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Real or nominal variables, does it matter for the impulse response?

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Real or nominal variables, does it matter for the impulse response?

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Abstract

This paper analyzes impulse response functions of vector autoregression models for variables that are linearly transformed. These impulse responses are equal to the linear transformation of the original impulse responses only if the shocks are equal to the linear transformation of the original shocks. Sufficient conditions are derived both for shocks in one error term only, orthogonalized shocks and generalized shocks. A vector autoregression model with inflation, the overnight target rate and a real interest rate that replaces the corresponding nominal interest rate, illustrates the applicability of our results for the empirical researcher.

Keywords: Impulse response, Linear transformation, Vector autoregression

JEL: C30, C32

1. Introduction

The impulse response function of a vector autoregression (VAR) model is an often used tool in macro-econometrics to analyze the response of the variables in the model to different types of shocks. For many applications, it is of interest to the empirical researcher to know how the impulse response functions would change if one or more variables in the VAR model are replaced by a linear transformation of the original variables. A first example of such a transformation is the replacement of a nominal growth rate variable in a VAR model that also includes inflation by its corresponding real growth rate variable, which is the difference between the nominal growth rate variable and inflation. A second example is the replacement of a variable in logs by its percentage of GDP counterpart in VAR models that also includes GDP in logs. A third example is the replacement of quarterly growth rates variables by their corresponding linear approximation of the annualized

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growth rates, which is four times the quarterly growth rate. A final example is the swapping of variables in the recursive ordering scheme for the identification of orthogonalized shocks.

The relationship between the impulse response functions of linearly transformed variables has been relevant for different economic studies. Using a recursive identification scheme in which inflation is ordered before house prices, Goodhart and Hofmann (2008) deduce the response of real house prices to different orthogonalized shocks as the difference between the response of the nominal house prices and the response of inflation. Similarly, using a recursive identification scheme in which money is ranked last, Baumeister *et al.* (2008) derive the response of the nominal asset prices to an orthogonalized shock in money as the sum of the response of the real asset prices and the response of inflation. The results in our paper will show that these relationships between impulse responses are valid for the above identification schemes, but invalid for some other recursive identification schemes. Next, for Israeli data, Kahn *et al.* (2002) separately analyze the response of nominal interest rates and real interest rates to a monetary policy shock using a VAR model that also includes expected inflation. As they assume a recursive identification scheme with expected inflation ranked before the interest rate, we will show that the response of each nominal interest rate should be the sum of the response of the corresponding real interest rate and the response of expected inflation. However, this expected relationship does not match with their reported impulse responses on pages 1504-1505. In contrast to Kahn *et al.* (2002), the responses of the real interest rate and nominal interest rate of Bhuiyan and Lucas (2007), who perform a similar analysis on Canadian data, exhibit the correct relationship.

As an illustrative example, we compare the impulse response functions for two estimated three-variable VAR models with lag length one, using the same monthly Canadian dataset for the period January 1994 until December 2002, as in Bhuiyan and Lucas (2007). The variables of the first VAR model (Model 1) are inflation (π_t), the overnight target rate (OT_t) and the one year *nominal* interest rate (N_t). The variables of the second VAR model (Model 2) are inflation, the overnight target rate and the one year *real* interest rate (R_t), which are linear transformations of the variables of Model 1.[1] First, Figure 1 shows the response of inflation and the interest rate to an orthogonalized shock in the overnight target rate, both for Model 1 and Model 2, where we assume the recursive ordering schemes (π_t, OT_t, N_t) for Model 1 and (π_t, OT_t, R_t) for Model 2. The figure shows that the response of π_t is the same for the two models and that the response of N_t is the sum of the

[1]Note that for this illustrative example, we use the actual inflation and the ex-post real interest rate instead of the expected inflation and the ex-ante real interest rate used by Bhuiyan and Lucas (2007).

responses of R_t and π_t, as expected. Second, Figure 2 shows the same responses but using the different recursive ordering schemes (N_t, OT_t, π_t) for Model 1 and (R_t, OT_t, π_t) for Model 2. In contrast to Figure 1, the response of π_t of the two models differs and the response of N_t is not equal to the sum of the response of R_t and any of the responses of π_t.

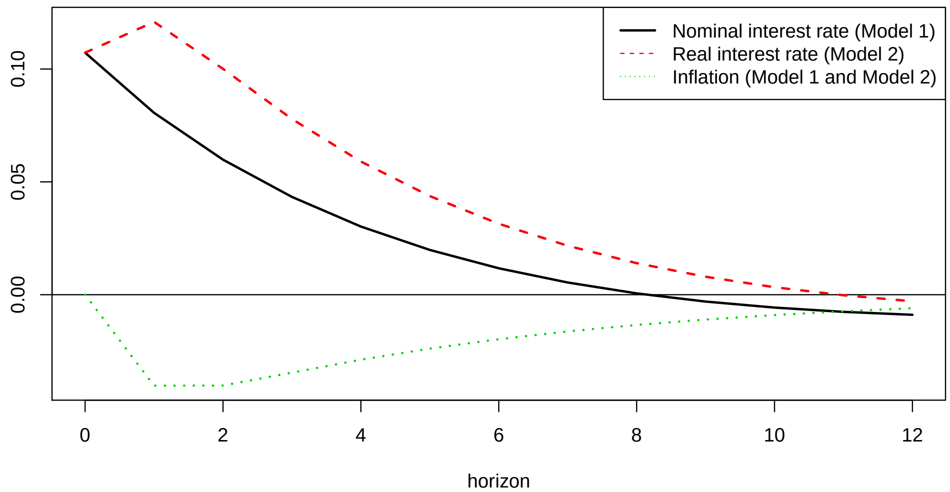

Figure 1: The response to an orthogonalized shock in the overnight target rate with recursive ordering scheme (π_t, OT_t, N_t) for Model 1 and (π_t, OT_t, R_t) for Model 2.

This paper analyzes the relationship between the impulse responses for the general case of linearly transformed variables. We name impulse responses to be economically equivalent if the impulse responses of the linearly transformed variables are equal to the linear transformation of the original impulse responses. As such, while the impulse responses of Model 1 (Figure 1) are economically equivalent, those of Model 2 (Figure 2) are not. Similarly, we denote shocks to be 'economically equivalent' if the shocks of the linearly transformed variables are respectively equal to the linear transformation of the original shocks. A first result of the paper is that impulse responses are economically equivalent if the shocks are economically equivalent. A second result is the derivation of easy to verify sufficient conditions for economic equivalence for three types of shocks that are commonly used by the empirical researcher. First, shocks in one error term only are discussed, which is a shock used by Lutkepohl (2005) among many others. Next, orthogonalized shocks in a recursive identification scheme are studied, which is proposed by Sims (1980) and

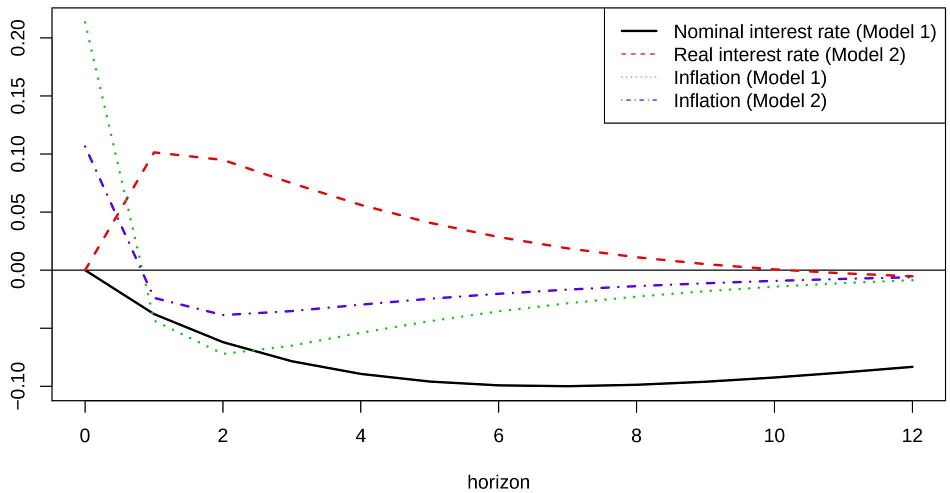

Figure 2: The response to an orthogonalized shock in the overnight target rate with recursive ordering scheme (N_t, OT_t, π_t) for Model 1 and (R_t, OT_t, π_t) for Model 2.

which is frequently interpreted as a structural shock to the economy. Finally, generalized shocks as proposed by Koop *et al.* (1996) and Pesaran and Shin (1998) are analyzed. For generalized shocks, the shock in the error term of the shocked variable is accompanied by shocks in the other error terms that are expected to occur based on the correlations of the error terms. Unlike the orthogonalized impulse response function, the generalized shock does not make any identification assumptions as their aim is not to identify the structural shocks.

This paper is organized as follows. Section 2 presents the vector autoregression model for both the original variables and the linearly transformed variables. Next, Section 3 discusses for different types of shocks the relationship between the shocks and impulse response functions of respectively the model with linear transformed variables and the model with the original variables. Section 4 then presents an example of a three variable VAR model with inflation, the overnight target rate and a real interest rate that replaces the corresponding nominal interest rate. Finally, Section 5 concludes our findings.

2. The vector autoregression model

Consider the vector autoregressive representation of the n-dimensional stationary time series y_t

$$\Phi(L)y_t = c + \epsilon_t \qquad \epsilon_t \sim N(0, \Sigma)$$

where $\Phi(L)$ is a lag polynomial of lag length p, c is a n-dimensional vector of constants and the n-dimensional vector ϵ_t are independent innovations with mean 0 and covariance matrix Σ. The moving average (MA) representation of y_t is then given by

$$y_t = \mu + \epsilon_t + \sum_{s=1}^{\infty} \Psi_s \epsilon_{t-s}$$

where μ is the mean of the stationary process and Ψ_s are absolute summable MA coefficients (Hamilton, 1994).

Next, we define the linearly transformed time series

$$y_t^* = Ay_t,$$

where A is assumed to be an invertible $n \times n$ matrix. The MA representation of y_t^* is given by

$$y_t^* = A\mu + \epsilon_t^* + \sum_{s=1}^{\infty} \Psi_s^* \epsilon_{t-s}^*,$$

with $\epsilon_t^* = A\epsilon_t \sim N(0, \Sigma^*)$ and

$$\Sigma^* = A\Sigma A^T \tag{1}$$

$$\Psi_s^* = A\Psi_s A^{-1}. \tag{2}$$

3. Impulse response function

Koop *et al.* (1996) and Pesaran and Shin (1998) provide a flexible framework to analyze the response of the variables to different types of shocks. In particular, the response at horizon s of y_t to a shock δ is given by

$$I_s = \Psi_s \delta \qquad s \geq 0, \tag{3}$$

where the l^{th} row of the n dimensional vector I_s represents the response of variable l to the shock δ, which is a $n \times 1$ vector representing the composition of the shock. Similarly, the response at horizon s of y_t^* to a shock δ^* is given by

$$I_s^* = \Psi_s^* \delta^* \qquad s \geq 0. \tag{4}$$

Definition 1. *Using the notation above, we define shocks δ and δ^* to be economically equivalent if*

$$\delta^* = A\delta. \tag{5}$$

Similarly, we define impulse responses I_s and I_s^* to be economically equivalent if

$$I_s^* = AI_s.$$

Proposition 1a then provides a sufficient condition for the economical equivalence of the impulse responses and Proposition 1b gives a necessary and sufficient condition for the economical equivalence both of the impulse responses and of the shocks.

Proposition 1. (a) *If δ^* and δ are economically equivalent, then the impulse responses I_s^* and I_s are also economically equivalent for each $s \geq 0$.*

(b) *For a given $s \geq 0$, if Ψ_s is invertible, then δ^* and δ are economically equivalent if and only if I_s^* and I_s are economically equivalent.*

We point out that at the sample level, the estimate of Ψ_s is invertible with probability one if the innovations follow a continuous distribution and that therefore, this invertibility condition of Proposition 1b can be safely assumed in most empirical work. Next, we emphasize that the composition of the shocks δ and δ^* is chosen by the empirical researcher. Sections 3.1, 3.2 and 3.3 present sufficient conditions for the economic equivalence for the three frequently used types of shocks which were discussed in Section 1: shocks in one error term only, orthogonalized shocks and generalized shocks. From Proposition 1a, it follows that these sufficient conditions for economic equivalence of the shocks are then also sufficient for economic equivalence of the impulse responses.

3.1. Shock in one error term only

Consider a one unit or a one standard deviation shocks in the j^{th} error term only. Then, the vectors δ and δ^* have δ_j and δ_j^* at the j^{th} position and zeros elsewhere, where δ_j and δ_j^* are either equal to one or equal to the standard deviation of the j^{th} error term.

Proposition 2. *The sufficient conditions for economic equivalence of 'shocks in one error term only' are:*

(a) *For a one unit shock in the error term of the j^{th} variable only: the j^{th} element of the j^{th} column of A is one and the other elements of the j^{th} column of A are zero.*

(b) *For a one standard deviation shock in the error term of the j^{th} variable only: the element of A at the j^{th} row and j^{th} column is greater than zero, all other elements of the j^{th} column of A are zero and all other elements of the j^{th} row of A are zero.*

3.2. Orthogonalized shock

For orthogonalized shocks, we assume that the recursive ordering of the variables is given by the ordering of y_t. First, consider the one standard deviation orthogonalized shocks to variable j. The vectors δ and δ^* are then given by the j^{th} column of respectively P and P^* of the Cholesky decompositions

$$\Sigma = PP^T \text{ and } \Sigma^* = P^* P^{*T}, \tag{6}$$

where P and P^* are lower triangular matrices.

Proposition 3. *A sufficient condition for economic equivalence of 'one standard deviation orthogonalized shocks' in the variable j is that either*

(a) *A is lower triangular or*

(b) *there exist indices i and k both smaller than j or both larger than j such that A^B is lower triangular, where A^B is obtained from A by swapping the i^{th} and k^{th} rows and the i^{th} and k^{th} columns.*

Second, consider the 'one unit orthogonalized' shocks δ and δ^*, which are given by the j^{th} column of respectively L and L^* of the decompositions

$$\Sigma = LDL^T \text{ and } \Sigma^* = L^* D^* L^{*T}, \tag{7}$$

where L and L^* are lower unitriangular matrices and D and D^* are diagonal matrices.

Proposition 4. *A sufficient condition for economic equivalence of 'one unit deviation orthogonalized shocks' is that either*

(a) *A is lower triangular with the j^{th} diagonal element equal to 1 or*

(b) *there exist indices i and k both smaller than j or both larger than j, such that A^B is lower triangular with the j^{th} diagonal element equal to 1, where A^B is obtained from A by swapping the i^{th} and k^{th} rows and the i^{th} and k^{th} columns.*

Note that Propositions 3 and 4 are very similar, with the only difference that the latter additionally requires that the j^{th} diagonal element of A, or of its swapped version A^B, to be equal to one. Condition (b) of Propositions 3 and 4 can be generalized to the swapping of multiple rows and corresponding columns, provided that the indices of each swap are either both smaller than j or both larger than j.

One important application of Propositions 3 and 4 is the detection of different recursive ordering schemes that generate the same orthogonalized shocks to the j^{th} variable, where j is a given number. From the perspective of the empirical researcher, the distinction between these identification schemes is then irrelevant and this knowledge allows the researcher to focus his attention on the economic validation of the relevant ordering assumptions. Let us consider an identification assumption where the ordering of the variables i and k is swapped, with indices i and k either both smaller than j or both larger than j. This corresponds to a transformation matrix A that is equal to the identity matrix with both rows i and k and columns i and k swapped. As A satisfies condition (b) of Propositions 3 and 4, the shocks are economically equivalent, which in this case means that δ^* is equal to δ with swapped rows i and k. This implies that the only relevant identification assumption is whether each non-shocked variable is ordered before or after the shocked variable. Hence, out of the $n!$ theoretically possible identification schemes, only 2^{n-1} are relevantly different. However, for the empirical researcher who analyzes multiple orthogonalized shocks to different variables, the number of relevantly different identification schemes becomes larger.

3.3. Generalized shock

As in Pesaran and Shin (1998), the one standard deviation generalized shocks in the j^{th} variable are given by the vectors

$$\delta = \frac{\Sigma_{.j}}{\sqrt{\Sigma_{jj}}} \text{ and } \delta^* = \frac{\Sigma^*_{.j}}{\sqrt{\Sigma^*_{jj}}}, \tag{8}$$

where the subscript '$.j$' denotes the j^{th} column of the matrix and the subscript 'jj' denotes the element of the j^{th} row and j^{th} column of the matrix. We obtain the following result.

Proposition 5. *A sufficient condition for economic equivalence of a 'one standard deviation generalized shock' is that the j^{th} row of A contains zeros for all except the j^{th} element and that the j^{th} element is greater than zero.*

The one unit generalized shocks in the j^{th} variable are defined as

$$\delta = \frac{\Sigma_{.j}}{\Sigma_{jj}} \text{ and } \delta^* = \frac{\Sigma^*_{.j}}{\Sigma^*_{jj}}.$$

We obtain the following result.

Proposition 6. *A sufficient condition for economic equivalence of a 'one unit generalized shock' is that the j^{th} row of A contains the value one at the j^{th} position and zeros elsewhere.*

Note that the conditions in Propositions 5 and 6 are very similar, with the only difference that the latter additionally requires the j^{th} diagonal element to be equal to 1.

4. Example: Real interest versus nominal interest rate in three variable VAR

This section applies the propositions in Section 3 to a three variable VAR example with inflation (π_t), the overnight target rate (OT_t) and a nominal interest rate (N_t). The transformed time series replaces N_t by a real interest rate (R_t). This continues the illustrative example given in Section 1. The vectors y_t and y_t^* are given by

$$y_t^T = \begin{pmatrix} \pi_t & OT_t & N_t \end{pmatrix} \text{ and } y_t^{*T} = \begin{pmatrix} \pi_t & OT_t & R_t \end{pmatrix}$$

with $R_t = N_t - \pi_t$. Hence, the linear transformation matrix A is given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

We apply the results of Section 3 to verify whether the shocks δ and δ^* are economically equivalent, and hence also whether the impulse responses of y_t and y_t^* are economically equivalent.

4.1. Shock in one error term only

Both for shocks in the overnight target rate ($j = 2$) and shocks in the interest rate ($j = 3$), the one unit shocks are economically equivalent since the conditions of Proposition 2a are then satisfied. For shocks in the overnight target rate ($j = 2$), also the one standard deviation shocks are economically equivalent, see Proposition 2b. For these shocks, both the response of π_t and OT_t will be identical in the model with linearly transformed variables and the original model, while the response of R_t will be equal to the difference between the response of N_t and the response of π_t.

4.2. Orthogonalized shock

For different recursive ordering schemes, we analyze orthogonalized shocks to each of the variables of this three variable example. The recursive ordering scheme for the model with the linearly transformed variables is the same as the ordering scheme of the original variables, but with R_t evidently replacing N_t. Although we discuss all six theoretically possible recursive orderings for this three variable example, we do not make a stance on the appropriateness of these choices. It is up to the empirical researcher to determine which recursive identification assumption, if any, is justified based on economic theory.

First, let us consider orthogonalized shocks with the recursive ordering scheme (π_t, OT_t, N_t). Note that both Kahn *et al.* (2002) and Bhuiyan and Lucas (2007) use this recursive identification assumption for identifying the shock in the overnight target rate. Since A is lower triangular with diagonal elements equal to one, condition (a) of Propositions 3 and 4 is satisfied for orthogonalized shocks in each of the variables. Hence, both the one unit shocks and the one standard deviation shocks will be economically equivalent Therefore, for an orthogonalized shock in each of the variables, both the response of π_t and OT_t will be identical in the model with linearly transformed variables and the original model and the response of R_t will equal the difference between the response of N_t and the response of π_t. Similarly, for recursive ordering schemes (π_t, N_t, OT_t) and (OT_t, π_t, N_t), the shocks are economically equivalent because the ranking of the interest rate after the inflation makes that the corresponding A matrix is a lower triangular matrix with diagonal elements equal to one. Second, let us consider the recursive ordering schemes (N_t, π_t, OT_t) and (OT_t, N_t, π_t). The conditions of Propositions 3 and 4 are not satisfied for shocks in the interest rate and shocks in inflation. However, both the one unit and one standard deviation shocks in the overnight target rate are economically equivalent since condition (b) of Propositions 3 and 4 is satisfied. Finally, for the recursive ordering scheme (N_t, OT_t, π_t), the conditions of Propositions 3 and 4 are not satisfied for any orthogonalized shock.

4.3. Generalized shock

The conditions in Propositions 5 and 6 are satisfied for generalized shocks in inflation and the overnight target rate. Therefore, for a generalized shock in each of the variables, both the response of π_t and OT_t will be identical in the model with linearly transformed variables and the original model and the response of R_t will equal the difference between the response of N_t and the response of π_t. In contrast, the conditions in Propositions 5 and 6 are not satisfied for generalized shocks in the interest rate.

5. Conclusion

Impulse responses functions are sometimes mechanically computed using a standard implementation of an econometric software package and the empirical researcher is not always aware of the exact composition of the shock he imposes. This evidently is not a good practise as different shocks generate different impulse responses. For linearly transformed variables, this paper can help to assess the change in the composition of the shock and the change in the corresponding impulse responses. A frequently encountered example of such a linear transformation is the replacement of a nominal growth rate variable in a VAR model that also includes inflation by its corresponding real growth rate variable, which is the difference between the nominal growth rate variable and inflation. We show that the intuitively expected result that the response of the real variable equals the difference between the response of the corresponding nominal variable and the response of inflation, only holds for specific settings.

For three types of shocks that are commonly used by the empirical researcher, this paper provides easy to check sufficient conditions for economic equivalence of shocks and impulse responses, by which we mean that the shocks and impulse response functions of the linearly transformed variables are equal to and the linear transformation of the original shocks and impulse responses, respectively. First, for a one standard deviation shock in the error term of the j^{th} variable only, a sufficient condition for economic equivalence is that the j^{th} variable cannot appear as a component of the linear combination of other variables and that no other variables can be part of the linear combination of the j^{th} variable. Second, for a one standard deviation orthogonalized shock to variable j, a sufficient condition for economic equivalence is that the ordering of the recursive identification scheme is such that each linearly transformed variable i only contains variables k with $i \geq k$, $i, k < j$ or $i, k > j$. Third, for a one standard deviation generalized shock in the j^{th} variable, a sufficient condition for economic equivalence is that the linear transformation of the shocked variable cannot contain other variables.

We have showed the applicability of these sufficient conditions to the empirical researcher. First, our sufficient condition for economic equivalence of orthogonalized shocks provides a tool to easily detect different recursive identification ordering schemes that generate the same shocks and impulse response functions. Second, we illustrate the economic equivalence of shocks and impulse responses for a vector autoregression model with inflation, the overnight target rate and a real interest rate that replaces the corresponding nominal interest rate.

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References

Baumeister, C., Durinck, E., and Peersman, G. (2008). Liquidity, inflation and asset prices in a timevarying framework for the euro area. NBB Working Papers. No. 142.

Bhuiyan, R. and Lucas, R. F. (2007). Real and nominal effects of monetary policy shocks. *Canadian Journal of Economics*, **40**(2), 679–702.

Goodhart, C. and Hofmann, B. (2008). House prices, money, credit, and the macroeconomy. *Oxford Review of Economic Policy*, **24**(1), 180–205.

Hamilton, J. (1994). *Time series analysis*. Princeton university press.

Kahn, M., Kandel, S., and Sarig, O. (2002). Real and nominal effects of central bank monetary policy. *Journal of Monetary Economics*, **49**(8), 1493–1519.

Koop, G., Pesaran, M. H., and Potter, S. M. (1996). Impulse response analysis in nonlinear multivariate models. *Journal of Econometrics*, **74**(1), 119–147.

Lutkepohl, H. (2005). *New introduction to multiple time series analysis*. Springer.

Pesaran, H. H. and Shin, Y. (1998). Generalized impulse response analysis in linear multivariate models. *Economics Letters*, **58**(1), 17–29.

Sims, C. A. (1980). Macroeconomics and reality. *Econometrica*, **48**(1), 1–48.

Appendix

Proof of Proposition 1

(a). Let us assume that δ^* and δ are economically equivalent, i.e. $\delta^* = A\delta$. Then, it follows from equations (2), (3) and (4) that, for each s,

$$I_s^* = \Psi_s^* \delta^* = A\Psi_s A^{-1}\delta^* = A\Psi_s A^{-1}A\delta = AI_s,$$

which proves that the impulse responses I_s^* and I_s are economically equivalent. □

(b). Assume that for a given $s > 0$, that I_s^* and I_s are economically equivalent and that Ψ_s is invertible. Then, Ψ_s^* is also invertible since A is assumed to be invertible. It follows from equations (2), (3) and (4) that

$$\delta^* = \Psi_s^{*-1} I_s^* = (A\Psi_s A^{-1})^{-1} AI_s = A\Psi_s^{-1} I_s = A\delta,$$

which proves that the shocks δ^* and δ are economically equivalent. □

Proof of Proposition 2

Take, without loss of generalization, $j = 1$. The shocks can be written as

$$\delta = (\delta_1, 0 \dots 0)^T \text{ and } \delta^* = (\delta_1^*, 0 \dots 0)^T.$$

(a). For a one unit shock, both δ_1 and δ_1^* are set to one. Under the conditions of Proposition 2a, $A\delta$ simplifies to the first column of A, which equals δ^*. Hence, by definition (5), the shocks δ and δ^* are economically equivalent. □

(b). For a one standard deviation shock, δ_1 and δ_1^* are set the standard deviation of the first error terms, denoted by $\sqrt{\Sigma_{11}}$ and $\sqrt{\Sigma_{11}^*}$, respectively. We have that $A\delta$ is equal to $A_{.1}\sqrt{\Sigma_{11}}$. Let us assume that the conditions of Proposition 2b are satisfied for $j = 1$. Since $A_{1.}$ has zero elements except for the first position, it follows from 1 that $A_{11}\sqrt{\Sigma_{11}} = \sqrt{\Sigma_{11}^*}$ and hence $(A\delta)_1 = \delta_1^*$. Since $A_{.1}$ has zero elements except for the first position, $A\delta = \delta^*$. □

Proof of Proposition 3a

Let us assume that A is lower triangular. From equations (1) and (6), it follows that $\Sigma^* = A\Sigma A^T = APP^TA^T = (AP)(AP)^T$. As both A and P are lower triangular, AP is also lower triangular and therefore, AP is equal to the unique lower triangular matrix P^* of the Cholesky decomposition of Σ^*. Therefore, the shock δ^* is equal to

$$\delta^* = P_{.j}^* = (AP)_{.j} = A(P)_{.j} = A\delta,$$

where the subscript '$.j$' denotes the j^{th} column of the matrix. This proves that the shocks δ^* and δ are economically equivalent. □

Before proving Proposition 3b, an intermediate result is needed.

Lemma 1. *Let B be the identity matrix of dimension n with rows i and k swapped, with either $i, k < j$ or $i, k > j$ and where j is a given number with $1 \leq j \leq n$. Let P and P^B be the lower triangular matrices of the Cholesky decompositions of the $n \times n$ dimensional symmetric matrices Σ and $B\Sigma B^T$, respectively. Then,*

$$P_{.j}^B = BP_{.j}.$$

Proof of Lemma 1. Let j be a given number with $1 \leq j \leq n$. B can be partitioned as

$$B = \left(\begin{array}{c|c|c} B_1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & B_2 \end{array}\right), \tag{9}$$

where the scalar 1 is at position (j, j) and B_1 and B_2 are square matrices respectively of size $j-1$ and $n-j$. Either B_1 or B_2 is an identity matrix and either B_2 or B_1 is an identity matrix where two rows are interchanged. Next, partition P in the same way as B:

$$P = \left(\begin{array}{c|c|c} P_1 & 0 & 0 \\ \hline P_2 & p & 0 \\ \hline P_3 & P_4 & P_5 \end{array}\right), \tag{10}$$

where p is a scalar and P_1 and P_5 are lower triangular matrices, respectively of size $(j-1)$ and $(n-j)$. Note that several submatrices in B and P disappear if $j = 1$ or $j = n$.

Let L_A and L_B be the lower triangular matrices and Q_A and Q_B the orthogonal matrices of decompositions $B_1P_1 = L_AQ_A$ and $B_2P_5 = L_BQ_B$. Then, BP can be written as

$$BP = \left(\begin{array}{c|c|c} B_1P_1 & 0 & 0 \\ \hline P_2 & p & 0 \\ \hline B_2P_3 & B_2P_4 & B_2P_5 \end{array}\right) = \tilde{L}Q, \tag{11}$$

with $\tilde{L}$ the lower triangular matrix

$$\tilde{L} = \left(\begin{array}{c|c|c} L_A & 0 & 0 \\ \hline P_2Q_A^{-1} & p & 0 \\ \hline B_2P_3Q_A^{-1} & B_2P_4 & L_B \end{array}\right), \tag{12}$$

and Q the orthogonal matrix

$$Q = \left(\begin{array}{c|c|c} Q_A & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & Q_B \end{array}\right).$$

From (6), (11) and the fact that Q is an orthogonal matrix, it follows that

$$B\Sigma B^T = BPP^TB^T = \tilde{L}QQ^T\tilde{L}^T = \tilde{L}\tilde{L}^T.$$

As $\tilde{L}$ is lower triangular, it is equal to the unique lower triangular matrix P^B of the Cholesky decomposition of the symmetric matrix $B\Sigma B^T$. Hence, using (9), (10) and (12), it follows that

$$P^B_{.j} = \tilde{L}_{.j} = \begin{pmatrix} 0 \\ p \\ B_2 P_4 \end{pmatrix} = B \begin{pmatrix} 0 \\ p \\ P_4 \end{pmatrix} = BP_{.j},$$

which proves Lemma 1. □

Proof of Proposition 3b

The matrix A can be made lower triangular by swapping both rows i and k and columns i and k. Let B be the identity matrix with rows i and k swapped. Then,

$$A^B = BAB^T \tag{13}$$

is lower triangular.

Denote

$$\Sigma^B = B\Sigma B^T \tag{14}$$

$$\Sigma^{*B} = B\Sigma^* B^T. \tag{15}$$

Applying Lemma 1 yields

$$P^B_{.j} = BP_{.j} \tag{16}$$

$$P^{*B}_{.j} = BP^*_{.j}, \tag{17}$$

where P^B and P^{*B} are the lower triangular matrices of the Cholesky decompositions of Σ^B and Σ^{*B}, respectively.

Since B is a 'swapping matrix', $B^T = B^{-1}$ and using (1), (13), (14) and (15), it follows that

$$\Sigma^{*B} = BA\Sigma A^T B^T = BAB^T B\Sigma B^T BA^T B^T = A^B \Sigma^B (A^B)^T = A^B P^B (A^B P^B)^T. \tag{18}$$

As lower triangularity of A^B and P^B implies that also $A^B P^B$ is lower triangular, it follows from (18) that

$$P^{*B} = A^B P^B. \tag{19}$$

From (13), (16) and (19) and using $B^T B = I$, it then follows that

$$P_{.j}^{*B} = A^B P_{.j}^B = BAP_{.j}. \tag{20}$$

Finally, equalling the expressions for $P_{.j}^{*B}$ in (17) and (20) and premultiplying both sides of the equation by B^{-1} gives

$$P_{.j}^{*} = AP_{.j},$$

which proves that shocks δ and δ^* are economically equivalent. $\square$

Proof of Proposition 4a

The proof is similar to the proof of Proposition 3a. Let us assume that A is lower triangular with the j^{th} diagonal element equal to 1. Next, define X as a diagonal matrix with diagonal elements equal to those of AL, which is a lower triangular matrix with the j^{th} diagonal element equal to 1. Note that since A is invertible, X^{-1} exists. Using (1) and (7) it then follows that

$$\Sigma^* = A\Sigma A^T = ALDL^T A^T = (ALX^{-1})(XDX)(ALX^{-1})^T.$$

Hence, ALX^{-1} and XDX are respectively equal to L^* and D^* of the unique decomposition $\Sigma^* = L^* D^* L^{*T}$. Since X is a diagonal matrix with j^{th} diagonal element equal to one, it follows that

$$L_{.j}^{*} = (ALX^{-1})_{.j} = AL(X^{-1})_{.j} = AL_{.j},$$

which proves the economic equivalence of the shocks δ and δ^*. $\square$

Before proving Proposition 4b, an intermediate result is needed.

Lemma 2. *Let B be the identity matrix of dimension n with rows i and k swapped, with either $i, k < j$ or $i, k > j$ and where j is a given number with $1 \leq j \leq n$. Let L and L^B be the lower unitriangular matrices and D and D^B be the diagonal matrices of the decompositions $\Sigma = LDL^T$ and $B\Sigma B^T = L^B D^B (L^B)^T$, where Σ is a $n \times n$ dimensional symmetric matrix. Then,*

$$L_{.j}^{B} = BL_{.j}.$$

Proof of Lemma 2. Let j be a given number with $1 \leq j \leq n$. B can be partitioned as

$$B = \left(\begin{array}{c:c:c} B_1 & 0 & 0 \\ \hdashline 0 & 1 & 0 \\ \hdashline 0 & 0 & B_2 \end{array}\right), \tag{21}$$

where the scalar 1 is at position (j, j) and B_1 and B_2 are square matrices respectively of size $j-1$ and $n-j$. Either B_1 or B_2 is an identity matrix and either B_2 or B_1 is an identity matrix where two rows are interchanged. Next, partition L and D in the same way as B

$$L = \left(\begin{array}{c:c:c} L_1 & 0 & 0 \\ \hdashline L_2 & 1 & 0 \\ \hdashline L_3 & L_4 & L_5 \end{array}\right), \tag{22}$$

$$D = \left(\begin{array}{c:c:c} D_1 & 0 & 0 \\ \hdashline 0 & d & 0 \\ \hdashline 0 & 0 & D_2 \end{array}\right),$$

where L_1 and L_5 are lower unitriangular matrices, respectively of size $j-1$ and $n-j$, d is a scalar and D_1 and D_2 are diagonal matrices, respectively of size $j-1$ and $n-j$. Note that several submatrices in B, L and D disappear if $j = 1$ or $j = n$.

Let $D^{1/2}$, $D_1^{1/2}$ and $D_2^{1/2}$ be the principal square root of the diagonal matrices D, D_1 and D_2 and let L_A and L_B be the lower triangular matrices and Q_A and Q_B the orthogonal matrices of the decompositions $B_1 L_1 D_1^{1/2} = L_A Q_A$ and $B_2 L_5 D_2^{1/2} = L_B Q_B$. Then,

$$BLD^{1/2} = \left(\begin{array}{c:c:c} B_1 L_1 D_1^{1/2} & 0 & 0 \\ \hdashline L_2 D_1^{1/2} & d^{1/2} & 0 \\ \hdashline B_2 L_3 D_1^{1/2} & B_2 L_4 d^{1/2} & B_2 L_5 D_2^{1/2} \end{array}\right) = \tilde{L} Q, \tag{23}$$

with $\tilde{L}$ the lower triangular matrix

$$\tilde{L} = \left(\begin{array}{c:c:c} L_A & 0 & 0 \\ \hdashline L_2 D_1^{1/2} Q_A^{-1} & d^{1/2} & 0 \\ \hdashline B_2 L_3 D_1^{1/2} Q_A^{-1} & B_2 L_4 d^{1/2} & L_B \end{array}\right), \tag{24}$$

and Q the orthogonal matrix

$$Q = \left(\begin{array}{c:c:c} Q_A & 0 & 0 \\ \hdashline 0 & 1 & 0 \\ \hdashline 0 & 0 & Q_B \end{array}\right).$$

Define $\tilde{X}$ as a diagonal matrix with diagonal elements equal to those of $\tilde{L}$. Since these diagonal elements are strictly greater than zero, $\tilde{X}^{-1}$ exists. From (21), (23) and the fact that Q is an orthogonal matrix, it follows that

$$B\Sigma B^T = BLDL^T B^T = \tilde{L}QQ^T\tilde{L}^T = \tilde{L}\tilde{L}^T = (\tilde{L}\tilde{X}^{-1})(\tilde{X}\tilde{X})(\tilde{L}\tilde{X}^{-1})^T.$$

where $\tilde{L}\tilde{X}^{-1}$ is a lower unitriangular matrix and $\tilde{X}\tilde{X}$ is a diagonal matrix. Hence, $\tilde{L}\tilde{X}^{-1}$ is equal to the lower unitriangular matrix L^B of the decomposition $\Sigma^B = L^B D^B (L^B)^T$. Therefore, using (21), (22) and (24), it follows that

$$L^B_{.j} = (\tilde{L}\tilde{X}^{-1})_{.j} = \begin{pmatrix} 0 \\ 1 \\ B_2 L_4 \end{pmatrix} = B \begin{pmatrix} 0 \\ 1 \\ L_4 \end{pmatrix} = BL_{.j},$$

which proves Lemma 2. □

Proof of Proposition 4b

The proof is similar to the proof of Proposition 3b. The matrix A can be made lower triangular with the j^{th} diagonal element equal to 1 by swapping rows i and k, and columns i and k. Let B be the identity matrix with rows i and k swapped. Then,

$$A^B = BAB^T \tag{25}$$

is lower triangular with the j^{th} diagonal element equal to 1.

Denote

$$\Sigma^B = B\Sigma B^T \tag{26}$$

$$\Sigma^{*B} = B\Sigma^* B^T. \tag{27}$$

Applying Lemma 2 yields

$$L^B_{.j} = BL_{.j} \tag{28}$$

$$L^{*B}_{.j} = BL^*_{.j}, \tag{29}$$

where L^B and L^{*B} are the lower unitriangular matrices of the decompositions $\Sigma^B = L^B D^B (L^B)^T$ and $\Sigma^{*B} = L^{*B} D^{*B} (L^{*B})^T$, where D^B and D^{*B} are diagonal matrices.

Since B is a 'swapping matrix', $B^T = B^{-1}$ and using (1), (25), (26) and (27), it follows that

$$\Sigma^{*B} = BA\Sigma A^T B^T = BAB^T B\Sigma B^T BA^T B^T = A^B \Sigma^B (A^B)^T.$$

Using the assumption that A^B is lower triangular with the j^{th} diagonal element equal to 1, a similar derivation as in the proof of Proposition 4a can show that

$$L^{*B}_{.j} = A^B L^B_{.j}. \tag{30}$$

From (25), (28) and (30) and using $B^T B = I$, it then follows that

$$L^{*B}_{.j} = A^B L^B_{.j} = BAB^{-1} BL_{.j} = BAL_{.j}. \tag{31}$$

Finally, equalling the expressions for $L^{*B}_{.j}$ in (29) and (31) and premultiplying both sides of the equation by B^{-1} gives

$$L^*_{.j} = AL_{.j},$$

which proves that shocks δ and δ^* are economically equivalent. □

Proof of Proposition 5

Let us assume that the j^{th} row of A contains zeros for all elements except for the j^{th} element which is equal to a strictly positive number c. Using (1), (8) and the assumed structure on the j^{th} row of A, δ^* can be written as

$$\delta^* = \frac{\Sigma^*_{.j}}{\sqrt{\Sigma^*_{jj}}} = \frac{[A\Sigma A^T]_{.j}}{\sqrt{[A\Sigma A^T]_{jj}}} = \frac{A\Sigma[A^T]_{.j}}{\sqrt{[A\Sigma A^T]_{jj}}} = \frac{Ac\Sigma_{.j}}{\sqrt{c^2\Sigma_{jj}}} = A\frac{\Sigma_{.j}}{\sqrt{\Sigma_{jj}}} = A\delta,$$

which proves the economic equivalence of the shocks δ and δ^*. □

Proof of Proposition 6 The proof is analogous to the proof of Proposition 5**.**

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