POLES OF ARCHIMEDEAN ZETA FUNCTIONS FOR ANALYTIC MAPPINGS

E. LEÓN-CARDENAL, WILLEM VEYS, AND W. A. ZÚÑIGA-GALINDO

ABSTRACT. Let $\mathbf{f} = (f_1, \dots, f_l) : U \rightarrow K^l$, with $K = \mathbb{R}$ or $\mathbb{C}$, be a K -analytic mapping defined on an open set $U \subset K^n$, and let Φ be a smooth function on U with compact support. In this paper, we give a description of the possible poles of the local zeta function attached to $(\mathbf{f}, \Phi)$ in terms of a log-principalization of the ideal $\mathcal{I}_{\mathbf{f}} = (f_1, \dots, f_l)$. When $\mathbf{f}$ is a non-degenerate mapping, we give an explicit list for the possible poles of $Z_{\Phi}(s, \mathbf{f})$ in terms of the normal vectors to the supporting hyperplanes of a Newton polyhedron attached to $\mathbf{f}$, and some additional vectors (or rays) that appear in the construction of a simplicial conical subdivision of the first orthant. These results extend the corresponding results of Varchenko to the case $l \geq 1$, and $K = \mathbb{R}$ or $\mathbb{C}$. In the case $l = 1$ and $K = \mathbb{R}$, Denef and Sargos proved that the candidates poles induced by the extra rays required in the construction of a simplicial conical subdivision can be discarded from the list of candidate poles. We extend the Denef-Sargos result arbitrary $l \geq 1$. This yields in general a much shorter list of candidate poles, that can moreover be read off immediately from $\Gamma(\mathbf{f})$.

1. INTRODUCTION

We take $K = \mathbb{R}$ or $\mathbb{C}$. Let $\mathbf{f} = (f_1, \dots, f_l) : U \rightarrow K^l$ be a K -analytic mapping defined on an open U in K^n . Let $\Phi : U \rightarrow \mathbb{C}$ be a smooth function on U with compact support. Then the local zeta function attached to $(\mathbf{f}, \Phi)$ is defined as

$$Z_{\Phi}(s, \mathbf{f}) = \int_{K^n \setminus \mathbf{f}^{-1}(0)} \Phi(x) |\mathbf{f}(x)|_K^s |dx|,$$

for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, where $|dx|$ is the Haar measure on K^n . The local zeta functions have a meromorphic continuation to the whole complex plane. In the case $l = 1$, the meromorphic continuation of $Z_{\Phi}(s, \mathbf{f})$ was established jointly by Bernstein and Gel'fand [3], independently by Atiyah [2], then by a different method by Bernstein [4]. In [10], see also [11], Igusa developed a uniform theory for local zeta functions over local fields of characteristic zero. In this context, there exist asymptotic expansions for oscillatory integrals depending on one parameter which are controlled by the poles of ‘twisted versions’ of $Z_{\Phi}(s, \mathbf{f})$, see also [1], [20]. In [18], with $l \geq 1$ and $K = \mathbb{C}$, Phong and Sturm studied the stability of the poles of $Z_{\Phi}(s, \mathbf{f})$ under small perturbations of $\mathbf{f}$.

In this paper, we give a geometric description of the possible poles of $Z_{\Phi}(s, \mathbf{f})$, including the largest one, in terms of a log-principalization of the ideal $\mathcal{I}_{\mathbf{f}} =$

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$(f_1, \dots, f_l)$, see Theorem 2 and Proposition 2. To the best of our knowledge a such description has not been reported before, in the Archimedean context. (See however Remark 1 and Proposition 1 when $K = \mathbb{R}$.) When $\mathbf{f}$ is a non-degenerate mapping in the sense of [21], we give an explicit list for the possible poles of $Z_\Phi(s, \mathbf{f})$ in terms of the normal vectors to the supporting hyperplanes of a Newton polyhedron attached to $\mathbf{f}$, and additional vectors (or rays) that appear in the construction of a simplicial conical subdivision of the first orthant, see Theorem 3. These results extend the corresponding results of Varchenko in [20] for $l = 1$ and $K = \mathbb{R}$ to the case $l \geq 1$, and $K = \mathbb{R}$ or $\mathbb{C}$. In the case $l = 1$ and $K = \mathbb{R}$, Denef and Sargos proved in [5] a strong and interesting result: the candidate poles induced by the extra rays required in the construction of a simplicial conical subdivision can be discarded from the list of candidate poles in Theorem 3. We extend the Denef-Sargos result to arbitrary $l \geq 1$, see Theorems 4, 5, and 6. This yields in general a much shorter list of candidate poles, that can moreover be read off immediately from $\Gamma(\mathbf{f})$.

At this point we must mention that in the case $K = \mathbb{R}$ a description of the poles of $Z_\Phi(s, \mathbf{f})$ can be obtained by using an embedded resolution of singularities of $\sum_{i=1}^l f_i^2$, see [1], [11], [20]. In particular, one could define $\mathbf{f}$ to be non-degenerate as meaning $\sum_{i=1}^l f_i^2$ to be non-degenerate in the usual sense, and thus one could use all the results of [20]. We note however that there exist many mappings $\mathbf{f}$ which are non-degenerate in the sense of our Definition 1 but such that $\sum_{i=1}^l f_i^2$ is degenerate in the usual sense, see Remark 4. Thus, our approach gives a finer explicit description of the poles of $Z_\Phi(s, \mathbf{f})$.

2. LOCAL ZETA FUNCTIONS FOR ANALYTIC MAPPINGS

2.1. Fixing the data. We take $K = \mathbb{R}$ or $\mathbb{C}$. For $a = (a_1, \dots, a_n) \in K^n$ we put $|a|_K = |a|_{\mathbb{R}}$ or $|a|_{\mathbb{C}}^2$, where $|\cdot|_{\mathbb{R}}$ and $|\cdot|_{\mathbb{C}}$ are the standard norms on $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively.

Let $f_1, \dots, f_l$ be polynomials in $K[x_1, \dots, x_n]$, or, more generally, K -analytic functions on an open set $U \subset K^n$. We consider the mapping $\mathbf{f} = (f_1, \dots, f_l) : K^n \rightarrow K^l$, respectively, $U \rightarrow K^l$. Let $\Phi : K^n \rightarrow \mathbb{C}$ be a smooth function with compact support, i.e. $\Phi \in C_0^\infty$, with support in U in the second case.

2.2. Log-principalization of ideals. We state the version of log-principalization of ideals that we will use in this paper, [7], see also [9], [22].

Theorem 1 ([7]). *Let $K = \mathbb{R}$ or $\mathbb{C}$ and let U be an open submanifold of K^n . Let $f_1, \dots, f_l$ be K -analytic functions on U such that the ideal $\mathcal{I}_{\mathbf{f}} = (f_1, \dots, f_l)$ is not trivial. Then there exists a log-principalization $h : X_K \rightarrow U$ of $\mathcal{I}_{\mathbf{f}}$, that is,*

(1) *X_K is an n -dimensional K -analytic manifold, h is a proper K -analytic map which can be chosen as a composition of a finite number of blow-ups in closed submanifolds, and which is an isomorphism outside of the common zero set Z_K of $f_1, \dots, f_l$;*

(2) *$h^{-1}(Z_K) = \cup_{i \in T} E_i$, where the E_i are closed submanifolds of X_K of codimension one, each equipped with a pair of positive integers (N_i, v_i) satisfying the following. At every point b of X_K there exist local coordinates $(y_1, \dots, y_n)$ on X_K around b such that, if $E_1, \dots, E_r$ are the E_i containing b , we have on some neighborhood of*

b that E_i is given by $y_i = 0$ for $i = 1, \dots, r$,

$$h^*(\mathcal{I}_{\mathbf{f}}) \text{ is generated by } \varepsilon(y) \prod_{i=1}^r y_i^{N_i},$$

and

$$h^*(dx_1 \wedge \dots \wedge dx_n) = \eta(y) \left(\prod_{i=1}^r y_i^{v_i-1} \right) dy_1 \wedge \dots \wedge dy_n,$$

where $\varepsilon(y)$, $\eta(y)$ are units in the local ring of X_K at b .

The (N_i, v_i) , $i \in T$, are called the numerical data of h for $\mathcal{I}_{\mathbf{f}}$.

2.3. Poles of local zeta functions. From now on we suppose that $\mathbf{f}^{-1}(0) \neq \emptyset$. To $\mathbf{f}$ and Φ as in 2.1 we associate the local zeta function $Z_{\Phi}(s, \mathbf{f})$, $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, defined in the introduction.

Remark 1. (1) When $K = \mathbb{R}$ the zeta function $Z_{\Phi}(s, f)$ of the mapping f is clearly equal to the zeta function $Z_{\Phi}(\frac{s}{2}, F)$ of the function $F := \sum_{i=1}^l f_i^2$. In particular, it is known that $Z_{\Phi}(s, f)$ has a meromorphic continuation to the whole complex plane, and that its poles are negative rational numbers and of order at most n , see e.g. [13]. Also the list of candidate poles in Theorem 2 below for $K = \mathbb{R}$ can in fact be derived from the function case, see Proposition 1. But since the proof of Theorem 2 for $K = \mathbb{C}$ is also valid for $K = \mathbb{R}$ we prefer to state and prove it simultaneously for both fields.

(2) Over $K = \mathbb{C}$ the meromorphic continuation can be analogously reduced to the case of one real-analytic function. The point of Theorem 2 is the description of the candidate poles in terms of a principalization of the ideal.

Theorem 2. Let $\mathbf{f}$ and Φ be as in 2.1. Let $h : X_K \rightarrow U$ be a fixed log-principalization of the ideal $\mathcal{I}_{\mathbf{f}} = (f_1, \dots, f_l)$, with numerical data (N_i, v_i) , $i \in T$, for $\mathcal{I}_{\mathbf{f}}$. Then $Z_{\Phi}(s, \mathbf{f})$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ and the poles are contained in the union of

$$-\frac{v_i}{N_i} - \frac{\mathbb{N}}{N_i}, \quad i \in T.$$

Therefore the poles are negative rational numbers. Moreover their orders are at most equal to n .

Proof. We use all the notations concerning the log-principalization h introduced in Theorem 1. Let $b \in X_K$ be a point, and (ϕ_V, V) a chart containing it. Let $E_1, \dots, E_r$ denote the components of $h^{-1}(\mathbf{f}^{-1}(0))$ passing through b . We set $\mathbf{f}^*(y) := \mathbf{f}(h(y))$.

If $r = 0$, i.e., $\mathbf{f}(h(b)) \neq 0$, then we can choose a small neighborhood V_b of b over which $|\mathbf{f}^*(y)|_K$ is positive and $\mathbb{R}$ -analytic, and thus

$$(2.1) \quad |\mathbf{f}^*(y)|_K^s = e^{s \ln |\mathbf{f}^*(y)|_K} \text{ is } \mathbb{R}\text{-analytic in } y \in V_b \text{ and holomorphic in } s \in \mathbb{C}.$$

In addition,

$$(2.2) \quad h^*(dx_1 \wedge \dots \wedge dx_n) = \eta(y) (dy_1 \wedge \dots \wedge dy_n),$$

where $\eta(y)$ is a unit of the local ring of X_K at b .

If $r \geq 1$, then we have in V that

$$(2.3) \quad f_i^*(y) = f_i(h(y)) = g(y) \tilde{f}_i(y), \quad i = 1, \dots, l,$$

$$(2.4) \quad g(y) = \varepsilon(y) \prod_{i=1}^r y_i^{N_i},$$

and

$$(2.5) \quad h^*(dx_1 \wedge \dots \wedge dx_n) = \eta(y) \prod_{i=1}^r y_i^{v_i-1} (dy_1 \wedge \dots \wedge dy_n),$$

where $\varepsilon(y)$ and $\eta(y)$ are units of the local ring of X_K at b . Furthermore, there exists an index i_0 such that $\tilde{f}_{i_0}(b) \neq 0$. Then

$$|\mathbf{f}^*(y)|_K^s = |\varepsilon(y)|_K^s \left(\prod_{i=1}^r |y_i|_K^{N_i} \right)^s |\tilde{\mathbf{f}}(y)|_K^s,$$

where $\tilde{\mathbf{f}}(y) := (\tilde{f}_1(y), \dots, \tilde{f}_l(y))$, in V .

We can choose a small neighborhood V_b of b over which (2.3)-(2.5) are valid, and $|\varepsilon(y)|_K$, $|\tilde{\mathbf{f}}(y)|_K$, $|\eta(y)|_K$ are $\mathbb{R}$ -analytic. Then $|\varepsilon(y)|_K^s$, $|\tilde{\mathbf{f}}(y)|_K^s$ are $\mathbb{R}$ -analytic in y for any $s \in \mathbb{C}$, and holomorphic in $s \in \mathbb{C}$, for any $y \in V_b$.

Since $h^{-1}(\text{support } \Phi)$ is compact, we can take a finite covering of the form $\{V_b\}$ where the V_b are homeomorphic under ϕ_V to the polydisc $P_\epsilon(0)$ in K^n defined by $|y_i|_K < \epsilon$, with ϵ sufficiently small and for $1 \leq i \leq n$. By picking a smooth partition of the unity subordinate to $\{V_b\}$, and using the previous discussion,

$$Z_\Phi(s, \mathbf{f}) = \int_{X_K \setminus h^{-1}(\mathbf{f}^{-1}(0))} \Phi^*(y) |\mathbf{f}^*(y)|_K^s |h^*(dx_1 \wedge \dots \wedge dx_n)|$$

becomes a finite sum of integrals of the following two types:

$$(2.6) \quad I(s) := \int_{K^n} \Psi(y) |\mathbf{f}^*(y)|_K^s |dy|,$$

where Ψ is a C_0^∞ function with support contained in a polydisc $P_\epsilon(0)$ and $e^{s \ln |\mathbf{f}^*(y)|_K}$ is $\mathbb{R}$ -analytic for $y \in V_b$ and holomorphic in $s \in \mathbb{C}$, or

$$(2.7) \quad J(s) := \int_{K^n} \Theta(y, s) \left(\prod_{i=1}^r |y_i|_K^{N_i s + v_i - 1} \right) |dy|,$$

where $\Theta(y, s)$ is a C_0^∞ function with support contained in a polydisc $P_\epsilon(0)$, depending holomorphically on $s \in \mathbb{C}$. By using the Dominated Convergence Lemma, we have that (2.6) defines a holomorphic function on the complex plane. The meromorphic continuation and the description of the corresponding poles for integrals $J(s)$ is known, see, for instance, the proofs of Theorem 5.4.1 in [13] or Theorem 1.6 in [11]. $\square$

We indicate now why also the list of candidate poles in Theorem 2 for $K = \mathbb{R}$ can be derived from the function case. We think this proposition has some independent interest. A note on terminology: we still use the term log-principalization starting with one function, i.e. with a principal ideal; usually one calls this an embedded resolution.

Proposition 1. *Let $\mathbf{f} = (f_1, \dots, f_l) : U \rightarrow \mathbb{R}^l$ be an $\mathbb{R}$ -analytic mapping on an open $U \subseteq \mathbb{R}^n$. Then $h : X_{\mathbb{R}} \rightarrow U$ is a log-principalization of the ideal $\mathcal{I}_{\mathbf{f}} = (f_1, \dots, f_l)$ if and only if it is a log-principalization of the function $F := \sum_{i=1}^l f_i^2$. Moreover, when $(N_i, v_i), i \in T$, are the numerical data of h for $\mathcal{I}_{\mathbf{f}}$, then $(2N_i, v_i), i \in T$, are the numerical data of h for F .*

Proof. We suppose that h is a log-principalization of the function F , with numerical data $(2N_i, v_i), i \in T$ ($2N_i$ will turn out to be even). We work in the local ring corresponding to some fixed point of $X_{\mathbb{R}}$; recall that this local ring is a unique factorization domain. Let $y_1, \dots, y_n$ be coordinates at this point, i.e., a system of parameters of the local ring. We know that

$$F^*(y) = \sum_{i=1}^l (f_i^*(y))^2 = \epsilon(y) \prod_{i=1}^r y_i^{2N_i},$$

where $\epsilon(y)$ is a unit and $r \leq n$. Let $g(y) := \gcd_{1 \leq i \leq l} f_i^*(y)$ and write $f_i^*(y) = g(y) \tilde{f}_i(y)$.

We claim that $\sum_{i=1}^l (\tilde{f}_i(y))^2$ is a unit $u(y)$. Assuming the claim, we have that at least one of the $\tilde{f}_i(y)$ is a unit. Hence

$$h^* \mathcal{I}_{\mathbf{f}} = (f_1^*(y), \dots, f_l^*(y)) = (g(y))$$

and $\epsilon(y) \prod_{i=1}^r y_i^{2N_i} = u(y)(g(y))^2$. So, up to a unit, $g(y)$ equals $\prod_{i=1}^r y_i^{N_i}$, and indeed h is a log-principalization of $\mathcal{I}_{\mathbf{f}}$.

We now prove the claim. We know that

$$\epsilon(y) \prod_{i=1}^r y_i^{2N_i} = (g(y))^2 \left(\sum_{i=1}^l (\tilde{f}_i(y))^2 \right).$$

Since we have unique factorization, either this sum is a unit, or it is divisible by one of the irreducible elements of the left hand side, i.e., by $y_1, y_2, \dots$ or y_r . Say $\sum_{i=1}^l (\tilde{f}_i(y))^2$ is divisible by y_1 . We can always write this sum as

$$\sum_{i=1}^l (\tilde{f}_i(0, y_2, \dots, y_n))^2 + y_1(\dots).$$

Then divisibility by y_1 implies that $\sum_{i=1}^l (\tilde{f}_i(0, y_2, \dots, y_n))^2 = 0$. Since we are working over $\mathbb{R}$ this can only happen if $\tilde{f}_i(0, y_2, \dots, y_n) = 0$ for all $i = 1, \dots, l$. But this is equivalent to y_1 dividing all $\tilde{f}_i(y_1, y_2, \dots, y_n)$, contradicting that the $\tilde{f}_i(y)$ are relatively prime.

The other implication is quite straightforward. $\square$

Remark 2. We recall the description of the (local and global) log canonical threshold, see e.g. [6]-[16]-[19], in terms of a log-principalization. Let $K = \mathbb{R}$ or $\mathbb{C}$ and U an open submanifold of K^n . Let $I_{\mathbf{f}} = (f_1, \dots, f_l)$ be a non-trivial ideal of K -analytic functions on U . Fix a log-principalization $h : X_K \rightarrow U$ of $I_{\mathbf{f}}$ as in Theorem 1.

(1) Let $p \in \mathbf{f}^{-1}(0)$. The (K) -log canonical threshold of $I_{\mathbf{f}}$ at p is $c_p(I_{\mathbf{f}}) = \min_{i \in T, p \in h(E_i)} \left\{ \frac{v_i}{N_i} \right\}$.

(2) The (K) -log canonical threshold of $I_{\mathbf{f}}$ is $c(I_{\mathbf{f}}) = \min_{i \in T} \left\{ \frac{v_i}{N_i} \right\}$.

Proposition 2. Let $\mathbf{f}$ and Φ be as in 2.1.

(1) Let $p \in \mathbf{f}^{-1}(0)$. If Φ is real and nonnegative with support in a small enough neighborhood of p (in particular $\Phi(p) > 0$), then $-c_p(\mathcal{I}_{\mathbf{f}})$ is a pole of $Z_{\Phi}(s, \mathbf{f})$, more precisely its largest pole.

Fix a log-principalization of $\mathcal{I}_{\mathbf{f}}$ as in Theorem 1.

(2) Say that $c(\mathcal{I}_{\mathbf{f}}) = \frac{v_i}{N_i}$ precisely for $i \in T_\lambda(\subset T)$. If Φ is real and nonnegative and its support intersects $h(\cup_{i \in T_\lambda} E_i)$, then $-c(\mathcal{I}_{\mathbf{f}})$ is a pole of $Z_\Phi(s, \mathbf{f})$, more precisely its largest pole.

(3) Let $r(\mathcal{I}_{\mathbf{f}})$ be the maximal number of $E_i, i \in T$, with $c(\mathcal{I}_{\mathbf{f}}) = \frac{v_i}{N_i}$, respectively $c_p(\mathcal{I}_{\mathbf{f}}) = \frac{v_i}{N_i}$, that have a nonempty intersection. Then $-c(\mathcal{I}_{\mathbf{f}})$, respectively $-c_p(\mathcal{I}_{\mathbf{f}})$, is a pole of order $r(\mathcal{I}_{\mathbf{f}})$ of $Z_\Phi(s, \mathbf{f})$ when Φ is as above.

Proof. One uses the case of monomial integrals like in the case of one analytic function, see e.g. [1, Chap. II, § 7]. The proof is a simple variation of the one given in [1, Chap. II, § 7, Lemme 4] and [12, pp. 32-33] for the case $l = 1$. For $K = \mathbb{R}$ one can alternatively use the case $l = 1$ and Proposition 1. $\square$

Note that this proposition gives an argument to see that these minima do not depend on the chosen log-principalization.

Corollary 1. *Let K be $\mathbb{R}$ or $\mathbb{C}$ as before, and $\mathbf{f}$ as in 2.1. Let D be a compact subset of K^n . Then*

(1) $|\mathbf{f}(x)|_K^\delta$ is locally integrable for $\delta > -c(\mathcal{I}_{\mathbf{f}})$,

(2) $\text{Vol}(\{x \in D \mid |\mathbf{f}(x)|_K \leq \alpha\}) \leq \alpha^{c(\mathcal{I}_{\mathbf{f}}) - \epsilon} \int_D |\mathbf{f}(x)|_K^{-c(\mathcal{I}_{\mathbf{f}}) + \epsilon} |dx|$,
for $\alpha > 0$ and any small $\epsilon > 0$.

Proof. The first part follows directly from Proposition 2. Then the second part follows via the Chebyshev inequality. $\square$

Such bounds on volumes have recently emerged as central to aspects of complex differential geometry, see [18] and references therein.

3. NEWTON POLYHEDRA AND LOG-PRINCIPALIZATIONS

We collect some results about Newton polyhedra and log-principalizations following [21] and the references therein. In this section we take again $K = \mathbb{R}$ or $K = \mathbb{C}$.

We set $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. Let G be a nonempty subset of $\mathbb{N}^n$. The *Newton polyhedron* $\Gamma = \Gamma(G)$ associated to G is the convex hull in $\mathbb{R}_+^n$ of the set $\cup_{m \in G} (m + \mathbb{R}_+^n)$. For instance classically one associates a *Newton polyhedron (at the origin) to* $g(x) = \sum_m c_m x^m$ ($x = (x_1, \dots, x_n)$, $g(0) = 0$), being a nonconstant polynomial function over K or a K -analytic function in a neighborhood of the origin, where $G = \text{supp}(g) := \{m \in \mathbb{N}^n \mid c_m \neq 0\}$. Further we associate more generally a Newton polyhedron to an analytic mapping.

We fix a Newton polyhedron Γ as above. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product of $\mathbb{R}^n$, and identify the dual space of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself by means of it.

For $a \in \mathbb{R}_+^n$, we define

$$d(a, \Gamma) = d(a) = \min_{x \in \Gamma} \langle a, x \rangle,$$

and the first meet locus $F(a)$ of a as

$$F(a) := \{x \in \Gamma \mid \langle a, x \rangle = d(a)\}.$$

The first meet locus is a face of Γ . Moreover, if $a \neq 0$, $F(a)$ is a proper face of Γ .

We define an equivalence relation in $\mathbb{R}_+^n$ by taking $a \sim a' \Leftrightarrow F(a) = F(a')$. The equivalence classes of $\sim$ are sets of the form

$$\Delta_\tau = \{a \in \mathbb{R}_+^n \mid F(a) = \tau\},$$

where τ is a face of Γ .

We recall that the cone strictly spanned by the vectors $a_1, \dots, a_r \in \mathbb{R}_+^n \setminus \{0\}$ is the set $\Delta = \{\lambda_1 a_1 + \dots + \lambda_r a_r \mid \lambda_i \in \mathbb{R}_+, \lambda_i > 0\}$. If $a_1, \dots, a_r$ are linearly independent over $\mathbb{R}$, Δ is called a *simplicial cone*. If $\{a_1, \dots, a_r\}$ is a subset of a basis of the $\mathbb{Z}$ -module $\mathbb{Z}^n$, we call Δ a *simple cone*.

A precise description of the geometry of the equivalence classes modulo $\sim$ is as follows. Each *facet* (i.e. a face of codimension one) γ of Γ has a unique vector $a(\gamma) = (a_{\gamma,1}, \dots, a_{\gamma,n}) \in \mathbb{N}^n \setminus \{0\}$, whose nonzero coordinates are relatively prime and which is perpendicular to γ . We denote by $\mathfrak{D}(\Gamma)$ the set of such vectors. The equivalence classes are rational cones of the form

$$\Delta_\tau = \left\{ \sum_{i=1}^r \lambda_i a(\gamma_i) \mid \lambda_i \in \mathbb{R}_+, \lambda_i > 0 \right\},$$

where τ runs through the set of faces of Γ , and γ_i , $i = 1, \dots, r$ are the facets containing τ . We note that $\Delta_\tau = \{0\}$ if and only if $\tau = \Gamma$. The family $\{\Delta_\tau\}_\tau$, with τ running over the proper faces of Γ , is a partition of $\mathbb{R}_+^n \setminus \{0\}$; we call this partition a *polyhedral subdivision of $\mathbb{R}_+^n$ subordinated to Γ* . We call $\{\overline{\Delta}_\tau\}_\tau$, the family formed by the topological closures of the Δ_τ , a *fan subordinated to Γ* .

Each cone Δ_τ can be partitioned into a finite number of simplicial cones $\Delta_{\tau,i}$. In addition, the subdivision can be chosen such that each $\Delta_{\tau,i}$ is spanned by part of $\mathfrak{D}(\Gamma)$. Thus from the above considerations we have the following partition of $\mathbb{R}_+^n \setminus \{0\}$:

$$(3.1) \quad \mathbb{R}_+^n \setminus \{0\} = \bigcup_{\tau} \left(\bigcup_{i=1}^{l_\tau} \Delta_{\tau,i} \right),$$

where τ runs over the proper faces of Γ , and each $\Delta_{\tau,i}$ is a simplicial cone contained in Δ_τ .

By adding new rays, each simplicial cone can be partitioned further into a finite number of simple cones. In this way we obtain a *simple polyhedral subdivision of $\mathbb{R}_+^n$ subordinated to Γ* and a *simple fan subordinated to Γ* (see e.g. [14]).

3.1. The Newton polyhedron associated to an analytic mapping. Let $\mathbf{f} = (f_1, \dots, f_l)$, $\mathbf{f}(0) = 0$, be a nonconstant analytic mapping defined on a neighborhood $U \subset K^n$ of the origin. In [21] the authors associated to $\mathbf{f}$ a Newton polyhedron $\Gamma(\mathbf{f}) := \Gamma(\cup_{i=1}^l \text{supp}(f_i))$, and a non-degeneracy condition to $\mathbf{f}$ and $\Gamma(\mathbf{f})$.

If $f_i(x) = \sum_m c_{m,i} x^m$, and τ is a face of $\Gamma(\mathbf{f})$, we set

$$f_{i,\tau}(x) := \sum_{m \in \text{supp}(f_i) \cap \tau} c_{m,i} x^m.$$

Definition 1. Let $\mathbf{f} = (f_1, \dots, f_l) : U \rightarrow K^l$ be a nonconstant analytic mapping satisfying $\mathbf{f}(0) = 0$. The mapping $\mathbf{f}$ is called *strongly non-degenerate at the origin with respect to $\Gamma(\mathbf{f})$* , if for any compact face $\tau \subset \Gamma(\mathbf{f})$ and any $z \in \{z \in (K^\times)^n \mid f_{1,\tau}(z) = \dots = f_{l,\tau}(z) = 0\}$ it satisfies $\text{rank} \left[\frac{\partial f_{i,\tau}}{\partial x_j}(z) \right] = \min\{l, n\}$.

Remark 3. (1) The above notion of non-degeneracy agrees with the one given by Varchenko for the case $l = 1$, see [20]. On the other hand, the previous notion does not agree with the non-degeneracy notion with respect to a collection of Newton

polyhedra given by Khovanskii in [15]. We refer the reader to [21] for a further discussion about the mentioned non-degeneracy conditions.

(2) If we fix l and Γ , then one can show, just as for the classical case $l = 1$, that ‘most’ mappings $\mathbf{f} = (f_1, \dots, f_l)$, with $\mathbf{f}(0) = 0$ and Newton polyhedron $\Gamma(\mathbf{f}) = \Gamma$, are strongly non-degenerate at the origin. One can easily generalize e.g. the proof of Lemme 1 in [1, p. 157]. However, just as for $l = 1$, we should say that there are many interesting non-generic mappings.

(3) In [15], Khovanskii established the existence of an embedded resolution for a variety using a collection of Newton polyhedra, see also [1] for the case $l = 1$. In [21], a log-principalization for an ideal with generators satisfying the above-mentioned notion of non-degeneracy and using one Newton polyhedron was established, see Proposition 3 below. This result agrees with Khovanskii’s result only in the case $l = 1$.

Remark 4. When $K = \mathbb{R}$ it is again natural to try to study the mapping $\mathbf{f} = (f_1, \dots, f_l)$ via the function $F := \sum_{i=1}^l f_i^2$. It is not difficult to verify that $\Gamma(F)$ is the ‘double’ of $\Gamma(\mathbf{f})$, i.e. obtained from it after scaling by a factor 2. Consider however the statements

- (i) F is non-degenerate at the origin with respect to $\Gamma(F)$, and
- (ii) $\mathbf{f}$ is strongly non-degenerate at the origin with respect to $\Gamma(\mathbf{f})$.

It is easy to verify that (i) implies (ii), but in general the converse is not true. Consider for instance any strongly non-degenerate $\mathbf{f}$ for which $\{f_{1,\tau} = \dots = f_{l,\tau} = 0\} \cap (\mathbb{R}^\times)^n \neq \emptyset$ for some compact face τ of $\Gamma(\mathbf{f})$, e.g. $\mathbf{f} = (x^2 - y^3, x^2 - z^3)$.

In such cases the ‘classical’ embedded resolution of a non-degenerate function is not helpful, but one can use the log-principalization of a strongly non-degenerate mapping of Proposition 3 below.

3.2. Log-principalizations.

Proposition 3 ([21, Prop. 3.9]). *Let $\mathbf{f} = (f_1, \dots, f_l) : U \subset K^n \rightarrow K^l$ be a nonconstant analytic mapping, strongly non-degenerate at the origin with respect to $\Gamma(\mathbf{f})$. Let $\mathcal{F}_{\mathbf{f}}$ be a simple fan subordinated to $\Gamma(\mathbf{f})$. Let Y_K be the toric manifold corresponding to $\mathcal{F}_{\mathbf{f}}$, and let*

$$\sigma_0 : Y_K \rightarrow U$$

be the restriction of the corresponding toric map to the inverse image of U . Denote by Z the set of common zeroes of $\mathcal{I}_{\mathbf{f}} = (f_1, \dots, f_l)$ in $U \cap (K^\times)^n$. When U is taken small enough, either $Z = \emptyset$ or it is a submanifold of codimension l . In this last case we have $l < n$ and we denote the closure of Z in Y_K by Z_Y .

(1) If $Z = \emptyset$ (or if $l = 1$), the ideal $\sigma_0^(\mathcal{I}_{\mathbf{f}})$ is principal (and monomial) in a sufficiently small neighborhood of $\sigma_0^{-1}\{0\}$.*

(2) If $Z \neq \emptyset$, we have that Z_Y is a closed submanifold of Y_K , having normal crossings with the exceptional divisor of σ_0 . Let $\sigma_1 : X_K \rightarrow Y_K$ be the blowing-up of Y_K with center Z_Y , and let $\sigma = \sigma_0 \circ \sigma_1 : X_K \rightarrow U$. Then the ideal $\sigma^(\mathcal{I}_{\mathbf{f}})$ is principal (and monomial) in a sufficiently small neighborhood of $\sigma^{-1}\{0\}$.*

4. POLES FOR LOCAL ZETA FUNCTIONS AND NEWTON POLYHEDRA

Given $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{N}^n \setminus \{0\}$, we put $\sigma(\xi) := \xi_1 + \dots + \xi_n$ and $d(\xi) = \min_{x \in \Gamma(\mathbf{f})} \langle \xi, x \rangle$ as before. We say that ξ is a primitive vector, if $\gcd(\xi_1, \dots, \xi_n) = 1$.

If $d(\xi) \neq 0$, we define

$$\mathcal{P}(\xi) = \left\{ -\frac{\sigma(\xi) + k}{d(\xi)} \mid k \in \mathbb{N} \right\}.$$

We also define

$$\gamma_0(\mathbf{f}) = \min_{\xi \in \mathfrak{D}(\Gamma(\mathbf{f}))} \left\{ \frac{\sigma(\xi)}{d(\xi)} \right\}.$$

Varchenko called $\gamma_0(\mathbf{f})$ the distance from the origin to $\Gamma(\mathbf{f})$. The number $\gamma_0(\mathbf{f})$ admits the following geometric interpretation. Let $(t_0, \dots, t_0)$ be the intersection point of the diagonal $\{(t, \dots, t) \in \mathbb{R}^n \mid t \in \mathbb{R}\}$ with the boundary of $\Gamma(\mathbf{f})$, then $\gamma_0(\mathbf{f}) = 1/t_0$.

Let $\mathcal{F}_{\mathbf{f}}$ be a simple fan subordinated to $\Gamma(\mathbf{f})$. Then the set of generators of the cones in $\mathcal{F}_{\mathbf{f}}$, i.e. the skeleton of $\mathcal{F}_{\mathbf{f}}$, can be partitioned as $\Lambda_{\mathbf{f}} \cup \mathfrak{D}(\Gamma(\mathbf{f}))$, where $\Lambda_{\mathbf{f}}$ is a finite set of primitive vectors, corresponding to the extra rays, induced by the subdivision into simple cones.

The numerical data of the log-principalizations constructed in Proposition 3 can be computed directly from the explicit expressions for the generators of $\sigma_0^*(\mathcal{I}_{\mathbf{f}})$, $\sigma^*(\mathcal{I}_{\mathbf{f}})$, and Lemme 8 in [1, p. 201]. The following theorem follows from the previous considerations by adapting the proof given by Varchenko for the case $l = 1$ to arbitrary $l \geq 1$.

Theorem 3. *Let $\mathbf{f} = (f_1, \dots, f_l) : U(\subset K^n) \longrightarrow K^l$, with $\mathbf{f}(0) = 0$, be an analytic mapping, strongly non-degenerate at the origin with respect to $\Gamma(\mathbf{f})$. There exists a neighborhood $V(\subset U)$ of the origin such that, if Φ is a smooth function with support contained in V , then the following assertions hold.*

- (1) *The function $Z_{\Phi}(s, \mathbf{f})$ is holomorphic on the complex half-plane $\operatorname{Re}(s) > \max\{-\gamma_0(\mathbf{f}), -l\}$.*
- (2) *The poles of $Z_{\Phi}(s, \mathbf{f})$ belong to the set $\cup_{\xi \in \mathfrak{D}(\Gamma(\mathbf{f}))} \mathcal{P}(\xi) \cup \cup_{\xi \in \Lambda_{\mathbf{f}}} \mathcal{P}(\xi) \cup -(l + \mathbb{N})$, where the last set may be discarded if $l \geq n$.*
- (3) *If $\gamma_0(\mathbf{f}) < l$, then $s = -\gamma_0(\mathbf{f})$ is a pole of $Z_{\Phi}(s, \mathbf{f})$ as a distribution on the space of smooth functions with compact support.*

In the case $l = 1$ and $K = \mathbb{R}$, Denef and Sargos proved in [5] that $\cup_{\xi \in \Lambda_{\mathbf{f}}} \mathcal{P}(\xi)$ may be discarded from the list of candidate poles in Theorem 3. This is a strong and interesting result, yielding in general a much shorter list of candidate poles, that can moreover be read off immediately from $\Gamma(\mathbf{f})$.

In the next sections, we extend the Denef-Sargos result to arbitrary $l \geq 1$. Actually, we follow reasonably closely the approach of [5]. However, a number of extra difficulties pop up, for which a careful analysis is needed. In order to produce a readable text, we have to recall in the sequel the main ideas of [5]. Our goal will be to show the following result, for which we already indicate the starting point of its proof.

Theorem 4. *Let $\mathbf{f} = (f_1, \dots, f_l) : U(\subset \mathbb{R}^n) \longrightarrow \mathbb{R}^l$, with $\mathbf{f}(0) = 0$, be an analytic mapping, strongly non-degenerate at the origin with respect to $\Gamma(\mathbf{f})$. There exists a neighborhood $V(\subset U)$ of the origin such that, if Φ is a smooth function with support contained in V , then the following assertions hold.*

- (1) *The poles of $Z_{\Phi}(s, \mathbf{f})$ belong to the set $\cup_{\xi \in \mathfrak{D}(\Gamma(\mathbf{f}))} \mathcal{P}(\xi) \cup -(l + \mathbb{N})$.*
- (2) *Let ρ be an integer satisfying $1 \leq \rho \leq n$, and let s_0 be a candidate pole of $Z_{\Phi}(s, \mathbf{f})$ with $s_0 \notin -(l + \mathbb{N})$ (respectively $s_0 \in -(l + \mathbb{N})$). A necessary condition*

for s_0 to be a pole of $Z_\Phi(s, \mathbf{f})$ of order ρ , is that there exists a face $\tau \subset \Gamma(\mathbf{f})$ of codimension ρ (respectively of codimension $\rho - 1$) such that $s_0 \in \mathcal{P}(\xi)$ for any facet $F(\xi) \supset \tau$.

Proof. Define

$$I_\Phi(s, \mathbf{f}) = I(s, \mathbf{f}, \Phi) = \int_{\mathbb{R}_+^n} \Phi(x) |\mathbf{f}(x)|_{\mathbb{R}}^s |dx|, \quad \operatorname{Re}(s) > 0.$$

This integral defines an analytic function for $\operatorname{Re}(s) > 0$; $I(s, \mathbf{f}, \Phi)$ and $Z_\Phi(s, \mathbf{f})$ are related by

$$Z_\Phi(s, \mathbf{f}) = \sum_{\theta \in \{-1, 1\}^n} I(s, \mathbf{f}(\theta \cdot x), \Phi(\theta \cdot x)),$$

where $\theta \cdot x = (\theta_1 x_1, \dots, \theta_n x_n)$. The result will follow from the meromorphic continuation of $I_\Phi(s, \mathbf{f})$, and the explicit description of its poles, cf. Theorem 6. $\square$

5. MONOMIAL INTEGRALS AND DECOUPAGES

From now on we take $K = \mathbb{R}$ and use all the notations introduced in the previous sections.

5.1. Some Monomial Integrals. We give some results about the meromorphic continuation of integrals attached to monomials that we will use later on. These results are easy variations of well-known results, see e.g. [1, Chap. II, § 7, Lemme 3], [5, Lemme 3.1], [8, Chap. I, Sect. 3.2], and [11, pp. 101-102].

Lemma 1. *Let Ω be an open neighborhood of the origin in $\mathbb{R}^k \times \mathbb{R}$ and let $g : \Omega \rightarrow \mathbb{R}$ be an $\mathbb{R}$ -analytic function. Let ϕ be a smooth function with support in Ω and containing the origin. Take $a = (a_1, \dots, a_{k+1}) \in (\mathbb{R}_+ \setminus \{0\})^{k+1}$ such that $[0, a_1] \times \dots \times [0, a_{k+1}]$ is contained in the support of ϕ . Assume that $g(y, z) > 0$ for (y, z) in the support of ϕ , and define*

$$J(s) = \int_0^{a_1} \dots \int_0^{a_{k+1}} \left(\prod_{j=1}^r y_j^{sm_j + \gamma_j - 1} \right) z^{s+l-1} g(y, z)^s \phi(y, z) |dy \wedge dz|, \quad \operatorname{Re}(s) > 0,$$

with $1 \leq r \leq k$ and $m_j, \gamma_j \in \mathbb{N} \setminus \{0\}$ for $j = 1, \dots, r$, and $l \in \mathbb{N} \setminus \{0\}$. Then the following assertions hold:

(1) $J(s)$ is convergent and defines a holomorphic function on

$$\operatorname{Re}(s) > \max\{-l, -\gamma_1/m_1, \dots, -\gamma_r/m_r\};$$

(2) $J(s)$ admits a meromorphic continuation to the whole complex plane, with poles of order at most $k + 1$. Furthermore, the poles belong to

$$\bigcup_{1 \leq i \leq r} \left(-\frac{\gamma_i + \mathbb{N}}{m_i} \right) \cup (-(l + \mathbb{N})).$$

(3) Let ρ be a positive integer and let s_0 be a candidate pole of $J(s)$ with $s_0 \notin -(l + \mathbb{N})$ (resp. $s_0 \in -(l + \mathbb{N})$). A necessary condition for s_0 to be a pole of $J(s)$ of order ρ , is that

$$\operatorname{Card} \left\{ i \mid s_0 \in -\frac{(\gamma_i + \mathbb{N})}{m_i} \right\} \geq \rho \quad (\text{resp. } \geq \rho - 1).$$

Corollary 2. *Let r, k and l be natural numbers such that $1 \leq r \leq k$ and $k+1-r-l \geq 0$. Let U be a neighborhood of the origin of $\mathbb{R}^l$. Let θ be a smooth function with compact support contained in $[0, 1]^r \times U \times [0, 1]^{k+1-r-l}$. Set*

$$J_1(s) := \int_{[0,1]^r \times U \times [0,1]^{k+1-r-l}} \prod_{j=1}^r y_j^{sm_j + \gamma_j - 1} \left(\sum_{i=r+1}^{r+l} y_i^2 \right)^{s/2} \theta(y) |dy|, \quad \operatorname{Re}(s) > 0.$$

Then $J_1(s)$ has a meromorphic continuation to $\mathbb{C}$ and its poles satisfy all the conclusions in Lemma 1.

Proof. The result follows from the previous lemma by using hyper-spherical coordinates. $\square$

5.2. Decoupages in $\mathbb{R}^n$ and compatible mappings. Let $\mathbf{f} = (f_1, \dots, f_l) : U(\subset \mathbb{R}^n) \rightarrow \mathbb{R}^l$ be an analytic mapping, strongly non-degenerate at the origin with respect to $\Gamma(\mathbf{f})$.

We fix a simplicial fan $\mathcal{F}_{\mathbf{f}}$ subordinated to $\Gamma(\mathbf{f})$, and assume that the cones of $\mathcal{F}_{\mathbf{f}}$ of dimension n are $\overline{\Delta}_1, \dots, \overline{\Delta}_m$.

Definition 2. Let $X \subseteq \mathbb{R}^n$ be a measurable set in the sense of Lebesgue. A *decoupage* of X is a finite family $\mathcal{D} = \{D_1, \dots, D_m\}$ of measurable sets satisfying

- (1) $D_i \cap D_j$ has measure zero for $i \neq j$, and
- (2) $X \setminus (\cup_{i=1}^m D_i)$ and $(\cup_{i=1}^m D_i) \setminus X$ have measure zero.

Define

$$\mathcal{L} : (0, 1]^n \rightarrow \mathbb{R}_+^n : (x_1, \dots, x_n) \rightarrow (-\ln x_1, \dots, -\ln x_n).$$

Then $\mathcal{L}$ is an $\mathbb{R}$ -analytic isomorphism, and $\{\mathcal{L}^{-1}(\overline{\Delta}_j)\}_{1 \leq j \leq m}$ is a decoupage of $[0, 1]^n$. In addition, for any smooth function $\phi(x)$,

$$\int_{[0,1]^n} \phi(x) |dx| = \sum_{j=1}^m \int_{\mathcal{L}^{-1}(\overline{\Delta}_j)} \phi(x) |dx|.$$

In the sequel we will use a decoupage of the domain of integration to give a short list of candidate poles for local zeta functions. It may happen that a mapping $(f_{1,\tau}(x), \dots, f_{l,\tau}(x))$, restricted to certain components of the boundary of a decoupage, has singularities. These cases will turn out to be annoying, but they are rare and they can be removed by using homotheties, cf. Lemma 3.

Take vectors $a_1, \dots, a_n \in \mathbb{N}^n$ that are linearly independent over $\mathbb{R}$, and denote

$$\overline{\Delta} := \left\{ \sum_{i=1}^n \lambda_i a_i \mid \lambda_i \in \mathbb{R}_+ \right\}$$

for the closed cone spanned by them. For $K \subset \{1, \dots, n\}$, set

$$\Delta_K := \left\{ \sum_{i \in K} \lambda_i a_i \mid \lambda_i \in \mathbb{R}_+, \lambda_i > 0 \right\}$$

and $S_K := \mathcal{L}^{-1}(\Delta_K)$, with the convention that $S_K = \{(1, \dots, 1)\}$ if $K = \emptyset$. We will call S_K a *sector*.

Definition 3. We say that $\mathbf{f}$ is *compatible with the sector S_K* if the following condition is satisfied. For each partition $\{I, J, K\}$ of $\{1, \dots, n\}$, with $I \neq \emptyset$, $J \neq \emptyset$, and such that the face $\tau = \cap_{i \in I} F(a_i)$ is compact and non-empty, the restriction

to S_K of the map $\mathbf{f}_\tau := (f_{1,\tau}, \dots, f_{l,\tau})$ does not admit $(0, \dots, 0) \in \mathbb{R}^l$ as a critical value.

We say that $\mathbf{f}$ is compatible with $\mathcal{L}^{-1}(\overline{\Delta})$, if $\mathbf{f}$ is compatible with the sector S_K for any $K \subset \{1, \dots, n\}$.

We say that $\mathbf{f}$ is *compatible with the decoupage* $\mathcal{L}^{-1}(\mathcal{F}_\mathbf{f}) = \{\mathcal{L}^{-1}(\overline{\Delta}) \mid \overline{\Delta} \in \mathcal{F}_\mathbf{f} \text{ with } \dim \overline{\Delta} = n\}$ (or *compatible with $\mathcal{F}_\mathbf{f}$*), if $\mathbf{f}$ is compatible with $\mathcal{L}^{-1}(\overline{\Delta})$, for any n -dimensional cone $\overline{\Delta}$ in $\mathcal{F}_\mathbf{f}$.

Remark 5. Note that the condition ‘the restriction to S_K of $\mathbf{f}_\tau$ does not admit $(0, \dots, 0) \in \mathbb{R}^l$ as a critical value’ is equivalent to the two following conditions:

- (1) if $\text{Card}(K) \geq l$, then $\text{rank} \left[\frac{\partial f_{i,\tau}}{\partial x_j}(z) \right] = l$, for any $z \in \{z \in S_K \mid f_{1,\tau}(z) = \dots = f_{l,\tau}(z) = 0\}$, and
- (2) if $0 \leq \text{Card}(K) \leq l-1$, then for any $z_0 \in S_K$, there exists an index $i_0 = i_0(z_0)$, such that $f_{i_0,\tau}(z_0) \neq 0$.

Remark 6. Set $n = l = 2$. Since any partition $\{I, J, K\}$ of $\{1, 2\}$, with $I \neq \emptyset$ and $J \neq \emptyset$, requires that $K = \emptyset$ and that $\text{Card}(I) = 1$, any τ in Definition 3 is a facet of $\Gamma(\mathbf{f})$. Therefore $\mathbf{f}$ is compatible with a simplicial fan $\mathcal{F}_\mathbf{f}$ if and only if $f_{1,\gamma}(1, 1) \neq 0$ or $f_{2,\gamma}(1, 1) \neq 0$, for any compact facet γ of $\Gamma(\mathbf{f})$.

Example 1. Set $\mathbf{f} = (x_2^a - x_1^b, x_1^c - x_2^d)$, with $a \leq d$ and $b \leq c$. Then $\mathbf{f}$ is not compatible with any simplicial fan $\mathcal{F}_\mathbf{f}$, since $f_{i,\gamma}(1, 1) = 0$ ($i = 1, 2$) for the facet γ containing $(0, a)$ and $(b, 0)$. Set $T_{(3,2)} : (\mathbb{R}_+ \setminus \{0\})^2 \rightarrow (\mathbb{R}_+ \setminus \{0\})^2$; $(x_1, x_2) \mapsto (3x_1, 2x_2)$. Then $\mathbf{f} \circ T_{(3,2)}$ is compatible with every simplicial fan $\mathcal{F}_\mathbf{f}$.

Example 2. Set $\mathbf{g} = (x_2^b + x_1^b, x_1^c)$, with $c \leq b$. Then $\mathbf{g}$ is compatible with every simplicial fan $\mathcal{F}_\mathbf{f}$.

5.3. Compatibility and homotheties. Subsequently we will use only the part of the following lemma, stating that the sets $T_{l,k'}$ have measure zero. But we think it is natural to state and prove it as below.

Lemma 2. Assume $l \leq n$. Let $W_0 \times W_1$ be an open neighborhood of $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$, and $\mathbf{h} : W_0 \times W_1 \rightarrow \mathbb{R}^l : (t, z) \mapsto \mathbf{h}(t, z) = (h_1(t, z), \dots, h_l(t, z))$ an analytic mapping such that $(0, \dots, 0) \in \mathbb{R}^l$ is not a critical value of $\mathbf{h}$. Set

$$M := \{(t, z) \in W_0 \times W_1 \mid h_1(t, z) = \dots = h_l(t, z) = 0\},$$

$\pi : M \rightarrow \mathbb{R}^n : (t, z) \mapsto t$, and

$$T_{l',k'} := \left\{ \begin{array}{l} t \in W_0 \mid \mathbf{h}(t, z) = 0 \text{ for some } z \in W_1 \text{ and} \\ \text{rank} \left[\frac{\partial h_i}{\partial z_k}(t, z) \right]_{\substack{1 \leq i \leq l' \\ k' \leq k \leq n}} < \min\{l', n - k' + 1\} \end{array} \right\}$$

for $1 \leq l' \leq l$ and $1 \leq k' \leq n$. Then $T_{l',k'}$ has measure zero, and the set of critical values of π is the union of the sets $T_{l',k'}$ and of the similar sets obtained by permuting the indices.

Proof. We establish the result by showing explicitly that the set of critical values of π restricted to a neighborhood $M_{(a,b)}$ of point (a, b) in M has the form $T_{l',k'}$. Then by Sard’s lemma $T_{l',k'}$ has measure zero, for any l', k' . This explicit description is achieved as follows: given $(a, b) \in M$, by the implicit function theorem, we can solve the system $\{h_1(t, z) = \dots = h_l(t, z) = 0\}$ locally around (a, b) , for l variables

in terms of the others. Depending on the selection made upon the set of variables several cases occur.

Case 1. All the depending variables are taken from the first n variables. After renaming the variables if necessary, we assume that

$$\det \left[\frac{\partial h_i}{\partial t_j} ((a, b)) \right]_{\substack{1 \leq i \leq l \\ 1 \leq j \leq l}} \neq 0.$$

Set $t = (t', t'')$, where $t' = (t_1, \dots, t_l)$ and $t'' = (t_{l+1}, \dots, t_n)$. We also set $(a, b) = (a', a'', b) \in (\mathbb{R}^l \times \mathbb{R}^{n-l} \times \mathbb{R}^n) \cap M$. Then there exist an open set $G \subset \mathbb{R}^{2n-l}$, containing (a'', b) , and a unique analytic function $\mathbf{g} : G \rightarrow \mathbb{R}^l$, such that $\mathbf{g}(a'', b) = a'$ and $\mathbf{h}(\mathbf{g}(t'', z), t'', z) = 0$ for any $(t'', z) \in G$.

Set $M_{(a,b)} = \{(t', t'', z) \in \mathbf{g}(G) \times G \mid t_1 = g_1(t'', z), \dots, t_l = g_l(t'', z)\} \subseteq M$. Then

$$\begin{aligned} \pi|_{M_{(a,b)}} : M_{(a,b)} &\rightarrow \mathbb{R}^n \\ (t'', z) &\mapsto (g_1(t'', z), \dots, g_l(t'', z), t''), \end{aligned}$$

and the Jacobian matrix of $\pi|_{M_{(a,b)}}$ is

$$J(t'', z) = \begin{bmatrix} \frac{\partial g_1}{\partial t_{l+1}} & \cdots & \frac{\partial g_1}{\partial t_n} & \frac{\partial g_1}{\partial z_1} & \cdots & \frac{\partial g_1}{\partial z_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_l}{\partial t_{l+1}} & \cdots & \frac{\partial g_l}{\partial t_n} & \frac{\partial g_l}{\partial z_1} & \cdots & \frac{\partial g_l}{\partial z_n} \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} (t'', z).$$

The point (t', t'', z) is a critical point for π if $\text{rank} J(t'', z) < n$; this means that $\text{rank} \left[\frac{\partial g_i}{\partial z_k} (t'', z) \right]_{\substack{1 \leq i \leq l \\ 1 \leq k \leq n}} < l$. Since $\sum_{j=1}^l \left(\frac{\partial h_i}{\partial t_j} \right) \left(\frac{\partial g_j}{\partial z_k} \right) = \frac{\partial h_i}{\partial z_k}$ and the matrix $\left[\frac{\partial h_i}{\partial t_j} \right]_{\substack{1 \leq i \leq l \\ 1 \leq j \leq l}}$ has maximal rank around (a, b) , this last condition is equivalent to

$$\text{rank} \left[\frac{\partial h_i}{\partial z_k} ((t', t'', z)) \right]_{\substack{1 \leq i \leq l \\ 1 \leq k \leq n}} < l.$$

Hence $T_{l,1}$ is the set of critical values of $\pi|_{M_{(a,b)}}$.

Case 2. Some of the depending variables are taken from the first n variables (t -variables) and the rest are taken from the last n variables (z -variables). After renaming the variables if necessary, we assume that

$$\det \left[\begin{array}{cc} \left[\frac{\partial h_i}{\partial t_j} ((a, b)) \right]_{\substack{1 \leq i \leq l \\ n-l'+1 \leq j \leq n}} & \left[\frac{\partial h_i}{\partial z_j} ((a, b)) \right]_{\substack{1 \leq i \leq l \\ 1 \leq j \leq l''}} \end{array} \right] \neq 0,$$

with $l = l' + l''$. Put $t' = (t_1, \dots, t_{n-l'})$, $t'' = (t_{n-l'+1}, \dots, t_n)$, $z' = (z_1, \dots, z_{l''})$ and $z'' = (z_{l''+1}, \dots, z_n)$, hence $t = (t', t'')$ and $z = (z', z'')$. Then the depending variables are t'' and z' . Consider a point $(a, b) = (a', a'', b', b'') \in (\mathbb{R}^{n-l'} \times \mathbb{R}^{l'} \times \mathbb{R}^{l''} \times \mathbb{R}^{n-l''}) \cap M$. Then there exist an open set $Q \subset \mathbb{R}^{2n-l}$, containing (a', b'') ,

and a unique analytic function $\mathbf{q} : Q \rightarrow \mathbb{R}^l$ such that $\mathbf{q}(a', b'') = (a'', b')$ and $\mathbf{h}(t', \mathbf{q}(t', z''), z'') = 0$, for any $(t', z'') \in Q$. Set

$$M_{(a,b)} :=$$

$$\left\{ \begin{array}{l} (t', t'', z', z'') \in W_0 \times W_1 \mid (t', z'') \in Q, (t'', z') \in \mathbf{q}(Q) \text{ and} \\ t_{n-l'+1} = q_1(t', z''), \dots, t_n = q_{l'}(t', z''), z_1 = q_{l'+1}(t', z''), \dots, z_{l''} = q_l(t', z'') \end{array} \right\}.$$

Then

$$\pi|_{M_{(a,b)}} : \begin{array}{ccc} M_{(a,b)} & \rightarrow & \mathbb{R}^n \\ (t', z'') & \mapsto & (t', q_1(t', z''), \dots, q_{l'}(t', z'')), \end{array}$$

and the corresponding Jacobian matrix is

$$J(t', z'') = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ \frac{\partial q_1}{\partial t_1} & \dots & \frac{\partial q_1}{\partial t_{n-l'}} & \frac{\partial q_1}{\partial z_{l'+1}} & \dots & \frac{\partial q_1}{\partial z_n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial q_{l'}}{\partial t_1} & \dots & \frac{\partial q_{l'}}{\partial t_{n-l'}} & \frac{\partial q_{l'}}{\partial z_{l'+1}} & \dots & \frac{\partial q_{l'}}{\partial z_n} \end{bmatrix} (t', z'').$$

The point (t', t'', z', z'') is a critical point for $\pi|_{M_{(a,b)}}$ if $\text{rank} J(t', z'') < n$, that is,

$$\text{rank} \left[\frac{\partial q_i}{\partial z_k}(t', z'') \right]_{\substack{1 \leq i \leq l' \\ k' \leq k \leq n}} < l' = \min\{l', n - l''\},$$

where $k' = l'' + 1$. As above, this is equivalent with $\text{rank} \left[\frac{\partial h_i}{\partial z_k}(t', t'', z', z'') \right]_{\substack{1 \leq i \leq l' \\ k' \leq k \leq n}} < l'$, hence $T_{l', k'}$ is the set of critical values of $\pi|_{M_{(a,b)}}$. $\square$

Given $b \in (\mathbb{R}_+ \setminus \{0\})^n$, we define

$$T_b : \begin{array}{ccc} (\mathbb{R}^\times)^n & \rightarrow & (\mathbb{R}^\times)^n \\ (x_1, \dots, x_n) & \mapsto & (b_1 x_1, \dots, b_n x_n). \end{array}$$

Lemma 3. *Let $\mathbf{f}$ be a strongly non-degenerate analytic mapping at the origin with respect to $\Gamma(\mathbf{f})$. Let $\mathcal{L}^{-1}(\mathcal{F}_{\mathbf{f}})$ be a decoupage induced by a simplicial fan subordinated to $\Gamma(\mathbf{f})$. Then for almost every $b \in (\mathbb{R}_+ \setminus \{0\})^n$, the mapping $\mathbf{f} \circ T_b$ is compatible with $\mathcal{L}^{-1}(\mathcal{F}_{\mathbf{f}})$.*

Proof. We use all the notation introduced in Section 5.2 and Definition 3. Set $\overline{\Delta} = \{\sum_{i=1}^n \lambda_i a_i \mid \lambda_i \in \mathbb{R}_+\}$ a cone in $\mathcal{F}_{\mathbf{f}}$, $\{I, J, K\}$ a partition of $\{1, \dots, n\}$ with $I \neq \emptyset$ and $J \neq \emptyset$, and $\Delta_K = \{\sum_{i \in K} \lambda_i a_i \mid \lambda_i \in \mathbb{R}_+, \lambda_i > 0\}$ for $K \subset \{1, \dots, n\}$. Assume $\tau = \cap_{i \in I} F(a_i)$ is non-empty and compact.

Case ($\text{Card}(K) \geq l$). Note that in this case necessarily $l < n$ (since $\text{Card}(K) < n$). We set $\psi := \mathcal{L}^{-1} \circ \theta$ where

$$\mathbb{R}^n \xrightarrow{\theta} \mathbb{R}^n \xrightarrow{\mathcal{L}^{-1}} (\mathbb{R}_+ \setminus \{0\})^n,$$

$\theta(z) = \sum_{i=1}^n z_i a_i$, and the extension of $\mathcal{L}$ to $(\mathbb{R}_+ \setminus \{0\})^n$ is denoted again by $\mathcal{L}$. We also set

$$U = \{z \in \mathbb{R}^n \mid z_j = 0 \text{ if } j \notin K, \text{ and } z_j > 0 \text{ if } j \in K\},$$

and

$$N = \left\{ \begin{array}{l} b \in (\mathbb{R}_+ \setminus \{0\})^n \mid (\mathbf{f}_\tau \circ T_b \circ \psi)(z) = 0 \quad \text{and} \\ \text{rank} \left[\frac{\partial(f_{i,\tau} \circ T_b \circ \psi)}{\partial z_k}(z) \right]_{\substack{1 \leq i \leq l \\ k \in K}} < \min\{l, \text{Card}(K)\} = l, \text{ for some } z \in U \end{array} \right\}.$$

To establish this case, we show that N has measure zero. We may assume that $K = \{k', k' + 1, \dots, n\}$, by renaming the coordinates, with $k' \leq n - l + 1$.

Note that for any $b \in (\mathbb{R}_+ \setminus \{0\})^n$ there exists a $t \in \mathbb{R}^n$ such that $b = \psi(t)$, and

$$\begin{aligned} (\mathbf{f}_\tau \circ T_b \circ \psi)(z) &= \mathbf{f}_\tau(T_{\psi(t)} \circ \psi)(z) = \mathbf{f}_\tau(\psi_1(t)\psi_1(z), \dots, \psi_n(t)\psi_n(z)) \\ &= \mathbf{f}_\tau(\psi_1(t+z), \dots, \psi_n(t+z)) = (\mathbf{f}_\tau \circ \psi)(t+z). \end{aligned}$$

Since $\mathbf{f}$ is strongly non-degenerate at the origin with respect to $\Gamma(\mathbf{f})$ and ψ is an analytic isomorphism over $(\mathbb{R}^\times)^n$, by taking $l' = l$ in Lemma 2 we conclude that

$$N' = \left\{ \begin{array}{l} t \in \mathbb{R}^n \mid (\mathbf{f}_\tau \circ \psi)(t+z) = 0 \quad \text{and} \\ \text{rank} \left[\frac{\partial(f_{i,\tau} \circ \psi)}{\partial z_k}(t+z) \right]_{\substack{1 \leq i \leq l \\ k' \leq k \leq n}} < l, \text{ for some } z \in U \end{array} \right\}$$

has measure zero. Since ψ is an analytic isomorphism, N has measure zero too.

Case ($0 \leq \text{Card}(K) \leq l - 1$). We show that the set of the b 's such that $(\mathbf{f}_\tau \circ T_b)(z_0) = 0$ for $z_0 \in S_K$ has measure zero. This is equivalent to show that $Y_{z_0} = \{t \in \mathbb{R}^n \mid (\mathbf{f}_\tau \circ \psi)(t+z_0) = 0\}$, for a $z_0 \in U$, has measure zero. Finally Y_{z_0} has measure zero, because it is a proper $\mathbb{R}$ -analytic subset of $\mathbb{R}^n$. $\square$

6. SHORT LIST OF CANDIDATE POLES

6.1. Integral restricted to a sector. Let $a_1, \dots, a_n \in \mathbb{N}^n$, determining linearly independent vectors of $\mathbb{R}^n$, and $\overline{\Delta}_\tau = \{\lambda_1 a_1 + \dots + \lambda_n a_n \mid \lambda_i \in \mathbb{R}_+\}$ a closed cone in a fixed simplicial fan $\mathcal{F}_\mathbf{f}$ subordinated to $\Gamma(\mathbf{f})$, such that $F(a) = \tau$ for any $a \in \Delta_\tau$. Set

$$I_{\overline{\Delta}_\tau}(s) := I_{\overline{\Delta}_\tau}(s, \mathbf{f}, \Phi) = \int_{\mathcal{L}^{-1}(\overline{\Delta}_\tau)} \Phi(x) |\mathbf{f}(x)|_{\mathbb{R}}^s |dx|,$$

for $\text{Re}(s) > 0$, where Φ is a smooth function with compact support, contained in a neighborhood of the origin.

6.1.1. The change of variables for a cone. Set $x \in (0, 1]^n$ such that $\mathcal{L}(x) \in \overline{\Delta}_\tau$. Then

$$\mathcal{L}(x) = (-\ln x_1, \dots, -\ln x_n) = \lambda_1 a_1 + \dots + \lambda_n a_n = (-\ln y_1) a_1 + \dots + (-\ln y_n) a_n,$$

for some unique $y = (y_1, \dots, y_n) \in (0, 1]^n$ satisfying $\mathcal{L}(y) = \lambda$. Thus

$$\ln x_i = \sum_{j=1}^n a_{ij} (\ln y_j) = \sum_{j=1}^n \ln(y_j^{a_{ij}}) = \ln \left(\prod_{j=1}^n y_j^{a_{ij}} \right),$$

i.e.

$$x_i = \prod_{j=1}^n y_j^{a_{ij}} \quad \text{and} \quad x^\alpha = \prod_{j=1}^n y_j^{\langle a_j, \alpha \rangle},$$

where $a_j = (a_{ij})_{1 \leq i \leq n}$. From these considerations, we define

$$\begin{aligned} w : [0, 1]^n &\rightarrow [0, 1]^n \\ y &\mapsto x = w(y), \quad \text{where } x_i = \prod_{j=1}^n y_j^{a_{ij}}. \end{aligned}$$

By using w as a change of variables in $I_{\overline{\Delta}_\tau}(s)$, one gets

$$(6.1) \quad I_{\overline{\Delta}_\tau}(s) = |\det[a_1, \dots, a_n]|_{\mathbb{R}} \int_{[0,1]^n} \phi(y) \left(\prod_{j=1}^n y_j^{\sigma(a_j)-1} \right) |\mathbf{f}(w(y))|_{\mathbb{R}}^s |dy|,$$

where $\det[a_1, \dots, a_n]$ denotes the determinant of the matrix with columns $a_1, \dots, a_n$ and $\phi(y) := \Phi(w(y))$.

6.1.2. *Description of $w^{-1}\{0\}$.* The set $w^{-1}\{0\}$ plays an important role in the considerations below. We give here some properties of this set, proven in [5, Lemma 5.1] and [5, Lemma 5.2], that we will use later on.

Lemma 4. (1) Take $y \in [0, 1]^n$. Set $I = \{i \in \{1, \dots, n\} \mid y_i = 0\}$. Assume that $I \neq \emptyset$ and that $\tau' = \cap_{i \in I} F(a_i) \neq \emptyset$. Then

$y \in w^{-1}\{0\}$ if and only if τ' is a compact face of $\Gamma(\mathbf{f})$.

(2) If V is a small enough neighborhood (resp. a neighborhood) of the origin in $[0, 1]^n$, then $w^{-1}(V)$ is a small enough neighborhood (resp. neighborhood) of $w^{-1}\{0\}$ in $[0, 1]^n$.

6.1.3. *Properties of $\mathbf{f} \circ w$.* Writing $f_i = \sum_m c_{m,i} x^m$ we note for any $i = 1, \dots, l$ that

$$(6.2) \quad f_i(w(y)) = \sum_m c_{m,i}(w(y))^m = \sum_m c_{m,i} \prod_{j=1}^n y_j^{\langle a_j, m \rangle},$$

what we can write as

$$(6.3) \quad f_i(w(y)) = \left(\prod_{j=1}^n y_j^{d(a_j)} \right) f_i^*(y),$$

where $d(a_j) = \min_{x \in \Gamma(\mathbf{f})} \langle a_j, x \rangle$. If $\text{supp}(f_i) \cap \tau \neq \emptyset$, then the minimum of all $\langle a_j, m \rangle$ is attained at τ and $f_i^*(0) \neq 0$. If $\text{supp}(f_i) \cap \tau = \emptyset$, then $f_i^*(0) = 0$.

From now on we pick a sufficiently small neighborhood W of $w^{-1}\{0\}$ on which the series in the right hand side of (6.2) converges for every i . Then for every i the restriction of this series to $W \cap [0, 1]^n$ agrees with $f_i \circ w$.

6.2. Meromorphic Continuation of $I_{\overline{\Delta}_\tau}(s)$.

Theorem 5. Let $\mathbf{f} : U(\subset \mathbb{R}^n) \rightarrow \mathbb{R}^l$, with $\mathbf{f}(0) = 0$, be an analytic mapping, strongly non-degenerate at the origin with respect to $\Gamma(\mathbf{f})$. Assume that $\tau = \cap_{i=1}^n F(a_i)$ is a 0-dimensional face of $\Gamma(\mathbf{f})$, and that $\mathbf{f}$ is compatible with $\mathcal{L}^{-1}(\overline{\Delta}_\tau)$. There exists a neighborhood $V(\subset U)$ of the origin such that, if Φ is a smooth function with support contained in V , the following assertions hold.

(1) $I_{\overline{\Delta}_\tau}(s)$ is convergent and defines a holomorphic function of s on $\text{Re}(s) > \max\{-l, -\frac{\sigma(a_1)}{d(a_1)}, \dots, -\frac{\sigma(a_n)}{d(a_n)}\}$,

(2) $I_{\overline{\Delta}_\tau}(s)$ admits a meromorphic extension to the complex plane with poles of order at most n . Furthermore, the poles belong to

$$\left(\bigcup_{1 \leq i \leq n} \left(-\frac{\sigma(a_i) + \mathbb{N}}{d(a_i)} \right) \right) \cup (-(l + \mathbb{N})).$$

(3) Let ρ be a positive integer and let s_0 be a candidate pole of $I_{\Delta_\tau}(s)$, with $s_0 \notin -(l + \mathbb{N})$ (resp. $s_0 \in -(l + \mathbb{N})$). A necessary condition for s_0 to be a pole of $I_{\Delta_\tau}(s)$ of order ρ , is that

$$\text{Card} \left\{ i \mid s_0 \in \left(-\frac{\sigma(a_i) + \mathbb{N}}{d(a_i)} \right) \right\} \geq \rho \text{ (resp. } \geq \rho - 1 \text{)}.$$

Proof. We first prove (1) and (2). By (6.1) it is sufficient to prove the result for

$$(6.4) \quad I_\Omega(s) = \int_{[0,1]^n} \Omega(y) \left(\prod_{j=1}^n y_j^{\sigma(a_j)-1} \right) |f(w(y))|_{\mathbb{R}}^s |dy|,$$

where Ω is a smooth function. By using a sufficiently fine partition of the unity, one can express $I_\Omega(s)$ as a finite sum of analogous integrals $I_{\Omega_p}(s)$, where Ω_p is a smooth function with support contained in a small ball around a point p belonging to the support of Ω . The relevant points p to consider belong to $w^{-1}\{0\}$. In the sequel we may and will assume that $\text{supp}(\Omega_p)$ is as small as necessary for the arguments that follow. Several cases occur.

Case 1 (p is the origin of $\mathbb{R}^n$).

Assume that Ω_0 is a smooth function containing the origin. There exists at least one index i_0 such that $\text{supp}(f_{i_0}) \cap \tau \neq \emptyset$. Hence, with the notation of (6.3), there exists a small neighborhood of the origin $V_{0,\tau}$ such that $\{0\} \subset \text{supp}(\Omega_0) \subset V_{0,\tau}$, and $f_{i_0}^*(y) \neq 0$ for any $y \in V_{0,\tau}$. Consequently, we have that $\sum_{i=1}^l (f_i^*(y))^2 > 0$ on $V_{0,\tau}$, and then by Lemma 1 the integral

$$(6.5) \quad I_{\Omega_0}(s) = \int_{[0,1]^n} \Omega_0(y) \left(\prod_{j=1}^n y_j^{sd(a_j)+\sigma(a_j)-1} \right) \left(\sum_{i=1}^l (f_i^*(y))^2 \right)^{s/2} |dy|$$

has a meromorphic continuation to $\mathbb{C}$, with poles (of order at most n) belonging to

$$\bigcup_{1 \leq i \leq n} \left(-\frac{\sigma(a_i) + \mathbb{N}}{d(a_i)} \right).$$

Case 2 (p has exactly r coordinates equal to zero, with $1 \leq r \leq n-1$).

After renaming the variables, we may suppose that the first r coordinates are zero, i.e. p is the form $p = (0, \dots, 0, p_{r+1}, \dots, p_n)$. Let τ' be the first meet locus of the cone $\Delta_{\tau'}$, spanned by $a_1, \dots, a_r$; it is a compact face of $\Gamma(f)$. Note that, similarly as in [1, Chap. II, § 8, Lemme 9],

$$(6.6) \quad f_i(w(y)) = \prod_{j=1}^r y_j^{d(a_j)} \left(\tilde{f}_i(y_{r+1}, \dots, y_n) + O_i(y_1, \dots, y_n) \right),$$

where the $\tilde{f}_i$ are polynomials in $y_{r+1}, \dots, y_n$ and the $O_i(y_1, \dots, y_n)$ are analytic functions in $y_1, \dots, y_n$ but belonging to the ideal generated by $y_1, \dots, y_r$. Here $\tilde{f}_i$ is identically zero if and only if $\text{supp}(f_i) \cap \tau' = \emptyset$. Furthermore,

$$(6.7) \quad f_{i,\tau'}(w(y)) = \prod_{j=1}^r y_j^{d(a_j)} \left(\tilde{f}_i(y_{r+1}, \dots, y_n) \right).$$

For the sequel we redefine $f_i^*(y)$ as

$$f_i^*(y) = \tilde{f}_i(y_{r+1}, \dots, y_n) + O_i(y_1, \dots, y_n).$$

Set $\tilde{p} = (p_{r+1}, \dots, p_n) \in (\mathbb{R}^\times)^{n-r}$. To accomplish the proof of Case 2, we need to study the following three subcases.

Subcase 2.1 (There exists an index i such that $\tilde{f}_i(\tilde{p}) \neq 0$). Then there exists a neighborhood $V_{p,\tau'}$ of $p = (0, \dots, 0, p_{r+1}, \dots, p_n)$, such that $f_i^*(y) \neq 0$ for any $y \in V_{p,\tau'}$ and such that $\text{supp}(\Omega_p) \subset V_{p,\tau'}$. Hence

$$g(y_1, \dots, y_n) := \left(\sum_{i=1}^l (f_i^*(y_1, \dots, y_n))^2 \right)^{1/2} > 0,$$

for any $y \in V_{p,\tau'}$, and, by Lemma 1,

$$(6.8) \quad I_{\Omega_p}(s) = \int_{[0,1]^n} \Omega_p(y) \left(\prod_{j=1}^r y_j^{sd(a_j)+\sigma(a_j)-1} \right) g(y)^s |dy|$$

has a meromorphic continuation to the whole complex plain with poles contained in

$$(6.9) \quad \bigcup_{1 \leq i \leq r} \left(-\frac{\sigma(a_i) + \mathbb{N}}{d(a_i)} \right).$$

Subcase 2.2 ($\tilde{f}_i(\tilde{p}) = 0$ for $i = 1, \dots, l$ and $\tilde{p} \in (0, 1)^{n-r}$). By the non-degeneracy condition, the fact that w is an analytic isomorphism on $(\mathbb{R}_+ \setminus \{0\})^n$ and (6.7), one gets for any $y = ((y_1, \dots, y_r), \tilde{y}) \in (\mathbb{R}_+ \setminus \{0\})^r \times (\mathbb{R}_+ \setminus \{0\})^{n-r} \cap \{y \mid \tilde{f}_1(\tilde{y}) = \dots = \tilde{f}_l(\tilde{y}) = 0\}$ that

$$\text{rank} \left[\frac{\partial f_{i,\tau'}}{\partial y_j}(w(y)) \right] = \text{rank} \begin{bmatrix} 0 & \dots & 0 & \prod_{j=1}^r y_j^{d(a_j)} \frac{\partial \tilde{f}_1}{\partial y_{r+1}}(\tilde{y}) & \dots & \prod_{j=1}^r y_j^{d(a_j)} \frac{\partial \tilde{f}_1}{\partial y_n}(\tilde{y}) \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \prod_{j=1}^r y_j^{d(a_j)} \frac{\partial \tilde{f}_l}{\partial y_{r+1}}(\tilde{y}) & \dots & \prod_{j=1}^r y_j^{d(a_j)} \frac{\partial \tilde{f}_l}{\partial y_n}(\tilde{y}) \end{bmatrix} = l.$$

Now this implies for $\tilde{y} = (y_{r+1}, \dots, y_n) \in (\mathbb{R}_+ \setminus \{0\})^{n-r} \cap \{y \mid \tilde{f}_1(\tilde{y}) = \dots = \tilde{f}_l(\tilde{y}) = 0\}$ that

$$\text{rank} \begin{bmatrix} \frac{\partial \tilde{f}_1}{\partial y_{r+1}} & \dots & \frac{\partial \tilde{f}_1}{\partial y_n} \\ \vdots & \dots & \vdots \\ \frac{\partial \tilde{f}_l}{\partial y_{r+1}} & \dots & \frac{\partial \tilde{f}_l}{\partial y_n} \end{bmatrix}(\tilde{y}) = l.$$

Then necessarily $l \leq n - r$, all $\tilde{f}_i$ are non-zero polynomials, and $\text{rank} \left[\frac{\partial \tilde{f}_i}{\partial y_j}(\tilde{p}) \right] = l$.

Since $\left[\frac{\partial \tilde{f}_i}{\partial y_j}(\tilde{p}) \right] = \left[\frac{\partial (\tilde{f}_i + O_i)}{\partial y_j}(p) \right]$, cf. (6.6), this last matrix having rank l implies that we can choose new coordinates $y' = (y_1, \dots, y_r, y'_{r+1}, \dots, y'_{r+l}, y_{r+l+1}, \dots, y_n)$ in a neighborhood V_p of p , with $\text{supp}(\Omega_p) \subset V_p$, such that

$$f_i(\omega(y')) = \prod_{j=1}^r (y'_j)^{d(a_j)} y'_{r+i}$$

for $i = 1, \dots, l$ and hence

$$|\mathbf{f}(w(y))|_{\mathbb{R}}^s = \left(\sum_{i=1}^l f_i^2(w(y')) \right)^{s/2} = \prod_{j=1}^r (y'_j)^{sd(a_j)} \left(\sum_{i=1}^l (y'_{r+i})^2 \right)^{s/2}.$$

Consequently

$$(6.10) \quad I_{\Omega_p}(s) = \int \tilde{\Omega}_p(y') \prod_{j=1}^r (y'_j)^{sd(a_j)+\sigma(a_j)-1} \left(\sum_{i=1}^l (y'_{r+i})^2 \right)^{s/2} |dy'|,$$

where the integration is performed over $[0, 1]^r \times U \times [0, 1]^{n-l-r}$, where U is a small neighborhood of the origin in $\mathbb{R}^l$. By applying Corollary 2, we get a meromorphic continuation to the whole complex plane for (6.10), with poles belonging to

$$\left(\bigcup_{1 \leq i \leq r} \left(-\frac{\sigma(a_i) + \mathbb{N}}{d(a_i)} \right) \right) \cup (-(l + \mathbb{N})).$$

Subcase 2.3 ($\tilde{f}_i(\tilde{p}) = 0$ for $i = 1, \dots, l$ and $\tilde{p} \in (0, 1]^{n-r}$ with at least one coordinate equal to 1). By renaming the variables we may assume that $\tilde{p}$ has the form $\tilde{p} = (p_{r+1}, \dots, p_{r+t}, 1, \dots, 1)$, with $p_{r+i} \in (0, 1)$ for each $i = 1, \dots, t$.

Claim. (1) If $0 \leq t \leq l-1$, there exists an index i_0 such that $f_{i_0}^*(p) \neq 0$.

(2) If $t \geq l$, then

$$\text{rank} \left[p_j(1-p_j) \frac{\partial f_i^*}{\partial y_j}(p) \right]_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}} = l.$$

Assuming the claim, we conclude when $0 \leq t \leq l-1$, by a similar argument as the one given for Case 1, that $I_{\Omega_p}(s)$ has a meromorphic continuation to $\mathbb{C}$ with poles contained in the set (6.9). When $t \geq l$, we can choose new coordinates $y' = (y_1, \dots, y_r, y'_{r+1}, \dots, y'_{r+l}, y_{r+l+1}, \dots, y_n)$ in a neighborhood V_p of p such that $y'_{r+i} = f_i^*(y)$ for $i = 1, \dots, l$. In this coordinate system $I_{\Omega_p}(s)$ has the form (6.10), and thus meromorphic continuation is obtained as in Subcase 2.2.

Proof of the claim. Set $I = \{i \mid p_i = 0\} = \{1, \dots, r\}$, $J = \{i \mid p_i = 1\} = \{r+t+1, \dots, n\}$ and $K = \{i \mid 0 < p_i < 1\} = \{r+1, \dots, r+t\}$. Set also

$$A = \{y \in [0, 1]^n \mid y_j = 0 \text{ if } j \in I, y_j = 1 \text{ if } j \in J, \text{ and } 0 < y_j < 1 \text{ if } j \in K\},$$

and

$$\hat{A} = \{z \in [0, 1]^n \mid z_j = 1 \text{ if } j \in I \cup J, \text{ and } 0 < z_j < 1 \text{ if } j \in K\}.$$

Given $y \in A$, we denote by $\hat{y}$ the element of $\hat{A}$ such that $y_k = \hat{y}_k$ for $k \in K$. Set $\Delta_K = \{\sum_{j \in K} \lambda_j a_j \mid \lambda_j \in \mathbb{R}_+ \setminus \{0\}\}$ and $S_K = \mathcal{L}^{-1}(\Delta_K)$ as before.

If $0 \leq t = \text{Card}(K) \leq l-1$, there exists an index i_0 , such that $\tilde{f}_{i_0}(p) \neq 0$, by the compatibility condition and (6.7). If $t = \text{Card}(K) \geq l$, the compatibility condition asserts, for

$$\begin{aligned} \hat{y} &= (1, \dots, 1, y_{r+1}, \dots, y_{r+t}, 1, \dots, 1) \\ &\in \{\hat{y} \in \hat{A} \mid (f_{1,\tau'} \circ w)(\hat{y}) = \dots = (f_{l,\tau'} \circ w)(\hat{y}) = 0\}, \end{aligned}$$

that

$$\text{rank} \left[\frac{\partial (f_{i,\tau'} \circ w)}{\partial y_j}(\hat{y}) \right]_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}} = \text{rank} \left[\frac{\partial (f_{i,\tau'} \circ w)}{\partial y_j}(\hat{y}) \right]_{\substack{1 \leq i \leq l \\ j \in K}} = l.$$

Note for the first equality that the restriction of ω to $\widehat{A}$ is a parametrization of S_K . By (6.6) and (6.7) this condition on y is equivalent, for

$$y = (0, \dots, 0, y_{r+1}, \dots, y_{r+t}, 1, \dots, 1) \in \{y \in A \mid f_1^*(y) = \dots = f_l^*(y) = 0\},$$

to

$$\text{rank} \left[\frac{\partial f_i^*}{\partial y_j}(y) \right]_{\substack{1 \leq i \leq l \\ j \in \bar{K}}} = l,$$

what can be rewritten as

$$l = \text{rank} \left[y_j(1 - y_j) \frac{\partial f_i^*}{\partial y_j}(y) \right]_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}} =$$

$$\text{rank} \begin{bmatrix} 0 & \dots & 0 & y_{r+1}(1 - y_{r+1}) \frac{\partial f_1^*}{\partial y_{r+1}}(y) & \dots & y_{r+t}(1 - y_{r+t}) \frac{\partial f_1^*}{\partial y_{r+t}}(y) & 0 & \dots & 0 \\ \vdots & & \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & y_{r+1}(1 - y_{r+1}) \frac{\partial f_l^*}{\partial y_{r+1}}(y) & \dots & y_{r+t}(1 - y_{r+t}) \frac{\partial f_l^*}{\partial y_{r+t}}(y) & 0 & \dots & 0 \end{bmatrix}.$$

This finishes the proof of the claim.

The third part of the theorem follow from (6.5), (6.8) and (6.10) by applying Lemma 1 and Corollary 2. $\square$

Finally we state in the following theorem the meromorphic continuation of

$$I(s) = I(s, \mathbf{f}, \Phi) = \int_{\mathbb{R}_+^n} \Phi(x) |\mathbf{f}(x)|_{\mathbb{R}}^s |dx|.$$

Recall the notation

$$\mathcal{P}(\xi) = \left\{ -\frac{\sigma(\xi) + k}{d(\xi)} \mid k \in \mathbb{N} \right\}.$$

for $\xi \in \mathfrak{D}(\Gamma(\mathbf{f}))$.

Theorem 6. *Let $\mathbf{f} : U(\subset \mathbb{R}^n) \rightarrow \mathbb{R}^l$, with $\mathbf{f}(0) = 0$, be an analytic mapping, strongly non-degenerate at the origin with respect to $\Gamma(\mathbf{f})$. There exists a neighborhood $V(\subset U)$ of the origin such that, if Φ is a smooth function with support contained in V , the following assertions hold:*

(1) *$I(s)$ converges and defines a holomorphic function on $\text{Re}(s) > \max\{-\gamma_0(\mathbf{f}), -l\}$, with*

$$\gamma_0(\mathbf{f}) = \min_{\xi \in \mathfrak{D}(\Gamma(\mathbf{f}))} \left\{ \frac{\sigma(\xi)}{d(\xi)} \right\},$$

(2) *$I(s)$ admits a meromorphic continuation to the whole complex plane with poles of order at most n , and the poles belong to*

$$\left(\bigcup_{\xi \in \mathfrak{D}(\Gamma(\mathbf{f}))} \mathcal{P}(\xi) \right) \cup (-(l + \mathbb{N})).$$

(3) *Let ρ be a positive integer and let s_0 be a candidate pole of $I(s)$, with $s_0 \notin -(l + \mathbb{N})$ (resp. $s_0 \in -(l + \mathbb{N})$). A necessary condition for s_0 to be a pole of $I(s)$ or order ρ , is that there exists a face τ of $\Gamma(\mathbf{f})$, of codimension ρ (resp. $\rho - 1$), such that $s_0 \in \mathcal{P}(\xi)$ for all facets of $\Gamma(\mathbf{f})$ containing τ .*

Proof. Pick a simplicial fan $\mathcal{F}_f$ subordinated to $\Gamma(f)$. By Lemma 3 there exists $b \in (\mathbb{R}_+ \setminus \{0\})^n$ such that $f \circ T_b$ is compatible with $\mathcal{F}_f$. Since the change of variables $x = T_b(y)$ does not affect the meromorphic continuation of $I(s)$, we may assume that f is compatible with the decoupage $\mathcal{L}^{-1}(\mathcal{F}_f)$. Parts (1) and (2) follow immediately from Theorem 5 by the decomposition

$$I(s) = \int_{\mathbb{R}_+^n} \Phi(x) |f(x)|_{\mathbb{R}}^s dx = \sum_{\Delta_\tau \in \mathcal{F}} I_{\Delta_\tau}(s).$$

Part (3) also follows from Theorem 5 by exactly the same argument as in the proof of [5, Théorème 6.1(iii)], using [5, Corollaire 2.2]. $\square$

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CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL INSTITUTO POLITÉCNICO NACIONAL, DEPARTAMENTO DE MATEMÁTICAS- UNIDAD QUERÉTARO, LIBRAMIENTO NORPONIENTE #2000, FRACC. REAL DE JURQUILLA. SANTIAGO DE QUERÉTARO, QRO. 76230, MÉXICO.

E-mail address: `eleon@math.cinvestav.mx`

UNIVERSITY OF LEUVEN, DEPARTMENT OF MATHEMATICS, CELESTIJNENLAAN 200 B, B-3001 LEUVEN (HEVERLEE), BELGIUM

E-mail address: `wim.veys@wis.kuleuven.be`

CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL INSTITUTO POLITÉCNICO NACIONAL, DEPARTAMENTO DE MATEMÁTICAS- UNIDAD QUERÉTARO, LIBRAMIENTO NORPONIENTE #2000, FRACC. REAL DE JURQUILLA. SANTIAGO DE QUERÉTARO, QRO. 76230, MÉXICO.

E-mail address: `wazuniga@math.cinvestav.edu.mx`