Matrix methods for quadrature formulas on the unit circle. A survey☆

Adhemar Bultheel[a], María José Cantero[b,1,*], Ruymán Cruz-Barroso[c,2]

[a] *Department of Computer Science, KU Leuven, Belgium*
[b] *Department of Applied Mathematics and IUMA, University of Zaragoza, Spain*
[c] *Department of Mathematical Analysis, La Laguna University, Spain*

Abstract

In this paper we give a survey of some results concerning the computation of quadrature formulas on the unit circle.

Like nodes and weights of Gauss quadrature formulas (for the estimation of integrals with respect to measures on the real line) can be computed from the eigenvalue decomposition of the Jacobi matrix, Szegő quadrature formulas (for the approximation of integrals with respect to measures on the unit circle) can be obtained from certain unitary five-diagonal or unitary Hessenberg matrices that characterize the recurrence for an orthogonal (Laurent) polynomial basis. These quadratures are exact in a maximal space of Laurent polynomials.

Orthogonal polynomials are a particular case of orthogonal rational functions with prescribed poles. More general Szegő quadrature formulas can be obtained that are exact in certain spaces of rational functions. In this context, the nodes and the weights of these rules are computed from the eigenvalue decomposition of an operator Möbius transform of the same five-diagonal or Hessenberg matrices.

Keywords: Szegő polynomials, orthogonal Laurent polynomials, orthogonal rational functions, Szegő quadrature formulas, Hessenberg matrices, CMV matrices, Givens transformation, Möbius transform.
2000 MSC: 65D32, 41A55, 65F15

1. Introduction

A well known procedure to approximate the integral $I_\sigma(f) = \int_a^b f(x)d\sigma(x)$, σ being a positive measure on $[a, b]$, is an n-point Gauss-Christoffel quadrature rule which takes the form $I_n^\sigma(f) = \sum_{j=1}^n A_j f(x_j)$ and is such that $I_\sigma(P) = I_n^\sigma(P)$ for any polynomial of degree up to $2n - 1$. The effective computation of the nodes $\{x_j\}_{j=1}^n$ and weights $\{A_j\}_{j=1}^n$ in $I_n^\sigma(f)$ has become an interesting matter of study both from a numerical and a theoretical point of view. As shown by Gautschi among others (see e.g. [30, 31]), the basic element is the three-term recurrence relation satisfied by the sequence of orthogonal polynomials for the measure σ. This recurrence is characterized by a tri-diagonal matrix (Jacobi matrix) so that the eigenvalues of the n-th principal submatrix coincides with the nodes $\{x_j\}_{j=1}^n$ i.e., with the zeros of the n-th orthogonal polynomial. Furthermore, the weights $\{A_j\}_{j=1}^n$ are precisely the first component of the normalized eigenvectors.

☆Dedicated to our friend and co-author of these results: Professor Pablo González-Vera. In Memoriam.

*Corresponding author
Email address: `mjcante@unizar.es` (María José Cantero)

[1]The work of this author was partially supported by the research project MTM2011-28952-C02-01 from the Ministry of Science and Innovation of Spain and the European Regional Development Fund (ERDF), and by project E-64 of Diputación General de Aragón (Spain).

[2]The work of this author was partially supported by research projects of Ministerio de Ciencia e Innovación under grant MTM2011-28781.

In this paper, we shall be concerned with the approximate calculation of integrals of 2π-periodic functions with respect to a positive measure μ on $[-\pi,\pi]$ or more generally, of integrals on the unit circle like $I_\mu(f) = \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) d\mu(\theta)$. In analogy with Gaussian rules, we propose an n-point quadrature rule $I_n(f) = \sum_{j=1}^{n} \lambda_j f(z_j)$ with distinct nodes on the unit circle but now imposing exactness not for algebraic polynomials but for trigonometric polynomials or more generally, Laurent polynomials. It should be recalled that Laurent polynomials on the real line were used by Jones and Thron in the early 1980's in connection with continued fractions and strong moment problems (see [43] and [45]) and also implicitly in [46]. Their study underwent a rapid development in the last decades giving rise to a theory of orthogonal Laurent polynomials on the real line (see e.g., [15, 24, 41, 42, 50, 54]), and it was extended to a theory of more general orthogonal rational functions (see [8]).

On the other hand, the rapidly growing interest in problems on the unit circle, like quadratures, Szegő polynomials and the trigonometric moment problem has suggested to develop a theory of orthogonal Laurent polynomials on the unit circle introduced by Thron in [54], continued in [16, 37, 44] and where the recent contributions of Cantero, Moral and Velázquez in [11–13] have meant an important and definitive impulse for the spectral analysis of certain problems on the unit circle. Here, it should be remarked that the theory of orthogonal Laurent polynomials on the unit circle establishes features totally different from the theory on the real line because of the close relation between orthogonal Laurent polynomials and the orthogonal polynomials on the unit circle (see [18]).

The orthogonal polynomials are a particular case of a more general kind of orthogonal functions with interest in many pure and applied sciences: the orthogonal rational functions with prescribed poles (see [8] and references therein). We may look at polynomials as a special case of rational functions having only a pole at ∞. The main difference between orthogonal polynomials on the real line and on the unit circle is not only the location of the support of the measure, but also the relative localization of the pole with respect to this support. The natural generalization of (orthogonal) polynomials with pole at infinity is to have rational functions with poles in a neighborhood of infinity. Hence on the real line, where polynomials are assumed to be real, it is natural to choose the poles of the (orthogonal) rational functions to be real as well, i.e., choose them on the extended real line. For the unit circle however, the polynomials depend on a complex variable, and are therefore also assumed to be complex functions, so that the generalization to rational functions will be to select poles that can be anywhere 'near' infinity, i.e., away from the support of the measure, which in our case will mean in the exterior of the closed unit disk.

A more general situation can also be considered with respect to the quadrature formulas: involving orthogonal rational functions on the unit circle with prescribed poles not on $\mathbb{T}$, but inside or outside of the unit disk. This situation, already studied in [6], gives rise to the rational Szegő quadrature formulas. In this case, the nodes are the zeros of the corresponding para-orthogonal rational functions and the quadrature formula is the integral of the rational Lagrange interpolant in these nodes, so that the weights are obtained from the integral of the corresponding rational Lagrange basis functions. An alternative approach to these quadrature formulas, using Hermite interpolation is considered in [6].

In the paper [2], an alternative way to calculate the nodes and the weights to the rational Szegő quadrature formulas was presented by using the recently obtained matricial representation for orthogonal rational functions on the unit circle with prescribed poles (see [55]). These matrices are the result of applying a matrix Möbius transformation of unitary Hessenberg and unitary five-diagonal matrices, having the same form as in the polynomial case. The five-diagonal matrices are also known as CMV matrices. The operator Möbius transformation of the unitary truncations of order n of these matrices, led us to obtain the nodes and the weights of rational Szegő quadrature formulas as in the polynomial case, by computing the eigenvalues and the first component of the normalized eigenvectors of such matrices.

The main purpose of this paper is to give a survey of recent results obtained in the study of matricial representations, both for orthogonal Laurent polynomials defined with respect to any order and for orthogonal rational functions, as well as the analysis and computation of the nodes and weights of Szegő quadrature formulas. Also, some new theoretical approaches along with new numerical experiments will be given.

The rest of the paper is organized as follows: In Section 2, sequences of orthogonal Laurent polynomials on the unit circle with respect to a general order and satisfying certain recurrence relations are constructed. The multiplication operator in the space of Laurent polynomials with a general nesting of the subspaces is considered in Section 3. On the unit circle, this operator plays a fundamental role in the five-diagonal or Hessenberg representation obtained in [12] that are the analogs of the Jacobi matrices on the real line. Our main result of this section is the minimal representation obtained with an approach that differs from the one in [13]. In Section 4 a matrix approach to Szegő quadrature formulas in the more natural framework of orthogonal Laurent polynomials on the unit circle is analyzed and we illustrate these results with a numerical example in Section 5. Section 6 is dedicated to the study of orthogonal rational functions for which in Section 7 we will introduce two different bases in the space of rational functions that will generalize the orthogonal polynomials and the orthogonal Laurent polynomials. This allows the derivation of two different matrix representations of the multiplication operator with respect to these bases that will be operator Möbius transformations of a Hessenberg and a five-diagonal matrices respectively. We also give the expressions for the sequences of para-orthogonal rational functions, whose zeros are eigenvalues of unitary truncations of the multiplication operator. In Section 8, we apply these results to the computation of the nodes and weights of rational Szegő quadrature formulas on the unit circle. Finally, in Section 9 we illustrate the preceding results with some numerical examples.

2. Orthogonal Laurent polynomials on the unit circle

We start this section with some convention for notation and some preliminary results that we will use in the rest of the paper. We denote by $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{E} := \{z \in \mathbb{C} : |z| > 1\}$ the unit circle, the open disk and the exterior of the closed unit disk in the complex plane, respectively. $\mathbb{P} = \mathbb{C}[z]$ is the vector space of polynomials in the variable z with complex coefficients, $\mathbb{P}_n := \text{span}\{1, z, z^2, \dots, z^n\}$ is the corresponding subspace of polynomials with degree less than or equal to n while $\mathbb{P}_{-1} := \{0\}$ is the trivial subspace. $\Lambda := \mathbb{C}[z, z^{-1}]$ denotes the complex vector space of Laurent polynomials in the variable z and for $m, n \in \mathbb{Z}$, $m \leq n$, we define the subspace $\Lambda_{m,n} := \text{span}\{z^m, z^{m+1}, \dots, z^n\}$. Also, for a given function f we define the "substar-conjugate" as $f_*(z) := \overline{f(1/\overline{z})}$. For a polynomial $P_n \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ its reversed (or reciprocal) polynomial is defined by $P_n^*(z) := z^n P_{n*}(z) = z^n \overline{P_n(1/\overline{z})}$.

Throughout the paper, we shall be dealing with a positive Borel measure μ supported on the unit circle $\mathbb{T}$, normalized by the condition $\int_{-\pi}^{\pi} d\mu(\theta) = 1$ (i.e, a probability measure). As usual, the inner product induced by μ is given by

$$\langle f, g \rangle_\mu = \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) \overline{g\left(e^{i\theta}\right)} d\mu(\theta).$$

For our purposes, we start constructing a sequence of subspaces of Laurent polynomials $\{\mathcal{L}_n\}_{n=0}^{\infty}$ satisfying

$$\dim(\mathcal{L}_n) = n + 1 \quad , \quad \mathcal{L}_n \subset \mathcal{L}_{n+1} \quad , \; n = 0, 1, \dots.$$

This can be done by taking a sequence $\{p_n\}_{n=0}^{\infty}$ of nonnegative integers such that $p_0 = 0$, $0 \leq p_n \leq n$ and $s_n = p_n - p_{n-1} \in \{0, 1\}$ for $n = 1, 2, \dots$. In the sequel, a sequence $\{p_n\}_{n=0}^{\infty}$ satisfying these requirements will be called a "generating sequence". Then, set

$$\mathcal{L}_n := \Lambda_{-p_n, q_n} = \text{span}\left\{z^j \mid -p_n \leq j \leq q_n\right\} \; , \; q_n := n - p_n.$$

Observe that $\{q_n\}_{n=0}^{\infty}$ is also a generating sequence and that $\Lambda = \bigcup_{n=0}^{\infty} \mathcal{L}_n$ if and only if $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \infty$. Moreover,

$$\mathcal{L}_{n+1} = \begin{cases} \mathcal{L}_n \oplus \text{span}\{z^{q_{n+1}}\} & \text{if } s_{n+1} = 0, \\ \mathcal{L}_n \oplus \text{span}\{z^{-p_{n+1}}\} & \text{if } s_{n+1} = 1. \end{cases}$$

In any case, we will say that $\{p_n\}_{n=0}^{\infty}$ has induced an "ordering" in Λ, i.e., a nesting of the subspaces $\mathcal{L}_n$. Sometimes we will need to define $p_{-1} = 0$ and hence $s_0 = 0$. Now, by applying the Gram-Schmidt orthogonalization procedure to $\mathcal{L}_n$, an orthogonal basis $\{\psi_0(z), \ldots, \psi_n(z)\}$ can be obtained. If we repeat the process for each $n = 1, 2, \ldots$, a sequence $\{\psi_n(z)\}_{n=0}^{\infty}$ of Laurent polynomials can be obtained, satisfying

$$\begin{aligned} &\psi_n \in \mathcal{L}_n \backslash \mathcal{L}_{n-1}, \ \ n = 1, 2, \ldots \quad , \quad \psi_0 \equiv c \neq 0 \\ &\langle \psi_n, \psi_m \rangle_\mu = \kappa_n \delta_{n,m}, \quad \kappa_n > 0 \quad , \quad \delta_{n,m} = \begin{cases} 0 & if \quad n \neq m, \\ 1 & if \quad n = m. \end{cases} \end{aligned} \tag{2.1}$$

$\{\psi_n(z)\}_{n=0}^{\infty}$ will be called a "sequence of orthogonal Laurent polynomials for the measure μ and the generating sequence $\{p_n\}_{n=0}^{\infty}$". It should be noted that the orders considered by Thron in [54] ("balanced" situation), defining the following orderings of nested subspaces of Λ

$$\Lambda_{0,0} \quad , \quad \Lambda_{-1,0} \quad , \quad \Lambda_{-1,1} \quad , \quad \Lambda_{-2,1} \quad , \quad \Lambda_{-2,2} \quad , \quad \Lambda_{-3,2} \quad , \quad \ldots$$

and

$$\Lambda_{0,0} \quad , \quad \Lambda_{0,1} \quad , \quad \Lambda_{-1,1} \quad , \quad \Lambda_{-1,2} \quad , \quad \Lambda_{-2,2} \quad , \quad \Lambda_{-2,3} \quad , \quad \ldots$$

correspond to $p_n = E\left[\frac{n+1}{2}\right]$ and $p_n = E\left[\frac{n}{2}\right]$ respectively, where as usual, $E[x]$ denotes the integer part of x (see [12, 18, 19] for other properties for these particular orderings). In the sequel we will denote by $\{\phi_n(z)\}_{n=0}^{\infty}$ the sequence of monic orthogonal Laurent polynomials for the measure μ and the generating sequence $\{p_n\}_{n=0}^{\infty}$. Here monic means that the leading coefficient is equal to 1, which is the coefficient of z^{q_n} in ϕ_n when $s_n = 0$ or of z^{-p_n} when $s_n = 1$. Moreover, we will denote by $\{\chi_n(z)\}_{n=0}^{\infty}$ the sequence of orthonormal Laurent polynomials for the measure μ and the generating sequence $\{p_n\}_{n=0}^{\infty}$, i.e. when $\kappa_n = 1$ for all $n \geq 0$ in (2.1). This sequence is also uniquely determined by assuming that the leading coefficient in χ_n is positive for each $n \geq 0$.

On the other hand, when taking $p_n = 0$ for all $n = 0, 1, \ldots$ then $\mathcal{L}_n = \Lambda_{0,n} = \mathbb{P}_n$, so that the n-th monic orthogonal Laurent polynomial coincides with the n-th monic Szegő polynomial (see e.g. [53]) which will be denoted by $\rho_n(z)$ for $n = 0, 1, \ldots$. This means that $\rho_0 \equiv 1$ and for each $n \geq 1$, $\rho_n \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ is monic and satisfies

$$\begin{aligned} &\langle \rho_n(z), z^s \rangle_\mu = \langle \rho_n^*(z), z^t \rangle_\mu = 0, \ s = 0, 1, \ldots, n-1 \ , \ t = 1, 2, \ldots, n \\ &\langle \rho_n(z), z^n \rangle_\mu = \langle \rho_n^*(z), 1 \rangle_\mu > 0. \end{aligned}$$

Moreover, we will denote by $\{\varphi_n(z)\}_{n=0}^{\infty}$ the sequence of orthonormal polynomials on the unit circle for μ, i.e., satisfying $\|\varphi_n\|_\mu = \langle \varphi_n, \varphi_n \rangle_\mu^{1/2} = 1$ for all $n \geq 0$. This family is uniquely determined by assuming that the leading coefficient in φ_n is positive for each $n \geq 0$ and it is related with the family of monic orthogonal polynomials by $\rho_0 \equiv \varphi_0 \equiv 1$ and $\rho_n = l_n \varphi_n$ with $l_n = \|\rho_n\|_\mu$ for all $n \geq 1$.

Explicit expressions for Szegő polynomials are in general not available and in order to compute them we can make use of the following (Szegő) forward recurrence relations (see e.g. [53]):

$$\begin{aligned} &\rho_0(z) = \rho_0^*(z) \equiv 1, \\ &\begin{pmatrix} \rho_n(z) \\ \rho_n^*(z) \end{pmatrix} = \begin{pmatrix} z & \delta_n \\ \overline{\delta_n} z & 1 \end{pmatrix} \begin{pmatrix} \rho_{n-1}(z) \\ \rho_{n-1}^*(z) \end{pmatrix}, \quad n \geq 1, \end{aligned} \tag{2.2}$$

where $\delta_n := \rho_n(0)$ for all $n = 1, 2, \ldots$ are the so-called *Schur parameters* (*Szegő*, *reflection*, *Verblunsky* or *Geronimus* parameters, see [52]) with respect to μ. Since the zeros of ρ_n lie in $\mathbb{D}$, the Schur parameters satisfy $|\delta_n| < 1$ for $n \geq 1$. Now, if we introduce the sequence $\{\eta_n\}_{n=1}^{\infty}$ by

$$\eta_n := \sqrt{1 - |\delta_n|^2} \in (0, 1], \ \ n = 1, 2, \ldots, \tag{2.3}$$

then, a straightforward computation from (2.2) yields $\eta_n^2 = \frac{\langle \rho_n, \rho_n \rangle_\mu}{\langle \rho_{n-1}, \rho_{n-1} \rangle_\mu}$ or $\eta_n = \frac{\|\rho_n\|_\mu}{\|\rho_{n-1}\|_\mu}$, and so, a forward recurrence for the family of orthonormal Szegő polynomials is given by:

$$\begin{pmatrix} \varphi_0 \\ \varphi_0^* \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \eta_n \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} = \begin{pmatrix} 1 & \delta_n \\ \overline{\delta_n} & 1 \end{pmatrix} \begin{pmatrix} z\varphi_{n-1}(z) \\ \varphi_{n-1}^*(z) \end{pmatrix}, \quad n \geq 1. \tag{2.4}$$

There are relations among the families of orthogonal Laurent polynomials and the Szegő polynomials. We consider two of them proved in [12] and [16]. The first one establishes the relation between the families of orthonormal Laurent polynomials with respect to the generating sequences $\{p_n\}_{n=0}^\infty$ and $\{q_n\}_{n=0}^\infty$ whereas the second one states the relation between the families of orthonormal and monic orthogonal Laurent polynomials for a generating sequence $\{p_n\}_{n=0}^\infty$ and the families of orthonormal and monic Szegő polynomials. This last result explains how to construct orthogonal Laurent polynomials on the unit circle from the sequence of Szegő polynomials. It should be remarked here that the situation on the real line, i.e. when dealing with sequences of orthogonal Laurent polynomials with respect to a positive measure supported on the real line, is totally different (for details, see e.g. [24]).

Proposition 2.1. *Let $\{\tilde{\chi}_n(z)\}_{n=0}^\infty$ and $\{\chi_n(z)\}_{n=0}^\infty$ be sequences of orthonormal Laurent polynomials for the measure μ and the generating sequences $\{q_n\}_{n=0}^\infty$ and $\{p_n\}_{n=0}^\infty$ respectively, where $p_n = n - q_n$ for all $n \geq 0$. Then, $\tilde{\chi}_n(z) = \chi_{n*}(z)$ for all $n \geq 0$.* □

Proposition 2.2. *The families $\{\phi_n(z)\}_{n=0}^\infty$ and $\{\chi_n(z)\}_{n=0}^\infty$ are the respective sequences of monic orthogonal and orthonormal Laurent polynomials on the unit circle for a measure μ and the ordering induced by the generating sequence $\{p_n\}_{n=0}^\infty$, if and only if,*

$$\phi_n(z) = \begin{cases} \frac{\rho_n(z)}{z^{p_n}} & \text{if} \quad s_n = 0, \\ \frac{\rho_n^*(z)}{z^{p_n}} & \text{if} \quad s_n = 1, \end{cases} \qquad \text{and} \qquad \chi_n(z) = \begin{cases} \frac{\varphi_n(z)}{z^{p_n}} & \text{if} \quad s_n = 0, \\ \frac{\varphi_n^*(z)}{z^{p_n}} & \text{if} \quad s_n = 1, \end{cases}$$

where $\{\rho_n\}_{n=0}^\infty$ and $\{\varphi_n\}_{n=0}^\infty$ are the respective monic orthogonal and orthonormal Szegő polynomial sequences for the same measure μ. □

The families of orthogonal Laurent polynomials with respect to any order $\{p_n\}_{n=0}^\infty$ and the Szegő polynomials satisfy some recurrence relations. In what follows we consider some of them which we will use in the next section.

We start with the next result which establishes a three-term recurrence relation for the monic orthogonal and orthonormal families of Laurent polynomials for the measure μ and the balanced generating sequences.

Proposition 2.3. *Consider the families $\{\phi_n(z)\}_{n=0}^\infty$ and $\{\tilde{\phi}_n(z)\}_{n=0}^\infty$ of monic orthogonal Laurent polynomials for the measure μ and the generating sequences $p_n = E\left[\frac{n+1}{2}\right]$ and $p_n = E\left[\frac{n}{2}\right]$ respectively. Set*

$$A_n = \begin{cases} \delta_n & \text{if} \quad n \text{ is even,} \\ \overline{\delta_n} & \text{if} \quad n \text{ is odd.} \end{cases}$$

Then,

$$\begin{gathered} \phi_n(z) = \left(A_n + \overline{A_{n-1}} z^{(-1)^n}\right) \phi_{n-1}(z) + \eta_{n-1}^2 z^{(-1)^n} \phi_{n-2}(z)\ , \ \ n \geq 2, \\ \phi_0(z) \equiv 1\ , \quad \phi_1(z) = \overline{\delta_1} + z^{-1}, \end{gathered} \tag{2.5}$$

$$\begin{gathered} \tilde{\phi}_n(z) = \left(\overline{A_n} + A_{n-1} z^{(-1)^{n+1}}\right) \tilde{\phi}_{n-1}(z) + \eta_{n-1}^2 z^{(-1)^{n+1}} \tilde{\phi}_{n-2}(z)\ , \ \ n \geq 2, \\ \tilde{\phi}_0(z) \equiv 1\ , \quad \tilde{\phi}_1(z) = \delta_1 + z. \end{gathered} \tag{2.6}$$

□

These recurrences were initially proved by Thron in [54] in the context of continued fractions. An alternative proof is given in [19]. The equivalence between the recurrences (2.2) and (2.5) (the same for (2.6)) was proved in [10]. Moreover, the following recurrence relations were proved in [12].

Proposition 2.4. *The family $\{\chi_n(z)\}_{n=0}^{\infty}$ of orthonormal Laurent polynomials for the measure μ and the generating sequence $p_n = E\left[\frac{n}{2}\right]$ satisfies*

$$\begin{aligned} z\chi_0(z) &= -\delta_1\chi_0(z) + \eta_1\chi_1(z) \quad , \\ z\begin{pmatrix} \chi_{2n-1}(z) \\ \chi_{2n}(z) \end{pmatrix} &= \begin{pmatrix} -\eta_{2n-1}\delta_{2n} & -\overline{\delta_{2n-1}}\delta_{2n} \\ \eta_{2n-1}\eta_{2n} & \overline{\delta_{2n-1}}\eta_{2n} \end{pmatrix}\begin{pmatrix} \chi_{2n-2}(z) \\ \chi_{2n-1}(z) \end{pmatrix} + \\ & \quad \begin{pmatrix} -\eta_{2n}\delta_{2n+1} & \eta_{2n}\eta_{2n+1} \\ -\overline{\delta_{2n}}\delta_{2n+1} & \overline{\delta_{2n}}\eta_{2n+1} \end{pmatrix}\begin{pmatrix} \chi_{2n}(z) \\ \chi_{2n+1}(z) \end{pmatrix} \quad , \quad n \geq 1. \end{aligned} \tag{2.7}$$

□

A similar matrix recurrence exists for the generating sequence $p_n = E\left[\frac{n+1}{2}\right]$. The equivalence of these recurrences with the recurrence given by Proposition 2.3 is proved in [10].

Remark 2.5. Note that the three-term recurrence given in Proposition 2.3 involves alternatingly a multiplication by z and by z^{-1} whereas the recurrence given in Poposition 2.4 involves only multiplication by z. These latter relations will play a fundamental role in the next section.

Until now we have deduced recurrences for the families of orthogonal Laurent polynomials when the generating sequences associated with the balanced orderings are considered. In the next result (see [10]) we will consider an arbitrary generating sequence, starting with a three-term recurrence relation for $\{\phi_n(z)\}_{n=0}^{\infty}$ involving multiplication by z and z^{-1}.

Theorem 2.6. *The family of (monic) orthogonal Laurent polynomials $\{\phi_n(z)\}_{n=0}^{\infty}$ with respect to the measure μ and the generating sequence $\{p_n\}_{n=0}^{\infty}$ satisfies for $n \geq 2$ the three-term recurrence relation*

$$\phi_n(z) = \left(A_n + B_n z^{1-2s_n}\right)\phi_{n-1}(z) + (-1)^{1+s_{n-2}-s_{n-1}} C_n \rho_{n-1}^2 \eta_{n-1}^2 z^{1-s_n-s_{n-2}} \phi_{n-2}(z),$$

with initial conditions $\phi_0(z) \equiv 1$ and $\phi_1(z) = k_1 + z^{1-2s_1}$, A_n, B_n, C_n depending on the Schur parameters δ_n and δ_{n-1} while k_1 is either δ_1 or $\overline{\delta}_1$ depending on s_1 being 0 or 1 and $s_0 = 0$. □

3. Matricial representations

Throughout this section, a fundamental role will be played by the multiplication operator defined on Λ, namely

$$M : \Lambda \to \Lambda, \qquad L(z) \mapsto zL(z).$$

As we have seen, if we consider the sequence of orthonormal Laurent polynomials with respect to the measure μ and the generating sequence $p_n = 0$ for all $n \geq 0$, then the n-th orthonormal Laurent polynomial coincides with the n-th orthonormal Szegő polynomial, for all $n \geq 0$. Since the operator M leaves $\mathbb{P}$ invariant, taking $\{\varphi_n(z)\}_{n=0}^{\infty}$ as a basis for $\mathbb{P}$, then the following matrix representation of the restriction of M to $\mathbb{P}$, with Hessenberg structure, is obtained (see e.g. [16], [17] or [38]):

$$\mathcal{H}(\delta) = \begin{pmatrix} h_{0,0} & h_{0,1} & 0 & 0 & 0 & \cdots \\ h_{1,0} & h_{1,1} & h_{1,2} & 0 & 0 & \cdots \\ h_{2,0} & h_{2,1} & h_{2,2} & h_{2,3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad h_{i,j} := \begin{cases} -\overline{\delta_j}\delta_{i+1}\prod_{k=j+1}^{i}\eta_k & \text{if } j = 0, 1, \ldots, i-1, \\ -\overline{\delta_i}\delta_{i+1} & \text{if } j = i, \\ \eta_{i+1} & \text{if } j = i+1. \end{cases} \tag{3.1}$$

If we consider now the generating sequence $p_n = E\left[\frac{n}{2}\right]$ then it follows from the recurrence given in Proposition 2.4 that we get the following five-diagonal matrix $\mathcal{C}(\delta)$ (CMV representation, see [52]) for the multiplication operator M

$$\mathcal{C}(\delta) = \begin{pmatrix} -\delta_1 & \eta_1 & 0 & 0 & 0 & 0 & \cdots \\ -\eta_1\delta_2 & -\overline{\delta_1}\delta_2 & -\eta_2\delta_3 & \eta_2\eta_3 & 0 & 0 & \cdots \\ \eta_1\eta_2 & \overline{\delta_1}\eta_2 & -\overline{\delta_2}\delta_3 & \overline{\delta_2}\eta_3 & 0 & 0 & \cdots \\ 0 & 0 & -\eta_3\delta_4 & -\overline{\delta_3}\delta_4 & -\eta_4\delta_5 & \eta_4\eta_5 & \cdots \\ 0 & 0 & \eta_3\eta_4 & \overline{\delta_3}\eta_4 & -\overline{\delta_4}\delta_5 & \overline{\delta_4}\eta_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.2}$$

and it can also be expressed as a product of two tri-diagonal ones $\mathcal{C}(\delta) = \mathcal{C}_e(\delta)\mathcal{C}_o(\delta)$, (see [12]), where

$$\mathcal{C}_e(\delta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\delta_2 & \eta_2 & 0 & 0 & \cdots \\ 0 & \eta_2 & \overline{\delta_2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\delta_4 & \eta_4 & \cdots \\ 0 & 0 & 0 & \eta_4 & \overline{\delta_4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{C}_o(\delta) = \begin{pmatrix} -\delta_1 & \eta_1 & 0 & 0 & 0 & \cdots \\ \eta_1 & \overline{\delta_1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & -\delta_3 & \eta_3 & 0 & \cdots \\ 0 & 0 & \eta_3 & \overline{\delta_3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & -\delta_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Moreover, it is easy to check that the matrix representation when the generating sequence $p_n = E\left[\frac{n+1}{2}\right]$ is considered is $\mathcal{C}(\delta)^T$.

Our aim now is to analyze the structure of the matrix representation for the multiplication operator M when an arbitrary generating sequence is considered. We start with a generalization of [12, Proposition 2.4] where the result was derived only in the case of a balanced generating sequence. It can be found as [10, Theorem 4.1] but because the proof is short and elegant, we repeat it for the convenience of the reader.

Theorem 3.1. *Let $\{\chi_n(z)\}_{n=0}^{\infty}$ be the sequence of orthonormal Laurent polynomials for the measure μ and the generating sequence $\{p_n\}_{n=0}^{\infty}$ and suppose that $\lim_{n\to\infty} q_n = \infty$. Then, for each $n \geq 0$ there exist $k = k_n \geq 1$ and $t = t_n \geq 1$ such that $z\chi_n(z) \in \operatorname{span}\{\chi_{n-t}(z), \cdots, \chi_{n+k}(z)\}$, i.e.*

$$z\chi_n(z) = \sum_{s=n-t}^{n+k} a_{n,s}\chi_s(z)\ , \qquad a_{n,s} = \langle z\chi_n(z), \chi_s(z)\rangle_\mu.$$

Moreover, $k = k_n$ and $t = t_n$ are defined as follows:

1. *$k = 1$ if $s_{n+1} = 0$ and otherwise $k \geq 2$ is defined satisfying $s_{n+1} = \cdots = s_{n+k-1} = 1, s_{n+k} = 0$.*
2. *$t = 1$ if $s_{n-1} = 1$ and otherwise $t \geq 2$ is defined satisfying $s_{n-1} = \cdots = s_{n+1-t} = 0, s_{n-t} = 1$.*

Proof. Since $\chi_n \in \mathcal{L}_n$ then

$$z\chi_n(z) \in z\mathcal{L}_n = \operatorname{span}\left\{\frac{1}{z^{p_n-1}}, \cdots, z^{q_n+1}\right\} \subset \mathcal{L}_{n+k}$$

with $k = k_n \geq 1$. Observe that the existence of k is guaranteed from the condition $\lim_{n\to\infty} q_n = \infty$. On the other hand, since $\chi_n \perp \mathcal{L}_{n-1}$ then

$$z\chi_n(z) \perp z\mathcal{L}_{n-1} = \operatorname{span}\left\{\frac{1}{z^{p_{n-1}-1}}, \cdots, z^{q_{n-1}+1}\right\} \supset \mathcal{L}_{n-1-t}$$

with $t = t_n \geq 1$. Since $z\chi_n(z) \subset \mathcal{L}_{n+k}$ and $z\chi_n(z) \perp \mathcal{L}_{n-1-t}$, the proof follows. □

Note that the indices k and t provide the *upper* width and *lower* width of the band matrices. Thus, from Theorem 3.1 we can analyze the matrix representation for the operator M with respect to a generating sequence with a minimal number of diagonals. As we have seen, the balanced orderings give rise to five-diagonal matrices. An obvious question is whether there exist other generating sequences that give rise to a matrix representation with five or less diagonals. We start by remarking that a five-diagonal representation is obtained if and only if one of the following cases is satisfied:

$$\begin{array}{ll} 1. & k_n = 1,\ t_n \leq 3 \ \text{ for all } n, \\ 2. & k_n \leq 2,\ t_n \leq 2 \ \text{ for all } n, \\ 3. & k_n \leq 3,\ t_n = 1 \ \text{ for all } n. \end{array} \tag{3.3}$$

Hence, we have the following considerations:

1. If the sequence $\{s_n\}_{n\geq 1}$ contains three or more consecutive zeros or ones, then the representation will not be five-diagonal. Suppose for example that there are 3 consecutive zeros: $(s_n, s_{n+1}, s_{n+2}, s_{n+3}, s_{n+4}) = (1, 0, 0, 0, 1)$. Then Theorem 3.1 implies that there is a block $(k_n, k_{n+1}, k_{n+2}) = (1, 1, 1)$ and $k_{n+3} \geq 2$ while $(t_{n+1}, t_{n+2}, t_{n+3}, t_{n+4}) = (1, 2, 3, 4)$. Similarly a block of 3 consecutive ones, say $(s_n, s_{n+1}, s_{n+2}, s_{n+3}, s_{n+4}) = (0, 1, 1, 1, 0)$, implies by the same theorem that $(k_n, k_{n+1}, k_{n+2}, k_{n+3}) = (4, 3, 2, 1)$ and $t_{n+1} \geq 2$ with $(t_{n+2}, t_{n+3}, t_{n+4}) = (1, 1, 1)$. In both cases the condition (3.3) fails. Thus, we obtain a matricial representation with more than five diagonals. In other words, to obtain a five-diagonal representation the number of consecutive zeros or ones in the sequence $\{s_n\}_{n\geq 1}$ must be at most 2.
2. If the number of consecutive zeros or ones is just one for all n then the situation corresponds to the generating sequences $p_n = E\left[\frac{n}{2}\right]$ or $p_n = E\left[\frac{n+1}{2}\right]$ and the five-diagonal matrix representations $\mathcal{C}(\delta)$ and $\mathcal{C}(\delta)^T$ are obtained, respectively.

Now, let us concentrate on what happens if two consecutive zeros or ones appear in the sequence $\{s_n\}_{n\geq 1}$. Indeed, a block of the form $(s_n, s_{n+1}, s_{n+2}, s_{n+3}) = (1, 0, 0, 1)$ implies $t_{n+2} = 2$, $k_{n+2} \geq 2$, $t_{n+3} = 3$, $k_{n+3} \geq 1$ and hence a five-diagonal representation is not obtained since condition (3.3) does not hold. Suppose now a block of the form $(s_n, s_{n+1}, s_{n+2}, s_{n+3}) = (0, 1, 1, 0)$. If $s_{n-1} = 0$ then $t_{n+1} = 3$ and $k_{n+1} = 2$ whereas if $s_{n-1} = 1$ then $t_n = 1$, $k_n = 3$ and $t_{n+1} = k_{n+1} = 2$, implying in both cases that the condition (3.3) is not satisfied. Observe that this argument is valid for all $n \geq 2$, but it really holds for $n \geq 0$. Indeed, if we consider the generating sequences $p_n = E\left[\frac{n+1}{2}\right]$ for all $n \geq 2$ with $p_0 = p_1 = 0$ or $p_n = E\left[\frac{n}{2}\right]$ for all $n \geq 2$ with $p_0 = 0$ and $p_1 = 1$ then it is easy to check that a non-five diagonal matrix representation is obtained. Summarizing, we can state:

Theorem 3.2. *The matrix representation for the multiplication operator M is a five-diagonal matrix if and only if $p_n = E\left[\frac{n}{2}\right]$ or $p_n = E\left[\frac{n+1}{2}\right]$. Moreover, this representation is the narrowest one in the sense that any matrix representation for another different generating sequence gives rise to a d-diagonal matrix representation with $d \geq 6$.* □

Remark 3.3. This proof of Theorem 3.2 based on orthogonality conditions was obtained in [10], but it had been deduced previously in [13] by using operator theory techniques.

Example 3.4. Suppose that we consider a generating sequence $\{p_n\}_{n\geq 1}$ corresponding to the following order of the natural basis for Λ $\{1, z, z^{-1}, z^2, z^3, z^4, z^5, z^{-2}, z^{-3}, z^6, \ldots\}$. By following similar arguments as in [10, Example 4.5], or alternatively from [17, Section 3], it is an exercise to check that the corresponding matrix representation has the following nine-diagonal structure:

$$\begin{pmatrix}
-\delta_1 & \eta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
-\eta_1\delta_2 & -\overline{\delta_1}\delta_2 & -\eta_2\delta_3 & \eta_2\eta_3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\eta_1\eta_2 & \overline{\delta_1}\eta_2 & -\overline{\delta_2}\delta_3 & \overline{\delta_2}\eta_3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -\eta_3\delta_4 & -\overline{\delta_3}\delta_4 & \eta_4 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -\eta_3\eta_4\delta_5 & -\overline{\delta_3}\eta_4\delta_5 & -\overline{\delta_4}\delta_5 & \eta_5 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -\eta_3\eta_4\eta_5\delta_6 & -\overline{\delta_3}\eta_4\eta_5\delta_6 & -\overline{\delta_4}\eta_5\delta_6 & -\overline{\delta_5}\delta_6 & \eta_6 & 0 & 0 & 0 & \cdots \\
0 & 0 & \eta_3\eta_4\eta_5\eta_6\delta_7 & -\overline{\delta_3}\eta_4\eta_5\eta_6\delta_7 & -\overline{\delta_4}\eta_5\eta_6\delta_7 & -\overline{\delta_5}\eta_6\delta_7 & -\overline{\delta_6}\delta_7 & -\eta_7\delta_8 & -\eta_7\eta_8\delta_9 & \eta_7\eta_8\eta_9 & \cdots \\
0 & 0 & \eta_3\eta_4\eta_5\eta_6\eta_7 & \overline{\delta_3}\eta_4\eta_5\eta_6\eta_7 & \overline{\delta_4}\eta_5\eta_6\eta_7 & \overline{\delta_5}\eta_6\eta_7 & \overline{\delta_6}\eta_7 & -\overline{\delta_7}\delta_8 & -\overline{\delta_7}\eta_8\delta_9 & \overline{\delta_7}\eta_8\eta_9 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_8 & -\overline{\delta_8}\delta_9 & \overline{\delta_8}\eta_9 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\eta_9\delta_{10} & -\overline{\delta_9}\delta_{10} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Hessenberg and CMV matrices are actually two particular cases of a more general structural theorem (see [17]). Indeed, denote by $G_{k-1,k}$ a Givens (or Jacobi) transformation

$$G_{k-1,k} = \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & \tilde{G}_{k-1,k} & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \tilde{G}_{k-1,k} = \begin{pmatrix} -\delta_k & \eta_k \\ \eta_k & \overline{\delta}_k \end{pmatrix}, \quad k \geq 1,$$

where I_{k-1} and I denotes identity matrices of sizes $k-1$ and ∞, respectively. Givens transformations can be considered as the most elementary type of unitary matrices and they can be used as building blocks to construct more general unitary matrices. Our interest is the fact that the infinite Hessenberg (3.1) and CMV (3.2) matrices allows a factorization as a product of Givens transformations in the form

$$\mathcal{H}(\delta) = G_{0,1}G_{1,2}G_{2,3}G_{3,4}G_{4,5}G_{5,6}\cdots, \quad \mathcal{C}(\delta) = (\cdots G_{7,8}G_{5,6}G_{3,4}G_{1,2}) \cdot (G_{0,1}G_{2,3}G_{4,5}G_{6,7}\cdots).$$

Note that the latter corresponds to the factorization $\mathcal{C}(\delta) = \mathcal{C}_e(\delta) \cdot \mathcal{C}_o(\delta)$ just after (3.2). This factorization must be understood in the sense that the principal $n \times n$ submatrices of $\mathcal{H}(\delta)$ and $\mathcal{C}(\delta)$, and their corresponding expansion in products of Givens transformations, coincide for each n. The following theorem, proved in [17], gives the recipe to construct the matrix representation of the multiplication operator on Λ when an arbitrary ordering induced by a generating sequence $\{p_n\}_{n=0}^{\infty}$ is considered.

Theorem 3.5. *Let $\{\chi_n(z)\}_{n=0}^{\infty}$ be the sequence of orthonormal Laurent polynomials on the unit circle for a measure μ and the ordering induced by a generating sequence $\{p_n\}_{n=0}^{\infty}$. Then, the matrix representation $\mathcal{S}(\delta)$ of the multiplication operator in Λ for the orthogonal basis corresponding to this ordering can be characterized by a 'snake-shaped' matrix factorization $\mathcal{S}(\delta) = \mathcal{S}^{(\infty)}(\delta)$ constructed by the following algorithm: We initialize with $\mathcal{S}^{(0)}(\delta) = G_{0,1}$. Then, for $k \geq 1$ we apply the following procedure:*

1. *If $s_k = 0$ then we multiply the matrix with a new Givens transformation on the right by setting $\mathcal{S}^{(k)}(\delta) = \mathcal{S}^{(k-1)}(\delta) \cdot G_{k,k+1}$.*
2. *If $s_k = 1$ then we multiply the matrix with a new Givens transformation on the left by setting $\mathcal{S}^{(k)}(\delta) = G_{k,k+1} \cdot \mathcal{S}^{(k-1)}(\delta)$.*

The factorization must be understood in the sense that the principal $n \times n$ submatrices of $\mathcal{S}^{(n-1)}(\delta)$ and $\mathcal{S}(\delta)$ coincide for all n. □

Example 3.6. Suppose that the generating sequence $\{p_n\}_{n\geq 1}$ is as in Example 3.4. Then the associated nine-diagonal matrix is factored as

$$\mathcal{S}(\delta) = \cdots G_{8,9}G_{7,8}G_{2,3}G_{0,1}G_{1,2}G_{3,4}G_{4,5}G_{5,6}G_{6,7}G_{9,10}\cdots.$$

The terminology 'snake-shaped' matrix factorization, duly explained in [17], arises from the distribution of the non-zero elements in this matrix representation in a zig-zag shape around the main diagonal.

4. Matricial computation of Szegő quadrature formulas

Throughout this section we shall be concerned with the estimation of integrals on $\mathbb{T}$ of the form,

$$I_\mu(f) = \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) d\mu(\theta). \tag{4.1}$$

For the polynomial case, the Szegő-quadrature formulas were introduced in [26]. Also it is remarkable the work of Geronimus (see [32]). A different approach, for Laurent polynomials, is considered in [44].

We shall first derive the formulas for nodes and weights of the Szegő quadrature formulas and then we shall show how these can be practically computed using our different matricial representations.

4.1. Formulas for the nodes and the weights of Szegő quadrature

As usual, estimates of $I_\mu(f)$ may be produced by replacing in (4.1), the integrand f by an appropriate approximating (or interpolating) function L for which the integral $I_\mu(L)$ is easily computed. Because the space of Laurent polynomials Λ is dense in the space $C(\mathbb{T})$ of continuous functions on $\mathbb{T}$ with respect to the uniform norm (see e.g., [22] and [53]), it seems reasonable to approximate f in (4.1) by some appropriate Laurent polynomial. The integral of this approximation gives a "quadrature rule on the unit circle". By an n-point quadrature formula on the unit circle we mean an expression like

$$I_n(f) := \sum_{j=1}^{n} \lambda_j f(z_j), \quad z_i \neq z_j \text{ if } i \neq j, \quad \{z_j\}_{j=1}^n \subset \mathbb{T}, \tag{4.2}$$

where the nodes $\{z_j\}_{j=1}^n$ and the coefficients or weights $\{\lambda_j\}_{j=1}^n$ are chosen so that $I_n(L) = I_\mu(L)$ for any $L \in \Lambda_{-p,q}$ with p and q nonnegative integers depending on n with $p+q$ as large as possible. If we first try with subspaces of the form $\Lambda_{-p,p}$, it can be easily checked that there can not exist an n-point quadrature formula $I_n(f)$ of the form (4.2) which is exact in $\Lambda_{-n,n}$. Hence, it holds that $p \leq n-1$. In [16] the following "necessary condition" on the nodal polynomial is proved:

Theorem 4.1. *For $n \geq 1$, let $I_n(f) = \sum_{j=1}^n \lambda_j f(z_j)$ with $z_j \in \mathbb{T}$, $j = 1, \ldots, n$ be exact in $\Lambda_{-(n-1),n-1}$, and set $P_n(z) = \prod_{j=1}^n (z - z_j)$. Then,*

$$P_n(z) = C_n \left[\rho_n(z) + u\rho_n^*(z)\right], \quad |u| = 1, \tag{4.3}$$

i.e., the nodal polynomial P_n is proportional to a para-orthogonal polynomial $P_n^u := \rho_n + u\rho_n^*$, where, for every n, the proportionality constant C_n depending on the Schur parameters δ_n. □

Moreover, in [44] the following converse result is proved (sufficient conditions on the nodal polynomial):

Theorem 4.2. *Consider the para-orthogonal polynomial $P_n^u = \rho_n + u\rho_n^*$, with $u \in \mathbb{T}$. Then,*

1. *$P_n^u(z)$ has exactly n distinct zeros $z_1, \ldots, z_n$ on $\mathbb{T}$.*
2. *There exist positive real numbers $\lambda_1, \ldots, \lambda_n$ such that*

$$I_n(f) = \sum_{j=1}^{n} \lambda_j f(z_j) = I_\mu(f), \quad \text{for all } f \in \Lambda_{-(n-1),n-1}. \tag{4.4}$$

I.e., the zeros of a para-orthogonal polynomial can be used as nodes in a quadrature formula satisfying (4.4). □

The quadrature formula $I_n(f)$ given by (4.4), and earlier introduced in [44] is called an "n-point Szegő quadrature formula". It is the analogue on the unit circle of the Gaussian formulas for intervals of the real axis. However, in this respect two big differences should be remarked: the nodes are not the zeros of the n-th orthogonal polynomial with respect to μ and the n-point Szegő formula is exact in $\Lambda_{-(n-1),n-1}$, whose dimension is $2n-1$ instead of $2n$. Observe that since the nodes are the zeros of an n-th para-orthogonal polynomial depending on a parameter $u \in \mathbb{T}$, a one-parameter family of quadrature formulas exact in $\Lambda_{-(n-1),n-1}$ arises. Moreover, it can be shown that there is a subspace of dimension $2n$, depending on u, for which the quadrature formula is exact (see [51]), but it is not of the form $\Lambda_{-r,s}$ with $r+s+1=2n$.

On the other hand, starting from a generating sequence $\{p_n\}_{n=0}^{\infty}$ we know that $\mathcal{L}_{n-1}$ is a Chebyshev system on $\mathbb{T}$ of dimension n (since $0 \notin \mathbb{T}$). Hence, for n distinct nodes $z_1, \ldots, z_n$ on $\mathbb{T}$, we can find a unique set of parameters $\lambda_1, \ldots, \lambda_n$ so that, $I_n(L) = I_\mu(L)$ for all $L \in \mathcal{L}_{n-1}$. In order to recover Szegő formulas in the natural framework of the orthogonal Laurent polynomials on the unit circle, and inspired by the ordinary polynomial situation, we will deal with subspaces of Λ of the form $\mathcal{L}_n\mathcal{L}_{r*} = \Lambda_{-(p_n+q_r),q_n+p_r}$ with dimension $n+r+1$ (observe that $\mathcal{L}_{n-1} \subset \mathcal{L}_n\mathcal{L}_{r*}$, $0 \le r \le n-1$). Now we might try to make $r = r(n)$ as large as possible. The first trial should be to consider $r = n-1$, but a negative answer is proved in [16]:

Theorem 4.3. *There cannot exist an n-point quadrature formula like (4.2) with nodes on $\mathbb{T}$ which is exact in $\mathcal{L}_n\mathcal{L}_{(n-1)*}$ for any given arbitrary generating sequence $\{p_n\}_{n=0}^{\infty}$.* □

The second step is to consider $r = n-2$. For this purpose, we set $ß_n = p_n - p_{n-2} \in \{0,1,2\}$. The results obtained in [16] are summarized in:

Theorem 4.4. *Let $\{\chi_n(z)\}_{n=0}^{\infty}$ be the sequence of orthonormal Laurent polynomials with respect to the measure μ and the ordering induced by the generating sequence $\{p_n\}_{n=0}^{\infty}$. Let $\{\delta_n\}_{n=0}^{\infty}$ be the sequence of Schur parameters associated with μ, η_n given by (2.3), suppose that $ß_n = p_n - p_{n-2} = 1$ and consider*

$$R_n^u(z) = C_n \left[\eta_n \chi_n(z) + \tau_n \chi_{n-1}(z)\right], \tag{4.5}$$

where $C_n \neq 0$ and

$$\tau_n = \begin{cases} \overline{u - \delta_n} & \text{if} \quad s_n = 1, \\ u - \delta_n & \text{if} \quad s_n = 0, \end{cases} \tag{4.6}$$

with $u \in \mathbb{T}$. Then,

1. *$R_n^u(z)$ has exactly n distinct zeros on $\mathbb{T}$.*
2. *If $z_1, \ldots, z_n$ are the zeros of $R_n^u(z)$, then there exist positive numbers $\lambda_1, \ldots, \lambda_n$ such that*

$$I_n(f) = \sum_{j=1}^{n} \lambda_j f(z_j) = I_\mu(f), \quad \text{for all} \ \ f \in \mathcal{L}_n\mathcal{L}_{(n-2)*}. \tag{4.7}$$

3. *There cannot exist an n-point quadrature formula with nodes on $\mathbb{T}$ which is exact in $\mathcal{L}_n\mathcal{L}_{(n-2)*}$ if $ß_n = 0$ or $ß_n = 2$.* □

Thus, under the assumption that $ß_n = p_n - p_{n-2} = 1$, we see that $\mathcal{L}_n\mathcal{L}_{(n-2)*} = \Lambda_{-(n-1),n-1}$. Therefore, the quadrature rule given by (4.7) coincides with an n-point Szegő quadrature formula for μ and, taking into account that the solutions of the finite difference equation $ß_n = p_n - p_{n-2} = 1$ for $n \ge 2$ are given by

$$p_n = \begin{cases} E\left[\frac{n}{2}\right] & \text{if} \quad p_0 = p_1 = 0, \\ E\left[\frac{n+1}{2}\right] & \text{if} \quad p_0 = 0\ , \quad p_1 = 1, \end{cases}$$

we see that the natural balanced orderings earlier introduced by Thron in [54] are again recovered. Furthermore, they are the only ones which produce quadrature formulas with nodes on $\mathbb{T}$ with a maximal domain of validity. On the other hand, as we have seen in Section 3, these orderings correspond with the narrowest matricial representation of a sequence of orthonormal Laurent polynomials.

We want to emphasize that the generating sequence $\{p_n\}_{n=0}^{\infty}$ necessary to provides the maximal domain of validity of Szegő quadrature rules coincides with the ordering for which we obtain the narrowest matricial representation for the multiplication operator defined in Λ.

In order to complete the construction of such quadrature formulas we give expressions for the weights also proved in [16]:

Theorem 4.5. *Let $\{\chi_n(z)\}_{n=0}^{\infty}$ be the sequence of orthonormal Laurent polynomials with respect to the measure μ and the ordering induced by a generating sequence $\{p_n\}_{n=0}^{\infty}$. Then, the weights $\{\lambda_j\}_{j=1}^{n}$ for the quadrature formula (4.7) are given for $j = 1, \ldots, n$ by*

$$\lambda_j = \frac{1}{\sum_{k=0}^{n-1} |\chi_k(z_j)|^2} \tag{4.8}$$

or equivalently, by

$$\lambda_j = \frac{(-1)^{s_n}}{2\Re\left[z_j \chi_n'(z_j)\overline{\chi_n(z_j)}\right] + (p_n - q_n)\,|\chi_n(z_j)|^2}\ , \tag{4.9}$$

where the nodes $\{z_j\}_{j=1}^{n}$ are the zeros of R_n^u given by (4.5), or the zeros of P_n given by (4.3). □

Once quadrature formulas on $\mathbb{T}$ with a maximal domain of validity have been constructed (nodes from (4.3) or (4.5) and weights by (4.8) or (4.9)) we need some efficient method to compute them. Traditionally the numerical values for nodes and weights of Szegő quadrature have mostly been obtained either for measures whose sequences of Szegő polynomials are explicitly known or by computing the polynomials by Levinson's algorithm (see e.g. [48], [21] and [37]). In these situations the zeros of (4.3) or (4.5) can be found by using any standard root finding method available in the literature (as for a specific procedure concerning rational modifications of the Lebesgue measure see also [36]).

In the rest of the section we shall review some alternative strategies to effectively compute the nodes $\{z_j\}_{j=1}^{n}$ and weights $\{\lambda_j\}_{j=1}^{n}$ for an n-point Szegő quadrature formula based on the matricial representations of Section 3. These will involve eigenvalue computations much like the traditional Golub-Welsch algorithm for Gauss-Christoffel quadrature formulas on the real line [35].

4.2. Computation using Hessenberg matrices

Recall the nodal polynomial $P_n(z) = \prod_{j=1}^{n}(z - z_j)$ given by (4.3) we can write from (2.2) (see [11]) as

$$P_n(z) = P_n^u(z) = z\rho_{n-1}(z) + u\rho_{n-1}^*(z)\ \ ,\ |u| = 1.$$

It can be easily checked that $\{z\varphi_0(z), \ldots, z\varphi_{n-2}(z), -u\varphi_{n-1}^*(z)\}$ is an orthonormal basis of $\mathbb{P}_{n-1}$ which must be related to $\{\varphi_0(z), \ldots, \varphi_{n-1}(z)\}$ by a unitary matrix U_n. Setting for $n \geq 0$, $e_n = \langle \rho_n, \rho_n \rangle_\mu$ (recall that $e_n = \prod_{k=1}^{n} \eta_k^2$ for $n \geq 1$ and that $\varphi_n = e_n^{-1/2}\rho_n$) it is straight forward to check that U_n is the n-truncation of the Hessenberg matrix (3.1) in which the Schur parameter δ_n is replaced by u. Therefore we shall denote it as $\mathcal{H}_n^u$:

$$\mathcal{H}_n^u := \begin{pmatrix} d_{0,0} & d_{0,1} & 0 & \cdots & 0 \\ d_{1,0} & d_{1,1} & d_{1,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{n-1,0} & d_{n-1,2} & d_{n-1,2} & \cdots & d_{n-1,n-1} \end{pmatrix}, \tag{4.10}$$

where all entries $d_{i,j}$ are equal to the corresponding entries $h_{i,j}$ of (3.1), except on the last row, i.e.,

$$h_{i,j} = \begin{cases} -\overline{\delta_j} u \sqrt{\frac{e_n}{e_j}} & \text{if} \quad i = n-1\ ,\ j \leq n-1, \\ -\overline{\delta_j}\delta_{i+1}\sqrt{\frac{e_i}{e_j}} & \text{if} \quad i \leq n-2\ ,\ j \leq i, \\ \eta_{i+1} & \text{if} \quad i \leq n-2\ ,\ j = i+1. \end{cases} \tag{4.11}$$

Hence, we can write

$$\begin{pmatrix} z\varphi_0(z) \\ \vdots \\ z\varphi_{n-2}(z) \\ -u\varphi_{n-1}^*(z) \end{pmatrix} = \mathcal{H}_n^u \begin{pmatrix} \varphi_0(z) \\ \vdots \\ \varphi_{n-2}(z) \\ \varphi_{n-1}(z) \end{pmatrix}.$$

Observe that, if we introduce the notation $\mathcal{H}_n(\delta_0, \ldots, \delta_n)$ to indicate the principal submatrix of $\mathcal{H}(\delta)$ (given by (3.1)) of order n, then $\mathcal{H}_n^u = \mathcal{H}_n(\delta_0, \ldots, \delta_{n-1}, u)$ (i.e., replace δ_n by u). Now,

$$\begin{pmatrix} z\varphi_0(z) \\ \vdots \\ z\varphi_{n-2}(z) \\ -u\varphi_{n-1}^*(z) \end{pmatrix} = \begin{pmatrix} z\varphi_0(z) \\ \vdots \\ z\varphi_{n-2}(z) \\ z\varphi_{n-1}(z) - z\varphi_{n-1}(z) - u\varphi_{n-1}^*(z) \end{pmatrix} = z \begin{pmatrix} \varphi_0(z) \\ \vdots \\ \varphi_{n-2}(z) \\ \varphi_{n-1}(z) \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{P_n^u(z)}{\sqrt{e_{n-1}}} \end{pmatrix}.$$

Thus, we have the identity

$$zV_n(z) = \mathcal{H}_n^u V_n(z) + b_n(z), \tag{4.12}$$

where

$$V_n(z) = (\varphi_0(z), \varphi_1(z), \cdots, \varphi_{n-1}(z))^T, \quad b_n(z) = \left(0, \cdots, 0, \frac{P_n^u(z)}{\sqrt{e_{n-1}}}\right)^T,$$

with $|u| = 1$. From (4.12) one sees that any zero ξ of $P_n^u(z)$ is an eigenvalue of $\mathcal{H}_n^u$, and vice versa, with associated eigenvector $V_n(\xi)$. So, let z_j be a zero of $P_n^u(z)$ for $j = 1, \ldots, n$ and consider the corresponding normalized eigenvector of $\mathcal{H}_n^u$:

$$W_n(z_j) = \frac{V_n(z_j)}{\left[\sum_{k=0}^{n-1} |\varphi_k(z_j)|^2\right]^{1/2}}.$$

Now, taking into account that for any $z \in \mathbb{T}$, $|\varphi_k(z)|^2 = |\chi_k(z)|^2$, from (4.8) it follows that

$$W_n(z_j) = \lambda_j^{1/2} V_n(z_j). \tag{4.13}$$

If we write $W_n(z_j) = (q_{0,j}, \ldots, q_{n-1,j})^T$ and select out the first components of both sides of (4.13), we obtain $q_{0,j} = \lambda_j^{1/2}\varphi_0(z_j)$. But, since we are dealing with a probability measure, then $\varphi_0 \equiv 1$ and hence, $\lambda_j = q_{0,j}^2$ for all $j = 1, \ldots, n$. In short, the following theorem has been proved:

Theorem 4.6. *Let $I_n(f)$ be the n-th Szegő quadrature formula (4.7) in a balanced situation. Then,*

1. *The nodes $\{z_j\}_{j=1}^n$ are the eigenvalues of $\mathcal{H}_n^u = \mathcal{H}_n(\delta_0, \ldots, \delta_{n-1}, u)$ given by (4.10)-(4.11), for all $u \in \mathbb{T}$.*
2. *The weights $\{\lambda_j\}_{j=1}^n$ are given by the squared modulus of the first component of the corresponding normalized eigenvectors.* □

This can be stated in an elegant formula following Golub and Meurant [33, 34] as follows:

$$I_n(f) = \mathbf{e}_1^T f(\mathcal{H}_n^u)\mathbf{e}_1,$$

where $\mathbf{e}_1 = [1, 0, \ldots, 0]^T$. Indeed, note that $\mathcal{H}_n^u$ is unitary so that with the previous notation, its eigenvalue decomposition can be written as $\mathcal{H}_n^u = \mathcal{W}_n Z_n \mathcal{W}_n^\dagger$ with $Z_n = \text{diag}(z_1, \ldots, z_n)$, $\mathcal{W}_n$ contains the normalized eigenvectors as columns (assume the first components are real), and † denotes the Hermitian transpose. Since $[\lambda_1^{1/2}, \ldots, \lambda_n^{1/2}] = \mathbf{e}_1^T \mathcal{W}_n$ it follows that

$$I_n(f) = \sum_{k=1}^n \lambda_k f(z_k) = \mathbf{e}_1^T \mathcal{W}_n f(Z_n) \mathcal{W}_n^\dagger \mathbf{e}_1 = \mathbf{e}_1^T f(\mathcal{H}_n^u)\mathbf{e}_1.$$

In this form there are many applications in linear algebra solving large scale problems with iterative methods (e.g., [40]).

Special purpose algorithms for eigenvalue computation of unitary Hessenberg matrices have been designed in the literature. See for example [25, 39].

4.3. Computation using CMV matrices

We give now an alternative approach to the computation of a Szegő quadrature formula (4.7) by using truncations of the five-diagonal matrix $\mathcal{C}(\delta)$. In the next result we will use the matrix $\mathcal{C}_n(\delta_1, \ldots, \delta_{n-1}, u)$, that is, the n-th principal submatrix of $\mathcal{C}(\delta)$ of order n where the Schur parameter δ_n is replaced by $u \in \mathbb{T}$. Without loss of generality, we can fix the ordering induced by $p_n = E\left[\frac{n}{2}\right]$ (recall that the matrix representation associated with the ordering induced by $p_n = E\left[\frac{n+1}{2}\right]$ is $\mathcal{C}(\delta)^T$).

Theorem 4.7. *Let $I_n(f)$ be the n-th Szegő quadrature formula (4.7) in a balanced situation. Then,*

1. *The nodes $\{z_j\}_{j=1}^n$ are the eigenvalues of $\mathcal{C}_n^u = \mathcal{C}_n(\delta_1, \ldots, \delta_{n-1}, u)$, for all $u \in \mathbb{T}$.*
2. *The weights $\{\lambda_j\}_{j=1}^n$ are given by the squared modulus of the first components of the corresponding normalized eigenvectors.*

The first part has been already deduced in [14] by using operator theory techniques while a proof based on the recurrence relations satisfied by the family of orthonormal Laurent polynomials is given in [10, Theorem 5.7].

Thus also in this case the Golub-Meurant formula holds $I_n(f) = \mathbf{e}_1^T f(\mathcal{C}_n^u)\mathbf{e}_1$.

4.4. Computation using 'snake-shaped' matrix factorizations

Finally, suppose we have a matrix representation $\mathcal{S}$ of the multiplication operator in Λ consisting in a factorization of $\prod_{k=1}^{\infty} G_{k-1,k}$, where the factors under the $\prod$-symbol may occur in a certain order, but by following the recipe given in Theorem 3.5. A drawback is that the principal $n \times n$ submatrix of $\mathcal{S}$ is in general not unitary and hence it has eigenvalues strictly inside the unit disk. A solution to this is to slightly modify the principal $n \times n$ submatrix of $\mathcal{S}$ in such a way that it becomes unitary. To do this, Gragg [38] and also Watkins [56] introduced the idea to redefine the 2×2 block in the n-th Givens transformation by

$$\tilde{G}_{n-1,n} := \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix},$$

where $u, v \in \mathbb{T}$ (v will actually be irrelevant for what follows), so that we can absorb u and v in the 2×2 blocks of the previous and next Givens transformations, respectively:

$$\tilde{G}_{n-2,n-1} := \begin{cases} \tilde{G}_{n-2,n-1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} & if \quad s_{n-1} = 0, \\ \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \cdot \tilde{G}_{n-2,n-1} & if \quad s_{n-1} = 1, \end{cases} , \tilde{G}_{n,n+1} := \begin{cases} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \cdot \tilde{G}_{n,n+1} & if \quad s_n = 0, \\ \tilde{G}_{n,n+1} \cdot \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} & if \quad s_n = 1. \end{cases}$$

We can then put $\tilde{G}_{n-1,n} = I_2$, and it is easily seen that $\mathcal{S} = UV$ with $U = \prod_{k=1}^{n-1} G_{k-1,k}$ (submatrix formed by rows and columns $1, \ldots, n$) and $V = \prod_{k=n+1}^{\infty} G_{k-1,k}$ (submatrix formed by rows and columns $n+1, \ldots, \infty$). U and V have complementary zero pattern, and hence they commute each other. The matrix U is a $n \times n$ unitary snake-shape matrix factorization and coincide with the $n \times n$ submatrix of $\mathcal{S}$ with $\delta_n \in \mathbb{D}$ replaced by $u \in \mathbb{T}$. Now, one could ask why there is such a similarity between finite Hessenberg and CMV matrices (Theorems 4.6 and 4.7). This is explained in the next result, which is essentially due to Ammar, Gragg and Reichel [1], and it is based on the general fact that AB and BA have the same eigenvalues (see [17]):

Theorem 4.8. *The eigenvalues and the first component of the normalized eigenvectors of U depend on the Schur parameters but not on the shape of the snake.*

5. A numerical example

In order to numerically illustrate the results given in the previous section we demonstrate the computation of nodes and weights of the Szegő quadrature formulas considering the absolutely continuous measure defined on $[-\pi, \pi]$ by $d\mu(\theta) = \omega(\theta)d\theta$ with

$$\omega(\theta) = \frac{1}{\sqrt{2\pi \log\left(\frac{1}{q}\right)}} \sum_{j=-\infty}^{\infty} \exp\left(\frac{-(\theta - 2\pi j)^2}{2\log\left(\frac{1}{q}\right)}\right) \quad , \quad q \in (0,1).$$

The corresponding monic orthogonal polynomials are the so-called Rogers-Szegő q-polynomials. Throughout this section we also fix the ordering induced by the generating sequence $p_n = E\left[\frac{n+1}{2}\right]$. An explicit expression for such polynomials is given in [52, Chapter 1] and so the following explicit expression for the corresponding monic orthogonal Laurent polynomials is deduced from Proposition 2.2:

$$\phi_n(z) = \begin{cases} \sum_{j=-k}^{k} (-1)^{j+k} \begin{bmatrix} 2k \\ k+j \end{bmatrix}_q q^{\frac{k-j}{2}} z^j & if \quad n = 2k, \\ \sum_{j=-(k+1)}^{k} (-1)^{j+k+1} \begin{bmatrix} 2k+1 \\ k-j \end{bmatrix}_q q^{\frac{j+k+1}{2}} z^j & if \quad n = 2k+1, \end{cases} \tag{5.1}$$

where, as usual, the q-binomial coefficients $\begin{bmatrix} n \\ j \end{bmatrix}_q$ are defined by

$$\begin{bmatrix} n \\ j \end{bmatrix}_q := \frac{(n)_q}{(j)_q(n-j)_q} = \frac{(1-q^n)\cdots(1-q^{n-j+1})}{(1-q)\cdots(1-q^j)}, \quad (n)_q := (1-q)(1-q^2)\cdots(1-q^n), \quad (0)_q \equiv 1.$$

Now, writing $\phi_{2k}(z) = \sum_{j=-k}^{k} a_j z^j$ and $\phi_{2k+1}(z) = \sum_{j=-(k+1)}^{k} b_j z^j$, then the coefficients $\{a_j\}_{j=-k}^{k}$ and $\{b_j\}_{j=-(k+1)}^{k}$ can be recursively computed by

$$a_k = 1, \quad a_j = -a_{j+1}\sqrt{q}\frac{1-q^{k+j+1}}{1-q^{k-j}}, \quad -k \le j \le k-1,$$

$$b_{-(k+1)} = 1, \quad b_{j+1} = -b_j\sqrt{q}\frac{1-q^{k-j}}{1-q^{k+j+2}}, \quad -(k+1) \le j \le k-1.$$

Also for this particular weight function the corresponding sequence of Schur parameters is given by $\delta_n = (-1)^n q^{\frac{n}{2}}$ for all $n \ge 1$ (see [52, Chapter 1]).

In order to illustrate our results we have computed the nodes and the weights of the n-point Szegő quadrature formula (4.7) with $n \in \{15, 16\}$ and $q \in \{0.1, 0.25, 0.5, 0.75, 0.9\}$ (our aim is merely illustrative and for that reason, a small number of nodes is used). We propose three methods:

1. The nodes are the zeros of $R_n^u(z)$ given by (4.5) (we have taken $u = 1$). As for the weights, we can make use of (4.9) taking into account that $\chi_n(z) = \frac{\phi_n(z)}{\sqrt{(1-q)\cdots(1-q^n)}}$, with ϕ_n explicitly given by (5.1).
2. From Theorem 4.6 by computing the eigenvalues and the first component of the normalized eigenvectors of $\mathcal{H}_{15}(\delta_0, \ldots, \delta_{14}, 1)$ and $\mathcal{H}_{16}(\delta_0, \ldots, \delta_{15}, 1)$ by using a standard eigenvalue-finding method.
3. As before but now from Theorem 4.7, by considering the matrices $\mathcal{C}_{15}(\delta_0, \ldots, \delta_{14}, 1)$ and $\mathcal{C}_{16}(\delta_0, \ldots, \delta_{15}, 1)$.

The results for the three methods by using MATLAB®[3] software for $n = 15$ are displayed in Table 1, whereas for $n = 16$ we have represented the locations of the nodes of the Szegő quadrature formula for the Rogers-Szegő case, in the plots of Figure 1. Since our aim is merely illustrative, a low number of nodes has been considered here and the results for the three methods coincide. We display them only once.

Table 1: Five different cases, depending on the value of q, for a 15-point Rogers-Szegő quadrature formula with $u = 1$. Since ω is a symmetric weight and u is real, the coefficients of the n-th orthonormal Laurent polynomial are real, and hence it follows from (4.8) that the weights corresponding to a pair of complex conjugate nodes are equal. The nodes are $e^{i\theta_k}$ and the left columns give the θ_k. Moreover, it is easy to check that for $u = 1$, $z = -1 = e^{i\pi}$ is always a node of the rule if n is odd whereas if n is even, then the rule has no real nodes (see Figure 1).

nodes $= z_j = \exp(i\theta_j),\ j = 1, \dots, 15$	
nodes(θ_j) | $q = 0.1$ |	weights
$\pm$2.020693660378099E − 01	1.053824229964858E − 01
$\pm$6.067535362211522E − 01	9.847839556827828E − 02
$\pm$1.013158403916661E + 00	8.603266363333939E − 02
$\pm$1.422746244907493E + 00	7.039800735830885E − 02
$\pm$1.837569086272958E + 00	5.433799575334528E − 02
$\pm$2.260636174383480E + 00	4.048668900300233E − 02
$\pm$2.695488166595799E + 00	3.104786939314506E − 02
3.141592653589793E + 00	2.767191258819483E − 02
nodes(θ_j) | $q = 0.25$ |	weights
$\pm$1.967461263446201E − 01	1.315190732477971E − 01
$\pm$5.909695184434383E − 01	1.180261484415183E − 01
$\pm$9.874875672980075E − 01	9.493562791891702E − 02
$\pm$1.388225415207939E + 00	6.827907861099329E − 02
$\pm$1.795982330151666E + 00	4.377944803893124E − 02
$\pm$2.215781208203728E + 00	2.509343083961357E − 02
$\pm$2.658750327222901E + 00	1.357168625676975E − 02
3.141592653589793E + 00	9.591013290928357E − 03
nodes(θ_j) | $q = 0.5$ |	weights
$\pm$1.850326604783442E − 01	1.730866443286772E − 01
$\pm$5.562044865371203E − 01	1.427872916287331E − 01
$\pm$9.308502697829589E − 01	9.676375236738939E − 02
$\pm$1.311877002500532E + 00	5.337006798489684E − 02
$\pm$1.703437459286332E + 00	2.356234786279431E − 02
$\pm$2.112739066667703E + 00	8.086491887354821E − 03
$\pm$2.557426706172959E + 00	2.052052160650692E − 03
3.141592653589793E + 00	5.827035590192061E − 04
nodes(θ_j) | $q = 0.75$ |	weights
$\pm$1.592753187503438E − 01	2.268901381737077E − 01
$\pm$4.793100886472880E − 01	1.605569608507605E − 01
$\pm$8.040304946336945E − 01	7.938814404567303E − 02
$\pm$1.137458980003955E + 00	2.665134725379743E − 02
$\pm$1.485540114688343E + 00	5.753874019524687E − 03
$\pm$1.859004158783506E + 00	7.188047679097230E − 04
$\pm$2.284765643114921E + 00	4.044806281122144E − 05
3.141592653589793E + 00	5.656516437058855E − 07
nodes(θ_j) | $q = 0.9$ |	weights
$\pm$1.171019738894465E − 01	2.699653291356543E − 01
$\pm$3.526366611998939E − 01	1.615275685720382E − 01
$\pm$5.923813640639528E − 01	5.647780298658640E − 02
$\pm$8.399840849035561E − 01	1.093217945976319E − 02
$\pm$1.100902826514202E + 00	1.055526272428221E − 03
$\pm$1.385141693734147E + 00	4.121083711475061E − 05
$\pm$1.718290179570013E + 00	3.827326472626507E − 07
3.141592653589793E + 00	3.530250294293521E − 12

[3] MATLAB is a registered trademark of The MathWorks, Inc.

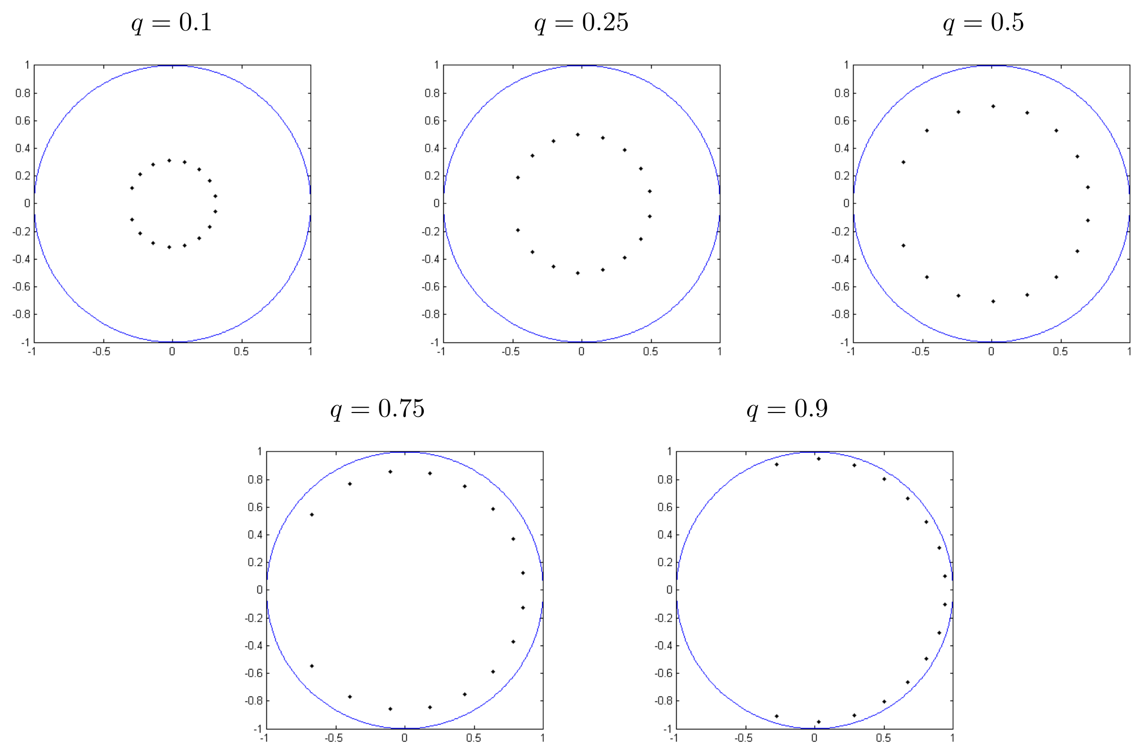


Figure 1: Five different distributions of nodes (depending on the value of q) in a 16-point Rogers-Szegő quadrature formula with $u = 1$. From Proposition 2.2 and since we have fixed $p_n = E\left[\frac{n+1}{2}\right]$, the Mazel-Geronimo-Hayes Theorem is clearly observed (see [52, Theorem 1.6.11]): the zeros of the Rogers-Szegő polynomials lie on the circle $\mathbb{T}_q = \{z \in \mathbb{C} : |z| = \sqrt{q}\}$. Moreover, it was proved in [20, Section 5] that if $q \to 0^+$ and $q \to 1^-$, then the Rogers-Szegő weight converges to the Lebesgue measure and to a δ-Dirac distribution at $z = 1$, respectively. This explains the behaviour of the distribution of the nodes in Table 1 and Figure 1, and also the reason why the weights corresponding to nodes with higher real parts are greater than those weights associated with nodes with lower real parts, specially as q increases (see Table 1).

Here, it should be remarked that if a prescribed precision is required, then a higher number of nodes should be used, and special eigenvalue-finding methods should be considered because of the error propagation. For reasons of efficiency and numerical stability, the eigenvalue computation should preferably be performed using their factorization as a product of Givens transformations, rather than using their entry-wise expansions. A whole variety of practical eigenvalue computation algorithms for unitary Hessenberg and CMV matrices has already been developed in the literature, e.g. LR-iterations, implicit QR-algorithms, divide and conquer algorithms, two half-size singular value decompositions, matrix pencils methods or the unitary equivalent of the Sturm sequence method (see [25]).

In the same way that we have analyzed the computation of Szegő quadrature formulas using the matricial representation of Szegő polynomials and orthonormal Laurent polynomials, we shall in what follows study the situation for the rational case. First of all we will introduce the orthogonal rational functions and their matricial representations.

6. Orthogonal rational functions

Polynomials can be considered as rational functions with all their poles at infinity. The idea of orthogonal rational functions is to generalize orthogonal polynomials by allowing other poles than infinity. In the context of this survey we will only consider orthogonal polynomials with respect to a measure on the unit circle $\mathbb{T}$, and since infinity is as far away from the unit circle as possible, the generalization will consist in considering poles in the "neighborhood of infinity" which is here loosely interpreted as all poles are outside the closed unit disk. But just as in the polynomial case the Laurent polynomials introduce also poles at the origin using the substar conjugate, also here we will be able to introduce poles inside the disk using the same substar conjugate.

In order to introduce our spaces of rational functions we use a Möbius transform as a building instrument. It is defined as

$$\zeta_\alpha(z) := \frac{\varpi_\alpha^*(z)}{\varpi_\alpha(z)} = \frac{z-\alpha}{1-\overline{\alpha}z}, \quad \alpha \in \mathbb{D},$$

where $\varpi_\alpha(z) = 1-\overline{\alpha}z$. This ζ_α is a bijection on $\overline{\mathbb{C}}$ that leaves the sets $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{E}$ invariant. Note that the inverse is given by

$$w = \zeta_\alpha(z) \Leftrightarrow z = \zeta_\alpha^{-1}(w) = \zeta_{-\alpha}(w).$$

If $\alpha \in \mathbb{D}$, then this first degree rational function ζ_α has a pole in $\mathbb{E}$. So, to introduce the rational functions with prescribed poles we shall define a sequence $\{\alpha_n\}_{n=0}^\infty \subset \mathbb{D}$ (we define $\alpha_0 = 0$ in the rest of this paper) and introduce elementary Blaschke factors as $\zeta_n = \zeta_{\alpha_n}$. We shall similarly denote ϖ_{α_n} as ϖ_n, etc. By defining Blaschke products $\{B_n\}_{n\geq 0}$ as

$$B_0 = 1 \text{ and } B_n = \zeta_1 \ldots \zeta_n = \frac{\pi_n^*}{\pi_n} \text{ for } n \geq 1,$$

where $\pi_n(z) = \varpi_1(z)\cdots\varpi_n(z)$, we can finally define a nested sequence of rational subspaces, defined by

$$\mathcal{R}_n := \operatorname{span}\{B_0, B_1, \ldots, B_n\} = \left\{ \frac{p(z)}{\pi_n(z)} : p \in \mathbb{P}_n \right\}.$$

If $f \in \mathcal{R}_n$ then it has all its poles in the set $\hat{\alpha} = \{\hat{\alpha}_i : i = 1, \ldots, n\} \subset \mathbb{E}$, with $\hat{\alpha}_i = 1/\overline{\alpha}_i$. Note that the polynomials are included as a special case because $\mathcal{R}_n = \mathbb{P}_n$ if all α_k are chosen to be zero. By our choice of the α_n sequence, the overall space $\mathcal{R} = \cup_{n\geq 0}\mathcal{R}_n$, i.e., $\mathcal{R} = \operatorname{span}\{B_n\}_{n\geq 0}$ will only contain functions analytic in $\mathbb{D}$. As mentioned above, poles inside the disk can be introduced using the substar conjugate:

$$\mathcal{R}_{n*} := \{f : f_* \in \mathcal{R}_n\} = \operatorname{span}\{B_{1*}, \ldots, B_{n*}\} = \left\{ \frac{q(z)}{\pi_n^*(z)} : q \in \mathbb{P}_n \right\},$$

and $\mathcal{R}_* = \cup_{n\geq 0}\mathcal{R}_{n*}$. For $m, n \in \mathbb{N}$ denote $\mathcal{L}_{-m,n} = \mathcal{R}_{m*} + \mathcal{R}_n$ and $\mathcal{L} = \mathcal{R}_* + \mathcal{R}$, which generalizes the space of all Laurent polynomials. Note that

$$\mathcal{L}_{-m,n} := \operatorname{span}\{B_{m*}, \ldots, B_{1*}, 1, B_1, \ldots, B_n\}.$$

We also need a reciprocal operation generalizing the polynomial reciprocal. To arrange that $\mathcal{R}_n^* \subset \mathcal{R}_n$, we shall define it for any $r_n = p_n/\pi_n \in \mathcal{R}_n \setminus \mathcal{R}_{n-1}$ as follows:

$$r_n^* = B_n r_{n*} = \frac{p_n^*}{\pi_n} \in \mathcal{R}_n.$$

Given the inner product induced by a probability measure μ on $\mathbb{T}$, we can construct the sequence $\{\phi_n\}_{n\geq 0}$ of orthogonal rational functions (ORF) according to the sequence $\{\alpha_n\}_{n\geq 0}$ satisfying $\mathcal{R}_{n-1} \perp \phi_n \in \mathcal{R}_n \setminus \mathcal{R}_{n-1}$ for $n \geq 1$ and $\phi_0 = 1$. These are not uniquely defined since we can always multiply with a unimodular constant. This constant can be fixed by making the leading coefficient of ϕ_n with respect to the basis $\{B_k\}_{k\geq 0}$ positive, i.e., $\overline{\phi_n^*(\alpha_n)} > 0$. In this paper, this constant is not essential and we prefer to use the following simplified recurrence which fixes these unimodular factors implicitly (see [8]):

$$\begin{pmatrix} \phi_n(z) \\ \phi_n^*(z) \end{pmatrix} = c_n \frac{\varpi_{n-1}(z)}{\varpi_n(z)} \begin{pmatrix} 1 & \delta_n \\ \overline{\delta_n} & 1 \end{pmatrix} \begin{pmatrix} \zeta_{n-1}(z)\phi_{n-1}(z) \\ \phi_{n-1}^*(z) \end{pmatrix}, \qquad n \geq 1, \tag{6.1}$$

with

$$\delta_n = \frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}, \qquad c_n = \sqrt{\frac{\varpi_n(\alpha_n)}{\varpi_{n-1}(\alpha_{n-1})} \frac{1}{1-|\delta_n|^2}}. \tag{6.2}$$

A first verification of the rational formulas is that at any moment, setting all $\alpha_n = 0$, one should recover the polynomial equivalent. In this case, (6.1) turns out to be exactly (2.4) since indeed then $\zeta_n(z) = z$, $\varpi_n(z) = 1$, $c_n = 1/\eta_n$ and the δ_n become the Schur parameters. For that reason we shall keep the notation δ_n since they are the direct generalizations of the Schur parameters in the rational case. As in the polynomial situation these parameters $\{\delta_n\}_{n\geq 1}$ of (6.1) lie in $\mathbb{D}$.

We should mention here that we have chosen the α_n to be all in $\mathbb{D}$ for convenience, but that it is possible to choose them anywhere outside $\mathbb{T}$ on condition that $\alpha_i \hat{\alpha}_j \neq 1$ for all i and j. This was used in the context of matrix valued ORF in the papers by Lasarow and coworkers (e.g., [27–29, 47]). A different approach to introduce poles in $\mathbb{D}$ and $\mathbb{E}$ for a balanced situation is explained in Section 7 where the CMV representation is derived. This ordering of the poles inside or outside the disk will influence the computation of sequence of ORF and the spaces that are spanned in the sequence just like in the Laurent case. We do not include a full generalization here for simplicity. Anyway, whatever the location and order of the α_n, the resulting rational Szegő quadrature formulas derived in Section 8 will be exact in spaces of rational functions with poles in the set of selected points α_k (in $\mathbb{D}$ and/or $\mathbb{E}$) and in the reflected points $\hat{\alpha}_k$, i.e., the space $\mathcal{R}_n + \mathcal{R}_{n*}$ and that does not depend on whether the $\mathcal{R}_n$ has or has not all its poles in $\mathbb{E}$.

7. Matricial representations of ORF

7.1. The multiplication operator in rational function spaces

Like in the polynomial case, it will be crucial to find the matrix representation for the multiplication operator $M : \mathcal{L} \to \mathcal{L}$, $f(z) \mapsto zf(z)$ with respect to some ORF basis. Choosing the ORF basis $\{\phi_n\}_{n\geq 0}$ that we just introduced, and noting that $\mathcal{L}$ is M-invariant we get a representation $\mathcal{V}$ that was obtained in [55] as

$$\mathcal{V} = \zeta_{\mathcal{A}}^{-1}(\mathcal{H}), \tag{7.1}$$

$\mathcal{A}$ being the diagonal matrix with the sequence $\{\alpha_n\}_{n\geq 0}$ on its diagonal, $\mathcal{H}$ is like the Hessenberg matrix given by (3.1) where $\eta_n = \sqrt{1-|\delta_n|^2}$ with δ_n as in (6.2) and $\zeta_{\mathcal{A}}^{-1}(\mathcal{H})$ is the inverse of the Möbius transformation $\zeta_\alpha(z)$ but generalized to a matrix form:

$$\zeta_{\mathcal{A}}^{-1}(\mathcal{H}) = \nu_{\mathcal{A}}^{-1}(\mathcal{I} + \mathcal{A}^\dagger \mathcal{H})(\mathcal{H} + \mathcal{A})^{-1} \nu_{\mathcal{A}^\dagger}, \tag{7.2}$$

where, $\nu_{\mathcal{A}} = \sqrt{1 - \mathcal{A}\mathcal{A}^\dagger}$ and $\mathcal{A}^\dagger$ is the adjoint of $\mathcal{A}$. (Note that in a scalar version ν-factors cancel out.) Thus where in the polynomial case, the unitary matrix $\mathcal{V}$ was a Hessenberg matrix, here it is the operator Möbius transformation of a Hessenberg matrix. There is some "rational intuition" for this formula. To go from the polynomial to the rational case, we have transformed products of z into products of the form $\zeta_i(z)$, e.g., powers of z are replaced by Blaschke products. Hence to find a multiplication with z in the rational setting (and this multiplication is what $\mathcal{H}$ represented in the polynomial case) we have to do the inverse transform to go from the $\zeta_i(z)$ back to z, and this is exactly what (7.2) does.

But there is an alternative basis for $\mathcal{L}$ that can be used. In the polynomial case, the basis of orthogonal polynomials in $\mathbb{P}$ could be replaced by a basis of orthogonal Laurent polynomials in Λ. Similarly, we can also in the rational case obtain a somewhat simpler matricial representation for M than the one given above.

However in the rational case, it is not so obvious how this should be done. Let us consider the balanced situation for simplicity. Then we can consider the rational generalization of the nesting of the Laurent polynomial subspaces of the form Λ_{-p_n,q_n} with $p_n = E\left[\frac{n}{2}\right]$ or

$p_n = E\left[\frac{n+1}{2}\right]$ and $p_n + q_n = n$. So that could be generalized to the rational spaces $\mathcal{L}_n = \mathcal{L}_{-p_n,q_n} = \text{span}\{B_{p_n*}, \ldots, B_0, \ldots, B_{q_n}\} = \text{span}\{\phi_{-p_n*}, \ldots, \phi_0, \ldots, \phi_{q_n}\}$. The problem is that the latter is not an orthogonal basis. Instead we will reshuffle the factors of the Blaschke products and define odd and even Blaschke products:

$$B_0^o = B_0^e = 1 \text{ while } B_n^o = \zeta_1\zeta_3 \ldots \zeta_{2n-1}, \quad B_n^e = \zeta_2\zeta_4 \ldots \zeta_{2n} \text{ for } n \geq 1,$$

and consider the rational functions given by

$$\chi_{2n} = B_{n*}^e \phi_{2n}^*, \qquad \chi_{2n+1} = B_{n*}^e \phi_{2n+1}, \qquad n \geq 0.$$

The subspaces $\mathcal{L}_n = \text{span}\,\{\chi_0, \chi_1, \ldots, \chi_n\}$ are

$$\mathcal{L}_{2n} = B_{n*}^e \mathcal{R}_{2n} = \text{span}\left\{B_{0*}^e, B_1^o, B_{1*}^e, \ldots, B_{n-1*}^e, B_n^o, B_{n*}^e\right\} = \mathcal{L}_{-n,n}$$

and

$$\mathcal{L}_{2n+1} = B_{n*}^e \mathcal{R}_{2n+1} = \text{span}\left\{B_{0*}^e, B_1^o, B_{1*}^e, \ldots, B_n^o, B_{n*}^e, B_{n+1}^o\right\} = \mathcal{L}_{-n,n+1}$$

i.e., $\mathcal{L}_{2n}$ and $\mathcal{L}_{2n+1}$ are the sets of rational functions whose poles, counted with multiplicity, are in $\{\hat{\alpha}_1, \alpha_2, \ldots, \alpha_{2n-2}, \hat{\alpha}_{2n-1}, \alpha_{2n}\}$ and in $\{\hat{\alpha}_1, \alpha_2, \ldots, \hat{\alpha}_{2n-1}, \alpha_{2n}, \hat{\alpha}_{2n+1}\}$ respectively.

The orthonormality of $\{\phi_n\}_{n\geq 0}$ is equivalent to the orthonormality of the sequence $\{\chi_n\}_{n\geq 0}$ (see [55]). So now we do have $\chi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ and $\chi_n \perp_\mu \mathcal{L}_{n-1}$. Another way of constructing this basis is introduced in [7].

This new basis $\{\chi_n\}_{n\geq 0}$ provides a simpler matricial representation for the multiplication operator under non-restrictive conditions. As in the Hessenberg case, if the sequence $\{\alpha_n\}_{n\geq 1}$ is compactly included in $\mathbb{D}$, the ORF $\{\chi_n\}_{n\geq 0}$ associated with $\{\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots\}$, form a basis for L^2_μ and the matrix of M with respect to $\{\chi_n\}_{n\geq 0}$ is

$$\mathcal{U} = \zeta_{\mathcal{A}}^{-1}(\mathcal{C}), \tag{7.3}$$

where again $\mathcal{A}$ is the diagonal matrix with the poles on its diagonal, $\mathcal{C}$ is given by (3.2) with $\eta_n = \sqrt{1 - |\delta_n|^2}$ with δ_n like on (6.2) and $\zeta_{\mathcal{A}}^{-1}(\mathcal{C})$ the inverse matrix Möbius transform given by (7.2). The unitary matrix $\mathcal{U}$ is not a five-diagonal matrix as in the polynomial case, but it is an (inverse) operator Möbius transformation of a five-diagonal matrix.

7.2. Para-orthogonal rational functions

The zeros of the ORF $\{\phi_n\}_{n\geq 1}$ and the zeros of $\{\chi_n\}_{n\geq 1}$ are the same, but they are all inside $\mathbb{D}$, and not on $\mathbb{T}$, so that they are not suitable as nodes for rational Szegő quadrature formulas. Thus we need a para-orthogonal rational functions (PORF) generalizing the para-orthogonal polynomials and that have simple zeros on $\mathbb{T}$. The next step is then to show that these zeros can be obtained by solving some eigenvalue problem.

Given a sequence of orthogonal rational functions $\{\phi_n\}_{n\geq 0}$ with poles $\{\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n\}$ in $\mathbb{E}$, the para-orthogonal rational functions (PORF) are defined by

$$P_n^v(z) := \phi_n(z) + v\phi_n^*(z), \qquad v \in \mathbb{T}. \tag{7.4}$$

The following result (see [8]) shows that these satisfy our needs perfectly.

Theorem 7.1. *The para-orthogonal rational function P_n^v with $v \in \mathbb{T}$ has n simple zeros which lie on the unit circle.* □

To arrive at the eigenvalue problem, we use the recurrence (6.1) for the ORF ϕ_n in the definition (7.4) of the PORF so that we can rewrite P_n^v as follows:

$$P_n^v(z) = (1 + \overline{\delta_n} v) c_n \frac{\varpi_{n-1}(z)}{\varpi_n(z)} [\zeta_{n-1}(z)\phi_{n-1}(z) + u\phi_{n-1}^*(z)], \qquad u = \zeta_{\delta_n}^{-1}(v).$$

Compare this with the recurrence relation for the ORF (6.1) then it is clear that P_n^v (making abstraction of the constant factor $(1 + \overline{\delta_n} v)$ which does not influence the zeros) can be obtained from the same recurrence relation as the ORF, except that in the last step we have to replace δ_n by $u = \zeta_{\delta_n}^{-1}(v) \in \mathbb{T}$.

We are now all set to mimic the polynomial situation and get unitary truncations of the matrix representation of the multiplication operator M restricted to $\mathcal{R}$ with respect to the basis $\{\phi_n\}_{n\geq 0}$ or with respect to the basis $\{\chi_n\}_{n\geq 0}$ in $\mathcal{L}$. I.e., we have to find unitary truncations of $\zeta_{\mathcal{A}}^{-1}(\mathcal{H})$ and $\zeta_{\mathcal{A}}^{-1}(\mathcal{C})$. Therefore we introduce the following notation for the truncations: $\mathcal{A}_n = \text{diag}(\alpha_0, \ldots, \alpha_{n-1})$, $\mathcal{H}_n^u$ stand for the unitary truncated Hessenberg matrix of dimension n based on the parameters $\delta_1, \ldots, \delta_{n-1}, u$ like in (3.1) and set $\mathcal{V}_n^u = \zeta_{\mathcal{A}_n}^{-1}(\mathcal{H}_n^u)$. Also $\mathcal{C}_n^u$ and $\mathcal{U}_n^u = \zeta_{\mathcal{A}_n}^{-1}(\mathcal{C}_n^u)$ are introduced in a similar way. Note that $\mathcal{H}_n^u$ as well as $\mathcal{C}_n^u$ are unitary matrices and this property is maintained by the Möbius transform. The n-th principal submatrices of the unitary $\mathcal{V}$ and $\mathcal{U}$ are not unitary, but replacing δ_n by $u \in \mathbb{T}$ turn the matrices $\mathcal{V}_n^u$ and $\mathcal{U}_n^u$ into unitary ones. Then the following result provides two different matricial representations of the PORF as well as of their zeros in terms of matrix Möbius transform of a Hessenberg matrices and CMV matrices respectively (see [55]).

Theorem 7.2. *Let $\{\alpha_n\}_{n\geq 1}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $\{\phi_n\}_{n\geq 0}$ the corresponding ORF and $\{\chi_n\}_{n\geq 0}$ the ORF associated with the sequence $\{\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots\}$. If $P_n^v = \phi_n + v\phi_n^*$ is the n-th PORF related to $v \in \mathbb{T}$, and if we introduce $u = \zeta_{\delta_n}^{-1}(v)$, then:*

(i.H) The zeros of P_n^v are the eigenvalues of $\mathcal{V}_n^u$ and if ξ is an eigenvalue then the related eigenvectors are spanned by $(\phi_0(\xi), \ldots, \phi_{n-1}(\xi))^\dagger$. Equivalently, this ξ is also a generalized eigenvalue of the pencil $(\mathcal{A}_n + \mathcal{H}_n^u, \mathcal{I}_n + \mathcal{A}_n^\dagger \mathcal{H}_n^u)$ and the corresponding left eigenvectors are spanned by $(\phi_0(\xi), \ldots, \phi_{n-1}(\xi))\nu_{\mathcal{A}_n}^{-1/2}$.

(i.C) The zeros of P_n^v are the eigenvalues of $\mathcal{U}_n^u$ and if ξ is an eigenvalue then the related eigenvectors are spanned by $(\chi_0(\xi), \ldots, \chi_{n-1}(\xi))^\dagger$. Equivalently, this ξ is also a generalized eigenvalue of the pencil $(\mathcal{A}_n + \mathcal{C}_n^u, \mathcal{I}_n + \mathcal{A}_n^\dagger \mathcal{C}_n^u)$ and the corresponding left eigenvectors are spanned by $(\chi_0(\xi), \ldots, \chi_{n-1}(\xi))\nu_{\mathcal{A}_n}^{-1}$.

(ii.H) $P_n^v = \dfrac{p_n^v}{\pi_n}$, with p_n^v proportional to the characteristic polynomial of $\mathcal{V}_n^u$.

(ii.C) $P_n^v = \dfrac{p_n^v}{\pi_n}$, with p_n^v proportional to the characteristic polynomial of $\mathcal{U}_n^u$. □

The factorization of the five-diagonal matrix as a product of two simple block diagonals as in (3.2) for the polynomial case will of course also hold for the rational case. When we truncate this factorization at the nth principal submatrices and replacing δ_n by $u \in \mathbb{T}$ in $\mathcal{C}$ as well as in the factors $\mathcal{C}_e$ and $\mathcal{C}_o$, i.e., $\mathcal{C}_{on}^u = \mathcal{C}_o(\delta_0, \ldots, \delta_{n-1}, u)$ and $\mathcal{C}_{en}^u = \mathcal{C}_e(\delta_0, \ldots, \delta_{n-1}, u)$, we obtain that $\mathcal{C}_n^u = \mathcal{C}_{on}^u \mathcal{C}_{en}^u$.

Since $\mathcal{U}_n^u$ is unitary, we can write $\mathcal{U}_n^u = [(\mathcal{U}_n^u)^\dagger]^{-1}$. And thus

$$\mathcal{U}_n^u = \nu_{\mathcal{A}_n}(\mathcal{C}_n^{u\dagger} + \mathcal{A}_n^\dagger)^{-1}(\mathcal{I}_n + \mathcal{C}_n^{u\dagger}\mathcal{A}_n)\nu_{\mathcal{A}_n}^{-1}.$$

Hence, the eigenvalue decomposition for $\mathcal{U}_n^u$, viz. $\mathcal{U}_n^u X_n = X_n \Lambda_n$ can be rewritten as

$$\begin{aligned} (\mathcal{C}_n^{u\dagger} + \mathcal{A}_n^\dagger)^{-1}(\mathcal{I}_n + \mathcal{C}_n^{u\dagger}\mathcal{A}_n)Y_n &= Y_n\Lambda_n, \quad Y_n = \nu_{\mathcal{A}_n}^{-1}X_n, \\ (\mathcal{C}_{on}^{u\dagger} + \mathcal{C}_{en}^u\mathcal{A}_n^\dagger)^{-1}(\mathcal{C}_{en}^u + \mathcal{C}_{on}^{u\dagger}\mathcal{A}_n)Y_n &= Y_n\Lambda_n. \end{aligned}$$

In other words, we have proved the following theorem which implies yet another computational scheme for the nodes and weights of the quadrature formulas as we shall elaborate in the next section.

Theorem 7.3. *Let $\{\alpha_n\}_{n\geq 1}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $\{\phi_n\}_{n\geq 0}$ the corresponding ORF and $\{\chi_n\}_{n\geq 0}$ the ORF associated with the sequence $\{\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots\}$. Furthermore* *let $P_n^v = \phi_n + v\phi_n^*$ be the n-th PORF related to $v \in \mathbb{T}$, and suppose that $\mathcal{C}_{on}^u$ and $\mathcal{C}_{en}^u$ are the unitary truncations of the matrices of (3.2) where δ_n is replaced by $u = \zeta_{\delta_n}^{-1}(v) \in \mathbb{T}$. Then the zeros of P_n^v are the generalized eigenvalues of the pencil $(\mathcal{C}_{en}^u + \mathcal{C}_{on}^{u\dagger}\mathcal{A}_n, \mathcal{C}_{on}^{u\dagger} + \mathcal{C}_{en}^u\mathcal{A}_n^\dagger)$ and if ξ is such an eigenvalue, then the corresponding right eigenvectors are spanned by $[(\chi_0(\xi), \dots, \chi_n(\xi))\nu_{\mathcal{A}_n}^{-1}]^\dagger$.* □

8. Matricial computation of rational Szegő quadrature formulas.

In this section we consider again the approximation of the integral

$$I_\mu(f) = \int_{-\pi}^{\pi} f(e^{i\theta})d\mu(\theta),$$

for any function f defined on $\mathbb{T}$, and for a finite Borel measure which we again assume to be a probability measure for simplicity of the formulas. As before we shall estimate it by a formula of the form

$$I_n(f) = \sum_{j=1}^{n} \lambda_j f(z_j), \tag{8.1}$$

with mutually distinct nodes z_j on $\mathbb{T}$ and positive weights λ_j, $j = 1, \dots, n,$. However instead of designing it in a way that it has a maximal domain of exactness in the space of Laurent polynomials, we will consider the more general rational functions introduced in the previous sections. This gives rise to the class of *rational* Szegő quadrature formulas considered for example in [4–7].

Like in Section 4 we may start from rational interpolatory type quadrature formulas. Since our space of rational functions has poles that are fixed by the $\{\alpha_n\}_{n\geq 0}$, we can still use the freedom in the numerator to construct a unique rational interpolant in a given set of interpolation points on $\mathbb{T}$ and approximate the integral by the integral of the interpolating rational function. Thus if $R \in \mathcal{L}_{-k,k}$ is the interpolant for f, defined by $2k+1$ interpolation conditions, then by construction its integral will be exact in that space $\mathcal{L}_{-k,k}$. The following theorem states this formally and it can be found in [5].

Theorem 8.1. *Let $n = 2k + 1$. Given the sequence $\{z_j\}_{j=1}^n \subset \mathbb{T}$ of mutually distinct points, then there exist unique positive numbers $\lambda_1, \dots, \lambda_n$ depending on μ, such that*

$$I_n(f) = \sum_{j=1}^{n} \lambda_j f(z_j) = I_\mu(f), \quad \forall f \in \mathcal{L}_{-k,k}.$$

Furthermore, $I_n(f) = I_\mu(R_n)$ where R_n is the unique rational function in $\mathcal{L}_{-k,k}$ satisfying the interpolation condition $R_n(z_j) = f(z_j)$, $j = 1, 2, \dots, n$. □

Similarly one may choose $2k$ interpolation points and construct a rational interpolant from $\mathcal{L}_{-k+1,k}$ or $\mathcal{L}_{-k,k-1}$ to construct a corresponding rational quadrature formula that is exact for these spaces. However, these interpolatory quadrature formulas with n nodes are exact in the space of dimension n. Rational Szegő quadrature formulas will generalize the polynomial Szegő quadrature formulas and rely on a special choice of the n nodes to obtain a quadrature formula that is exact in a rational space of the form $\mathcal{L}_{-p,q}$ with $p + q = 2n - 2$, i.e., of dimension $2n - 1$. Therefore, the nodes should be chosen as the zeros of para-orthogonal rational functions that we introduced earlier. Thus, we can improve on the previous theorem as follows.

Theorem 8.2. *Let $z_1, z_2, \dots, z_n$ be the zeros of the n-th PORF P_n^u (with $u \in \mathbb{T}$). Then, there exist positive numbers $\lambda_1, \dots, \lambda_n$ such that*

$$I_n(f) = \sum_{j=1}^{n} \lambda_j f(z_j) = I_\mu(f), \quad \forall f \in \mathcal{L}_{-(n-1),n-1},$$

i.e., the rational Szegő quadrature formula is exact on the space $\mathcal{L}_{-(n-1),n-1}$, which has dimension $2n - 1$. □

In this case, $\mathcal{L}_{-(n-1),n-1}$ is the maximal domain of validity in the sense that the formula can not be exact in $\mathcal{L}_{-n,n-1}$ nor in $\mathcal{L}_{-(n-1),n}$. Moreover, it was also proved in [6] that the only quadrature formulas with such a maximal domain of validity are just the ones given in Theorem 8.2, i.e., with weights given by

$$\lambda_j = \int_{\mathbb{T}} L_{j,n}(e^{i\theta}) d\mu(\theta)$$

with $L_{j,n} \in \mathcal{L}_{0,n-1}$ the rational form of the Lagrangian interpolation basis, that is, defined by the interpolation conditions $L_{j,n}(z_i) = \delta_{ij}$. The quadrature formulas of the previous theorem are called rational Szegő quadrature formulas. It should be noted that there is some subspace $\mathcal{L}_n^u$ of dimension $2n$, depending on u, such that $\mathcal{L}_{-(n-1),n-1} \subset \mathcal{L}_n^u \subset \mathcal{L}_{-n,n}$ in which the quadrature formula is exact (see [9]), but it is not of the form $\mathcal{L}_{-p,q}$.

Given the result by Theorem 8.2 that the nodes of the rational Szegő quadrature have to be the zeros of a PORF, and given our matricial representations of Theorem 7.2, we can immediately state the following corollaries.

Corollary 8.3. *The nodes of rational Szegő quadrature formulas are the eigenvalues of matrix Möbius transformations*

(H) $\mathcal{V}_n^u$ *of unitary truncations of Hessenberg matrices* $\mathcal{H}_n^u$*,*

(C) $\mathcal{U}_n^u$ *of unitary truncations of five-diagonal matrices* $\mathcal{C}_n^u$*.* □

For the weights we need to check whether they can be obtained from corresponding eigenvectors. Therefore we first have to verify whether the weights are given by formulas similar to the polynomial case and secondly we should check whether these results are obtainable from the eigenvectors corresponding to the eigenvalues defining the nodes in a way similar to the polynomial case.

With respect to the first problem we refer to [8, Theorem 5.4.2] where it has been proved that the weights λ_j of the n-point rational Szegő quadrature formulas are given by

$$\lambda_j = \left(\sum_{k=0}^{n-1} |\phi_k(z_j)|^2 \right)^{-1}, \tag{8.2}$$

where $(\phi_k)_{k\geq 0}$ are the ORF for the measure μ and the sequence $\{\alpha_n\}_{n\geq 0}$.

So now it has to be verified if these values are computable from the eigenvectors given in Theorem 7.2. There it was shown that if z_j is an eigenvalue of $\mathcal{V}_n^u$ then $V_n(z_j) = (\phi_0(z_j), \cdots, \phi_{n-1}(z_j))^\dagger$ is a corresponding eigenvector. Thus a normalized eigenvector $W(z_j) = (w_0(z_j), \cdots, w_{n-1}(z_j))^\dagger$ is given by

$$W_n(z_j) = \frac{V_n(z_j)}{\|V_n\|} \quad \text{with} \quad \|V_n\| = \left(\sum_{k=0}^{n-1} |\phi_k(z_j)|^2 \right)^{1/2},$$

or equivalently, using (8.2),

$$(w_0(z_j), \cdots, w_{n-1}(z_j))^\dagger = \lambda_j^{1/2} (\phi_0(z_j), \cdots, \phi_{n-1}(z_j))^\dagger .$$

Equating the first component of both sides gives

$$w_0(z_j) = \lambda_j^{1/2} \phi_0(z_j).$$

If we are dealing with a probability measure, then $\phi_0(z_j) \equiv 1$ so that $\lambda_j = w_0(z_j)^2$, showing that the weight corresponding to the node z_j is indeed the squared modulus (the eigenvectors can be complex) of the first component of the corresponding normalized eigenvector.

Similarly, it also follows from Theorem 7.2 that if z_j is an eigenvalue of $\mathcal{U}_n^u$ then the corresponding eigenvector is $(\chi_0(z_j), \cdots, \chi_{n-1}(z_j))^\dagger$, so that it suffices to remind that

$$\sum_{k=0}^{n-1} |\chi_k(z_j)|^2 = \sum_{k=0}^{n-1} |\phi_k(z_j)|^2,$$

to obtain the same result for this situation as in the case of $\mathcal{V}_n^u$.

In short the following Proposition has been proved:

Proposition 8.4. *If $z_1, \ldots, z_n$ are the nodes of the rational Szegő quadrature formula, i.e., the eigenvalues of $\mathcal{V}_n^u$ (resp. $\mathcal{U}_n^u$), then the weights are given by the squared modulus of the first components of the corresponding normalized eigenvectors.* □

We have formulated the previous results in terms of $\mathcal{V}_n^u$ and $\mathcal{U}_n^u$ but in practice we may of course implement the computations using the pencils that are mentioned in Theorems 7.2 and 7.3.

This new point of view avoids the calculation of the nodes of rational quadrature formulas on the unit circle by first constructing the corresponding PORF and then computing its zeros. It provides a method to calculate the nodes as eigenvalues of the above mentioned matrices or pencils, that require only the knowledge of the parameters δ_n and the poles of the corresponding orthogonal rational functions. The practical drawback is that there are only few examples where the expressions for the δ_n are known, and they depend moreover on the choice of the $\{\alpha_n\}$ sequence and these are to be chosen in function of the problematic points in the integrand. Hence the $\{\delta_n\}$ differ for each problem. The latter dependency kills the hope for simple generic expressions for these rational Schur parameters. They are rather trivial exceptions for cases like the Lebesgue or Poisson measure. Other known cases where the rational Schur parameters are computable are Chebyshev rational functions (e.g., [3]), or rational modification of simpler measures for which the δ_n are known (e.g., [23]).

Exactly like it was done for the polynomial case in Sections 4.3 and 4.4, our results of Corollarie 8.3 in combination with Proposition 8.4 can be summarized in the Golub-Meurant formulas

$$I_n(f) = \mathbf{e}_1^T f(\mathcal{V}_n^u)\mathbf{e}_1 = \mathbf{e}_1^T f(\mathcal{U}_n^u)\mathbf{e}_1.$$

Such formulas for rational Gauss formulas on the real line can be found in [49]. They received much attention recently in the context of rational Krylov methods for large scale problems for example in the analysis of complex networks (see e.g. [40]).

9. Numerical examples

As in the Szegő quadratures on the unit circle, for rational Szegő quadrature formulas we also will illustrate the preceding results with some numerical examples. Again the algorithms were implemented in MATLAB.

Example 9.1. In this first example we will consider the function f given by

$$f(z) = \frac{2+7z}{1-z/3} + \frac{1-z}{3-z} + \frac{z(1+2z^2)}{(z-5)(1-6z)(7-z)} \tag{9.1}$$

and we will approximate $I_\mu(f)$, μ being the Chebyshev measure $(1-\cos\theta)d\theta/(2\pi)$. For a matricial computation using the operator Möbius transform for Hessenberg and five-diagonal matrices, we need the poles and the sequences $\{\delta_n\}$ and $\{\eta_n\}$ that parameterize these matrices.

We consider $n = 7$ and we choose $\alpha_n = 1/(n+1)$, $n = 1, \ldots, 7$ and $\alpha_0 = 0$ as usual. For the Chebyshev measure the δ_n can be derived from the explicit expressions for the ϕ_k in [3] as shown in [2]. We list the first seven parameters in Table 2.

Solving, in this case, the generalized eigenvector problem for the pencil with an appropriate $v \in \mathbb{T}$,

$$\left(\mathcal{A}_n + \mathcal{H}_n^u, \mathcal{I}_n + \mathcal{A}_n^\dagger \mathcal{H}_n^u\right),$$

we obtain the nodes and weights of the Table 3.

If we compute them via the eigenvalues and eigenvectors of the matrix $\mathcal{V}_n^u$, the results are as accurate, showing possibly a slight difference in the last digit.

Table 2: The Schur parameters for the Chebyshev measure $\frac{(1-\cos\theta)d\theta}{2\pi}$ when the poles are given by $\alpha_k = 1/(k+1)$, $k = 1, \dots, 7$.

n	δ_n
1	$6.666666666666666\text{E}-01$
2	$5.555555555555556\text{E}-01$
3	$2.682926829268292\text{E}-01$
4	$1.844660194174757\text{E}-01$
5	$1.433278418451400\text{E}-01$
6	$1.184971098265896\text{E}-01$
7	$1.017011834319527\text{E}-01$

Table 3: The nodes and weights for the Chebyshev measure $\frac{(1-\cos\theta)d\theta}{2\pi}$ when the poles are given by $\alpha_k = 1/(k+1)$, $k = 1, \dots, 7$ and $v = -1$.

θ_j of nodes $z_j = \exp(i\theta_j)$	weights
0	$3.594351732991019\text{E}-01$
$\pm 1.086641439916068\text{E}+00$	$2.348319150907187\text{E}-01$
$\pm 1.169153308542978\text{E}+00$	$7.385174323898922\text{E}-02$
$\pm 5.136477064591293\text{E}-01$	$1.159875502074117\text{E}-02$

Similarly one can use the generalized eigenvalues and vectors for the pencils

$$\left(\mathcal{A}_n + \mathcal{C}_n^u, \mathcal{I}_n + \mathcal{A}_n^\dagger \mathcal{C}_n^u\right), \quad \text{or} \quad (\mathcal{C}_{en}^u + \mathcal{C}_{on}^{u\dagger}\mathcal{A}_n, \mathcal{C}_{on}^{u\dagger} + \mathcal{C}_{en}^u \mathcal{A}_n^\dagger),$$

or compute the eigenvalues and eigenvectors of $\mathcal{U}_n^u$. The numerical values of the nodes and weights are again the same except for some rounding which only affects one or at most two of the last digits.

Because the given function belongs to the domain of validity, the quadrature formula should give the exact integral, viz. -1.385700268332733, which is indeed the case within machine precision, because the relative error is $O(10^{-16})$.

For a different, nonreal, choice of $v \in \mathbb{T}$, the symmetry is lost, as can be seen in Table 4 where we listed the nodes and weight for $v = i = \sqrt{-1}$.

Table 4: The nodes and weights for the Chebyshev measure $\frac{(1-\cos\theta)d\theta}{2\pi}$ when the poles are given by $\alpha_k = 1/(k+1)$, $k = 1, \dots, 7$ and $v = i$.

θ_j of nodes $z_j = \exp(i\theta_j)$	weights
$-9.852343382366260\text{E}-01$	$5.111787059177298\text{E}-02$
$-8.551217874454911\text{E}-01$	$2.792640925331908\text{E}-01$
$-3.467594087977110\text{E}-01$	$5.837499871365035\text{E}-03$
$2.602181673212490\text{E}-01$	$3.533166059075208\text{E}-01$
$6.297216857300113\text{E}-01$	$1.869962102376529\text{E}-02$
$1.319929652919720\text{E}+00$	$1.907125846597272\text{E}-01$
$1.339860693483318\text{E}+00$	$1.010517254126578\text{E}-01$

Although the symmetry is lost, the integral is still approximated with the same accuracy as in the symmetric case.

In practice, of course one usually does not know the poles exactly, but if some good approximations are known, one gets very fast very good approximations. For example if we replace the estimated poles by $\alpha_k = 1/(k+1.1)$, we get with the same routine using 7 nodes an error of $O(10^{-6})$, but if we take $n = 13$ and repeat our approximate poles cyclically as $\alpha_{k+7} = 1/(k+1.1)$, $k = 1, 2, \ldots$, then the error is already $O(10^{-12})$ and for $n = 19$, it is $O(10^{-15})$ which means an exact result up to machine precision.

Example 9.2. In the second example we will consider the Poisson weight, which is a rational modification of the Lebesgue measure

$$d\mu(\theta) = \frac{1-|r|^2}{|z-r|^2}\frac{d\theta}{2\pi}, \quad r \in \mathbb{D}, \quad z = e^{i\theta},$$

and we shall calculate the integral of $f(z) = \sin(|R(z)|^2)$ with $R(z)$ a rational function given by

$$R(z) = \frac{(z-c_r)(z-c_t)}{z-c_p}, \quad c_r = 0.8e^{i\pi/3}, \quad c_t = 0.8e^{-i\pi/3}, \quad c_p = 1.1i.$$

We know the Schur parameters for the corresponding ORF which are particularly simple since they are all equal to zero except the first one, which is $\delta_1 = -r$. This can be derived from the formula (3.1) in [6] giving explicit expressions for the ORF, and knowing that $\delta_n = \phi_n(\alpha_{n-1})/\phi_n^*(\alpha_{n-1})$. Note that this can also be obtained by considering this weight as a rational modification of the Lebesgue measure, [23, Theorem 7].

The function $f(z)$ for $z \in \mathbb{T}$ is plotted as a function of $\theta \in [-\pi, \pi]$ where $z = e^{i\theta}$ in Figure 2. It is clear that the essential singularity at $z = c_p$ causes some problems near $\theta = \pi/2$, i.e., at $z = i$ on $\mathbb{T}$. When we use the Szegő quadrature formula in the polynomial case, i.e., setting all $\alpha_k = 0$ we get no convergence. The 10-log of the relative error is seen in Figure 3. It is the graph that stagnates with no more than two to three digits correct in the result. However when placing the α_k all at $1/c_p \approx 0.91i$, corresponding to the singularity at c_p (recall that all our α_k are in $\mathbb{D}$), we see in the same Figure 3 that the method converges linearly to machine precision for $n = 400$. The reason is that for all $\alpha_k = 0$, the nodes are almost uniformly distributed over $\mathbb{T}$, while if we set all $\alpha_k = 1/c_p$, then this will attract most of the nodes to the problematic neighborhood of $z = i$ on $\mathbb{T}$ where the function f is highly oscillating. This is illustrated on Figure 4. The explanation of the periodic dips in the error curve can then be explained by how the nodes align with these oscillations. If we distribute K nodes in the region with N oscillations, then the accuracy of the quadrature formula will depend on whether this distribution is in phase with the oscillations or not. When they are in phase we get a much better approximation, than when they are not.

In the rational case, the previous examples show that these (generalized) eigenvalue methods for computing the nodes and weights of rational Szegő quadrature formulas are very efficient and stable as are the polynomial counterparts. The main purpose of the second part of this paper was to show that also in the case of quadrature formulas on the unit circle based on orthogonal rational functions, the nodes and weights can be computed by solving an eigenvalue problem for a (structured) matrix. In this case, this matrix is a matrix Möbius transform of a unitary Hessenberg matrix $\mathcal{H}_n^u$ or a five-diagonal matrix $\mathcal{C}_n^u$.

To end this section it should be remarked that we have some indications that these results can be also stated for matrix Möbius transform of snake-shaped matrices. More details on this issue are postponed to a future paper.

References

[1] G. Ammar, W.B. Gragg, and L. Reichel. On the eigenproblem for orthogonal matrices. In *Proceedings of the 25th Conference on Decision and Control Athens*, pages 1963–1966. IEEE, 1986.

Figure 2: The function of example 9.2 plotted as a function of θ on the unit circle

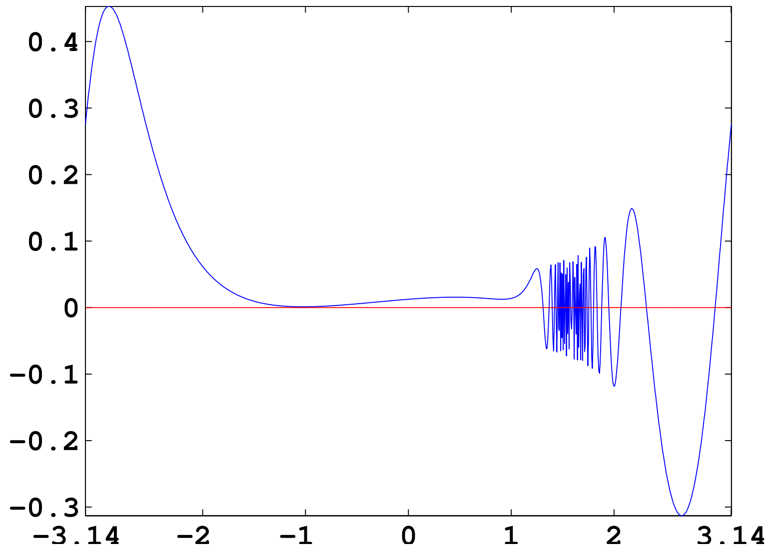


Figure 3: The 10-log of the relative error as a function of the number of nodes for example 9.2. The stagnating curve is when all $\alpha_k = 0$, while linear convergence is obvious when all $\alpha_k = 1/c_p$.

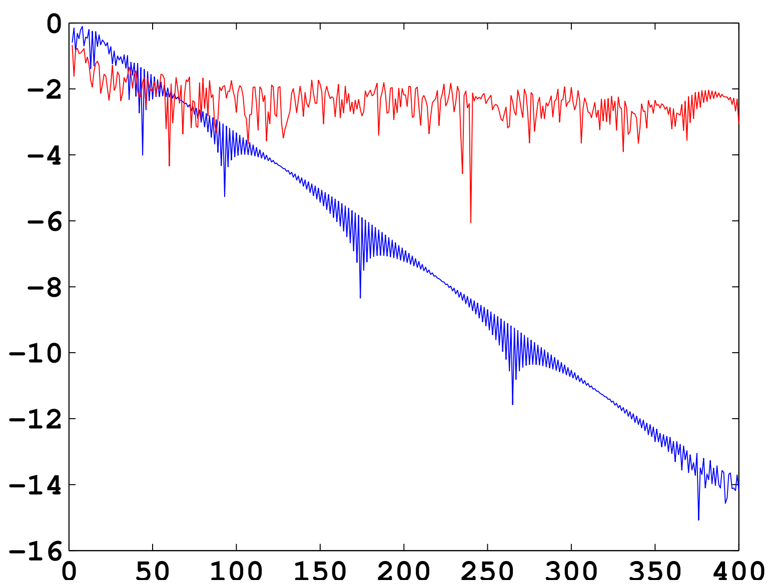
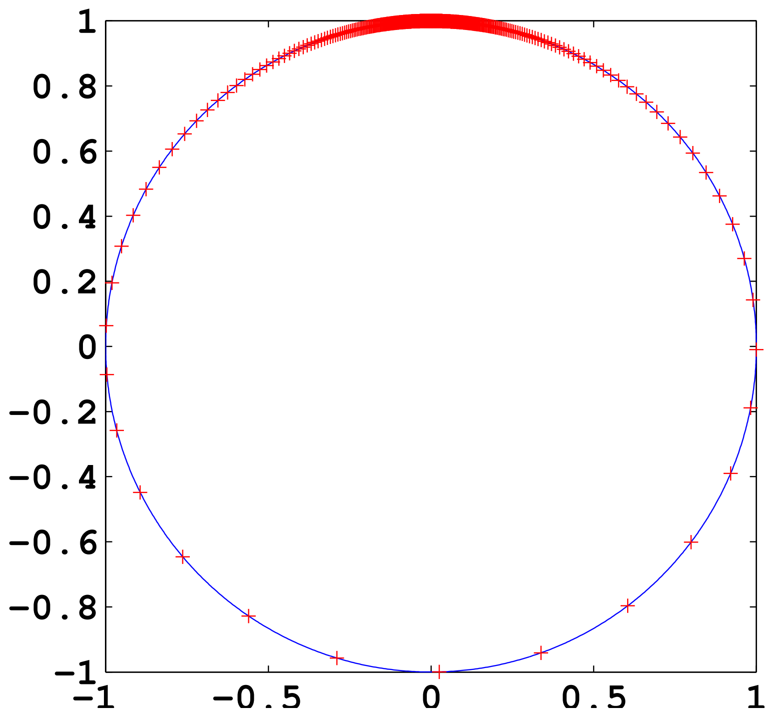


Figure 4: The location of the 400 nodes for example 9.2. The choice $\alpha_k = 1/c_p$ attracts most of the nodes to the problematic region on $\mathbb{T}$ where the integrand is highly oscillating.



[2] A. Bultheel and M.J. Cantero. A matricial computation of rational quadrature formulas on the unit circle. *Numerical Algorithms*, 52(1):47–68, 2009.

[3] A. Bultheel, R. Cruz Barroso, K. Deckers, and P. González-Vera. Rational Szegő quadratures associated with Chebyshev weight functions. *Mathematics of Computation*, 78:1031–1059, 2009.

[4] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. A Szegő theory for rational functions. Technical Report TW131, Department of Computer Science, K.U. Leuven, May 1990.

[5] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions and quadrature on the unit circle. *Numerical Algorithms*, 3:105–116, 1992.

[6] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Quadrature formulas on the unit circle based on rational functions. *Journal of Computational and Applied Mathematics*, 50:159–170, 1994.

[7] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions and interpolatory product rules on the unit circle. II. Quadrature and convergence. *Analysis*, 18:185–200, 1998.

[8] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. *Orthogonal rational functions*, volume 5 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, 1999.

[9] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Rational quadrature formulas on the unit circle with prescribed nodes and maximal domain of validity. *IMA Journal of Numerical Analysis*, 30(4):940–963, 2010.

[10] M.J. Cantero, R. Cruz Barroso, and P. González-Vera. A matrix approach to the computation of quadrature formulas on the interval. *Applied Numererical Mathematics*, 58(3):296–318, 2008.

[11] M.J. Cantero, L. Moral, and L. Velázquez. Measures and para-orthogonal polynomials on the unit circle. *East Journal on Approximations*, 8:447–464, 2002.

[12] M.J. Cantero, L. Moral, and L. Velázquez. Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle. *Linear Algebra and Its Applications*, 362:29–56, 2003.

[13] M.J. Cantero, L. Moral, and L. Velázquez. Minimal representations of unitary operators and orthogonal polynomials on the unit circle. *Linear Algebra and Its Applications*, 408:40–65, 2005.

[14] M.J. Cantero, L. Moral, and L. Velázquez. Measures on the unit circle and unitary truncations of unitary operators. *Journal of Approximation Theory*, 139(1-2):430–468, 2006.

[15] L. Cochran and S.C. Cooper. Orthogonal Laurent polynomials on the real line. In S.C. Cooper and W.J. Thron, editors, *Continued fractions and orthogonal functions*, pages 47–100, New York, 1994. Marcel Dekker.

[16] R. Cruz Barroso, L. Daruis, P. González-Vera, and O. Njåstad. Sequences of orthogonal Laurent polynomials, bi-orthogonality and quadrature formulas on the unit circle. *Journal of Computational and Applied Mathematics*, 206:950–966, 2007.

[17] R. Cruz Barroso and S. Delvaux. Orthogonal Laurent polynomials on the unit circle and snake-shaped matrix factorizations. *Journal of Approximation Theory*, 161(1):65–87, 2009.

[18] R. Cruz Barroso and P. González-Vera. A Christoffel-Darboux formula and a Favard's theorem for orthogonal Laurent polynomials on the unit circle. *Journal of Computational and Applied Mathematics*, 179(1-2):157–173, 2005.

[19] R. Cruz Barroso and P. González-Vera. Orthogonal Laurent polynomials and quadratures on the unit circle and the real half-line. *Electronic Transactions on Numerical Analysis*, 19:113–134, 2005.

[20] R. Cruz Barroso, P. González Vera, and F. Perdomo Pío. Quadrature formulas associated with Rogers-Szegő polynomials. *Computers and Mathematics with Applications*, 57:308–323, 2009.

[21] L. Daruis, P. González-Vera, and O. Njåstad. Szegő quadrature formulas for certain Jacobi-type weight functions. *Mathematics of Computation*, 71(238):683–701, 2002.

[22] P.J. Davis and P. Rabinowitz. *Methods of numerical integration.* Academic Press, 2nd ed edition, 1984. First edition 1967.

[23] K. Deckers and A. Bultheel. Orthogonal rational functions and rational modifications of a measure on the unit circle. *Journal of Approximation Theory*, 157:1–18, 2009.

[24] C. Díaz Mendoza, P. González-Vera, and M. Jiménez Páiz. Strong Stieltjes distributions and orthogonal Laurent polynomials with applications to quadratures and Padé approximation. *Mathematics of Computation*, 74(252):1843–1870, 2005.

[25] Y. Eidelman, I. Gohberg, and I. Haimovici. *Separable type representations of matrices and fast algorithms. Volume II Eigenvalue method*, volume 235 of *Operator Theory: Advances and Applications.* Springer Verlag, 2014.

[26] G. Freud. *Orthogonal polynomials.* Pergamon Press, Oxford, 1971.

[27] B. Fritzsche, B. Kirstein, and A. Lasarow. Orthogonal rational matrix-valued functions on the unit circle: Recurrence relations and a Favard-type theorem. *Mathematische Nachrichten*, 279:513–542, 2006.

[28] B. Fritzsche, B. Kirstein, and A. Lasarow. Para-orthogonal rational matrix-valued functions on the unit circle. *Operator and Matrices*, 6(4):631–680, 2012.

[29] B. Fritzsche, B. Kirstein, and A. Lasarow. On canonical solutions of a moment problem for rational matrix-valued functions. *Operator Theory: Advances and Applications*, Springer, pages 323-372, 2012.

[30] W. Gautschi. On the construction of Gaussian quadrature rules from modified moments.ances and Applications. *Mathematics of Computation*, 24:245–260, 1970.

[31] W. Gautschi. On generating orthogonal polynomials. *SIAM Journal on Scientific and Statistical Computation*, 3:289–317, 1982.

[32] Ja.L. Geronimus. Polynomials orthogonal on a circle and their applicantions (Russian. *Zap.Naučno-Issled. Inst. Mat. Meh. Har'kov. Mat. Obšč.*, 19:35–120, 1948.

[33] G.H. Golub and G. Meurant. Matrices, moments and quadrature. In *Numerical analysis 1993*, volume 303 of *Pitman Research Notes in Mathematics*, pages 105–146. Longman Scientific & Technical, 1994.

[34] G.H. Golub and G. Meurant. *Matrices, Moments and Quadrature with Applications.* Princeton Series in Applied Mathematics. Princeton University Press, 2009.

[35] G.H. Golub and J.H. Welsch. Calculation of Gauss quadrature rules. *Mathematics of Computation*, 23(106):221–230, 1969.

[36] P. González-Vera, H. Martínez, and J.J. Trujillo. An application of Szegő quadratures to the computation of the Fourier transform. *Applied Mathematics and Computation*, 187(1):183–194, 2007.

[37] P. González-Vera, J.C. Santos León, and O. Njåstad. Some results about numerical quadrature on the unit circle. *Advances in Computational Mathematics*, 5:297–328, 1996.

[38] W.B. Gragg. Positive definite Toeplitz matrices, the Arnoldi process for isometric operators, and Gaussian quadrature on the unit circle. *Journal of Computational and Applied Mathematics*, 46:183–198, 1993. Russian original in Numerical methods in linear algebra, E.S. Nikolaev (ed.) pp. 16-32, Moscow university press, 1982.

[39] M. Gu, R. Guzzo, X.B. Chi, and X.Q. Cao. A stable divide and conquer algorithm for the unitary eigenproblem. *SIAM Journal on Matrix Analysis and its Applications*, 25(2):385–404, 2003.

[40] S. Güttel. Rational Krylov approximation of matrix functions: Numerical methods and optimal pole selection. *GAMM Mitteilungen*, 36(1):8–31, 2013.

[41] E. Hendriksen and C. Nijhuis. Laurent-Jacobi matrices and the strong Hamburger moment problem. *Acta Applicandae Mathematicae*, 61:119–132, 2000.

[42] W.B. Jones, O. Njåstad, and W.J. Thron. Two-point Padé expansions for a family of analytic functions. *Journal of Computational and Applied Mathematics*, 9:105–124, 1983.

[43] W.B. Jones, O. Njåstad, and W.J. Thron. Orthogonal Laurent polynomials and the strong Hamburger moment problem. *Journal of Mathematical Analysis and its Applications*, 98:528–554, 1984.

[44] W.B. Jones, O. Njåstad, and W.J. Thron. Moment theory, orthogonal polynomials, quadrature and continued fractions associated with the unit circle. *Bull. London Math. Soc.*, 21:113–152, 1989.

[45] W.B. Jones and W.J. Thron. Orthogonal Laurent polynomials and Gaussian quadrature. In K. Gustafson and W.P. Reinhardt, editors, *Quantum mechanics in mathematics, chemistry and physics*, pages 449–455, New York, 1984. Plenum.

[46] W.B. Jones, W.J. Thron, and H. Waadeland. A strong Stieltjes moment problem. *Transactions of the American Mathematical Society*, 206:503–528, 1980.

[47] A. Lasarow. *Aufbau einer Szegő-Theorie für rationale Matrixfunktionen.* PhD thesis, Universität Leipzig, Fak. Mathematik Informatik, 2000.

[48] N. Levinson. The Wiener rms (root mean square) error criterion in filter design and prediction. *Journal of Mathematical Physics*, 25:261–278, 1947.

[49] G. López Lagomasino, L. Reichel, and L. Wunderlich. Matrices, moments, and rational quadrature. *Linear Algebra and Its Applications*, 429(10):2540–2554, 2008.

[50] O. Njåstad and W.J. Thron. The theory of sequences of orthogonal L-polynomials. In H. Waadeland and H. Wallin, editors, *Padé Approximants and Continued Fractions*, Det Kongelige Norske Videnskabers Selskab Skrifter (No. 1), pages 54–91, 1983.

[51] J.C. Santos Léon and O. Njåstad. Domain of validity of Szegő quadrature formulas. *Journal of Computational and Applied Mathematics*, 2002(2):440–449, 2007.

[52] B. Simon. *Orthogonal polynomials on the unit circle. Part 1: Classical theory*, volume 54 of *Colloquium Publications*. AMS, 2005.

[53] G. Szegő. *Orthogonal polynomials*, volume 23 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, Rhode Island, 4th edition, 1975. First edition 1939.

[54] W.J. Thron. L-polynomials orthogonal on the unit circle. In A.M. Cuyt, editor, *Nonlinear Numerical Methods and Rational Approximation*, pages 271–278. Kluwer, 1988.

[55] L. Velázquez. Spectral methods for orthogonal rational functions. *Journal Functional Analysis*, 254(4):954–986, 2008.

[56] D.S. Watkins. Some perspectives on the eigenvalues problem. *SIAM Review*, 35:430–471, 1993.