The Geometric Matrix Mean: an Adaptation for Structured Matrices

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Geometric matrix mean

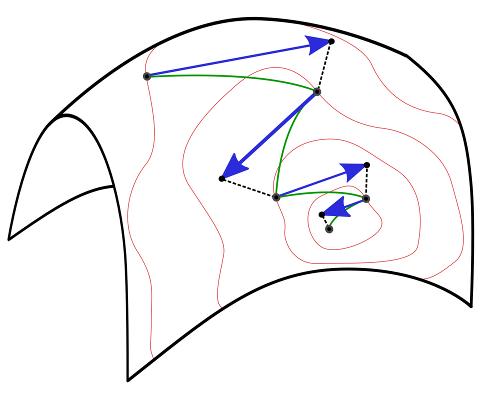
Barycenter for the following settings

- Domain: set of (symmetric) positive definite (PD) matrices $\mathcal{P}_n$;
- Distance measure: based on natural PD metric:

$$d(A,B) = \left\| \log \left(A^{-1/2} B A^{-1/2} \right) \right\|_F .$$

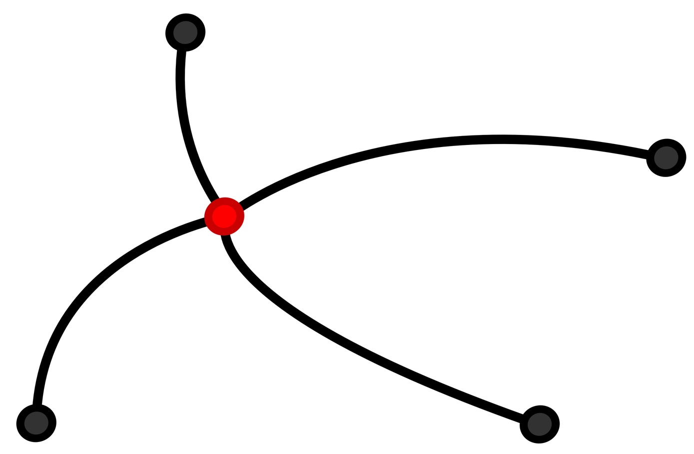
Computed using manifold optimization

- Gradient, Hessian, ... incorporate the Riemannian geometry of the set;
- Iterative steps are taken along the manifold.



Barycenter

Minimizer of the sum of squared distances to given elements A_i



Depends on

- Search space/Domain S: restriction to the desired matrix structure;
- Distance measure d: influences additional properties of the barycenter.

$$\Rightarrow B_S(A_1,\ldots,A_k) = \arg\min_{X\in S} \sum_{i=1}^{k} d^2(X,A_i)$$

Kähler metric mean

Barycenter for the following settings

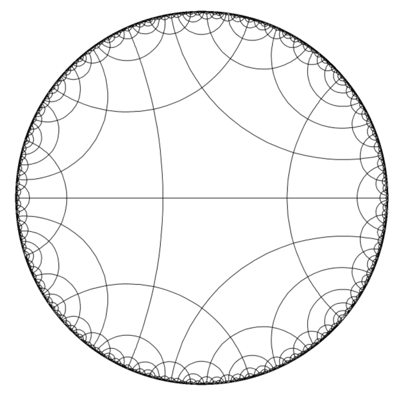
- Domain: set of (Hermitian) PD Toeplitz matrices $\mathcal{T}_n$;
- Distance measure: Kähler metric.

The metric is defined using an application-inspired transformation of the elements of $\mathcal{T}_n$:

$$T \in \mathcal{T}_n \to (P, \mu_1, \ldots, \mu_{n-1}) \in R_*^+ \times D^{n-1},$$

with D the open complex unit disk.

- R_*^+: the geometry of positive numbers;
- D: hyperbolic geometry (Poincaré disk).



Structured matrix adaptations

Why?

- Many applications use a matrix structure which is an intersection of $\mathcal{P}_n$ and an additional (vectorspace) structure;
- The geometric mean generally destroys additional structure.

How?

- Domain restriction: minimize over $S \subset \mathcal{P}_n$ instead of $\mathcal{P}_n$ itself.

Properties?

Unstructured

- *Permutation invariance:* A permutation of the elements A_i does not change the result;
- *Joint homogeneity:* for $\alpha_1,\ldots,\alpha_k > 0$,

$$A_i \to \alpha_i A_i$$
$$B_{\mathcal{P}_n} \to (\alpha_1 \cdots \alpha_k)^{1/k} B_{\mathcal{P}_n};$$

- *Inversion invariance:*

$$B_{\mathcal{P}_n}(A_1^{-1},\ldots,A_k^{-1}) = B_{\mathcal{P}_n}(A_1,\ldots,A_k)^{-1}.$$

Structured

- *Permutation invariance:* Still holds because of the definition of the cost function;
- *Joint homogeneity:* Remains valid if S is the intersection of $\mathcal{P}_n$ with a vectorspace;
- *Inversion invariance:* Define S^{-1} as the set of inverses of the nonsingular elements in S, then

$$B_{S^{-1}}(A_1^{-1},\ldots,A_k^{-1}) = B_S(A_1,\ldots,A_k)^{-1}.$$

Computation and properties

Transformation to product space $R_*^+ \times D^{n-1}$, optimization can be decoupled into n separate averaging operations.

- Barycenter of the coefficients P_i in R_*^+: exactly the (scalar) geometric mean $(P_1 \cdots P_k)^{1/k}$;
- Barycenter of the coefficients $\mu_{j,i}$ (separate for each j) in D:
  - Real matrices: $\mathcal{C}\left((\mathcal{C}(\mu_{j,1}) \cdots \mathcal{C}(\mu_{j,k}))^{1/k} \right)$, where $\mathcal{C}$ is the Cayley transform, $\mathcal{C}(z) = (1-z)/(1+z)$;
  - Complex matrices: no explicit formula, but fast scalar optimization algorithms can be constructed.

Properties?

- Barycenter properties (permutation invariance, repetition invariance, ...) hold by construction;
- Properties of the geometric mean which translate well through the transformation are preserved, e.g., joint homogeneity:

$$T \to \alpha T$$
$$(P, \mu_1, \ldots, \mu_{n-1}) \to (\alpha P, \mu_1, \ldots, \mu_{n-1})$$

- Many other geometric mean properties do not hold or are not well-defined because of the transformation, showing the clear distinction between the means;
- Useful because of its decoupling property and its close relation to the application via the transformation.

Two preconditioners

Suppose S is the intersection of a vectorspace $\mathcal{V}$ and $\mathcal{P}_n$, then parametrize $\mathcal{V}$ using $\sigma : \mathbb{R}^q \to \mathbb{R}^{n\times n}$, $q = \dim(\mathcal{V})$, such that $\mathrm{vec}(\sigma(t)) = Ut$. Denote by $\Gamma(X)$ the Euclidean gradient of the unstructured barycenter cost function $f_{B_{\mathcal{P}_n}}$ at $X \in \mathcal{P}_n$.
From this, the gradient for the structured barycenter cost function f_{B_S} can be obtained in both the Euclidean and the Riemannian geometry.

- Euclidean geometry:

$$\left(U^T U\right)^{-1} U^T \,\mathrm{vec}(\Gamma(\sigma(t)));$$

- Riemannian geometry:

$$\left(U^T \left(\sigma(t)^{-1} \otimes \sigma(t)^{-1}\right) U\right)^{-1} U^T \,\mathrm{vec}(\Gamma(\sigma(t))).$$

The pre-multiplied terms in the expressions can be interpreted as preconditioners to the Euclidean gradient. Using the geometric information of $\mathcal{P}_n$, a more effective gradient and more efficient algorithms are obtained.

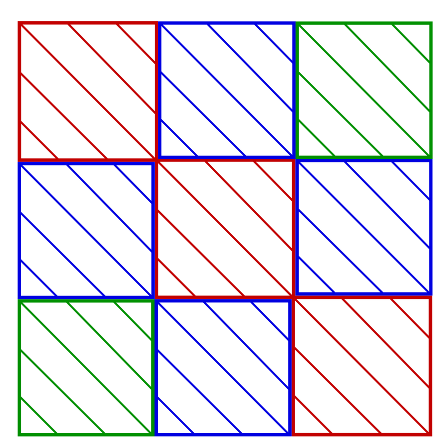
TBBT matrices

Generalize the theory of the Kähler mean to the set of PD Toeplitz-block Block-Toeplitz (TBBT) matrices $\mathcal{B}_{p,n}$ (an $n \times n$ block structure of $p \times p$ matrix blocks):

- Adaptation of the transformation to the block structure:

$$T \in \mathcal{B}_{p,n} \to (P, U_1, \ldots, U_{n-1}) \in \mathcal{P}_p \times \mathcal{D}_p^{n-1},$$

where $\mathcal{D}_p$ is a generalization of D;
- The computational advantage of decoupling the separate coefficients can again be exploited to average large matrices.



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