega = 0.45 \pm 0.1$ and $\lambda = 0.44 \pm 0.05$. The tricritical exponents are therefore in fair agreement with the gaussian fixed-point exponents, with the possibility of a logarithmic correction to α . The amplitudes of the series are not accurate enough to determine the crossover exponent.

TABLE VII

Exponents and amplitudes for specific heat, concentration of ^3He and concentration susceptibilities. The notation corresponds to the formulas (11)–(13).

βH_1	α	A	ω	B	λ	C
5	0.12 ± 0.02	25 ± 2	0.88 ± 0.02	0.0001	—	—
0	0.15 ± 0.05	15 ± 1	0.88 ± 0.02	1.6 ± 0.4	—	—
-0.2	0.15 ± 0.05	14 ± 1	0.8 ± 0.1	2.5 ± 0.5	0.2 ± 0.1	0.2 ± 0.1
-0.4	0.175 ± 0.03	10 ± 1	0.85 ± 0.05	3 ± 1	0.15 ± 0.05	0.4 ± 0.1
-0.6	0.25 ± 0.05	5 ± 1	0.82 ± 0.02	2.2 ± 0.2	0.16 ± 0.02	0.65 ± 0.05
-0.8	0.37 ± 0.02	2.0 ± 0.2	0.60 ± 0.05	0.9 ± 0.1	0.28 ± 0.02	0.34 ± 0.04
-1.0	0.65 ± 0.05	0.5 ± 0.1	0.5 ± 0.1	0.6 ± 0.1	0.41 ± 0.02	0.18 ± 0.02
-1.05	0.70 ± 0.05	0.37 ± 0.05	0.45 ± 0.1	0.6 ± 0.1	0.44 ± 0.02	0.15 ± 0.02
-1.1	0.75 ± 0.05	0.28 ± 0.02	0.42 ± 0.05	0.5 ± 0.1	0.48 ± 0.02	0.12 ± 0.02

Fig. 4. Variation of the exponents α , γ , λ and ω with βH_1 for $\bar{\mu}_4 = 0$. The horizontal lines correspond to the mean-field values $\gamma = 1$, $\alpha = \lambda = \omega = 0.5$.

Fig. 5. Variation with the concentration x_3 of the reduced critical temperature $T_c(x)/T_c(0)$ for several values of the density ρ . The circles correspond to the values for $\rho = 0.65$ in the analysis of Plischke and Betts¹⁵) of the Cheng-Schick model, and seem to coincide with the experimental values for $^3\text{He}-^4\text{He}$.

By elimination of H_1 and H_2 between the fluctuation series $Y(\beta, H_1, H_2)$ and the series for the density $\varrho(\beta, H_1, H_2)$ and the concentration $x_3(\beta, H_1, H_2)$, we derived the series $Y(\beta, \varrho, x_3)$ for several fixed values of ϱ and x_3 . From the Neville tables and the Padé approximants to these series, we determined the critical temperatures. The results are plotted in fig. 5 in such a way that they may be compared with the experimental results for the x_3 dependence of the reduced critical temperature $T_c(x_3)/T_c(0)$. Although there is a certain spread in the curves for different ϱ values, we cannot match the experimental values, which are reproduced in the analysis of the Cheng-Schick model¹⁵) at a value of $\varrho = 0.65$.

5. Conclusion

We have analysed the fluctuation of the order parameter, the specific heat, the concentration and the concentration-susceptibility high-temperature series for the Takagi model, which we derived in a previous paper. The critical exponents γ and α are found to be approximately $\frac{4}{3}$ and 0. The tricritical point was determined from the condition that γ_t should be 1. We then found that α_t , ω_t and λ_t were consistent with the gaussian fixed-point exponents $\frac{1}{2}$. The x_3 dependence of the critical temperature does not completely match the experimental results.

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