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FLEXIBLE UNIT-LINKED LIFE INSURANCE CONTRACTS AND EXOTIC OPTIONS

by

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Flexible Unit-Linked Life Insurance Contracts and Exotic Options

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Abstract

In this paper we discuss the pricing of unit linked contracts that offer the insured the posibility to change some features of his contract at a prespecified point in time. This will be done by means of a generalized chooser option.

Keywords: Unit-linked life insurance contracts, insurance premiums, exotic options

1 Introduction

The evolution in the pricing of derivative securities, which we have seen the last decades, has been applied in actuarial science in the pricing of unit-linked life insurance contracts, see for instance Brennan & Schwartz (1976), Aase & Persson (1994), Nielsen & Sandmann (1995) and Ekern & Persson (1996) and in the valuation of minimum guarantees in life insurance contracts under a stochastic interest rate, see for instance Persson & Aase (1997) and Persson (1998). A special type of unit-linked contracts are the so-called flexible unit-linked contracts or FUL contracts, see for instance Ekern & Persson (1996), these offer the insured the possibility to change the characteristics of his contract at a certain point in time.

One reason for considering FUL contracts is, that life insurance contracts are very often long term contracts. A maturity of 20 or even 40 years is

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not unusual, and over such a long period the insured's risk attitude might change, for example as a result of a changed financial situation. That is, at inception, the insured is probably not able to foresee what type of insurance contract he or she might prefer at a later date, especially with respect to an endowment part. Therefore, the logical thing to do, is to offer the insured a second change at a future date, at which he or she can alter the insurance contract. In this article, we will derive pricing formulas for a fairly large class of FUL contracts.

In the following section we will give a short overview of the pricing of unit-linked insurance contracts in order to be able to define flexible unit-linked products in section 3. After section 4, in which we will demonstrate how unit-linked contracts and a small class of FUL contracts can be priced by means of European put an call options, we will discuss the pricing of a rather general type of FUL contracts in section 5. And in section 6 we will illustrate these results with a numerical example.

2 Unit-Linked Contracts

The insurance benefit will be priced within the traditional financial framework, see Black & Scholes (1973). That is, we consider a securities market consisting of one risky asset $S(t)$ and a riskless money-market account in which money can be invested at a fixed spot rate r. The risky asset $S(t)$ is assumed to be defined on a filtered probability space $(\Omega_1, \mathcal{F}_t, P_1)$ with $\mathcal{F}_t$ the filtration generated by $S(t)$. Furthermore, we assume that there exists a single equivalent martingale measure Q, i.e. that we are dealing with a complete and arbitrage free financial market, see Harrison and Kreps (1979) and Harrison and Pliska (1981).

The main issue in pricing unit-linked life insurance contracts is the fact that the pay-off is not a contingent claim with respect to the financial market, as it depends on the remaining lifetime of the insured. Therefore, we introduce a second filtered probability space $(\Omega_2, \mathcal{G}_t, P_2)$ with $\mathcal{G}_t$ the filtration generated by the mortality process $I_\rho(t)$ given by:

$$\begin{aligned} I_\rho(t) &= 0 \text{ if the status } \rho \text{ is still alive at time } t \\ &= 1 \text{ if the status } \rho \text{ is no more alive at time t .} \end{aligned} \tag{1}$$

Here, ρ is a status defined in terms of a life x. For example: $\rho = \left(\overset{1}{x}\colon \overline{n|}\right)$ or

$\rho = \left(x : \overline{n|}\right)$. See Bowers et al. (1997) for a discussion of the concept of a status.

The fact that the life insurance contract is unit-linked, means that the benefit $B(t)$ is a function of the value of the underlying asset $S(t)$. For example the benefit could consist of the market value of an investment in the risky asset $S(t)$, with a time-dependent guaranteed minimum pay-off $K(t)$ and a maximum pay off $G(t)$, i.e. $B(t) = \max\left[K(t), \min(S(t), G(t))\right]$. The time-dependence of the strike prices enables us on the one hand to model for instance a guaranteed minimum return as an increasing function $G(t)$ and on the other to distinguish, in case of an n-year endowment insurance, between the guaranteed (maximum) amount payable upon the death of the insured during the n-year period and the guaranteed (maximum) amount payable when the insured survives the n-year period.

As such, the unit-linked insurance contract B_{T_ρ} consists of the payment of a stochastic amount $B(t + T_\rho)$ at a stochastic moment in time $t + T_\rho$. Note that we distinguish between the time varying benefit $B(\tau)$,which is a purely financial product and a contingent claim within $(\Omega_1, \mathcal{F}_t, P_1)$, and the unit-linked life insurance product B_{T_ρ}, which is a contingent claim within the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_t \times \mathcal{G}_t, P_1 \times P_2)$ and of which the benefit scheme $B(\tau)$ is one component. However, as mortality risk can not be hedged away, we are no longer dealing with a complete market. Therefore there will exist more than one equivalent martingale measure, i.e. prices are not uniquely defined. In order to obtain a unique price for this type of financial products, we will have to make some additional assumptions. In the actuarial literature one usually assumes, see Brennan & Schwartz (1976), Aase & Persson (1994), Nielsen & Sandmann (1995) and Ekern & Persson (1996):

1. *The mortality process $I_\rho(t)$ and the risky asset $S(t)$ are independent processes.*

2. *The insurer is risk neutral with respect to mortality risk.*

The first assumption is intuitively acceptable. The second one can be justified by the fact that mortality risk is considered to be diversifiable, this in contrast to financial risk.

We will now turn to how a complete and arbitrage free financial market together with these two assumptions leads to a unique price for a financial product of the type described above. Throughout this paper we will distinguish between $V(t)$: the price of the unit-linked contract B_{T_ρ} at time t and

$V(t,\tau)$: the price of the unit-linked contract B_{T_ρ} at time t conditional upon $T_\rho = t+\tau$. Under the above assumptions we obtain the following expression for $V(t,\tau)$:

$$\begin{aligned}
& V(t,\tau) \\
= \; & e^{-(\tau-t)r} E^Q \left[E^{P_1 \times P_2} \left(B(t+T_\rho) | T_\rho = \tau - t \right) \middle| \mathcal{F}_t \right] \\
= \; & e^{-(\tau-t)r} E^Q \left[E^{P_1 \times P_2} \left(B(\tau) | T_\rho = \tau - t \right) \middle| \mathcal{F}_t \right] \\
= \; & e^{-(\tau-t)r} E^Q \left[B(\tau) | \mathcal{F}_t \right]. \qquad (2)
\end{aligned}$$

Where the last equality holds because of the assumption that the mortality process and the risky asset are independent processes and with Q the unique equivalent martingale measure in the financial market $(\Omega_1, \mathcal{F}_t, P_1)$. As the insurer is risk neutral with respect to mortality risk, we obtain that $V(t)$ is given by:

$$\begin{aligned}
V(t) &= E\left[V(t, t+T_\rho)\right] \qquad (3) \\
&= \int_0^\infty V(t,\tau) f_{T_\rho}(\tau - t)\, d\tau. \qquad (4)
\end{aligned}$$

with f_{T_ρ} the mortality density of the status ρ, i.e. $P(T_\rho \leq t) = \int_0^t f_{T_\rho}(u)\, du$.

Note that formula 3 can be interpreted as the single premium of a life insurance with a deterministic benefit scheme $e^{(\tau-t)r}V(t,\tau)$. Another possibility, is to see formula 3 as the price of a purely financial product that guarantees a continuous cash flow of which the intensity $\beta(\tau)$ is given by:

$$\beta(\tau) = e^{(\tau-t)r} V(t,\tau) f_{T_\rho}(\tau - t). \qquad (5)$$

As such, under assumptions 1 and 2, pricing a unit-linked product in the incomplete market $(\Omega_1 \times \Omega_2, \mathcal{F}_t \times \mathcal{G}_t, P_1 \times P_2)$ turns out to be equivalent to pricing an adjusted benefit scheme $\beta(\tau)$ in the complete and arbitrage free financial market $(\Omega_1, \mathcal{F}_t, P_1)$.

As, in general, we will not be able to evaluate the integral in expression 3 analytically, we will restrict ourselves to deriving an analytical expression for $V(t,\tau)$.

3 Flexible Unit-Linked Contracts

Let us consider a unit-linked contract B_{T_ρ} with an initial benefit scheme $B_1(\tau)$ and an optional benefit scheme $B_2(\tau)$. With this type of product, the insured has the possibility to change the (initial) benefit scheme $B_1(\tau)$ for the optional scheme $B_2(\tau)$ at a time θ. However, the amount that will be invested at time θ is the maximum of the two investment schemes. Therefore, for $\tau > \theta$ we obtain that $V(t,\tau)$ is given by:

$$V(t,\tau) = e^{-(\tau-t)r} E^Q \left[\max\left(E^Q\left[B_1(\tau)|\,\mathcal{F}_\theta\right], E^Q\left[B_2(\tau)|\,\mathcal{F}_\theta\right]\right)\big|\,\mathcal{F}_t\right] \quad (6)$$

This pay off has the form of a generalized chooser option, where the two underlying derivatives are given by $B_1(\tau)$ and $B_2(\tau)$.

As in the previous section, we obtain the price for the FUL life insurance contract B_{T_ρ} by taking the expectation of expression 6 with respect to T_ρ. In this case, we obtain the following expression for the price of B_{T_ρ} :

$$\begin{aligned}
& E\left[V(t,T_\rho)\right] \\
= \; & E\left[V(t,T_\rho) : T_\rho \le \theta\right] + E\left[V(t,T_\rho) : T_\rho > \theta\right] \\
= \; & \int_0^\theta V(t,\tau)_{\tau-t}p_\rho \; \mu_\rho(\tau-t)\, d\tau + \int_\theta^\infty V(t,\tau)_{\tau-t}p_\rho \; \mu_\rho(\tau-t)\, d\tau. \quad (7)
\end{aligned}$$

Where in case $\tau \le \theta$, we have that $V(t,\tau)$ is given by:

$$V(t,\tau) = e^{-(\tau-t)r} E^Q\left[B_1(\tau)|\,\mathcal{F}_t\right]. \quad (8)$$

Note that the formula 6 is a generalization of formula 2. Indeed, if we set $B_2(t) \equiv 0$, then expression 6 becomes:

$$\begin{aligned}
V(t,\tau) &= e^{-(\tau-t)r} E^Q\left(\left[E^Q\left[B_1(\tau)|\,\mathcal{F}_\theta\right]\right] \mathcal{F}_t\right) \\
&= e^{-(\tau-t)r} E^Q\left(B_1(\tau)|\,\mathcal{F}_t\right). \quad (9)
\end{aligned}$$

In section 5 we obtain explicit expressions for formula 6 for some possible choices of $B_1(t)$ and $B_2(t)$.

4 Applications of European Call and Put Options

In this section we will discuss the possible applications of European put and call options in the pricing of flexible and non-flexible unit-linked insurance contracts.

4.1 Unit-Linked Contracts with Guarantees

The first way in which put and call options can be used, is to guarantee a minimum return or to impose a maximum return or cap, see Brennan & Schwartz (1976), Asse & Persson (1994) and Ekern & Persson (1996). In the general case with a minimum and a maximum return, the benefit scheme $B(\tau)$ is given by:

$$B(\tau) = \max\left[K(\tau), \min(S(\tau), G(\tau))\right]. \tag{10}$$

Therefore, for $V(t,\tau)$ we immediately obtain:

$$V(t,\tau) = S(t) + P(t, S(t), \tau, K(\tau)) - C(t, S(t), \tau, G(\tau)). \tag{11}$$

Where $P(t, S(t), \tau, K(\tau))$ and $C(t, S(t), \tau, G(\tau))$ are the prices of a European put and call option respectively. For a generalization with two underlying assets, that is:$B(\tau) = \max\left[K(\tau), \min\left(\max\left[S_1(\tau), S_2(\tau)\right], G(\tau)\right)\right]$, we refer the interested reader to Ekern & Persson (1996).

4.2 Flexible Unit-Linked Contracts

Here we will turn to the pricing of a first type of FUL contracts. Consider a contract with an initial benefit scheme $B_1(t) \equiv S(t)$, where the insured has the possibility to swap this plain vanilla unit-linked contract for a deterministic benefit scheme $B_2(t)$ at a prespecified date θ. For $V(t,\tau)$ (with $\tau > \theta$) we obtain:

$$\begin{aligned} & V(t,\tau) \\ = \ & e^{-(\theta-t)r} E^Q\left[\max\left(E^Q\left[e^{-(\tau-\theta)r} S(\tau)\middle|\, \mathcal{F}_\theta\right], e^{-(\tau-\theta)r} B_2(\tau)\right)\middle|\, \mathcal{F}_t\right]. \end{aligned} \tag{12}$$

As the discounted price process is a martingale under the measure Q we obtain:

$$
\begin{aligned}
& V(t,\tau) \\
= \;& S(\tau) + e^{-(\theta-t)r} E^Q \left[\max\left(0, e^{-(\tau-\theta)r} B_2(\tau) - S(\tau)\right) \middle| \mathcal{F}_t \right] \\
= \;& S(\tau) + EC\left(t, S(t), \tau, e^{-(\tau-\theta)r} B_2(\tau)\right), \qquad (13)
\end{aligned}
$$

with $EC\left(t, S(t), \tau, e^{-(\tau-\theta)r} B_2(\tau)\right)$ the price at time t of a European put option on S with exercise price $e^{-(\tau-\theta)r} B_2(\tau)$ and exercise date τ.

For $\tau \leq \theta$ we immediately have:

$$
V(t,\tau) = S(t). \qquad (14)
$$

5 General Pricing Formula

In the previous section we showed how European puts and calls can be used to price unit-linked contracts and a limited class of FUL contracts. Here we will discuss how pricing formulas can be obtained for a larger class of flexible unit-linked contracts.

5.1 The Generalized Flexible Unit-Linked Product

In section 4 we looked at a first, simple FUL product: at a given time τ, the insured could choose between a plain-vanilla unit linked contract and a deterministic benefit scheme. We were able to derive a price for this type of contract in terms of European put and call options. In this section, we will derive a pricing formula for a wider class of FUL contracts. With this type of contract the insured can choose between two unit linked contracts. That is, both the initial benefit scheme $B_1(\tau)$ and the optional scheme $B_2(\tau)$ are unit linked contracts themselves. The one condition we impose on these two schemes, is that they are both of the form as discussed in section 4.1. That is, the two benefit schemes $B_1(\tau)$ and $B_2(\tau)$ are both of the following form:

$$
\begin{aligned}
B_1(\tau) &= \max\left[K_1(\tau), \min(S(\tau), G_1(\tau))\right] \qquad (15) \\
B_2(\tau) &= \max\left[K_2(\tau), \min(S(\tau), G_2(\tau))\right], \qquad (16)
\end{aligned}
$$

for some functions $K_1(\tau)$, $G_1(\tau)$, $K_2(\tau)$ and $G_2(\tau)$ on $\mathbb{R}^+$. We will impose either one of the two following conditions on these exercise prices:

$$\forall\tau \in \mathbb{R}^+ \; K_2(\tau) < K_1(\tau) < G_1(\tau) < G_2(\tau) \tag{17}$$

$$\forall\tau \in \mathbb{R}^+ \; K_1(\tau) < K_2(\tau) < G_2(\tau) < G_1(\tau). \tag{18}$$

The reason for this is, that if neither of the two above conditions is met, the pay-off of one of the benefit schemes dominates the pay-off of the other. In which case the FUL contract is essentially a (non-flexible) unit-linked contract. By formula 6 we obtain for $\tau > \theta$:

$$\begin{aligned}
& V(t,\tau) \\
= \; & e^{-(\tau-t)r} E^Q \left[\max\left(E^Q\left[B_1(\tau)|\mathcal{F}_\theta\right], E^Q\left[B_2(\tau)|\mathcal{F}_\theta\right]\right)\middle|\mathcal{F}_t\right] \\
= \; & e^{-(\theta-t)r} E^Q \left[\max\left\{P(\theta,S(\theta),\tau,K_1(\tau)) + S(\theta) - C(\theta,S(\theta),\tau,G_1(\tau))\right.\right., \\
& \left.\left.P(\theta,S(\theta),\tau,K_2(\tau)) + S(\theta) - C(\theta,S(\theta),\tau,G_2(\tau))\right\}\middle|\mathcal{F}_t\right].
\end{aligned} \tag{19}$$

With $P(\theta,S(\theta),\tau,K)$ $(C(\theta,S(\theta),\tau,G))$ the price at time θ of a put (call) option with maturity date τ and exercise price K (G). Note that, as already mentioned in section 3 evaluating $V(t,\tau)$ comes down to calculating the price of a generalized chooser option.

Conditional upon $\tau < \theta$ the problem of pricing the FUL contract is equivalent to pricing a (non-flexible) unit linked contract with benefit scheme $B_1(\tau)$ and we obtain:

$$\begin{aligned}
V(t,\tau) &= e^{-(\tau-t)r} E^Q \left[E^Q \, B_1(\tau)|\mathcal{F}_t\right] \\
&= P(t,S(t),\tau,K_2(\tau)) + S(t) - C(t,S(t),\tau,G_2(\tau)).
\end{aligned} \tag{20}$$

5.2 Pricing the Generalized Flexible Unit-Linked Product

In this section we will derive a more explicit expression for $V(t,\tau)$, the conditional price of a FUL contract for $\tau > \theta$ as it is given by formula 19. We will follow a similar approach as the one that was used in M. Rubinstein (1991) and M. Rubinstein (1992) for pricing chooser and compound options respectively. The pay-off of a chooser option with maturity date T is of the following form:

$$\max\left[P(\theta, S(\theta), T_1, K), C(\theta, S(\theta), T_2, G)\right]. \tag{21}$$

In Rubinstein (1991) the price at time t ($t < \theta < T$) for such an option is derived on basis of the fact that there exists a single value $\varsigma \in \mathbb{R}^+$ for $S(\theta)$ such that: $\forall s \leq \varsigma \;\; P(\theta, s, T_1, K) \geq C(\theta, s, T_2, G)$ and $\forall s \geq \varsigma \;\; P(\theta, s, T_1, K) \leq C(\theta, s, T_2, G)$. For a discussion of compound and chooser options we refer the reader to R. Geske (1979) where prices of compound options were derived for the first time and M. Rubinstein (1992). More information on a wide variety of exotic options can be found in Nelken (1996) and Briys et al (1998).

As the discounted price process $e^{-rt}S(t)$ is a martingale under the measure Q, we immediately obtain from formula 19 :

$$\begin{aligned} & V(t,\tau) \\ = \;& S(t) + e^{-(\theta-t)r} E^Q \left[\max\left\{P(\theta, S(\theta), \tau, K_1(\tau)) - C(\theta, S(\theta), \tau, G_1(\tau))\right.\right. , \\ & \left.\left. P(\theta, S(\theta), \tau, K_2(\tau)) - C(\theta, S(\theta), \tau, G_2(\tau))\right\} \middle| \mathcal{F}_t\right]. \end{aligned} \tag{22}$$

As such, the pricing problem turns out to be equivalent to pricing an exotic chooser option, of which the two underlying derivatives are now not European options, but spreads. We will derive an expression for $V(t,\tau)$ under the assumption that the second condition on the exercise prices, as it is given by formula 18, holds. The case where the first condition, given by formula 17, holds, is completely analogues. In order that there would exist a single $\varsigma \in \mathbb{R}_0^+$ such that:

$$\begin{array}{rcl} P(\theta, s, \tau, K_1(\tau)) - C(\theta, s, \tau, G_1(\tau)) & > & P(\theta, s, \tau, K_2(\tau)) - C(\theta, s, \tau, G_2(\tau)) \\ & \Updownarrow & \\ s & > & \varsigma \end{array} \tag{23}$$

we will need some regularity conditions on the prices of the European call and put options. First, we have the following properties of European option prices which hold irrespective of the stochastic process used to model the price of the underlying asset $S(t)$:

$$\lim_{S(\theta)\downarrow 0} \left(S(\theta) - C(\theta, S(\theta), \tau, G)\right) = 0, \tag{24}$$

$$\lim_{S(\theta)\downarrow 0}(P(\theta,S(\theta),\tau,K)) = Ke^{-(\tau-\theta)r}, \tag{25}$$

$$\lim_{S(\theta)\uparrow\infty}(S(\theta) - C(\theta,S(\theta),\tau,G)) = Ge^{-(\tau-\theta)r}, \tag{26}$$

$$\lim_{S(\theta)\uparrow\infty}(P(\theta,S(\theta),\tau,K)) = 0. \tag{27}$$

From the formulas 24 to 27 it follows immediately that there exists a $\varsigma \in \mathbb{R}_0^+$ such that: $P(\theta,\varsigma,\tau,K_1(\tau)) - C(\theta,\varsigma,\tau,G_1(\tau)) = P(\theta,\varsigma,\tau,K_2(\tau)) - C(\theta,\varsigma,\tau,G_2(\tau))$. Furthermore, we make the following assumption about the price of a call option:

$$\frac{\partial}{\partial S(\theta)}C(\theta,S(\theta),\tau,G) \text{ is decreasing in } G$$

It is clear that under this extra condition there exists exactly one $\xi \in \mathbb{R}_0^+$ for which 23 holds and that ξ is the solution of the following equation:

$$P(\theta,\xi,\tau,K_1(\tau)) - C(\theta,\xi,\tau,G_1(\tau)) = P(\theta,\xi,\tau,K_2(\tau)) - C(\theta,\xi,\tau,G_2(\tau)). \tag{28}$$

As such, we obtain from formula 22 that for $\tau > \theta$, $V(t,\tau)$ is given by:

$$\begin{aligned}
& V(t,\tau) \\
= \; & S(t) + e^{-(\theta-t)r}E^Q\left[P(\theta,S(\theta),\tau,K_1(\tau)) : S(\theta) > \xi\right] \\
& -e^{-(\theta-t)r}E^Q\left[C(\theta,S(\theta),\tau,G_1(\tau)) : S(\theta) > \xi\right] \\
& +e^{-(\theta-t)r}E^Q\left[P(\theta,S(\theta),\tau,K_2(\tau)) : S(\theta) \le \xi\right] \\
& -e^{-(\theta-t)r}E^Q\left[C(\theta,S(\theta),\tau,G_2(\tau)) : S(\theta) \le \xi\right].
\end{aligned} \tag{29}$$

In order to derive explicit expressions for expected values in the above equality, one has to specify the dynamics of the price process of the underlying asset. We will derive such expressions in case of the Black and Scholes model in the following section.

5.3 Black and Scholes model

Here we will derive an explicit expression for formula 29 for the case where $S(t)$ follows an exponential Brownian motion, i.e. that the price process $S(t)$ satisfies the following equation:

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dW(t), \tag{30}$$

with μ and σ given constants and $W(t)$ a standard Brownian motion.

In this case we obtain the four following equalities:

$$\begin{aligned} & e^{-(\theta-t)r} E^Q \left[P(\theta, S(\theta), \tau, K_2(\tau)) : S(\theta) \leq \xi\right] \\ = \; & e^{-(\tau-t)r} K_2(\tau) N_2 \left(x - \sigma\sqrt{\theta - t}, -y(K_2(\tau)) + \sigma\sqrt{\tau - t}; -\rho\right) \\ & -S(t) N_2 \left(x, -y(K_2(\tau)), -\rho\right), \end{aligned} \tag{31}$$

$$\begin{aligned} & e^{-(\theta-t)r} E^Q \left[C(\theta, S(\theta), \tau, G_2(\tau)) : S(\theta) \leq \xi\right] \\ = \; & -e^{-(\tau-t)r} G_2(\tau) N_2 \left(-x + \sigma\sqrt{\theta - t}, y(G_2(\tau)) - \sigma\sqrt{\tau - t}; -\rho\right) \\ & +S(t) N_2 \left(-x, y(G_2(\tau)); -\rho\right) \end{aligned} \tag{32}$$

$$\begin{aligned} & e^{-(\theta-t)r} E^Q \left[P(\theta, S(\theta), \tau, K_1(\tau)) : S(\theta) > \xi\right] \\ = \; & e^{-(\tau-t)r} K_1(\tau) N_2 \left(-x + \sigma\sqrt{\theta - t}, -y(K_1(\tau)) + \sigma\sqrt{\tau - t}; \rho\right) \\ & -S(t) N_2 \left(-x, -y(K_1(\tau)), \rho\right) \end{aligned} \tag{33}$$

$$\begin{aligned} & e^{-(\theta-t)r} E^Q \left[C(\theta, S(\theta), \tau, G_1(\tau)) : S(\theta) > \xi\right] \\ = \; & -e^{-(\tau-t)r} G_1(\tau) N_2 \left(x - \sigma\sqrt{\theta - t}, y(G_1(\tau)) - \sigma\sqrt{\tau - t}; \rho\right) \\ & +S(t) N_2 \left(x, y(G_1(\tau)); \rho\right). \end{aligned} \tag{34}$$

With:

$$x=\frac{\log(S(t)/\xi)+(r+\sigma^2/2)(\theta-t)}{\sigma\sqrt{\theta-t}}, \tag{35}$$

$$y(\alpha)=\frac{\log(S(t)/\alpha)+(r+\sigma^2/2)(\tau-t)}{\sigma\sqrt{\tau-t}},\ \alpha=K_1(\tau),G_1(\tau),K_2(\tau),G_2(\tau) \tag{36}$$

and:

$$\rho=\sqrt{\frac{\theta-t}{\tau-t}}. \tag{37}$$

Where ξ is the unique solution of:

$$\begin{aligned}
& K_1(\tau)e^{-(\tau-t)r}N\left(-z(K_1(\tau))+\sigma\sqrt{\theta-t}\right)-\xi N\left(-z(K_1(\tau))\right) && (38)\\
& +G_1(\tau)e^{-(\tau-t)r}N\left(z(G_1(\tau))-\sigma\sqrt{\theta-t}\right)-\xi N\left(z(G_1(\tau))\right) && (39)\\
= \;& K_2(\tau)e^{-(\tau-t)r}N\left(-z(K_2(\tau))+\sigma\sqrt{\theta-t}\right)-\xi N\left(-z(K_2(\tau))\right) && (40)\\
& +G_2(\tau)e^{-(\tau-t)r}N\left(z(G_2(\tau))-\sigma\sqrt{\theta-t}\right)-\xi N\left(z(G_2(\tau))\right). && (41)
\end{aligned}$$

With:

$$z(\alpha)=\frac{\log(S(t)/\alpha)+(r+\sigma^2/2)(\tau-\theta)}{\sigma\sqrt{\tau-\theta}},\ \alpha=K_1(\tau),G_1(\tau),K_2(\tau),G_2(\tau). \tag{42}$$

Here $N_2(x,y;\rho)$ is the cumulative bivariate normal distribution with correlation coefficient ρ.

In the next section we illustrate our results by means of an example.

6 Example: Unit-linked contract with guarantee versus deterministic benefit

First we will define a specific FUL contract, and in the second subsection we will give some numerical results for this type of contract.

6.1 The FUL Contract

The FUL contract we will discuss in this section is of a very specific form. The initial benefit scheme $B_1(\tau)$ is a unit-linked contract of the usual form as given by formula 15, but with $G_1(\tau) \equiv +\infty$ That is, there is no call option that limits the return on the investment. However, the optional benefit scheme $B_2(\tau)$ is, independent of the asset $S(t)$. Note that any deterministic scheme $D(\tau)$ is of the form as given by formula 16. By setting $G_2(\tau) = K_2(\tau) = D(\tau)$ and using the put call parity, we immediately obtain:

$$\begin{aligned} B_1(\tau) &= \max\left[D(\tau), \min(S(\tau), D(\tau))\right] \\ &= D(\tau). \end{aligned} \tag{43}$$

In this case the second condition on the exercise prices becomes:

$$K_1(\tau) < B_2(\tau). \tag{44}$$

This type of contract could for example be applied in an n-year endowment insurance. At a date θ, close towards the end of the contract, the insured would then be given the possibility to reinvest the value of his risky assets in a deterministic investment scheme B_2.

6.2 The Price of the Contract

From equation 22, and using the put-call parity, we obtain for $\tau > \theta$:

$$\begin{aligned} V(t,\tau) &= e^{-(\theta-t)r} E^Q \left[\max\left\{e^{-(\tau-\theta)r} K_1(\tau) + C(\theta, S(\theta), \tau, K_1(\tau))\right.\right., \\ &\quad \left.\left. e^{-(\tau-\theta)r} B_2(\tau)\right\} \middle| \mathcal{F}_t\right]. \end{aligned} \tag{45}$$

Which is equivalent to:

$$\begin{aligned} & V(t,\tau) \\ =\ & e^{-(\theta-t)r} E^Q \left[\max\left\{C(\theta, S(\theta), \tau, K_1(\tau))\right.\right. \\ -\ & \left.\left. e^{-(\tau-\theta)r} \left(B_2(\tau) - K_1(\tau)\right), 0\right\} \middle| \mathcal{F}_t\right] + e^{-(\tau-\theta)r} B_2(\tau). \end{aligned} \tag{46}$$

And in case the insured dies before he can change the characteristics of the contract, that is $\tau \leq \theta$, we obtain from formula 11 that $V(t,\tau)$ is given by:

$$V(t,\tau) = S(t) + P(t, S(t), \tau, K_1(\tau)).$$

Note that the first term in equation 46 is equal to the price of a compound call with as underlying option a call and of which the exercise price is given by: $e^{-(\tau-\theta)r}\left(B_2(\tau) - K_1(\tau)\right)$. Using the pricing formula for compound options, see Rubinstein (1992), we immediately obtain the following result. Note that one could as well start from formulas 31,...34, but this would have been a much longer way to obtain this result.

Corollary 1 *In an economy as described above, the conditional price $V(t,\tau)$, for $\tau > \theta$, of a FUL product B_{T_ρ} with a deterministic optional benefit scheme $B_2(\tau)$ and with $G_1(\tau) \equiv +\infty$, such that condition 18 is met, is given by:*

$$\begin{aligned} & V(t,\tau) \\ = \; & S(t)N_2\left(d1, d2, \rho\right) - K_1(\tau)e^{(\tau-t)r}N_2\left(d_1 - \sigma\sqrt{\theta - t}, d_2 - \sigma\sqrt{\tau - t}, \rho\right) \\ & + e^{-(\tau-t)r}B_2(\tau)N\left(-d_2\right) - e^{-(\tau-t)r}K_1(\tau)N\left(d_2\right). \end{aligned} \tag{47}$$

with:

$$d_1 = \frac{\log\left(S/\xi\right) + (\theta - t)\left(r + \sigma^2/2\right)}{\sigma\sqrt{\theta - t}} \tag{48}$$

$$d_2 = \frac{\log(S/K_1(\tau)) + (\tau - t)\left(r + \sigma^2/2\right)}{\sigma\sqrt{\tau - t}} \tag{49}$$

and:

$$\rho = \sqrt{\frac{\theta - t}{\tau - t}}. \tag{50}$$

And with ξ the solution to the following equation:

$$\xi\, N(d_3) - K_1(\tau)e^{-(\tau-\theta)r}N\left(d_3 - \sigma\sqrt{\tau - \theta}\right) = e^{-(\tau-\theta)r}\left(B_2(\tau) - K_1(\tau)\right), \tag{51}$$

with:

$$d_3 = \frac{\log(\xi/K(\tau)) + (\tau - \theta)(r + \sigma^2/2)}{\sigma\sqrt{\tau - \theta}}. \tag{52}$$

6.3 Numerical Illustration

Let us consider a 25-year endowment insurance, the insured being a woman, age 35. Therefore we have that: $\rho = \left(x : \overline{n|}\right)$. Let us set $S(t) = 50000BEF$ and $K_1(\tau) \equiv 50000BEF$. That is, the initial amount is guaranteed. Furthermore, we have: $B_2(\tau) = S(t)e^{-(\tau-t)\rho}$. For ρ we will consider two values: 0.025375 (annual interest of 2.75%) and 0.035078 (annual interest of 3.75%). Finally, the value of σ was set equal to 0.24202, and the interest rate r was set equal to 0.084274.

We will model the mortality risk by means of the Makeham law, that is:

$$P\left(T_x > t\right) = s^t g^{c^x(c^t-1)}.$$

The values of the three parameters were set equal to those of the Belgian mortality table FR:

$$\begin{aligned} s &= 0{,}999669730966 \\ g &= 0{,}999951440172 \\ c &= 1{,}116792453830. \end{aligned}$$

To model the mortality risk, we will make the assumption that the amount payable upon death of the insured will be payed at the end of the year during which the death occurred. Therefore, from formula 7 we obtain that the price of this FUL contract is given by:

$$\sum_{i=0}^{24} {}_{i|}q_{35} V(t, t+i+1) + {}_{25}p_{35}\, V(t, t+25).$$

Where ${}_{i|}q_{35}$ is the probability that the insured dies in the $i+1$the year of the contract and ${}_{25}p_{35}$ denotes the probability that the insured survives the period of 25 years. For $V(t,\tau)$ we obtain the following values:

TABLE 1
Value of $V(t,\tau)$

τ	$\rho = 0.025375$	$\rho = 0.035078$
1	52876.6	52876.6
2	53125.0	53125.0
3	53063.4	53063.4
4	52891.4	52891.4
5	52677.8	52677.8
6	52452.7	52452.7
7	52230.7	52230.7
8	52019.1	52019.1
9	51821.4	51821.4
10	51638.9	51638.9
11	51472.0	51472.0
12	51320.1	51320.1
13	51182.6	51182.6
14	51058.4	51058.4
15	50946.6	50946.6
16	50846.1	50846.1
17	50756.0	50756.0
18	50675.2	50675.0
19	50602.8	50602.8
20	50538.0	50538.0
21	51820.8	52905.1
22	51591.9	52611.3
23	51391.8	52345.1
24	51216.6	52104.3
25	51063.0	51886.7

Note that the $V(t,\tau)$ is fairly stable over time and that changing the guaranteed return in the optional benefit scheme $B_2(t)$ seems to have rather little impact. Finely, we obtain the following values for the price of the FUL contract.

TABLE 2
Price of the FUL contract

ρ	Price
0.025375	51075
0.035078	51880

7 Conclusion

In this paper we derived a pricing formula for a fairly general class of flexible (exotic) unit-linked products. However, in order to be able to price any unit-linked product one has to make some assumptions on financial risk and mortality. First of all, we assumed that for any insurance company these two risks are independent. We saw two reasons to withhold this assumption. First off all, and most importantly, there is very little evidence that this assumption would not hold in the real world. And secondly, if one wants to model any dependency between those two risks, for instance to allow for a catastrophic event, it is far from clear how this should be done.

Furthermore, we assumed that the insurer is risk neutral with respect to mortality risk. However, this restriction can be abolished without any difficulty by using an adjusted mortality law to compute the single premium.

Under those two assumptions, we were able to show that the price of a FUL contract can be calculated by using results from the pricing of exotic options, more specific compound options and chooser options.

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